REMARKS ON BANACH SPACES OF S-COTYPE $p$

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Abstract. The paper continues the work of [10]. There are examined the relations between the class of Banach spaces of $S$-cotype $p$, the class of Banach spaces of $M$-cotype $p$ in the sense of Mouchta-ri [7] and the class $V_p$ of Banach spaces defined by Tien and Werorn [11].

1. Introduction. Let $E$ be a Banach space with dual $E'$. $E$ is said to be of stable type $p$ ($0 < p \leq 2$) if, for every sequence $(x_n)$ in $E$ with $\sum \|x_n\|^p < \infty$, $\sum x^* \theta_n^{(p)}$ converges a.s., where $\theta_n^{(p)}$ are i.i.d. symmetric $p$-stable random variables. For $p = 2$ stable type 2 is equivalent to type 2. $E$ is said to be of cotype 2 if, for every sequence $(x_n)$ in $E$ such that $\sum x_n \theta_n^{(2)}$ converges a.s., $\sum \|x_n\|^2 < \infty$. It is known that an analogous definition of stable cotype $p$ ($0 < p < 2$) by replacing the sequence $\{\theta_n^{(2)}\}$ by the sequence $\{\theta_n^{(p)}\}$ does not restrict the class of Banach spaces, since the a.s. convergence of $\sum x_n \theta_n^{(p)}$ implies that $\sum \|x_n\|^p$ is finite for $p < 2$.


Our aim is to examine the relation between the class $M_p$ of spaces of $M$-cotype $p$, the class $S_p$ of spaces of $S$-cotype $p$ and the class $V_p$. The main result of the paper, the inclusions $M_p \subset V_p \subset S_p \subset \bigcap_{e > 0} M_{p+\varepsilon}$ ($1 < p < 2$), allows us to obtain the conclusion $V_p \subset V_q$ for $p < q$ (going up phenomenon). By this phenomenon we can refer to a Banach space in the class $V_p$ as a Banach space of $V$-cotype $p$. It is interesting to know whether the three possible notions of cotype coincide.

2. Preliminaries and notation. Let $E$ be a Banach space with dual $E'$. We say that $E$ is a Sazonov space if there exists a topology $\mathcal{T}$ on $E$ such that a

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positive definite function $f$ with $f(0) = 1$ is $\mathcal{T}$-continuous iff it is a characteristic function (ch. f.) of a probability measure on $E$. It has been shown [6] that every Sazonov space can be embedded into $L_0$ and, conversely, if a Banach space with the metric approximation property embeds in $L_0$, then it is a Sazonov space. In particular, every closed subspace of $L_p$, $1 \leq p \leq 2$, is a Sazonov space, while, for $p > 2$, $L_p$ is not a Sazonov space.

For a real number $p$ ($0 < p \leq 2$) we denote by $X_p$ a closed subspace of $L_p$. $\mathcal{A}_p(E', X_p)$ denotes the set of linear continuous operators $T$ from $E$ into $X_p$ for which the function $f(a) = \exp \{-||Ta||^p\}$, $a \in E'$, is the ch. f. of a probability measure on $E$. An operator $T$ in $\mathcal{A}_p(E', X_p)$ for some $X_p$ is called a $\mathcal{A}_p$-operator on $E'$.

Let $\mathcal{T}_p$ denote the coarsest topology on $E$ for which all the ch. f. of symmetric $p$-stable measure are continuous. A Banach space $E$ is said to be of $M$-cotype $p$ ($0 < p \leq 2$), provided the function $f: E' \to C$ is the ch. f. of a probability measure on $E$, if it is positive definite, $\mathcal{T}_p$-continuous and $f(0) = 1$. Equivalently, a Banach space $E$ is of $M$-cotype $p$ iff any $\mathcal{T}_p$-continuous linear mapping $A$ from $E'$ into $L_0(\Omega, P)$ is decomposable.

