SOME CHARACTERIZATIONS OF
THE EXPONENTIAL DISTRIBUTION FUNCTION

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Abstract. Let $X$ be a nonnegative random variable and let $[x]$ denote the integer part of $x$. The main result of the paper is the following characterization: $X$ is exponentially distributed iff $[\alpha X]$ and $\alpha X - [\alpha X]$ are mutually independent for every $\alpha > 0$. Some modifications of this theorem are also considered.

1. Results. Let $X$ be a nonnegative random variable, and let $F(x) = \Pr(X < x)$ be its probability distribution function. Assume that the distribution is not concentrated at one atom. We say that $X$ is exponentially distributed if $F(x) = 1 - e^{-\lambda x}$ ($x > 0$) for some $\lambda > 0$. We say that $X$ is geometrically distributed if $\Pr(X = k) = pq^k$, $k = 0, 1, \ldots$, for some $0 < p < 1$, $q = 1 - p$. Denote by $[x]$ the integer part of $x$.

The main result of the paper is the following characterization of the exponential probability distribution function:

**Theorem 1.** $X$ is exponentially distributed iff for every $\alpha > 0$, $[\alpha X]$ and $\alpha X - [\alpha X]$ are mutually independent.

The random variables $[\alpha X]$ and $\alpha X - [\alpha X]$, separately considered, may be used to the characterization of the exponential probability distribution function.

**Theorem 2 (Bosch [1]).** $X$ is exponentially distributed iff, for every $\alpha > 0$, $[\alpha X]$ is geometrically distributed.

**Theorem 3.** $X$ is exponentially distributed iff, for every $\alpha > 0$, $\alpha X - [\alpha X]$ has the truncated exponential probability distribution function.

The modified version of Bosch's theorem is given by Riedl [3]. Theorem 1 has its discrete version and its continuous version formulated in terms of the renewal theory.

**Theorem 4.** Let $X$ be a nonnegative integer-valued random variable. $X$ is
geometrically distributed iff, for every \( a = 1, 2, \ldots, [X/a] \) and \( X - a[X/a] \) are mutually independent.

**Theorem 5.** Let \( X, Y_1, Y_2, \ldots \) be independent nonnegative random variables, let \( X \) have an absolutely continuous probability distribution function with bounded and continuous density, and let \( Y_1, Y_2, \ldots \), have a common probability distribution function with the finite expected value. Let \( N(t) = \max(n: Y_1 + \ldots + Y_n \leq t) \) and \( R(t) = t - (Y_1 + \ldots + Y_{N(t)}) \), \( t \geq 0 \), be the renewal process and the residual life process, respectively. \( X \) is exponentially distributed iff, for every \( \alpha > 0 \), \( N(\alpha X) \) and \( R(\alpha X) \) are mutually independent.

In the proofs which now follow, we limit our considerations merely to the "only if" part.

2. **Proof of Theorem 1.** Write \( N = [\alpha X] \) and \( R = \alpha X - N \). Let \( \mathcal{B} \) be the \( \sigma \)-field of Borel sets on \([0, 1]\). Define \( \alpha(B + \beta) \) for \( \alpha > 0 \), \( -\infty < \beta < \infty \), in such a manner that \( x \in B \) iff \( \alpha(x + \beta) \in \alpha(B + \beta) \). We have

\[
\Pr(N = n) = \Pr(n \leq \alpha X < n + 1) = F\left(\frac{n + 1}{\alpha}\right) - F\left(\frac{n}{\alpha}\right),
\]

\[
\Pr(R \in B) = \Pr(X \in \bigcup_{n=0}^{\infty} B + \frac{n}{\alpha}) = \sum_{n=0}^{\infty} \Pr(X \in \frac{B + n}{\alpha}),
\]

\[
\Pr(N = n, R \in B) = \Pr\left(X \in \frac{B + n}{\alpha}\right), \quad n = 0, 1, \ldots, B \in \mathcal{B}, \alpha > 0.
\]

