SEQUENTIAL ESTIMATION IN RANDOM FIELDS

BY

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Abstract. Absolute continuity of measures $\mu_\theta$, generated by a random field and a Markov stopping set $\tau$, is considered. The analogue of Sudakov lemma is proved. Moreover, with some additional assumptions on $\tau$, the author proves the absolute continuity of the measure $\mu_\theta$ with respect to the measure $\mu_{\theta_0}$ on the $\sigma$-algebra $F_\tau$.

Results obtained in the paper make it possible to characterize efficient (in the sense of Cramer-Rao-Wolfowitz inequality) sequential plans for some random fields.

INTRODUCTION

Suppose we observe a realization $w_t \in W$ of the stochastic process $X_t$ till a random Markov moment $\tau$ defined on the space of realizations $W$. The process $X_t$ generates in the space $W$ the measure $\mu_\theta$ depending on the unknown parameter $\theta$.

In papers [8], [11], [13], and [16] it was proved, under some additional assumptions, that the distribution $m_\theta$ of the random vector $(\tau, S(\tau))$ is absolutely continuous with respect to $m_{\theta_0}$ ($S$ is a sufficient statistic for the parameter $\theta$, defined on the space of realizations $w_s, s \in [0, \tau]$). It was given the form of the density function $dm_\theta/dm_{\theta_0}$. Moreover, in [3], it was proved that $\mu_\theta$ is absolutely continuous with respect to $\mu_{\theta_0}$ on the $\sigma$-algebra $F_\tau$. Thus $(\tau, S(\tau))$ is sufficient for the parameter $\theta$. These theorems made it possible to characterize efficient, in the sense of the Cramer–Rao–Wolfowitz inequality, sequential plans for some classes of stochastic processes (see [6], [9], [11]–[15], [17], and [18]).

In this paper we consider the problem of sequential estimation for random fields. We define a Markov stopping set $\tau$ (Definition 1) as a random compact set on which we observe the realization $w_t \in W$ of the random field $X_s, s \in R^2$. It is proved that the distribution $m_\theta$ of the random vector $(p(\tau), S(\tau))$ is absolutely continuous with respect to $m_{\theta_0}$ ($S(K)$ is a sufficient statistic defined on the set of realizations $w_s, s \in K$, where $K$ is a compact set.
contained in \( R^2 \), \(|\tau|\) is the Lebesgue measure of the set \( \tau \), and \( p \) is some Borel real function. The form of the density function \( dm_{\theta}/dm_{\theta_0} \) is given (Theorem 1). Moreover, it is proved (Theorem 2) that the measure \( \mu_\theta \), generated by the random field \( X_\tau \), is absolutely continuous with respect to \( \mu_{\theta_0} \) on the \( \sigma \)-algebra \( F_\tau \) generated by the Markov stopping set \( \tau \). The statistic \( (p(|\tau|), S(\tau)) \) is, therefore, sufficient for the parameter \( \theta \). Although the thesis of Theorem 1 may be deduced from Theorem 2, they are formulated separately because the first theorem is proved with less restrictive assumptions on \( \mathcal{H} \) and \( \tau \) (without conditions (5) and (6)). Theorems 1 and 2 allow to prove the Cramer–Rao–Wolfowitz inequality. Afterwards, we consider the random field of the Ornstein–Uhlenbeck type, which is a generalization of the stochastic process examined in [11]. For this random field, using results of Stefanov [15], we have proved theorems characterizing efficient sequential plans.

The problem of efficient sequential plans for a random Wiener field and a Poisson field is reduced to the problem of efficient sequential plans for the Wiener process and the Poisson homogeneous process.

In Sections I and II we prove some auxiliary lemmas extending the well-known results from the 1-dimensional time-parameter case to the 2-dimensional case, which seems to be new in the sense of Definition 1.

I. SUDAKOV LEMMA FOR RANDOM FIELDS

Let us consider a random field \( X_s, s \in R^2 \). Let \( W \) be a set of realizations of a random field \( X_s \). This random field generates the measure \( \mu_\theta \) defined on \((W, F)\), where \( F \) is a \( \sigma \)-algebra of subsets of \( W \), generated by the cylindrical sets. \( \theta \in A \subset R^l \) is a parameter.

Let \( \mathcal{X} \) be a family of compact sets \( K \) contained in \( R^2 \) satisfying the following condition:

1. There exists a countable family of compact sets \( P_i(n) \) with a diameter \( \delta [P_i(n)] \to 0 \) for \( n \to \infty \) such that for every \( K \in \mathcal{X} \) there exists a finite minimal unique covering \( C_n \) of \( K \) by the sets \( P_i(n) \) for which \( C_{n+1} \subset C_n \) and \( \bigcap_{n=1}^{\infty} C_n = K \).

**Remark 1.** Condition (1) is satisfied if we take

\[
P_i(n) = [x^n_{k_i}, x^n_{k_i+1}] \times [y^n_{l_i}, y^n_{l_i+1}],
\]

\[
(x^n_{k_i}, y^n_{l_i}) = \left( \frac{k_i}{2^n}, \frac{l_i}{2^n} \right), \quad k_i, l_i = 0, \pm 1, \pm 2, \ldots \quad (i = 1, 2, \ldots).
\]

By \( F \) we denote a \( \sigma \)-algebra of subsets of \( W \) generated by the
cylindrical sets \( \{ w: (w(s_1), w(s_2), \ldots, w(s_n)) \in B \} \), where \( B \in \mathcal{B}_{\mathbb{R}^n} \) and \( s_i \in \mathcal{K} \) \((i = 1, 2, \ldots, n)\).  

\( \mu_\theta^k \) is a restriction of the measure \( \mu_\theta \) to the \( \sigma \)-algebra \( F_k \).  

Assume that \( \mu_\theta^k \) is absolutely continuous with respect to the measure \( \mu_\theta^0 \) and that  

\[
\frac{d\mu_\theta^k}{d\mu_\theta^0} = g(p(|K|), S(K, w), \theta, \theta_0),
\]

where \( g \) is a continuous function; \( S(K, \cdot) \) is a mapping which is \( F_k \)-measurable and, for every nonincreasing sequence of compact sets \( K_n \searrow K \), \( S(K_n, w) \to S(K, w) \mu_\theta \) a.s. for every \( \theta \in \Lambda \); \( p \) is a Borel function from \( \mathbb{R}_+ \) into \( R; |K| \) is the Lebesgue measure of the compact set \( K \).  

**Definition 1** (see [4]). A Markov stopping set \( T \) is the mapping \( z: \mathbb{W} \to \mathcal{X} \) such that \( \{ w: z(w) \leq K \} \in F_k \) for every \( K \in \mathcal{X} \).  

