HAAR SYSTEM AND NONPARAMETRIC DENSITY ESTIMATION IN SEVERAL VARIABLES

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$P_m: D(Q) \to D(Q)$, where $D(Q)$ is the set of all probability densities concentrated on $Q$, i.e.

$$D(Q) = \{f \in L^1(Q): f \geq 0 \text{ on } Q, \int_Q f = 1\}.$$  

It is also clear from (1.2) that $P_m(x, \cdot) \in D(Q)$ for fixed $x \in Q$.

We assume that we are given a probability space $(\Omega, \mathcal{F}, \Pr)$ and a simple sample of size $n$, i.e. a sequence $X_1, X_2, \ldots$ of i.i.d. random vectors with values in $Q$ and such that their common distribution has a density $f \in D(Q)$. The standard way of producing estimators for $f$ is given by formula (1.5)

$$f_{m,n}(x) = \frac{1}{n} \sum_{j=1}^{n} P_m(x, X_j),$$

which can be written in the form (1.6)

$$f_{m,n}(x) = \sum_{J \in \mathcal{Q}_m} n(J) h_J(x)$$

with

$$n(J) = \frac{1}{n} |\{j: X_j \in J\}|, \quad h_J(x) = \frac{1}{|J|} \chi_J(x).$$

Thus, the diagram of $f_{m,n}: Q \to R$ is simply the histogram. Our aim is to investigate the rate of convergence of $f_{m,n}$ to $f$ as $m$ and $n$ go to infinity and $f$ is a Lipschitz class. For those classes the optimal relation between $m$ and $n$ will be described. The first results in this direction we find in Glivenko's book [7] (see also [10]).

2. Preliminaries. We are going to discuss probability densities from $D(Q) \cap C(Q)$ and from $D(Q) \cap L^p(Q)$ with $1 \leq p < \infty$. To this end we need some properties of the operator $P_m$. The most elementary are the following:

$$P_m \geq 0,$$

$$P_m^2 = P_m,$$

$$P_m 1 = 1,$$

$$\|P_m f\|_p \leq \|f\|_p \quad \text{for } 1 \leq p \leq \infty, f \in L^p(Q),$$

where

$$\|f\|_p = \|f\|_{L^p(Q)} = (\int_Q |f|^p)^{1/p}, \quad \|f\|_\infty = \|f\|_{L^\infty(Q)} = \text{ess sup } \{|f(x)|: x \in Q\}.$$  

The modulus of smoothness of $f \in L^p(Q)$ is defined as

$$\omega_p(f; \delta) = \sup_{|h|_\infty \leq \delta} \left( \int_{Q(h)} (f(x + h) - f(x))^p \, dx \right)^{1/p},$$
where $Q(h) = \{ x \in Q : x+h \in Q \}$ and, for $f \in C(Q)$,

$$\omega_p(f; \delta) = \sup_{|x-y|_\infty < \delta, x, y \in Q} |f(x) - f(y)|,$$

where $|x|_\infty = \max(|x_1|, \ldots, |x_d|)$.

**Proposition 2.5.** For $f \in C(Q)$ we have

$$\|f - P_m f\| \leq \omega_\infty \left( f; \frac{1}{2^m} \right). \quad (2.6)$$

Conversely, let for some nondecreasing $\omega : R_+ \to R_+$

$$\|f - P_m f\|_\infty \leq \omega \left( \frac{1}{2^m} \right) \quad \text{for } m = 0, 1, \ldots \quad (2.7)$$

Then

$$\omega_\infty(f; \delta) \leq 4d \omega(2\delta) \quad \text{for } \delta > 0. \quad (2.8)$$

**Proof.** Inequality (2.6) is a simple consequence of (1.1). The converse can be proved as follows. If for some $J \in Q_m$ the points $x', x''$ are in $J$, then $P_m f(x') = P_m f(x'')$ and, by (2.7),

$$|f(x') - f(x'')| \leq |f(x') - P_m f(x')| + |f(x'') - P_m f(x'')| \leq 2 \omega \left( \frac{1}{2^m} \right). \quad (2.9)$$