The independence condition for \( N \) and \( R \) may be written as

\[
(1) \quad \Pr\left(X \in \frac{B + n}{\alpha}\right) = \left(F\left(\frac{n + 1}{\alpha}\right) - F\left(\frac{n}{\alpha}\right)\right) \sum_{k=0}^{\infty} \Pr(X \in \frac{B + k}{\alpha}),
\]

\( n = 0, 1, \ldots, B \in \mathcal{B}, \alpha > 0. \)

If \( B = [0, y) \), \( 0 \leq y \leq 1 \), then (1) has the form

\[
(2) \quad F\left(\frac{n + y}{\alpha}\right) - F\left(\frac{n}{\alpha}\right) = \left(F\left(\frac{n + 1}{\alpha}\right) - F\left(\frac{n}{\alpha}\right)\right) \sum_{k=0}^{\infty} \left(F\left(\frac{k + y}{\alpha}\right) - F\left(\frac{k}{\alpha}\right)\right),
\]

\( n = 0, 1, \ldots, 0 \leq y \leq 1, \alpha > 0. \)

For \( n = 0 \) we have

\[
(3) \quad F\left(\frac{y}{\alpha}\right) = F\left(\frac{1}{\alpha}\right) \sum_{k=0}^{\infty} \left(F\left(\frac{k + y}{\alpha}\right) - F\left(\frac{k}{\alpha}\right)\right), \quad 0 \leq y \leq 1, \alpha > 0.
\]

For \( \alpha \) such that \( F(1/\alpha) > 0 \) we have

\[
(4) \quad F\left(\frac{n + y}{\alpha}\right) - F\left(\frac{n}{\alpha}\right) = \left(F\left(\frac{n + 1}{\alpha}\right) - F\left(\frac{n}{\alpha}\right)\right) \frac{F\left(\frac{y}{\alpha}\right)}{F\left(\frac{1}{\alpha}\right)},
\]

\( n = 1, 2, \ldots, 0 \leq y \leq 1, \alpha > 0. \).
Let \( F = pF_d + qF_s + rF_a \), where \( F_d, F_s \) and \( F_a \) are discrete, singular and absolutely continuous components, \( p \geq 0, q \geq 0 \) and \( r \geq 0 \), \( p + q + r = 1 \), are the weights of one.

Let \( B = \{x_i, i = 1, 2, \ldots\} \) be the support of the discrete component of the distribution function \( F \), \( \Pr(X \in B) = p \). Consider \( \alpha > 0 \) such that \( 0 < F(1/\alpha) < 1 \). For nondegenerate \( F \) the set \( \alpha \) which satisfies that condition contains some interval. Define \( B_\alpha = \{\alpha x_i - \lfloor \alpha x_i \rfloor, i = 1, 2, \ldots\} \). We have

\[
\Pr \left( X = \frac{1}{\alpha}(B_\alpha + n) \right) = \Pr \left( X = x_i; \frac{n}{\alpha} \leq x_i < \frac{n+1}{\alpha} \right)
= pF_d \left( \frac{n+1}{\alpha} \right) - pF_d \left( \frac{n}{\alpha} \right), \quad n = 0, 1, \ldots
\]

From (1) for \( n = 0 \) and \( B = B_\alpha \) it follows that

\[
pF_d \left( \frac{1}{\alpha} \right) = \left( pF_d \left( \frac{1}{\alpha} \right) + qF_s \left( \frac{1}{\alpha} \right) + rF_a \left( \frac{1}{\alpha} \right) \right) p,
\]

which implies \( p = 0 \) or \( p = 1 \).

Let \( p = 0 \) and \( B \) be the support of the singular component of the distribution \( F \) (e.g. \( B \) is a set of the Lebesgue measure zero), \( B \subset [0, \infty) \), and \( \Pr(X \in B) = q \). Let

\[
B_\alpha = \bigcup_{k=0}^{\infty} \alpha \left( B \cap \left[ \frac{k}{\alpha}, \frac{k+1}{\alpha} \right) - \frac{k}{\alpha} \right).
\]

We have

\[
\Pr \left( X = \frac{1}{\alpha}(B_\alpha + n) \right) = \Pr \left( X \in B_\alpha \cap \left[ \frac{n}{\alpha}, \frac{n+1}{\alpha} \right) \right)
= qF_s \left( \frac{n+1}{\alpha} \right) - qF_s \left( \frac{n}{\alpha} \right), \quad n = 0, 1, \ldots
\]

From (1) for \( n = 0 \) and \( B = B_\alpha \) it follows that

\[
qF_s \left( \frac{1}{\alpha} \right) = \left( qF_s \left( \frac{1}{\alpha} \right) + rF_a \left( \frac{1}{\alpha} \right) \right) q,
\]

which implies \( q = 0 \) or \( q = 1 \).