**Remark 2.** Let \((\Omega, F)\) be a measurable space. \( F_z, z \in \mathbb{R}^2 \), is a family of sub-\( \sigma \)-algebras of \( F \) which satisfy some other conditions specified in [5] and [10]. These papers define a Markov stopping point as a mapping \( Z: \Omega \to \mathbb{R}_+ \cup \{ \infty \} \) for which \( \{ w: Z(\omega) \leq z \} \in F_z \) \((\leq\) is a relation of partial ordering in \( \mathbb{R}^2 \)). If \( \mathcal{X} = \{ K_z, z \in \mathbb{R}^2 \} = \{ [0, z], z \in \mathbb{R}_+^2 \}, F_z = F_{K_z} \) and \( (w) = [0, Z(w)] \), then \( \tau \) is a Markov stopping set \([0, z] = \{ s \in \mathbb{R}_+: s \leq z \} \).  

**Lemma 1.** For every Markov stopping set \( \tau \) there exists a sequence of Markov stopping sets \( \tau_n \) such that \( \tau_{n+1}(w) \subset \tau_n(w) \) and \( \tau_n(w) \searrow \tau(w) \).  

**Proof.** Since \( \tau(w) \in \mathcal{X} \) for every \( w \in \mathbb{W} \), let \( \tau_n(w) = C_n(w), \) where \( C_n(w) \in \mathcal{X} \) is the unique covering \( C_n \in \mathcal{X} \) of the set \( \tau(w) \). Let \( K \in \mathcal{X} \). We have  

\[
\{ w: \tau_n(w) \leq K \} = \bigcup_{C_n \in \mathcal{X}} \{ w: \tau(w) \leq C_n \} \in F_K, \quad \text{where } C_n \in \mathcal{X}.
\]

Hence \( \tau_n \) is a Markov stopping set. By (1) we infer that \( \tau_{n+1}(w) \subset \tau_n(w) \) and \( \tau_n(w) \searrow \tau(w) \) for every \( w \in \mathbb{W} \).  

**Lemma 2.** \( \{ w: \tau_n(w) = C_n^0 \} \in F_{C_n^0} \) for every \( C_n^0 \in \mathcal{X} \).  

**Proof.** We have  

\[
\{ w: \tau_n(w) = C_n^0 \} = (\{ w: \tau(w) \leq C_n^0 \} \cap \bigcap_{C_n \in \mathcal{X}} \{ w: \tau(w) \geq C_n \}) \in F_{C_n^0}, \quad \text{where } C_n \subset C_n^0.
\]

**Lemma 3.** The mapping \( S(\cdot, \cdot): \mathbb{W} \to \mathbb{R}^r \) is \( F \)-measurable.  

**Proof.** Since  

\[
S(\tau_n(w), w) = \sum_{C_n \in \mathcal{X}} S(C_n, w) 1_{[w \in \tau_n(w) = C_n]}(w),
\]
the mapping \( S(\tau_n(\cdot), \cdot) \) is \( F \)-measurable. By (2) and Lemma 1 we have
\[
\lim_{n \to \infty} S(\tau_n(w), w) = S(\tau(w), w) \mu_\theta \text{ a.s. for every } \theta \in A.
\]

**Remark 3.** The mapping \([\tau]: W \to R_+\) is \( F \)-measurable.

For any Markov stopping set \( \tau \) we define the mapping \( f = (p(|\tau|), S(\tau, \cdot)) \) and the measure \( m_\theta \) on \((R \times R', \mathcal{B}_R \times \mathcal{B}_R')\) by the formula
\[
m_\theta(B) = \mu_\theta\left( \left\{ w: (p(|\tau(w)|), S(\tau(w), w)) \in B \right\} \right)
\]
for every \( B \in \mathcal{B}_U \) and \( u = (p(u), s(u)) \in R \times R' \).

**Theorem 1.** The measure \( m_\theta \) is absolutely continuous with respect to the measure \( m_{\theta_0} \) and
\[
\frac{dm_\theta}{dm_{\theta_0}}(u) = g(p(u), s(u), \theta, \theta_0).
\]

**Proof.** Let \( p^n_z, z = 1, 2, \ldots, \) denote the values of the function \( p(|\tau_n|) \), and
\[
Y_{C_n, p^n_z} = \left\{ w: \tau_n(w) = C_n \land p(|\tau_n(w)|) = p^n_z \right\}.
\]

We define the measure \( m^n_{\theta, z} \) by
\[
m_{\theta, z}(B) = m_\theta\left( B \cap \left[ (p(u), s(u)): p(u) = p^n_z \right] \right) \text{ for every } B \in \mathcal{B}_U,
\]
where \( m^n_{\theta} \) is the probability measure defined by (3) for the Markov stopping set \( \tau_n \).

We have
\[
m_{\theta, z}(B) = \sum_{C_n \in \mathcal{X}} \mu_\theta\left( \left\{ w: (p^n_z, S(C_n, w)) \in B \right\} \right) \cap Y_{C_n, p^n_z}.
\]

Introducing the function \( S_n: W \to R \times R' \) such that, for every \( w \in W \), \( S_n(w) = (p^n_z, S(C_n, w)) \), we get
\[
m_{\theta, z}(B) = \sum_{C_n \in \mathcal{X}} \mu_\theta\left( S^{-1}_n(B) \cap Y_{C_n, p^n_z} \right) = \sum_{C_n \in \mathcal{X}} \int g(p^n_z, S(C_n, w)) d\mu_\theta_0(w)
\]
\[
= \int_{I_1} g(p^n_z, S(\tau_n(w), w)) d\mu_\theta_0(w) = \int_{I_2} g(p^n_z, s, \theta, \theta_0) d\mu^n_{\theta_0, z},
\]
where
\[
I_1 = S^{-1}_n(B) \cap Y_{C_n, p^n_z}, \quad I_2 = S^{-1}_n(B) \cap \left\{ w: p(|\tau_n(w)|) = p^n_z \right\}.
\]

We have thus proved that \( m^n_{\theta, z} \ll m^n_{\theta_0, z} \). Hence
\[
m_{\theta}(B) = \sum_z m^n_{\theta}(B \cap \left[ (p, s): p = p^n_z \right]) = \sum_z m^n_{\theta, z}(B)
\]
\[
= \sum_{z \in \mathcal{B}} \int g(p^n_z, s, \theta, \theta_0) d\mu^n_{\theta_0, z} = \int g(p, s, \theta, \theta_0) d\mu^n_{\theta_0}.
\]
The theorem is proved for a Markov stopping set $\tau_n$.