Since $f$ is continuous, it follows that (2.9) holds for $x', x'' \in J$. Let now $x', x'' \in Q$ be arbitrary two different points and let $m$ be such that

$$\frac{1}{2^m} \geq |x' - x''|_\infty > \frac{1}{2^{m+1}}. \quad (2.10)$$

Since

$$x'' - x' = \sum_{j=1}^d (y^{(j)} - y^{(j-1)}), \quad \text{where } y^{(j)} = \sum_{k=1}^f (x'_k - x''_k) e_k,$$

with $e_k$ being the $k$-th unit vector in $R^d$, we find by (2.10) that $y^{(j)}$ and $y^{(j-1)}$ belong to two neighbouring cubes from $Q_m$ and therefore, by (2.9),

$$|f(x') - f(x'')| \leq 4d \omega(2 |x' - x''|_\infty).$$

The converse part of Proposition 2.5 for $d = 1$ was proved in [2].

**Corollary 2.11.** Let $0 < \alpha \leq 1$ and $f \in C(Q)$ be given. Then the following conditions are equivalent:

$$\|f - P_m f\|_\infty = O \left( \frac{1}{2^m} \right) \quad \text{as } m \to \infty, \quad (2.12)$$

$$\omega_\infty(f; \delta) = O(\delta^\alpha) \quad \text{as } \delta \to 0. \quad (2.13)$$
The $L^p$-case is little more complicated. We have the following direct result:

**Proposition 2.14.** Let $1 \leq p < \infty$ and let $f \in L^p(Q)$. Then

\begin{equation}
\|f - P_m f\|_p \leq 2^{d/p} \omega_p \left( f; \frac{1}{2^m} \right).
\end{equation}

**Proof.** For $J \in Q_m$ we have

\begin{equation}
\left\| \frac{1}{|J|} \int_J f(y) \, dy \right\|^p \lesssim \frac{1}{|J|^2} \int_J \int_J |f(x) - f(y)|^p \, dx \, dy.
\end{equation}

It follows that $J(h) = \{ x \in J : x + h \in J \}$. Then

\begin{equation}
\|f - P_m f\|_p \leq 2^{d/m} \int_{2^m \|h\|_\infty \leq 1} dh \int_{Q(h)} |A_h f(y)|^p \, dy \lesssim 2^d \left( \omega_p \left( f; \frac{1}{2^m} \right) \right)^p.
\end{equation}

The converse result depends on the following Bernstein type inequality (in case $d = 1$, see [3,4], and for $d > 1$, [5]).

**Proposition 2.16.** Define

\[ S_m(Q) = \text{span} \{ \chi_J : J \in Q_m \}. \]

Then, for $1 \leq p < \infty$ and for $f \in S_m(Q)$, we have

\begin{equation}
\|A_h f\|_{L^p(Q(h))} \lesssim 2d \cdot 3^{d/p} (2^m \|h\|_\infty)^{1/p} \|f\|_{L^p(Q)} \quad \text{for } |h|_\infty \leq \frac{1}{2^m}.
\end{equation}

**Proof.** Let $e_1, \ldots, e_d$ be the basic unit vectors in $R^d$ and, for $h = (h_1, \ldots, h_d)$, let $h(j) = h_1 e_1 + \ldots + h_j e_j$. Since

\[ A_h f(x) = \sum_{j=1}^d A_{h^j} f(x + h(j-1)), \quad h(0) = 0, \]

we obtain, for $J \in Q_m$,

\begin{equation}
\int_{Q(h)} |\Delta_h f|^p \lesssim d^{p-1} \sum_{j=1}^d \int_{J \cap Q(h)} |\Delta_{h^j} f(x + h(j-1))|^p \, dx
\end{equation}

\[ = d^{p-1} \sum_{j=1}^d \int_{J \cap Q(h)} |f(x + h(j)) - f(x + h(j-1))|^p \, dx. \]

Now, $f(x + h(j)) = f(x + h(j-1))$ for $x \in (J - h(j)) \cap (J - h(j-1))$ and, therefore,

\begin{equation}
\int_{Q(h)} |f(x + h(j)) - f(x + h(j-1))|^p \, dx
\end{equation}

\[ = \int_E |f(x + h(j)) - f(x + h(j-1))|^p \, dx \lesssim 2^{p-1} \left( \int_{E_{j-1}} |f|^p + \int_{E_j} |f|^p \right), \]

for $E_{j-1} \cap E_j = \emptyset$. 

where
\[ E = J \cap Q(h) \backslash (J - h(j)) \cap (J - h(j-1)) \]
and
\[ E_j = E + h(i) = (J \cap Q(h) + h(j)) \backslash J \cap (J + h_e) \].