Now we prove that \( 0 < F(x) < 1 \) for \( x > 0 \). The conditions \( F(a) = 0, F(a+0) > 0 \) for some \( a > 0 \) and (4) imply that \( F \) is discrete and generated by \( \Pr(X = ka) = p_k \geq 0, k = 1, 2, \ldots, p_1 + p_2 + \ldots = 1 \). Putting \( 1/\alpha > a \) and such that \( a/\alpha \) is irrational, from (1) for \( n = 0 \) and \( B = \{xa\} \) it
follows that $\Pr(X = k/\alpha + a, k = 1, 2, \ldots) > 0$, which does not hold. The conditions $F(a) < 1$, $F(a+0) = 1$ and (3) imply

$$F\left(\frac{a}{2} - \varepsilon\right) = F\left(\frac{a}{2} + 2\varepsilon\right)\left(1 - F\left(\frac{a}{2} + 2\varepsilon\right)\right), \quad \text{where } 0 < \varepsilon < \frac{a}{4}.$$ 

Hence $F(\frac{a}{2} - \varepsilon) = 1$ or $F(\frac{a}{2} + 2\varepsilon) = 0$, which does not hold.

It is obvious that the derivative of the probability distribution function exists almost surely. From (4) it follows that $f^+(0) = \lim_{t \downarrow 0} F(t)/t$ exists.

Now we prove that $f^+(0) > 0$.

Write $1 - F = \bar{F}$. From (3) it follows that

$$F\left(\frac{y}{\alpha}\right) \geq F\left(\frac{1}{\alpha}\right)\left(F\left(\frac{y}{\alpha}\right) + F\left(\frac{1+y}{\alpha}\right) - F\left(\frac{1}{\alpha}\right)\right),$$

that is

$$F\left(\frac{y}{\alpha}\right) \geq \left(F\left(\frac{1+y}{\alpha}\right) - F\left(\frac{1}{\alpha}\right)\right)\frac{F\left(\frac{1}{\alpha}\right)}{\bar{F}\left(\frac{1}{\alpha}\right)}, \quad 0 \leq y \leq 1, \alpha > 0.$$

For fixed $x > 0$ substitute $y = ax$, $1/\alpha = a$. If $0 \leq y \leq 1$, then $0 \leq x \leq 1/\alpha$, and we have

$$F(x) \geq (F(a+x) - F(a))F(a)/\bar{F}(a), \quad 0 \leq x \leq a,$$

which implies

$$F(x) \geq \sup_{a \geq x} (F(a+x) - F(a))/\bar{F}(a) \geq \sup_{x > A \geq x} F(A)/\bar{F}(A) \sup_{A < B-x} (F(a+x) - F(a)) \geq \sup_{B-x > A \geq x} \frac{F(A)F(B)-F(A)}{B-A} x,$$

and, finally,

$$\frac{F(x)}{x} \geq \sup_{B-x > A \geq x} \frac{F(A)F(B)-F(A)}{B-A} > 0.$$ 

From (4) it follows that if $n/\alpha$ is the point of existence of the derivative of $F$, then

$$f\left(\frac{n}{\alpha}\right) = \left(F\left(\frac{n+1}{\alpha}\right) - F\left(\frac{n}{\alpha}\right)\right)f^+(0)/F\left(\frac{1}{\alpha}\right).$$
Hence the absolute continuous component of \( F \) has the positive weight. There remains the case \( p = 0, q = 0, r = 1 \) (e.g. \( F = F_a \)).