By (2) we have
\[
\lim_{n \to \infty} S(\tau_n(w), w) = S(\tau(w), w) \mu_0 \quad \text{a.s.}
\]

Let $h$ be a real function defined on the set $U$. Assuming $h$ to be continuous and bounded, by the Lebesgue theorem we have
\[
\lim_{n \to \infty} \int_U h(u) m_n^u(du) = \int_U h(p(\tau_n(w))), S(\tau_n(w), w)) d\mu_0(w)
\]
\[
= \int_U h(p(\tau(w))), S(\tau(w), w)) d\mu_0(w) = \int_U h(u) m_0(du).
\]

Hence the sequence of measures $m_n^u$ is weakly converging to the measure $m_0$.

Now we use the following lemma (see [8]):

Let $m_n$ be a sequence of probability measures, weakly converging to the measure $m$, and let $g(p, s)$ be a continuous nonnegative function. Then the sequence of measure $m_n$, having densities $g(p, s)$ with respect to the measure $m_n$, is weakly converging to the measure $m'$ with the density $g(p, s)$ with respect to the measure $m$.

By this lemma the proof of Theorem 1 is completed.

II. ABSOLUTE CONTINUITY OF THE MEASURE $\mu_0$ ON THE $\sigma$-ALGEBRA $F$.

We assume that for every $K_1$ and $K_2$ belonging to $\mathcal{H}$, $K_1 \subset K_2$, there exist an $n$ and a finite minimal covering $C_n \in \mathcal{H}$ of $K_1$ by the sets $P_k(n)$ such that

(5) $K_1 \subseteq C_n \subseteq K_2$

and

(6) $\{w: \tau(w) = K\} \in F_K$ for every $K \in \mathcal{H}$.

**Lemma 4.** If the family $\mathcal{H}$ satisfies (5), then $\tau$ is a Markov stopping set if and only if $\{w: \tau(w) \subset K\} \in F_K$ for every $K \in \mathcal{H}$.

**Proof.** Let $\tau$ be a Markov stopping set. By (5) we have
\[
\{w: \tau(w) \subset K\} = \bigcup_{n} \bigcup_{C_n \in \mathcal{H}} \{w: \tau(w) \subseteq C_n\},
\]
where $C_n \subset K$, $\{w: \tau(w) \subseteq C_n\} \in F_{C_n} \subset F_K$.

So $\{w: \tau(w) \subset K\} \in F_K$ for every $K \in \mathcal{H}$.

Now let $\{w: \tau(w) \subset K\} \in F_K$ for every $K \in \mathcal{H}$. Then $\{w: \tau(w) \subset C_n\} \in F_{C_n}$ for every $C_n \in \mathcal{H}$ and $\{w: \tau(w) \subseteq K\} = \bigcap_n \{w: \tau(w) \subset C_n\}$.

Since $F_{C_n} \subset F_K$, we have $\bigcap_n \{w: \tau(w) \subset C_n\} \in F_K$.
Definition 2. Let $\tau$ be a Markov stopping set. By $F_\tau$ we denote a $\sigma$-algebra of the sets $V \in F$ such that $V \cap \{w: \tau(w) \leq K\} \in F_K$ for every $K \in \mathcal{K}$.

Lemma 5. If the family $\mathcal{K}$ satisfies (5), then $V \in F_\tau$ if and only if $\{(w: \tau(w) \leq K) \cap V\} \in F_K$ for every $K \in \mathcal{K}$.

Proof. Let $V \in F_\tau$. We have $V \cap \{w: \tau(w) \leq K\} \in F_K$ for every $K \in \mathcal{K}$.

By (5),

$$V \cap \{w: \tau(w) \leq K\} = V \cap \bigcup_{n} \bigcup_{C_n \in \mathcal{K}} \{w: \tau(w) \leq C_n\}$$

$$= \bigcup_{n} \bigcup_{C_n \in \mathcal{K}} \{V \cap \{w: \tau(w) \leq C_n\}\} \in F_K,$$

where $C_n \subset K$.

Now assume that $V \cap \{w: \tau(w) \leq K\} \in F_K$ for every $K \in \mathcal{K}$. We have

$$V \cap \{w: \tau(w) \leq K\} = V \cap \bigcap_n \{w: \tau(w) \leq C_n\} = \bigcap_n \{V \cap \{w: \tau(w) \leq C_n\}\}$$

$$\in F_K \ni V \cap \{w: \tau(w) \leq K\} \in F_K,$$

hence $V \cap \{w: \tau(w) \leq K\} \in F_K$ for every $K \in \mathcal{K}$.

Lemma 6. Let $\tau$ be a Markov stopping set and $\tau_n$ be a sequence of Markov stopping sets defined as in Lemma 1. If the family $\mathcal{K}$ satisfies (5), then $F_{\tau_{n+1}} \subset F_{\tau_n}$, $F_\tau = \bigcap_n F_{\tau_n}$.

Proof. We have:

$$\tau(w) \leq \tau_{n+1}(w) \leq \tau_n(w) \quad \text{for every } w \in W,$$

$$\{w: \tau_n(w) \leq K\} \subset \{w: \tau_{n+1}(w) \leq K\} \quad \text{for every } K \in \mathcal{K},$$

$$\{w: \tau_n(w) \leq K\} \subset \{w: \tau(w) \leq K\} \quad \text{for every } n \text{ and } K \in \mathcal{K}.$$

Let $V \in F_{\tau_{n+1}}$. We have

$$V \cap \{w: \tau_n(w) \leq K\} = \{V \cap \{w: \tau_{n+1}(w) \leq K\} \cap \{w: \tau_n(w) \leq K\}\} \in F_K.$$ 

So $V \in F_{\tau_n}$ and $F_{\tau_{n+1}} \subset F_{\tau_n}$.

Let $V \in F_\tau$, which means that, for every $K \in \mathcal{K}$,

$$V \cap \{w: \tau(w) \leq K\} \in F_K,$$

$$V \cap \{w: \tau_n(w) \leq K\} = V \cap \{w: \tau(w) \leq K\} \cap \{w: \tau_n(w) \leq K\} \in F_K.$$ 

Hence, for every $n$, $V \in F_{\tau_n}$ and $F_\tau \subset \bigcap_n F_{\tau_n}$.

Now let $V \in \bigcap_n F_{\tau_n}$. We have

$$V \cap \{w: \tau(w) \subset K\} = V \cap \{w: \tau_n(w) \subset K\} = \bigcup_n \{V \cap \{w: \tau_n(w) \subset K\}\} \in F_K.$$ 

By Lemma 5 we see that $V \in F_\tau$. So $\bigcap_n F_{\tau_n} \subset F_\tau$ and $F_\tau = \bigcap_n F_{\tau_n}$. 


