PICKANDS–PITERBARG CONSTANTS FOR SELF-SIMILAR GAUSSIAN PROCESSES

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Abstract. For a centered self-similar Gaussian process \( \{Y(t) : t \in [0, \infty)\} \) and \( R \geq 0 \) we analyze the asymptotic behavior of

\[
H^R_Y(T) = \mathbb{E} \exp \left( \sup_{t \in [0, T]} \left( \sqrt{2} Y(t) - (1 + R) \sigma_Y^2(t) \right) \right)
\]

as \( T \to \infty \). We prove that \( H^R_Y \to \lim_{T \to \infty} H^R_Y(T) \in (0, \infty) \) for \( R > 0 \) and

\[
H_Y = \lim_{T \to \infty} \frac{H^0_Y(T)}{T^\gamma} \in (0, \infty)
\]

for suitably chosen \( \gamma > 0 \). Additionally, we find bounds for \( H^R_Y, \ R > 0 \), and a surprising relation between \( H_Y \) and the classical Pickands constants.

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1. INTRODUCTION

For a centered Gaussian process \( \{Y(t) : t \in [0, \infty)\} \) with a.s. continuous sample paths, \( \text{Var}(Y(t)) = \sigma^2_Y(t) \) and \( Y(0) = 0 \) a.s., let

\[
H^R_Y(T) = \mathbb{E} \exp \left( \sup_{t \in [0, T]} \left( \sqrt{2} Y(t) - (1 + R) \sigma^2_Y(t) \right) \right).
\]

(1.1)

where \( R \geq 0 \) and let \( H_Y(T) := H^0_Y(T) \).

The functionals \( H^R_Y(T), H_Y(T) \) play an important role in many areas of probability theory. For example, consider a fractional Brownian motion \( \{B_\kappa(t) : t \in [0, \infty)\} \) with Hurst parameter \( \kappa/2 \in (0, 1] \), i.e. a centered Gaussian process with stationary increments, continuous sample paths a.s. and variance function \( \text{Var}(B_\kappa(t)) = t^\kappa \). Then, for \( \kappa \in (0, 2] \), the Pickands constants \( \mathcal{H}_{B_\kappa} \) defined
as

\[ H_{B^\kappa} = \lim_{T \to \infty} \frac{H_{B^\kappa}(T)}{T}, \tag{1.2} \]

and the Piterbarg constants \( H_{B^\kappa}^R \), for \( R > 0 \), defined as

\[ \mathcal{H}_{B^\kappa}^R = \lim_{T \to \infty} \mathcal{H}_{B^\kappa}^R(T) \tag{1.3} \]

play a key role in the extreme value theory of Gaussian processes; see, e.g., \([25]\), \([26]\), \([27]\) or more recent contributions \([18]\), \([24]\). In \([6]\) it was observed that the notion of Pickands and Piterbarg constants can be extended to generalized Pickands and Piterbarg constants, defined as

\[ \mathcal{H}_{\eta} = \lim_{T \to \infty} \frac{\mathcal{H}_{\eta}(T)}{T} \quad \text{and} \quad \mathcal{H}_{\eta}^R = \lim_{T \to \infty} \mathcal{H}_{\eta}(T) \]

respectively, where \( R > 0 \) and \( \{ \eta(t) : t \in (0, \infty) \} \) is a centered Gaussian process with stationary increments. We refer to \([2]\), \([3]\), \([5]\), \([12]\), \([14]\), \([16]\) for properties and other representations of \( \mathcal{H}_{B^\kappa} \), \( \mathcal{H}_{B^\kappa}^R \) and generalized Pickands–Piterbarg constants, and to \([10]\), \([11]\) for multidimensional analogs of Pickands–Piterbarg constants.

Recently (see e.g. \([12]\)), it was found that for general Gaussian processes \( Y \) (satisfying some regularity conditions) the functionals \( \{1.1\} \) appear in the formulas for exact asymptotics of suprema of some Gaussian processes (see Proposition \( 2.1)\).

The interest in \( \{1.1\} \) also stems from an important contribution \([16]\) which established a direct connection between Pickands constants and max-stationary stable processes (see also \([7]\), \([8]\), \([9]\)).

The constants \( \mathcal{H}_Y(T) \) also appear in the context of convex geometry where they are known as Wills functionals (see \([28]\)).

In this contribution we analyze the properties of \( \mathcal{H}_Y^R(T) \) and \( \mathcal{H}_Y(T) \) for a class of general self-similar Gaussian processes \( Y \) with non-stationary increments. In particular, we find analogs of limits \( \{1.2\}, \{1.3\} \) and give some bounds for them. Surprisingly, it appears that, up to some explicitly given constant, \( \mathcal{H}_Y \) is equal to the classical \( \mathcal{H}_{B^\kappa} \) for some appropriately chosen \( \kappa \).