From (4) we get
\[
f\left(\frac{n+y}{\alpha}\right) = \left( F\left(\frac{n+1}{\alpha}\right) - F\left(\frac{n}{\alpha}\right)\right) f\left(\frac{y}{\alpha}\right) / F\left(\frac{1}{\alpha}\right)
\]
which implies
\[
(5) \quad f\left(\frac{n+y}{\alpha}\right) f^+ (0) = f\left(\frac{n}{\alpha}\right) f\left(\frac{y}{\alpha}\right), \quad n = 1, 2, \ldots; 0 < y < 1, \alpha > 0.
\]

It is obvious that the unique solution of equation (5) in the class of integrable functions is \( f(x) = \lambda e^{-\lambda x} \) \((x > 0)\) for some \( \lambda > 0 \).

3. Proof of Theorem 3. Let \( X \) have the probability distribution function \( F \). Then, for \( R = \alpha X - [\alpha X] \), we have
\[
H(y) = \Pr(R < y) = \sum_{n=0}^{\infty} \left( F\left(\frac{n+y}{\alpha}\right) - F\left(\frac{n}{\alpha}\right)\right), \quad 0 \leq y \leq 1, \alpha > 0.
\]

We have assumed that \( H \) is a truncated exponential probability distribution function, e.g.
\[
H(y) = \frac{1-e^{-\lambda(\alpha)y}}{1-e^{-\lambda(\alpha)}}, \quad 0 \leq y \leq 1,
\]
where \( \lambda(\alpha) \) is a parameter which depends on \( \alpha \) only.

Taking the limit \( H(y) \) if \( \alpha \to 0 \) and \( y/\alpha \to x \), since \( F(y/\alpha) \leq H(y) \leq F(y/\alpha) + 1 - F(1/\alpha) \), we get
\[
F(x) = \lim_{\alpha \to 0} \frac{1-e^{-\lambda(\alpha)\alpha x}}{1-e^{-\lambda(\alpha)}}.
\]

Hence the limits, for \( \alpha \to 0 \), \( \lim \lambda(\alpha) \alpha = \lambda \) and \( \lim \lambda(\alpha) = \infty \) exist. Finally, we have \( F(x) = 1 - e^{-\lambda x} (x \geq 0) \), where \( \lambda > 0 \) for the nondegenerate case.

4. Proof of Theorem 4. Let \( \Pr(X = k) = p_k, k = 0, 1, \ldots \) Then, for \( a = 1, 2, \ldots \), we have \( \Pr([X/a] = n, X - a[X/a] = i) = \Pr(X = an+i) = p_{an+i}, n = 0, 1, \ldots; i = 0, 1, \ldots, a-1 \). The independence condition for \([X/a]\) and \( X - a[X/a]\) is equivalent to
\[
p_{an+i} = \left( \sum_{j=0}^{a-1} p_{an+j} \right) \left( \sum_{k=0}^{\infty} p_{ak+i} \right), \quad n = 0, 1, \ldots; i = 0, 1, \ldots, a-1; a = 1, 2, \ldots,
\]
whence
\[
p_{an+i+1} = p_{an+i}, q_i, \quad i = 0, 1, \ldots, a-2; m = 0, 1, \ldots; a = 1, 2, \ldots,
\]
where \( q_i \) does not depend on \( m \).
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In particular, for $a = 2$, we have

\begin{equation}
\label{eq:11}
p_{2n+1} = p_{2n} q, \quad n = 0, 1, \ldots,
\end{equation}

where $q$ does not depend on $n$.

Let $a = 4k + 1$. If $n = 0, 1, \ldots, 2k - 1$ in (7), and $m = 0, i = 0, 2, \ldots, 4k - 2$ in (6), then $q_i = q$ for $i = 0, 2, \ldots, 4k - 2$. If $n = 2k + 1, 2k + 2, \ldots, 4k$ in (7), and $m = 1, i = 1, 3, \ldots, 4k - 1$ in (6), then $q_i = q$ for $i = 1, 3, \ldots, 4k - 1$. We have $p_{i+1} = p_i q$ for $i = 0, 1, \ldots, 4k - 1$. Since $k$ is arbitrary, we have $p_i = pq^i$, $i = 0, 1, \ldots$.