Lemma 7. If $\tau$ is a Markov stopping set for which (6) is satisfied, then
\[ V \in F_{\tau} \Rightarrow \{ \omega: \tau(\omega) = K \} \in F_K \quad \text{for every } K \in \mathcal{K}. \]

Proof. We have
\[ V \cap \{ \omega: \tau(\omega) = K \} = (V \cap \{ \omega: \tau(\omega) \leq K \} \cap \{ \omega: \tau(\omega) = K \}) \in F_K. \]

Lemma 8. If $\tau$ is a Markov stopping set, then $\{ \omega: \tau_n(\omega) = C_n \} \in F_{\tau_n}$.

Proof. Let $K \in \mathcal{K}$. We have
\[ \{ \omega: \tau_n(\omega) = C_n \} \cap \{ \omega: \tau_n(\omega) \leq K \} = \{ \omega: \tau_n(\omega) = C_n \land C_n \leq K \} \]
\[ = \begin{cases} \emptyset & \text{if } C_n \notin K, \\ \{ \omega: \tau_n(\omega) = C_n \} & \text{if } C_n \in K. \end{cases} \]

Lemma 9. If the family $\mathcal{K}$ and a Markov stopping set $\tau$ satisfy (5) and (6), then the mapping $S(\tau(\cdot), \cdot): W \to \mathbb{R}$ is $F_{\tau}$-measurable.

Proof. We will prove that $S(\tau_n(\cdot), \cdot)$ is $F_{\tau_n}$-measurable.

Let $K \in \mathcal{K}$ and $x \in \mathbb{R}$. We have
\[ \{ \omega: S(\tau_n(\omega), \omega) < x \} \cap \{ \omega: \tau_n(\omega) \leq K \} \]
\[ = \bigcup_{C_n \subseteq K} \{ \omega: S(\tau_n(\omega), \omega) < x \} \cap \{ \omega: \tau_n(\omega) = C_n \} \cap \{ \omega: \tau_n(\omega) \leq K \} \]
\[ = \bigcup_{C_n \subseteq K} \{ \omega: S(\tau_n(\omega), \omega) < x \land \tau_n(\omega) = C_n \land \tau_n(\omega) \leq K \} \]
\[ = \bigcup_{C_n \subseteq K} \{ \omega: S(\tau_n(\omega), \omega) < x \land \tau_n(\omega) = C_n \} \]
\[ = \bigcup_{C_n \subseteq K} \{ \omega: S(C_n, \omega) < x \} \cap \{ \omega: \tau_n(\omega) = C_n \} \in F_K, \quad C_n \subseteq K. \]

Now we will show that $S(\tau(\cdot), \cdot)$ is $F_{\tau}$-measurable. We can assume that the $\sigma$-algebras $F_K$ are $\sigma$-complete $\sigma$-algebras for every $\theta \in A$ and every $K \in \mathcal{K}$. We have
\[ \{ \omega: S(\tau(\omega), \omega) < x \} \cap \{ \omega: \tau(\omega) \leq K \} \]
\[ = \left( \{ \omega: S(K, \omega) < x \} \cap \{ \omega: \tau(\omega) = K \} \right) \cup \\
\[ \quad \cup \{ \omega: S(\tau(\omega), \omega) < x \} \cap \{ \omega: \tau(\omega) \leq K \}. \]

By (6) and $F_K$-measurability of the mapping $S(K, \cdot)$ we infer that
\[ \{ \omega: S(K, \omega) < x \} \cap \{ \omega: \tau(\omega) = K \} \in F_K. \]

Since $S(\tau_n(\omega), \omega) \to S(\tau(\omega), \omega) \mu_0$ a.s. as $n \to \infty$, there exist sets $C_\theta$ and $D_\theta \subseteq W$ of the measure $\mu_0$ equal to zero such that
\[ \{ \omega: S(\tau(\omega), \omega) < x \} \cup D_\theta = \bigcup_{C_n \subseteq K \atop \frac{1}{N} \leq \frac{1}{N}} \{ \omega: S(\tau_n(\omega), \omega) < x - \frac{1}{N} \} \cup C_\theta. \]
Hence \( w : S(\tau(w), w) < x \) \( \cap \{ w : \tau(w) \in K \} \in F_K \) if and only if
\[
\bigcup_{l \in N} \bigcap_{n > N} \{ w : S(\tau_n(w), w) < x - \frac{1}{l} \} \cap \{ w : \tau(w) \in K \} \in F_K.
\]
But
\[
\bigcup_{l \in N} \bigcap_{n > N} \{ w : S(\tau_n(w), w) < x - \frac{1}{l} \} \cap \{ w : \tau(w) \in K \} \in F_K.
\]
\[
= \bigcup_{l \in N} \bigcap_{n > N} \{ w : S(\tau_n(w), w) < x - \frac{1}{l} \} \cap \bigcup_{N \in N} \{ w : \tau_n(w) \in K \}
\]
\[
= \left( \bigcup_{l \in N} \bigcap_{n > N} \{ w : S(\tau_n(w), w) < x - \frac{1}{l} \} \cap \{ w : \tau_n(w) \in K \} \right) \in F_K,
\]
\( S(\tau_n(\cdot), \cdot) \) being \( F_{\tau_n} \)-measurable.

By \( \mu_\theta^* \) we denote the restriction of the measure \( \mu_\theta \) to the \( \sigma \)-algebra \( F_\tau \).

**Theorem 2.** Assume that for every \( K \in \mathcal{K} \) the measure \( \mu_\theta^K \) is absolutely continuous with respect to the measure \( \mu_{\theta_0}^K \) and that
\[
d\mu_\theta^K(w) = g(K, w, \theta, \theta_0),
\]
where \( g \) is such that, for every nonincreasing sequence \( K_n \searrow K \) \( (K_n, K) \in \mathcal{K} \),
\[
\lim_{n \to \infty} g(K_n, \cdot, \theta, \theta_0) = g(K, \cdot, \theta, \theta_0) \mu_{\theta_0} \text{ a.s.}
\]
If \( \tau \) is a Markov stopping set satisfying (5) and (6), then \( \mu_\theta^\tau \) is absolutely continuous with respect to \( \mu_{\theta_0}^\tau \) and
\[
d\mu_\theta^\tau(w) = g(\tau, w, \theta, \theta_0).
\]