We remind that a linear mapping $A$ from $E'$ into $L_0(\Omega, P)$ is said to be decomposable if there exists an $E$-valued random variable $\varphi$ such that $\mathbb{P}\{w: A\varphi(w) = (\varphi(\omega), a)\} = 1$ for all $a \in R$.

Mouchtari [7] has shown that $M$-cotype 2 spaces are exactly cotype 2 spaces, and $M$-cotype $p$ spaces, for some $p < 1$, are exactly Sazonov spaces.

Following [11] we say that a Banach space $E$ is in the class $V_p$ ($0 < p \leq 2$) if for every symmetric $p$-stable measure $\mu$ and for every symmetric $p$-stable cylindrical measure $v$ the inequality $|1 - \tilde{\mu}(a)| \leq |1 - \hat{\mu}(a)|$ for all $a \in E'$ implies that $v$ is a Radon measure, where $\tilde{\mu}(a)$ and $\hat{\mu}(a)$ are the ch. f. of $\mu$ and $v$, respectively.

Finally, a Banach space $E$ is said to be of $S$-cotype $p$ ($0 < p \leq 2$) if for every sequence $(x_n)$ in $E$ and every symmetric $p$-stable measure $\mu$ on $E$ the inequality

$$1 - \exp \left\{ - \sum |\langle x_n, a \rangle|^p \right\} \leq 1 - \hat{\mu}(a)$$

implies $\sum ||x_n||^p$ is finite.

In [10] it was shown that $E$ is of $S$-cotype 2 iff it is of cotype 2. A Banach space with the approximation property is of $S$-cotype $p$ for $p < 1$ iff it is a Sazonov space.


1. Theorem. Let $M_p$ and $S_p$ denote the class of spaces of $M$-cotype $p$ and the class of spaces of $S$-cotype $p$, respectively. Then $M_p \subset V_p \subset S_p \subset \bigcap_{\varepsilon > 0} M_{p+\varepsilon}$ $(1 \leq p < 2)$. 
Proof. (a) $M_p \subset V_p$. Let $E$ be a Banach space of $M$-cotype $p$ and suppose that $\mu$ is a symmetric $p$-stable measure on $E$, and $\nu$ is a symmetric $p$-stable cylindrical measure on $E$ such that $|1 - \hat{\nu}(a)| \leq |1 - \hat{\mu}(a)|$ for all $a \in E'$. From this inequality it follows that $\hat{\nu}(a)$ is $\mathcal{T}_p$-continuous. Since $E$ is of $M$-cotype $p$, $\hat{\nu}(a)$ is a ch. f. of a Radon measure on $E$. This shows that $E$ belongs to the class $V_p$.

(b) $V_p \subset S_p$. Let $E$ be in the class $V_p$ and let $(x_n)$ be a sequence in $E$ such that

$$1 - \exp \left\{ - \sum |\langle x_n, a \rangle|^p \right\} \leq 1 - \hat{\mu}(a) \quad \text{for all } a \in E',$$

where $\mu$ is a symmetric $p$-stable measure on $E$.

Let $\nu$ be the $p$-stable cylindrical measure with the ch. f. $\hat{\nu}(a) = \exp \left\{ - \sum |\langle x_n, a \rangle|^p \right\}$.

By the assumption that $E$ belongs to $V_p$, $\hat{\nu}(a)$ is a ch. f. of a Radon measure on $E$. From the Ito-Nisio theorem it follows that the series $\sum x_n \theta_n^{(p)}$ converges a.s. Since $p < 2$, we have $\sum \|x_n\|^p < \infty$. Hence $E$ is of $S$-cotype $p$.

(c) $S_p \subset \bigcap_{\epsilon > 0} N_{p+\epsilon}$.

We split the proof into two steps.

Step one. Suppose that $E$ is of $S$-cotype $p$ ($1 \leq p < 2$). Then every symmetric $q$-stable measure on $E$ ($q > p$) is the continuous image of a symmetric $q$-stable measure on some Sazonov space.