Let now \( J^* = \bigcup \{ J_e : \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d), \varepsilon_j = 0, 1, -1 \} \cap Q \), where \( J_e = J + \varepsilon / 2^m \). It should be clear that \( E_j = J^* \cap E_j = \bigcup J_e \cap E_j \). Now \( J_e = J \) for \( \varepsilon = (0, \ldots, 0) \) and then
\[ |J \cap E_j| \leq |J| - |J \cap (J + |h|_\infty \varepsilon_j)| = |J| 2^m |h|_\infty \]

For \( \varepsilon \neq 0, |J_e \cap J| = 0 \) and
\[ |J \cap E_j| = |J \cap Q(h) \cap h(i)) \cap J_e| \]
\[ \leq |(J + h(j)) \cap J_e| \leq \frac{|h|_\infty}{2^{(d-1)m}} = |J| (2^m |h|_\infty) \]

therefore, for \( J_e \subset Q \),
\[ \int_{J \cap E_j} |f|^p \leq \frac{|E_j \cap J_e|}{|J|} \int_{J_e} |f|^p \leq (|h|_{\infty} 2^m) \int_{J_e} |f|^p \]

whence we infer that
\[ \int_{Q(h)} |D_h f|^p \leq (2d)^{p-1} |h|_{\infty} \sum_{j=1}^{d} \sum_{J_e \subset Q} \left( \int_{J \cap E_j} |f|^p + \int_{J \cap E_{j-1} \cap J_e} |f|^p \right) \]
\[ \leq (2d)^{p} 3d |h|_{\infty} 2^m \int_{Q(h)} |f|^p \]

We are now in position, using a standard method from approximation theory, to prove the main converse result.

**Theorem 2.18.** Let \( 1 \leq p < \infty \) and let \( f \in L^p(Q) \). Then
\[ \omega_p \left( f; \frac{1}{2^m} \right) \leq \frac{6d \cdot 3^{d/p}}{2^{m/p}} \sum_{i=0}^{m} 2^{i/p} \| f - P_m f \|_p. \]  

**Proof.** We have
\[ f = P_1 f + \sum_{j=1}^{m} f_j + (f - P_m f) \]
with \( f_j = P_j f - P_{j-1} f \), whence, for \( |h|_{\infty} \leq 1/2^m \),
\[ D_h f = \sum_{j=1}^{m} D_h f_j + D_h (f - P_m f), \]
\[ \| D_h f \|_{L^p(Q(h))} \leq \sum_{j=1}^{m} \| D_j f \|_{L^p(Q(h))} + 2 \| f - P_m f \|_p. \]
Now, (2.17) gives

\[ \|\Delta_k f\|_{L^p(Q(n))} \leq 2d \cdot 3d/p (|h| \|f\|_{L^p(Q(n))}) \leq 2d \cdot 3d/p (|h| \|f\|_{L^p(Q(n))}) \leq 2d \cdot 3d/p (|h| \|f\|_{L^p(Q(n))}) \]

Combining these inequalities, we get (2.19).

**Corollary 2.20.** Let \( f \in L^p(Q) \) and let \( \alpha \) and \( p \) be such that \( 0 < \alpha < 1/p \leq 1 \). Then the following conditions are equivalent:

(i) \[ \omega_p(f; \delta) = O(\delta^\alpha) \quad \text{as} \quad \delta \to 0^+ , \]

(ii) \[ \|f - P_m f\|_p = O \left( \frac{1}{2^m} \right) \quad \text{as} \quad m \to \infty . \]

This result in the 1-dimensional case we find in [8] and in [4]. It should be also mentioned here that Properties (2.2) and (2.4) imply

(2.21) \[ E_{m,p}(f) \leq \|f - P_m f\|_p \leq 2E_{m,p}(f) \quad \text{for} \quad 1 \leq p \leq \infty , \]

where \( E_{m,p}(f) = \inf \{ \|f - g\|_p : g \in S_m(Q) \} \).