5. Proof of Theorem 5. Let $G_0(x) = 1_{[0, \infty)}(x)$, $G(x) = P(Y_1 < x)$, $G_n(x) = P(Y_1 + Y_2 + \ldots + Y_n < x)$, $x > 0$, $n = 1, 2, \ldots$, $G = 1 - G$, $EY_1 = \mu_1$. Assuming the existence of the probability density function $f$, we improve the joint density of $N(aX)$ and $R(aX)$:

\begin{equation}
\frac{d}{dy} Pr(N(aX) = n, R(aX) < y) = \int_0^\infty f \left( \frac{u+y}{\alpha} \right) G(y) dG_n(u),
\end{equation}

\begin{align*}
Pr(N(aX) = n) &= \int_0^\infty \int_0^\infty f \left( \frac{u+y}{\alpha} \right) G(y) dy dG_n(u) \\
&= \int_0^\infty \int_0^\infty f \left( \frac{y}{\alpha} \right) (G_s(z) - G_{s+1}(z)) dz,
\end{align*}

\begin{equation}
\frac{d}{dy} Pr(R(aX) < y) = \int_0^\infty \int_0^\infty f \left( \frac{u+y}{\alpha} \right) G(y) dH_G(u), \quad n = 0, 1, \ldots, y \geq 0, \alpha > 0,
\end{equation}

where

\begin{equation}
H_G(u) = \sum_{k=0}^\infty G_k(u) = EN(u), \quad u \geq 0.
\end{equation}

The independence condition of $N(aX)$ and $R(aX)$ has the form

\begin{equation}
\int_0^\infty f \left( \frac{u+y}{\alpha} \right) G(y) dG_n(u) = \left( \int_0^\infty f \left( \frac{y}{\alpha} \right) (G_n(y) - G_{n+1}(y)) dy \right) \int_0^\infty f \left( \frac{u+y}{\alpha} \right) G(y) dH_G(u), \quad n = 0, 1, \ldots, y \geq 0, \alpha > 0.
\end{equation}

For $n = 0$ we have

\begin{equation}
f \left( \frac{y}{\alpha} \right) G(y) = \left( \int_0^\infty f \left( \frac{y}{\alpha} \right) G(y) dy \right) \left( \int_0^\infty f \left( \frac{u+y}{\alpha} \right) G(y) dH_G(u) \right).
\end{equation}
Hence, for $G(y) > 0$,

$$f(y/x) = \frac{\int_0^\infty f(y/x)G(y)\,dy}{\int_0^\infty f((u+y)/x)dG_n(u)} = \frac{\int_0^\infty f(y/x)(G_n(y) - G_{n+1}(y))\,dy}{\int_0^\infty f(y/x)\,dy}.$$  \hspace{1cm} (8)

Putting $y = 0$ in (8), we get a further simplification:

$$f(y/x) = \frac{f(0)}{\int_0^\infty f(u/x)dG_n(u)}, \quad n = 1, 2, \ldots, y \geq 0, x > 0, G(y) > 0.$$ \hspace{1cm} (9)

Let $x > 0$, $m = [n\mu_1/x]$. We have

$$G_n(mu) = \Pr(Y_1 + \ldots + Y_n < mu) = \Pr\left(\frac{1}{n}(Y_1 + \ldots + Y_n) < \frac{mu}{n}\right).$$

Since $m/n \rightarrow \mu_1/x$, we have $G_n(mu) \rightarrow 1_{(x,\infty)}(u)$. For bounded and continuous $f$ (see [2], p. 254) we have

$$f(y/x) = \frac{f(0)}{f((y+x)/x)}, \quad y \geq 0, x > 0, x > 0.$$  \hspace{1cm} (10)

Substituting $u := mu$, $\alpha := mx$, $y := my$ in (9) and taking the limit if $n \rightarrow \infty$, we get

$$f(y/x) = \frac{f(0)}{f((y+x)/x)}, \quad y \geq 0, x > 0, x > 0.$$  \hspace{1cm} (10)

The unique continuous solution of (10) is $f(x) = \lambda e^{-\lambda x}$ ($x > 0$) for some $\lambda > 0$.

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