Indeed, let $\mu$ be a symmetric $q$-stable measure on $E$ ($q > p$) with the ch. f. $\hat{\mu}(a) = \exp \left\{ - \|Ta\|^q \right\}$, where $T \in A_q(E', L_q)$. Because of $q > p$, by Theorem 2 in [7], the function $\exp \left\{ - \|Ta\|^p \right\}$ is also the ch. f. of a Radon measure on $E$. Thus $T \in A_p(E', L_2)$. Since $E$ is of $S$-cotype $p$ by Theorem 3.3 in [10], the adjoint $T^* \colon L_q' \to E$ is $p$-summing. By the Pietsch factorization theorem, there exists a factorization

$$T^* \colon L_q \overset{\mu}{\to} S \overset{\nu}{\to} E,$$

where $S$ is a closed subspace of $L_p$, $V \colon S \to E$ is a linear continuous operator, and $U \colon L_q' \to S$ is a $p$-summing operator.

The operator $U$, being $p$-summing, is also $r$-summing for $1 \leq p < r < q$. Let $\gamma_q$ be the canonical cylindrical $q$-stable measure on $L_q$ with the ch. f. $\exp \left\{ - \|x\|^q \right\}$, $x \in L_q$. $\gamma_q$ is of scalar order $r$, i.e.

$$\sup_{\|x\| \leq 1} \int \|\langle x, y \rangle\|^r d\gamma_q(y) < \infty.$$

As $U$ is $r$-summing ($r > 1$) in view of the Schwartz Radonification theorem [9], $\nu = U(\gamma_q)$ is a Radon measure on $S$. We have $\mu = T^*(\gamma_q) = V[U(\gamma_q)] = V(\nu)$. 
v is a symmetric q-stable measure on S and S is a Sazonov space (since every closed subspace of \( L_p \) (\( 1 \leq p \leq 2 \)) is a Sazonov space).

**Step two.** Suppose that every symmetric p-stable measure on a Banach space E is a continuous image of a symmetric p-stable measure on some Sazonov space. Then E must be of M-cotype p.

Indeed, let A be a \( \mathcal{T}_p \)-continuous linear mapping from E' into \( L_0(\Omega, P) \). Then given \( \varepsilon > 0 \), there exists a \( A_p \)-operator \( T_\varepsilon \) on E' such that \( ||T_\varepsilon a|| \leq 1 \) implies \( ||Aa||_0 < \varepsilon \), where \( ||\cdot||_0 \) is the F-norm in \( L_0(\Omega, P) \) metrizing the topology of convergence in probability.

By Lemma 5.2 in [3], we can choose a single \( A_p \)-operator T on E' satisfying the following condition:

\[
(1.1) \text{For every } \varepsilon > 0 \text{ there exists a } \delta > 0 \text{ such that } ||Aa||_0 < \varepsilon, \text{ whenever } ||Ta|| < \delta.
\]

Let \( \mu \) be a symmetric p-stable measure generated by T, i.e. \( \tilde{\mu}(a) = \exp\{ -||Ta||^p \} \), \( a \in E' \). By the assumption, there exists a Sazonov space S, a linear continuous operator \( V : S \rightarrow E \) and a symmetric p-stable measure \( \nu \) on S such that \( \mu = V(\nu) \). Without loss of generality we can assume that V is 1-1. Let H be a \( A_p \)-operator on S' generating \( \nu \), i.e. \( \tilde{\nu}(b) = \exp\{ -||Hb||^p \} \), \( b \in S' \). We have \( \tilde{\mu}(a) = V(\nu)(a) = \tilde{\nu}(V^* a) = \exp\{ -||HV^* a||^p \} \). Hence

\[
(1.2) ||Ta|| = ||HV^* a|| \quad \text{for all } a \in E'.
\]