**Proof.** First we prove that \( \mu_\theta^\tau \ll \mu_{\theta_0}^\tau \). We have, for every \( B \in F_{\tau_n} \),
\[
\mu_\theta^\tau(B) = \sum_{C_n \in \mathcal{K}} \mu_\theta^\tau(B \cap \{ w : \tau_n(w) = C_n \})
\]
\[
= \sum_{C_n \in \mathcal{K}} \int_{I_1} g(C_n, w, \theta, \theta_0) d\mu_\theta = \int_B g(\tau_n(w), w, \theta, \theta_0) d\mu_{\theta_0},
\]
where \( I_1 = B \cap \{ w : \tau_n(w) = C_n \} \). Therefore
\[
d\mu_\theta^\tau(w) = g(\tau_n(w), w, \theta, \theta_0).
\]

Further we use Döhler's idea [3]. Let us define the sequence \((\xi_n, F_{\tau_n})\)
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This sequence is a martingale. Since

\[
\lim_{n \to \infty} E_{\mu_{\theta_0}}^{T_n} \xi_n = 1 > -\infty,
\]
we see that \( d\mu_{\theta_0}/d\mu_{\theta_0}^{T_n} \) is covering in \( L_1 \) and with probability 1 (see [7]). By the assumptions on \( g \) we infer that

\[
\lim_{n \to \infty} \frac{d\mu_{\theta_0}^{T_n}}{d\mu_{\theta_0}} = g(\tau, \cdot, \theta, \theta_0).
\]

Looking at \( g(\tau(\cdot), \cdot, \theta, \theta_0) \) as \( S(\tau(\cdot), \cdot) \), we conclude from Lemma 9 that \( g(\tau, \cdot, \theta, \theta_0) \) is \( F_\tau \)-measurable, which gives the thesis.

**Remark 4.** The thesis of Theorem 1 may be obtained as a consequence of Theorem 2, but Theorem 1 is proved with less restrictive assumptions on \( X \) and \( \tau \) (without conditions (5) and (6)). Therefore we have formulated both these theorems separately.

**III. THE CRAMER-RAO-WOLFOWITZ INEQUALITY**

In this section we assume that, for every \( K \in X, \mu_{\theta}^K \ll \mu_{\theta_0}^K \) and that the density function takes the form (2).

**Definition 3.** Let \( h: A \to R, h \neq \text{const.} \) An estimator \( f \) of the parameter \( h(\theta) \) is a function \( f: U \to R \) which is \( \mathcal{B}_U \)-measurable.

**Remark 5.** Theorem 2 implies that the random vector \((p(\|\tau\|), S(\tau))\) is a sufficient statistic and we can restrict ourselves to estimators \( f \) being only a function of \( u \).

**Definition 4.** By a sequential plan we call the pair \((\tau, f(p(\|\tau\|), S(\tau)))\), where \( \tau \) is a Markov stopping set and \( f(p(\|\tau\|), S(\tau)) \) is an estimator of the parameter \( h(\theta) \).

**Theorem 3.** Let \((\tau, f(p(\|\tau\|), S(\tau)))\) be a sequential plan, where \( f(p(\|\tau\|), S(\tau)) \) is an unbiased estimator for the function \( h(\theta) \) i.e.

\[
E_{\mu_{\theta}} f(p(\|\tau\|), S(\tau)) = h(\theta) \quad \text{and} \quad \text{Var}_{\mu_{\theta}} f(p(\|\tau\|), S(\tau)) < \infty.
\]

We also assume that the density function \( g(u, \theta, \theta_0) \) satisfies some regularity conditions which guarantee the following equalities:

\[
\int_U \nabla\theta (g(u, \theta, \theta_0)) m_{\theta_0}(du) = 0,
\]

\[
\int_U \nabla\theta (f(u)) g(u, \theta, \theta_0) m_{\theta_0}(du) = \int_U f(u) \cdot \nabla\theta (\ln g(u, \theta, \theta_0)) m_{\theta_0}(du).
\]

Then

\[
\text{Var}_{\theta} f(p(\|\tau\|), S(\tau)) \geq (\nabla\theta(h))^* I^{-1}(\theta)(\nabla\theta(h))^*.
\]
Equality in (11) holds at some \( \theta \) if and only if
\[
(12) \quad f(u) = (\nabla_\theta h)^{-1}(-\ln g(u, \theta, \theta_0)) \text{ is a vector-row, } I(\theta) = E_\theta V^*(\cdot, \theta) V(\cdot, \theta), \text{ and } \ast \text{ denotes the transposition of a matrix.}
\]

The proof of this theorem, being analogous to that in [1], is omitted.

**Definition 5.** A sequential plan \( \tau, f(p(\tau)), S(\tau) \) is said to be efficient at \( \theta \) if (12) holds at this \( \theta \).

**Definition 6.** A sequential plan \( \tau, f(p(\tau)), S(\tau) \) is efficient for \( \theta \in A \) if (12) holds for all \( \theta \in A \). In this case the estimator \( f \) is called efficient, and the function \( h(\theta) \) — efficiently estimable.

**IV. EFFICIENT SEQUENTIAL PLANS FOR RANDOM FIELDS OF THE ORNSTEIN-UHLENBECK TYPE**

Let \( X(t_1, t_2), t_1, t_2 \geq 0 \), be a homogeneous Gaussian random field with the mean value \( \theta \) and correlation function
\[
R((t_1, t_2, t_1 + h_1, t_2 + h_2)) = \exp(-\alpha|t_1| - \beta|h_2|).
\]

The measure generated by this field is denoted by \( \mu_\theta \). This measure is defined on the space \( (W, F) \), where \( W \) is the set of continuous functions \( w \) defined on \( R^2 \).

By \( K_t \) we shall denote the rectangle \([0, t] \times [0, t] \), \( t \geq 0 \).

The measure \( \mu_{\theta_0}^{K_t} \) is absolutely continuous with respect to the measure \( \mu_\theta \) for \( \theta_0 = 0 \) and in [19] it is derived that
\[
(13) \quad \frac{d\mu_{\theta_0}^{K_t}}{d\mu_0}(w) = \exp \left( \frac{\theta}{4} S(K_t, w) - \frac{\theta^2}{8} (\alpha t + 2)(\beta t + 2) \right),
\]

where
\[
S(K_t, w) = w(0, 0) + w(t, 0) + w(0, t) + w(t, t) + \frac{\alpha}{0} \int_0^t w(u, 0) du +
\]
\[
+ \frac{\beta}{0} \int_0^t w(0, v) dv + \frac{\beta}{0} \int_0^t w(t, v) dv + \frac{\alpha}{0} \int_0^t \int_0^t w(u, v) dudv.
\]

Further, we assume that \( X(0, 0, 0) = 0 \) with probability 1. Under this condition the random variable \( X(t_1, t_2) \) has the normal distribution
\[
N(\theta(1 - \exp(-\alpha|t_1| - \beta|t_2|)), 1 - \exp(-2\alpha|t_1| - 2\beta|t_2|)).
\]

The considered random field generates the measure \( \mu_\theta \) on the space
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$(W, F)$, where $W$ is the set of continuous functions $w$ defined on $R^2$, for which $w(0, 0) = 0$.