3. Estimation of continuous densities. As in Introduction, we are given a sequence \( (X_1, X_2, \ldots) \) of i.i.d. random vectors with values in \( Q \) and with the common density \( f \in C(Q) \cap D(Q) \). The random function \( f_{m,n} \) is defined as in (1.5). It will be shown that, for suitable dependence of \( n \) on \( m \), the function \( f_{m,n} \) is a good estimator for \( f \). In what follows it is assumed that the sample size \( n \) is a dyadic natural. For given positive \( \beta \) the dependence of \( m \) on \( n \) is defined by

(3.1) \[ n = 2^v \quad \text{and} \quad m = [\beta v/d] , \]

where \( v \) is natural and \([x]\) is the integer part of \( x \). In this particular situation the \( f_{m,n} \) is denoted by \( f_{\nu,\beta} \). It is important that \( \beta \) is asymptotically \( \log N/\log n \) for large \( v \), \( N \) being the number of elements in \( Q_m \) and \( n \) the size of the sample. Our aim is, given properties of \( f \), to determine the best \( \beta \) and then to compute \( N \).

The main tool in the following discussion is the Bernstein inequality (cf. [9], p. 19);

**Lemma 3.2.** Let \( Y_j (j = 1, 2, \ldots , n) \) be independent random variables such that \( \Pr \{ Y_k = 1 \} = y, \Pr \{ Y_k = 0 \} = z, y + z = 1 \). Then

\[ \Pr \left\{ \left| \sum_{j=1}^{n} (Y_j - y) \right| \geq 2\omega (n y z)^{1/2} \right\} \leq 2e^{-\omega^2} \quad \text{for} \quad 0 \leq \omega \leq \frac{3}{2} (n y z)^{1/2} . \]

The rate of convergence of \( \|f - f_{\nu,\beta}\|_\infty \) to zero as \( \nu \to \infty \) can be investigated with the help of inequalities (\( m = [\beta v/d] , 1 \leq \beta \leq \infty \))

(3.4) \[ \frac{1}{2}\|f - P_m f\|_p \leq E_{m,p} \leq \|f - f_{\nu,\beta}\|_p \leq \|f - P_m f\|_p + \|P_m f - f_{\nu,\beta}\|_p , \]

which hold with probability 1 by the definition of \( E_{m,p} \) and by (2.21).
Lemma 3.5. Let \( f \in C(Q) \cap D(Q) \) and let \( k > 0, \lambda > 0, 0 < \beta < \frac{1}{2} \). Then

\[
\Pr \left\{ \left\| \frac{P_m f - f_{\gamma, \beta}}{P_m f} \right\|_\infty > \lambda \right\} = O(\varepsilon + \varepsilon^k 2^{md}), \quad m = \left[ \frac{\beta \varepsilon}{d} \right],
\]

where \( \varepsilon = \lambda^{-1} \cdot 2^{md(1-1/2\beta)} \) and the big \( O \) is independent of \( \lambda \).

Proof. Note that

\[
\left\| \frac{P_m f - f_{\gamma, \beta}}{P_m f} \right\|_\infty = \sup_{J \in Q_m} \left\{ \frac{\int f - n^{-1} \sum_{j=1}^n \chi_j(X_j)}{\int f} \right\}.
\]

Now, for \( J \in Q_m \), we put \( Y_j = \chi_j(X_j) \), \( y = \int f \), \( y + \varepsilon = 1 \), and then apply Lemma 3.2 to get (3.3) with \( \omega = \lambda ny/2\sqrt{nyz} \leq \frac{3}{2} \sqrt{nyz} \), provided that \( \lambda \leq 3z \). This condition holds in particular for \( \lambda \) and \( m \) satisfying

\[
2^{md} \geq \frac{3}{2} \|f\|_\infty, \quad \lambda \leq 1.
\]

Now,

\[
\omega^2 = \frac{\lambda^2 \gamma}{4z} \geq \lambda_1 \frac{1}{|J|} \left\{ \int f \right\} \lambda_1 = \frac{\lambda^2 n}{4 \cdot 2^{md}},
\]

and, therefore, by Jensen's inequality

\[
\sum_{J \in Q_m} \exp(-\omega^2) \leq \sum_{J \in Q_m} \exp\left(-\lambda_1 J/J \right) f^2 \leq 2^{md} \sum_{J \in Q_m} \int e^{-\lambda_1 f(a)} dx
\]

\[
= 2^{md} \int e^{-\lambda_1 f(a)} dx = 2^{md} \int_0^\infty e^{-\lambda_1 s} dF_f(s),
\]