Define a linear mapping G from \( V^*(E') \) into \( L_0(\Omega, P) \) by \( G(V^* a) = Aa \). G is well-defined on \( V^*(E') \). Indeed, if \( V^* a_1 = V^* a_2 \), then by (1.2) we have \( ||T(a_1 - a_2)|| = 0 \), which, together with (1.1), enables us to conclude that \( ||A(a_1 - a_2)||_0 = 0 \), i.e. \( Aa_1 = Aa_2 \) in \( L_0(\Omega, P) \). In view of (1.1), for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( ||G(b)||_0 < \varepsilon \), whenever \( ||Hb|| < \delta \) for all \( b \in V^*(E') \). In other words, G is \( \mathcal{T}_p \)-continuous on \( V^*(E') \). The linearity of G is obvious. Since \( V^*(E') \) is dense in \( S' \), G admits a \( \mathcal{T}_p \)-continuous linear extension on the entire \( S' \). As S is of M-cotype p (every Sazonov space is of M-cotype p for all p), G is decomposed by an S-valued random variable \( \varphi \), i.e. \( G(b)(\omega) = \langle \varphi(\omega), b \rangle \) P-a.s. for all \( b \in S' \). Hence, for all \( a \in E' \),

\[
A(a)(\omega) = G(V^* a)(\omega) = \langle \varphi(\omega), V^* a \rangle = \langle V\varphi(\omega), a \rangle \quad \text{P-a.s.,}
\]

which shows that A is decomposable, as desired.

Thus the proof of Theorem 1 is completed.

From Theorem 1 we derive:

2. **Corollary.** If a Banach space E belongs to the class \( V_p \), then it also belongs to the class \( V_q \) for \( 1 \leq p < q \).

3. **Corollary.** The space \( l_s(l_t) \), where \( 1 \leq p < t < s < q \), is in the class \( V_q \) but is not in the class \( V_p \).
Proof. By Theorem 7 in [7], $l_q (l_t)$ is of $M$-cotype $q$, hence it is in the class $V_q$ by Theorem 1. Assume that $l_q (l_t)$ is in the class $V_p$. By Proposition 8 in [7], $l_q (l_t)$ is of stable type $p$, so it imbeds in $L_p$ by Theorem 4.5 in [10]. But this contradicts the Proposition 9 in [7].

Thus, it is reasonable to refer to a Banach space in the class $V_p$ as a Banach space of $V$-cotype $p$.

4. Concluding remarks. 1. If $E$ is of stable type $p$ ($1 \leq p < 2$), then, by Proposition 4.8 in [10] and Theorem 1, the following statements are equivalent:

1) $E$ is of $M$-cotype $p$.
2) $E$ is of $V$-cotype $p$.
3) $E$ is of $S$-cotype $p$.

It is natural to ask

Problem 1. Are the three possible notions of cotype equivalent in general?

2. Garling [2] characterized spaces of cotype 2 by the following property:

A Banach space $E$ is of cotype 2 iff every symmetric Gaussian measure on $E$ is the continuous image of a symmetric Gaussian measure on a Hilbert space.

It is known that every Hilbert space is a Sazonov space. On the other hand, since every Sazonov space $S$ is of cotype 2, every symmetric Gaussian measure on $S$ is the continuous image of a symmetric Gaussian measure on a Hilbert space. Then Garling's theorem can be stated as follows:

A Banach space $E$ is of cotype 2 iff every symmetric Gaussian measure on $E$ is the continuous image on a symmetric Gaussian measure on a Sazonov space.

We want to extend this fact to spaces of $S$-cotype $p$.

Problem 2. Is it true that a Banach space $E$ is of $S$-cotype $p$ iff every symmetric $p$-stable measure on $E$ is the continuous image of a symmetric $p$-stable measure on a Sazonov space?

In the proof of Theorem 1 we have shown that:

1° if every symmetric $p$-stable measure on $E$ is the continuous image of a symmetric $p$-stable measure on a Sazonov space, then $E$ must be of $S$-cotype $p$;

2° if $E$ is of $S$-cotype $p - \varepsilon$ ($p > 1$), then every symmetric $p$-stable measure on $E$ is the continuous image of a symmetric $p$-stable measure on a Sazonov space.

It should be noted that if the answer to Problem 2 is positive, then the answer to Problem 1 is also positive.

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