It is easy to see that the measure $\mu_{0}^{K_t}$ is absolutely continuous with respect to the measure $\mu_{0}$ for $\theta_0 = 0$ and the density function takes the form

$$
\frac{d\mu_{0}^{K_t}}{d\mu_{0}}(w) = \exp \left( \frac{\theta}{4} S(K_t, w) - \frac{\theta^2}{2} p(|K_t|) \right),
$$

where

$$
S(K_t, w) = w(t, 0) + w(0, t) + w(t, t) + \alpha \int_0^t w(u, 0) du + \alpha \int_0^t w(u, t) du + \beta \int_0^t w(0, u) du + \beta \int_0^t w(t, u) du + \alpha \beta \int_0^t \int_0^t w(u, v) du dv,
$$

$$
p(|K_t|) = \frac{\alpha + \beta}{2} t + \frac{\alpha \beta}{4} t^2.
$$

Remark 6. If this random field varies along one axis only, then it is identical to the process considered in [11].

Let $\mathcal{K} = \{K_t, t \geq 0\}$. By $\mathcal{F}$ we denote the class of Markov stopping sets $\tau$ with respect to the family of $\sigma$-algebras $\{\mathcal{F}_t\}_{t \geq 0}$. Let us observe that the Markov stopping set $\tau$ must not be defined by $(S(K_t), p(|K_t|))$ only.

Our aim is to examine the properties of efficient sequential plans $(\tau, f(p(|\tau|), S(\tau)))$ for $\tau \in \mathcal{F}$.

IV.1. The Wald identities. We assume that the function $\psi: U \times A \to R$, $m_\theta$-integrable for every $\theta \in A$, satisfies some regularity conditions which guarantee that

$$
\frac{d}{d\theta} \int_u \psi(p(\theta), s(u), \theta) g(u, \theta) m_\theta(du) = \int_u \frac{d}{d\theta} (\psi(u, \theta) g(u, \theta)) m_\theta(du).
$$

We then have

$$
E_\theta \psi(p(|\tau|), S(\tau), \theta) \frac{d}{d\theta} \ln g(p(|\tau|), S(\tau), \theta)
$$

$$
= \frac{d}{d\theta} E_\theta \psi(p(|\tau|), S(\tau), \theta) - E_\theta \frac{d}{d\theta} \psi(p(|\tau|), S(\tau), \theta).
$$

If $\psi(p(|\tau|), S(\tau), \theta) = 1$, we get the first Wald identity:

$$
\frac{1}{4} E_\theta S(\tau) = \theta E_\theta p(|\tau|).
$$
Let
\[ \psi(p(|\tau|), S(\tau), \theta) = \frac{d}{d\theta} \ln g(p(|\tau|), S(\tau), \theta). \]

Then
\[ \mathbb{E}_\theta \left[ \frac{1}{4} S(\tau) - \theta p(|\tau|) \right]^2 = \mathbb{E}_\theta p(|\tau|). \quad (18) \]

Let \( \psi(p(|\tau|), S(\tau), \theta) = S(\tau) \). Then
\[ \mathbb{E}_\theta S(\tau) \left[ \frac{1}{4} S(\tau) - \theta p(|\tau|) \right] = \frac{d}{d\theta} \mathbb{E}_\theta S(\tau). \quad (19) \]

So we have
\[ \text{Var}_\theta S(\tau) = 16 \mathbb{E}_\theta p(|\tau|) + 32 \frac{d}{d\theta} \mathbb{E}_\theta p(|\tau|) + 16 \theta^2 \text{Var}_\theta p(|\tau|). \quad (20) \]

Putting \( \psi(p(|\tau|), S(\tau), \theta) = p(|\tau|) \), we obtain
\[ \mathbb{E}_\theta p(|\tau|) \left[ \frac{1}{4} S(\tau) - \theta p(|\tau|) \right] = \frac{d}{d\theta} \mathbb{E}_\theta p(|\tau|). \quad (21) \]

**IV.2. Properties of efficient sequential plans.** By Theorem 3 and formula (18) we have
\[ \text{Var}_\theta f(p(|\tau|), S(\tau)) \geq \left[ \frac{d}{d\theta} h(\theta) \right]^2 / \mathbb{E}_\theta p(|\tau|). \quad (22) \]

This inequality becomes equality at some \( \theta \) if and only if
\[ f(u) = k(\theta) \left[ \frac{1}{4} s(u) - \theta p(u) \right] + h(\theta) \quad \text{m}_0 \text{ a.s.}, \quad (23) \]
where \( k(\theta) \neq 0 \).

**Theorem 4.** If a sequential plan \( (\tau, f(p(|\tau|), S(\tau)), \tau \in \mathcal{F}) \), is efficient at \( \theta_1 \), then there exists a constant \( k_1 \) such that
\[ h(\theta) = k_1 (\theta - \theta_1) \mathbb{E}_\theta p(|\tau|) + h(\theta_1). \quad (24) \]

**Proof.** By (23) we have
\[ f(u) = k(\theta_1) \left[ \frac{1}{4} s(u) - \theta_1 p(u) \right] + h(\theta_1) \quad \text{m}_0 \text{ a.s.} \]

Taking the expected value \( \mathbb{E}_\theta \) and using (17) we obtain the thesis.
THEOREM 5. If a sequential plan \((τ, f(p(τ)), S(τ))\), \(τ \in \mathcal{F}\), is efficient, then there are constants \(a_1, a_2, \) and \(a_3\) such that

\[ a_1 s(u) + a_2 p(u) + a_3 = 0 \text{ a.s.} \]

where \(a_1^2 + a_2^2 \neq 0\) and \(a_3 \neq 0\).

Proof. If a sequential plan \((τ, f(p(τ)), S(τ))\) is efficient, then it is efficient at some \(θ_1\) and \(θ_2\). By (23) the following equalities hold:

\[
\begin{align*}
    f(u) &= k(θ_1) \left[ \frac{1}{4}s(u) - θ_1 p(u) \right] + h(θ_1), \\
    f(u) &= k(θ_2) \left[ \frac{1}{4}s(u) - θ_2 p(u) \right] + h(θ_2),
\end{align*}
\]

where \(k(θ_1) \neq 0, k(θ_2) \neq 0\).