where \( F_f \) is the distribution of \( f \) on \( (0, \infty) \) with respect to the Lebesgue measure on \( Q \). Now,

\[
\lambda_1 \geq \frac{\lambda^2}{4} 2^{md(1/\beta - 1)},
\]

whence, for \( \gamma > 0 \),

\[
\sum_{0}^{\infty} e^{-\lambda^2 N^{1/\beta - 1} s/4} dF_f(s) \leq N^{1-\gamma - 1} \sum_{N^{-\gamma \lambda - 1}}^{\infty} \left( \int_0^{\infty} e^{-\lambda^2 N^{1/\beta - 1} s/4} dF_f(s) \right)
\]

\[
\leq \frac{1}{\lambda} N^{1-\gamma} + Ne^{-\lambda N^{1/\beta - 1} - \gamma/4} \leq \frac{1}{\lambda} N^{1-\gamma} + N \left( \frac{\lambda}{4} N^{1/\beta - 1 - \gamma} \right)^{-k} \sup_{0 < \alpha < \infty} \alpha^k e^{-\alpha x}
\]

\[
= O \left( \frac{1}{\lambda} N^{1-\gamma} + \frac{1}{\lambda^k} N^{1+k(1+\gamma - 1/\beta)} \right),
\]
where \( N = 2^{md} \), \( k \) is any positive number and the \( O \) depends on \( \beta, d, \) and \( k \) only. Combining (3.7)-(3.10) with \( \gamma = 1/2\beta \) we obtain (3.6).

**Proposition 3.11.** Let \( f \in C(Q) \cap D(Q) \) and let \( 0 < \beta < 1/2 \). Then

\[
\Pr \{ \| f - f_{v,\beta} \|_\infty = o(1) \text{ as } v \to \infty \} = 1. 
\]

**Proof.** It follows by (3.6) that taking \( k > 0 \) such that \( 1/2\beta - 1 > 1/k \), we obtain with probability 1 for large \( m \)

\[
\bigcap_{x \in Q} | P_m f(x) - f_{v,\beta}(x) | \leq \lambda | P_m f(x) |
\]

whence \( \| P_m f - f_{v,\beta} \|_\infty \leq \lambda \| f \|_\infty \), and therefore

\[
\Pr \{ \| P_m f - f_{v,\beta} \|_\infty = o(1) \text{ as } v \to \infty \} = 1.
\]

On the other hand, according to (2.15), \( \| f - P_m f \|_\infty = o(1) \) as \( m \to \infty \).

Thus, (3.4) implies (3.12).

**Theorem 3.13.** Let \( f \in C(Q) \cap D(Q) \). Then for \( 0 < \alpha \leq 1 \), \( 0 < \beta < d/2(\alpha + d) \) the following conditions are equivalent:

(i) \( \omega_\infty (f; \delta) = O(\delta^{\alpha}) \) \quad as \( \delta \to 0_+ \),

(ii) \( \Pr \left\{ \| f - f_{v,\beta} \|_\infty = O \left( \frac{1}{2^{ma}} \right) \text{ as } m \to \infty \right\} = 1, \quad m = \left[ \frac{v\beta}{d} \right] \).

**Proof.** (i) \( \Rightarrow \) (ii). According to Corollary 2.5 we have \( \| f - P_m f \|_\infty = O(1/2^{ma}) \), and (3.6) with \( \lambda = 1/2^{ma} \) and \( k \) such that \((k-1)(1/2\beta - 1 - \alpha/d) \geq 1\) gives

\[
\Pr \left\{ \| P_m f - f_{v,\beta} \|_\infty > \frac{1}{2^{ma}} \right\} = O(2^{md(\alpha/d + 1 - 1/2\beta)}).
\]

Combining these inequalities with (3.4) we complete this part of the proof.

(ii) \( \Rightarrow \) (i). Using (3.4) we find that \( \| f - P_m \|_\infty = O(1/2^{ma}) \), whence by Proposition 2.5 the required result follows.

4. Estimation of densities in \( L^p \). Like in the previous section we consider densities concentrated on the \( d \)-dimensional cube \( Q \). It is also assumed that (3.1) is satisfied. The expectation of an r.v. \( Y \) with respect to the given probability space \((\Omega, \mathcal{F}, \Pr)\) is denoted by \( \mathbb{E} Y \).