Subtracting one equality from the other, we obtain

\[
\frac{1}{4}(k(θ_1) - k(θ_2))s(u) + (θ_2 k(θ_2) - θ_1 k(θ_1)) p(u) + h(θ_2) - h(θ_1) = 0,
\]

which gives the thesis.

Let \(G_{K_t}\) denote the \(σ\)-algebra generated by the process \((S(K_t), p(K_t))\), \(t \in \mathcal{F}\). Evidently, \(G_{K_t} \subset F_{K_t}\).

By \(\mathcal{F}\) we denote the class of Markov stopping sets with respect to the family of \(σ\)-algebras \(\{G_{K_t}\}_{t \geq 0}\). Using the same arguments as in [17] and [15], we obtain the following

COROLLARY 1. If a sequential plan \((τ, f(p(τ)), S(τ))\), \(τ \in \mathcal{F}\), satisfying regularity conditions (which guarantee the Wald identities), is efficient, then \(τ \in \mathcal{F}\) and the measure \(m_0\) is accumulated on the line (25).

Let

\[ τ_r = \inf\{ t \in [t_0, t_1) : p(\{K_t\} \geq r) \} \text{ and } (S_r, G_r, μ_0)_{r \geq 0} = (S(K_t), G_{K_t}, μ_0). \]

The process \((S_r, G_r, μ_0)_{r \geq 0}\) belongs to the exponential class of processes considered in [15], where a full characterization of efficient sequential plans for this class of process was given (Theorem 2). Since \(p(\{K_t\})\) is a strictly increasing function of \(t\), we see that the analysis of the efficient sequential plans \((τ, f(p(τ)), S(τ))\), \(τ \in \mathcal{F}\), is equivalent to the analysis of the efficient sequential plans for the process \((S_r, G_r, μ_0)\). Therefore, we can use the mentioned result of Stefanov [15].

Definition 7. We say that \((τ, f(p(τ)), S(τ))\) is a simple plan if \(τ(w) = K_t\) for almost all \(w \in \mathcal{W}\).
Definition 8. We say that \((\tau_0, f(p(\tau_0)), S(\tau_0))\) is an oblique plan if 
\[
\tau_0(w) = \min \{K_t \in \mathcal{K} : S(K_t, w) = \gamma p(|K_t|) + \delta\}
\]
for almost all \(w \in W\) with respect to \(\mu_0\).

Definition 9. We say that \((\tau_{x_0}, f(p(\tau_{x_0})), S(\tau_{x_0}))\) is an inverse plan if 
\[
\tau_{x_0}(w) = \min \{K_t \in \mathcal{K} : S(K_t, w) = x_0\}
\]
for almost all \(w \in W\).

Lemma 10. If \(\theta = 0\) and \(\delta \gamma < 0\), then \(\mu_0(\bigcup_i \{w : \tau_0(w) = K_i\}) = 1\).

Proof. We introduce the Markov stopping time \(\tau_{1,0}\):
\[
\tau_{1,0}(w) = \inf \{t : S(K_t, w) = \gamma p(|K_t|) + \delta\}
\]
for almost all \(w \in W\).

We consider only the case \(\delta > 0\) and \(\gamma < 0\) (the second is analogous).

We have
\[
\mu_0(\bigcup_i \{w : \tau_0(w) = K_i\}) = \mu_0(\{w : \tau_{1,0}(w) < \infty\})
\]
\[
\geq \mu_0(\bigcup_i \{w : S(K_t, w) > \gamma p(|K_t|) + \delta\}) \geq \mu_0(\{w : S(K_t, w) > \gamma p(|K_t|) + \delta\}).
\]

Simple but uphill calculations lead to the conclusion that the random variable \(S(K_t, \cdot)\) has the normal distribution with \(E_0 S(K_t) = 0\) and \(\text{Var}_0 S(K_t) = O(t^2)\). Thus
\[
\mu_0(\{w : S(K_t, w) > \gamma p(|K_t|) + \delta\})
\]
\[
= \mu_0 \left( \left\{ w : \frac{S(K_t, w)}{\sqrt{\text{Var}_0 S(K_t)}} > \frac{\gamma \left(\frac{1}{2} \alpha + \beta t + \frac{1}{2} \alpha \beta t^2 + \delta\right)}{\sqrt{\text{Var}_0 S(K_t)}} \right\} \right)
\]
\[
= 1 - \Phi \left( \frac{\gamma \left(\frac{1}{2} \alpha + \beta t + \frac{1}{2} \alpha \beta t^2 + \delta\right)}{\sqrt{O(t^2)}} \right).
\]

Thus
\[
1 \geq \mu_0(\bigcup_i \{w : \tau_0(w) = K_i\}) \geq \mu_0(\bigcup_i \{w : S(K_t, w) > \gamma p(|K_t|) + \delta\})
\]
\[
\geq \lim_{n \to \infty} \mu_0(\{w : S(K_t, w) > \gamma p(|K_t|) + \delta\}) = 1,
\]
which completes the proof.

Lemma 11. If \(\theta = 0\), then \(\mu_0(\bigcup_i \{w : \tau_{x_0}(w) = K_i\}) = 1\).

Proof. Let \(\tau_{1,x_0}(w) = \inf \{t : S(K_t, w) = x_0\}\) and \(x_0 > 0\). We choose the sequence \(\gamma_n \to 0_+\) as \(n \to \infty\). By \(\tau_{1,0}^n\) we denote the Markov stopping time:
\[
\tau_{1,0}^n(w) = \inf \{t : S(K_t, w) = - \gamma_n p(|K_t|) + x_0\}.
\]

We have, for all \(w \in W\), \(\tau_{1,0}^n(w) \leq \tau_{1,0}^{n+1}(w)\) and \(\lim_{n \to \infty} \tau_{1,0}^n(w) = \tau_{1,x_0}(w)\).
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By Lemma 10 we have \( \mu_0(\{w: \tau_{1,0}^n(w) < \infty\}) = 1 \). Hence
\[
\mu_0(\{w: \tau_{x_0}(w) = K_i\}) = \mu_0(\{w: \tau_{1,x_0}(w) < \infty\})
\]
\[
= \mu_0(\{w: \tau_{1,0}^n(w) < \infty\}) = \lim_{n \to \infty} \mu_0(\{w: \tau_{1,0}^n(w) < \infty\}) = 1.
\]

Thus Lemma 11 is proved.

Let \( \tau_{2,0}(w) = \inf \{r: S_r(w) = \gamma r + \delta\} \), \( \tau_{2,x_0}(w) = \inf \{r: S_r(w) = x_0\} \).

By previous considerations and the mentioned result of Stefanov [15],
we can formulate the following corollaries.

**COROLLARY 2.** If a sequential plan \((\tau, f(p(|\tau|), S(\tau))\), \(\tau \in \mathcal{F}\), is efficient,
then there exist constants \(a_1, a_2, b_1, b_2\) such that \(h(\theta) = (a_1 \theta + a_2)/(b_1 \theta + b_2)\) is efficiently estimable function for this plan.

**COROLLARY 3.** If \(\theta \neq 0\) and \(\delta(4\theta - \gamma) > 0\), then \(E_\theta |\tau_0|^2 < \infty\) and the
regularity conditions, guaranteeing the Wald identities, hold.

**COROLLARY 4.** If \(\theta \neq 0\) and \(x_0 \theta > 0\), then \(E_\theta |\tau_{x_0}|^2 < \infty\) and the regularity
conditions, guaranteeing the Wald identities, hold.

**COROLLARY 5.** A simple plan is efficient.
Indeed, \(f(p(|\tau|), S(\tau)) = aS(K_\theta) + b\), \(h(\theta) = 4a\theta p(|K_\theta|) + b\), \(Var_\theta f(p(|K_\theta|), S(K_\theta)) = 16a^2 p(|K_\theta|)\).

**COROLLARY 6.** An oblique plan is efficient.
Indeed, \(f(p(|\tau_0|), S(\tau_0)) = ap(|\tau_0|) + b\), \(h(\theta) = a\delta/(4\theta - \gamma)\), \(Var_\theta f(p(|\tau|), S(\tau_0)) = 16a^2 \delta/(4\theta - \gamma)^3\).

**COROLLARY 7.** An inverse plan is efficient.
Indeed, \(f(p(|\tau_{x_0}|), S(\tau_{x_0})) = ap(|\tau_{x_0}|) + b\), \(h(\theta) = ax_0/4\theta + b\), \(Var_\theta f(p(|\tau_{x_0}|), S(\tau_{x_0})) = a^2 x_0/4\theta^3\).

**IV.3.** Sequential plans efficient at a given value \(\theta_1\). By the previous
theorems, the sequential plan \((\tau, f(p(|\tau|), S(\tau))\) is efficient at a given value \(\theta_1\)
if and only if
\[
f(p(|\tau|), S(\tau)) = k(\theta_1)\left[\frac{1}{4} S(\tau) + \theta_1 p(|\tau|) \right] + h(\theta_1), \quad \tau \in \mathcal{F},
\]
and if the function \(h(\theta)\) is efficiently estimable at \(\theta_1\), then \(h(\theta) = k_1(\theta - \theta_1) E_\theta p(|\tau|)\).

Consider the class of all sequential plans, efficient at \(\theta_1\), for the function
\(h(\theta)\) for which \(E_\theta p(|\tau|)\) is the same. We denote this class by \(\mathcal{E}_0\). So, if the
sequential plan \((\tau, f(p(|\tau|), S(\tau))\) belongs to \(\mathcal{E}_0\), then
\[
Var_\theta f(p(|\tau|), S(\tau)) = Var_\theta \left[ k(\theta_1) \left(\frac{1}{4} S(\tau) + \theta_1 p(|\tau|) \right) + h(\theta_1) \right]
\]
\[
= (k(\theta_1))^2 \left[ \frac{1}{16} Var_\theta S(\tau) + \theta_1^2 Var_\theta p(|\tau|) - \frac{1}{2} \theta_1 E_\theta p(|\tau|) S(\tau) + \frac{1}{2} \theta_1 E_\theta p(|\tau|) E_\theta S(\tau) \right].
\]
By (17)-(19) we have
\[ \text{Var}_\theta f(p(|\tau|), S(\tau)) \]
\[ = \left( k(\theta_1) \right)^2 \left[ E_\theta p(|\tau|) + 2(\theta - \theta_1) \frac{d}{d\theta} E_\theta p(|\tau|) + (\theta - \theta_1)^2 \text{Var}_\theta p(|\tau|) \right] \]
\[ = A + B \text{Var}_\theta p(|\tau|). \]

For all plans \((\tau, f(p(|\tau|), S(\tau)))\), belonging to \(\mathcal{E}_0\), the constants \(A\) and \(B\) are the same. So we can say that the smaller at \(\theta\) is \(\text{Var}_\theta p(|\tau|)\) the better is the plan \((\tau, f(p(|\tau|), S(\tau)))\) belonging to \(\mathcal{E}_0\), because the variance at \(\theta\) of the estimator \(f(p(|\tau|), S(\tau))\) is then getting smaller.

V. EFFICIENT SEQUENTIAL PLANS FOR POISSON AND WIENER FIELDS

Definition 10. Let \(\mathcal{B}^b \subset R^2\) be a family of bounded Borel subsets of \(R^2\). Assume that the family \(\{N(A)_{A \in \mathcal{B}^b} \}_{R^2}\) of random variables has the following properties:

1\(^o\) for an arbitrary set of disjoint bounded Borel subsets \(A_1, A_2, \ldots, A_n\) of \(R^2\) the random variables \(N(A_1), N(A_2), \ldots, N(A_n)\) are independent;

2\(^o\) \(P(N(A) = k) = (\lambda |A|)^k \exp(-\lambda |A|)/k!\).

\(N_x = N(K_x)\) is called the Poisson random field if \(K_x = [0, t_1] \times [0, t_2]\) for any \(z = (t_1, t_2) \in R^2_+\).

Definition 11. Assume that the collection of random variables \(\{W(A)_{A \in \mathcal{B}^b} \}_{R^2}\) has the following properties:

1\(^o\) for an arbitrary set of disjoint bounded Borel subsets \(A_1, A_2, \ldots, A_n\) of \(R^2\) the random variables \(W(A_1), W(A_2), \ldots, W(A_n)\) are independent;

2\(^o\) the random variable \(W(A)\) is normally distributed with \(EW(A) = 0\) and \(EW^2(A) = |A|\).

Then \(W_z = W(K_z), z \in R^2_+\), is called the Wiener random field.

Let \(\mathcal{K}\) be the family of the squares \(K_v \subset R^2_+\) with \(|K_v| = v\). In this case \(N_v = N(K_v)\) is the homogeneous Poisson process and \(W_v = W(K_v)\) is the Wiener process. With the same arguments as previously, the problem of characterization of efficient sequential plans for the random Poisson field and the Wiener field can be reduced to the problem of efficient sequential plans for the homogeneous Poisson process and the Wiener process.

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