The following result from Lorentz and Berens [1] plays an important role in our considerations:

**Proposition 4.1.** Let \( g \in C(I), I = (0, 1) \). Then, for \( x \in I \),

\[
\left| g(x) - \sum_{j=0}^{n} g \left( \frac{j}{n} \right) \binom{n}{j} x^j (1-x)^{n-j} \right| \leq 3\omega_{2,\infty} \left( g; \frac{x}{n} \right) \left( 1 - x \right) \sqrt{\frac{1}{n}}.
\]
where

\begin{equation}
\omega_{2,\alpha}(g; \delta) = \sup_{x_1, x_2 \in I, \delta < \frac{1}{2}} \left| g \left( \frac{x_1 + x_2}{2} \right) - \frac{g(x_1) + g(x_2)}{2} \right|, \quad 0 < \delta \leq \frac{1}{2}.
\end{equation}

The following elementary inequalities are well known.

**Proposition 4.3.** Let \( I = (-1, 1), R = (-\infty, \infty) \). Then

(i) \( 0 \leq |x + h|^p + |x - h|^p - 2|x|^p \leq 2|h|^p \) for \( 1 \leq p \leq 2, x + h, x - h \in R \),

(ii) \( 0 \leq |x + h|^p + |x - h|^p - 2|x|^p \leq (p-1)|h|^2 \) for \( p > 2, x + h, x - h \in I \).

**Proposition 4.4.** Let \( \beta > 0, 1 \leq p < \infty \) and let \( f \in L^p(Q) \cap D(Q) \). Then

\begin{equation}
\|f - P_m f\|_p \leq (E \|f - f_{v, \beta}\|_p^p)^{1/p} \leq \|f - P_m f\|_p + (E \|P_m f - f_{v, \beta}\|_p^p)^{1/p}.
\end{equation}

**Proof.** Since \( E f_{v, \beta}(x) = P_m f(x) \), Jensen's inequality implies the first inequality in (4.5). The second one follows by the triangle inequality.

**Lemma 4.6.** Let \( 1 \leq p < \infty, p^{-1} + q^{-1} = 1, \beta > 0 \) and let \( f \in L^p(Q) \cap D(Q) \).

Then, under (3.1),

\begin{equation}
(E \|P_m f - f_{v, \beta}\|_p^p)^{1/p} \leq C \cdot 2^{-md}\gamma \text{ for } v \to \infty,
\end{equation}

where

\[ \gamma = \frac{1}{\beta} \frac{1}{2 \vee p} - \frac{1}{2 \wedge q} \]

\((a \wedge b = \min(a, b), a \vee b = \max(a, b)), and C depends on p only. \)

**Proof.** Notice that with \( N = 2^m \) we have

\begin{equation}
E \|P_m f - f_{m, \beta}\|_p^p = N^{p-1} \sum_{J \in \mathcal{Q}_m} E \left( \frac{1}{n} \sum_{j=1}^n (Y_j(J) - y(J))^p \right),
\end{equation}

where \( Y_j(J) = \chi_f(x_j), y(J) = \text{Pr}(Y_j(J) = 1) = EY_j(J) = \int f \). Applying Propositions 4.1 and 4.3 to \( g(x) = |x - y(J)|^p \), we obtain

\begin{equation}
E \left( \frac{1}{n} \sum_{j=1}^n (Y_j(J) - y(J))^p \right)^{1/p} \leq C_1 \left( \frac{y(J)(1 - y(J))}{n} \right)^{(2 \wedge p)/2}.
\end{equation}

The combination of (4.9) and (4.8) gives

\begin{equation}
E \|P_m f - f_{m, \beta}\|_p^p \leq C_2 N^{p-1} n^{-\frac{(2 \wedge p)}{2}} \sum_{J \in \mathcal{Q}_m} (y(J)(1 - y(J)))^{(2 \wedge p)/2}.
\end{equation}

Now, \( 1 - y(J) \leq 1 \) and in addition, by concavity, for \( 1 \leq p \leq 2 \) we have

\[ \sum_{J \in \mathcal{Q}_m} y(J)(1 - y(L))^{(2 \wedge p)/2} \leq \sum_{J \in \mathcal{Q}_m} y(J)^{p/2} \leq N^{1-p/2} \left( \sum_{J \in \mathcal{Q}_m} y(J)^{p/2} \right) \leq N^{1-p/2}, \]
and, for $p > 2$,
\[
\sum_{J \in \Omega_m} (y(J)(1-y(J)))^{2 \wedge p/2} \leq \sum_{J \in \Omega_m} y(J) = 1.
\]

Both these inequalities and (4.10) give
\[
\text{E}\|P_m f - f_{v, \beta}\|_p^p \leq C_0^p N^{-p'}. \tag{4.11}
\]

Now, we are in position to state our main theorem for the $L^p$-case, namely

**Theorem 4.12.** Assume (3.1) and

\[
0 < \alpha \leq 1, \quad 1 \leq p < \infty, \quad 0 < \beta < \frac{d}{(2 \vee p) + ((2 \vee p) - 1)d}. \tag{4.13}
\]

Then, for $f \in L^p(\Omega) \cap D(\Omega)$, the following conditions are equivalent:

(i) \quad $\|f - P_k f\|_p = O\left(\frac{1}{2^{\alpha k}}\right)$ as $k \to \infty$,

(ii) \quad $\text{E}\|f - f_{v, \beta}\|_p^{1/p} = O\left(\frac{1}{2^{\beta v/d}}\right)$ as $v \to \infty$.

Moreover, next condition implies (i):

(iii) \quad $\omega_p(f; \delta) = O(\delta^p)$ as $\delta \to 0_+$.

If, in addition to (4.13), $0 < \alpha < 1/p$, then also (i) implies (iii).

**Proof.** In view of Corollary 2.7 it is sufficient to show that (a) the model is regular, and (b) each limit of UBE’s is admissible.

\[
\frac{1}{2} \|f - P_k f\|_p \leq \text{E}\|f - f_{v, \beta}\|_p^{1/p} \leq 2 (\|f - P_m f\|_p + \text{E}\|P_m f - f_{v, \beta}\|_p^{1/p}). \tag{4.14}
\]

This, (3.1), (i) and Lemma 4.6 imply

\[
\text{E}\|f - f_{v, \beta}\|_p^{1/p} \leq O\left(\frac{1}{2^{\alpha k}} + \frac{1}{2^{\beta v/d}}\right),
\]

where $\gamma = 1/(2 \vee p)/\beta - 1/(2 \wedge q)$. From this (ii) follows.

(ii) $\Rightarrow$ (i). It follows by (4.14) that (i) is satisfied for $k = m = [\beta v/d]$. However, by (4.13), $\beta/d < 1$ and therefore each $k$ is of the form $[\beta v/d]$.

(iii) $\Rightarrow$ (i). This implication holds true by Proposition 2.14. Its converse in case $0 < \alpha < 1/p$ follows by Corollary 2.20.

**Corollary 4.15.** Let $f \in L^p(\Omega) \cap D(\Omega)$ for some $p (1 \leq p < \infty)$ and let (iii) hold for some $\alpha$ ($0 < \alpha \leq 1$). Then, for each $\beta$ satisfying (4.13), we have

\[
\Pr \{\|f - f_{v, \beta}\|_p \to 0 \text{ as } v \to \infty\} = 1.
\]
Corollary 4.16. For given $\alpha$, $p$ and $\beta$ satisfying (4.13) the best choice for $\beta$ with respect to (ii) is

$$\beta = \frac{d}{(2 \vee p) \alpha + ((2 \vee p) - 1)d}.$$ 

Examples. 1. Let $\alpha$, $\beta$ and $p$ be as in (4.13). Let $f \in W^1_p(Q)$ for some $p'$ satisfying the inequalities $1 \leq p' \leq p < \infty$ and $d(1/p' - 1/p) < 1$. Then $\omega_p(f; \delta) = O(\delta^\alpha)$ with $\alpha = 1 - d(1/p' - 1/p)$, and the $\beta$ can be easily computed. This is actually an embedding theorem which can be derived for instance from [6].

2. Let $d = 1$ and $1 \leq p < 2$. Then the density for the arcsin law is given by the formula

$$f(x) = \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}, \quad x \in Q = \langle 0, 1 \rangle.$$ 

One checks that $f \in L^p, f \notin L^2$ and $\omega_p(f; \delta) = O(\delta^{1/p - 1/2})$. In this case $\alpha = 1/p - 1/2$ and $\beta = p/2$.

References


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Received on 28. 3. 1987