2. NOTATION AND PRELIMINARY RESULTS

Let \( \{ Y(t) : t \geq 0 \} \) be a centered Gaussian process with a.s. continuous sample paths and let

\[ V_Y(s, t) := \text{Var}(Y(s) - Y(t)), \quad R_Y(s, t) := \text{Cov}(Y(s), Y(t)). \]

We say that a stochastic process \( Y(\cdot) \) is self-similar with index \( H > 0 \) if for all \( a > 0 \),

\[ \{Y(at) : t \geq 0\} \overset{D}{=} \{a^H Y(t) : t \geq 0\}. \tag{2.1} \]
A straightforward consequence of (2.1) is that for self-similar Gaussian processes, \( \sigma_Y^2(t) = \sigma_Y^2(1)t^{2H} \) for \( t \geq 0 \).

We write \( Y \in S(\alpha, \kappa, c_Y) \) if

**S1.** \( Y(\cdot) \) is self-similar with index \( \alpha/2 > 0 \) and \( \sigma_Y^2(1) = 1 \);

**S2.** there exist \( \kappa \in (0, 2] \) and \( c_Y > 0 \) such that

\[ \text{Var}(Y(1) - Y(1 - h)) = c_Y |h|^\kappa + o(|h|^\kappa) \quad \text{as } h \to 0. \]

It is well known (see Lamperti [21]) that \( \{Y(t) : t \geq 0\} \) is a self-similar Gaussian process with index \( \alpha/2 \) if and only if its Lamperti transform \( X(t) = e^{-(\alpha/2)t}Y(e^t) \) is a stationary Gaussian process. Thus, there is a unique correspondence between self-similar Gaussian processes and stationary Gaussian processes. In fact condition **S2** relates to regularity of the covariance function of the stationary counterpart of \( Y \). More precisely, let \( \{X(t) : t \in \mathbb{R}\} \) be a stationary Gaussian process such that \( R_X(t, 0) = 1 - a|t|^\kappa + o(|t|^\kappa) \) as \( t \to 0 \) with \( \kappa \in (0, 2], a > 0 \). Then one can check that the self-similar process \( Y(t) := t^{\alpha/2}X(\log t) \) for \( \alpha \in (0, 2] \) is \( S(\alpha, \kappa, c_Y) \) with

\[
c_Y = \begin{cases} 2a & \text{for } \kappa < 2, \\ \alpha^2/4 + 2a & \text{for } \kappa = 2. \end{cases}
\]

Below we specify some important classes of self-similar Gaussian processes that satisfy **S1–S2**.

- **Fractional Brownian motion** \( B_\alpha \) is in \( S(\alpha, \alpha, 1) \) with \( \alpha/2 \in (0, 1] \).

- **Bifractional Brownian motion** \( \{Y(1)(t) : t \geq 0\} \) with parameters \( \alpha \in (0, 2) \) and \( K \in (0, 1] \) is a centered Gaussian process with covariance function

  \[
  R_{Y(1)}(t, s) = \frac{1}{2^K}((t^\alpha + s^\alpha)^K - |t - s|^\alpha K)
  \]

(see e.g. [19], [22]). We have \( Y(1) \in S(\alpha K, \alpha K, 2^{1-K}) \).

- **Sub-fractional Brownian motion** \( \{Y(2)(t) : t \geq 0\} \) with parameter \( \alpha \in (0, 2) \) is a centered Gaussian process with covariance function

  \[
  R_{Y(2)}(t, s) = \frac{1}{2 - 2\alpha - 1} \left( t^\alpha + s^\alpha - \frac{(t + s)^\alpha + |t - s|^\alpha}{2} \right)
  \]

(see [4], [17]). Then \( Y(2) \in S(\alpha, \alpha, (2 - 2^\alpha)^{-1}) \).
\diamond \textit{k-fold integrated fractional Brownian motion} \{Y^{(3),k}(t) : t \geq 0\} with parameters \(k \in \mathbb{N} := \{1, 2, \ldots\}\) and \(\alpha \in (0, 2]\) is a Gaussian process defined as

\[
Y^{(3),1}(t) = \sqrt{\alpha + 2} \int_0^t B_\alpha(s) \, ds,
\]

\[
Y^{(3),k}(t) = \sqrt{\frac{k(\alpha + 2k)(\alpha + k - 1)}{\alpha + 2k - 2}} \int_0^t Y^{(3),k-1}(s) \, ds \quad \text{for } k \geq 2.
\]

Then \(Y^{(3),k} \in \mathbb{S}(\alpha + 2k, 2, \frac{k(\alpha + 2k)(\alpha + k - 1)}{\alpha + 2k - 2})\) for \(k \geq 1\).

\diamond \textit{Time-average of fractional Brownian motion} \{Y^{(4)}(t) : t \geq 0\} with parameter \(\alpha \in (0, 2]\) is a Gaussian process defined as

\[
Y^{(4)}(t) = \sqrt{\alpha + 2} \frac{1}{t} \int_0^t B_\alpha(s) \, ds.
\]

Its covariance function is

\[
R_{Y^{(4)}}(t, s) = \frac{(\alpha + 2)(s^{\alpha+1}t + st^{\alpha+1}) + |t-s|^{\alpha+2} - t^{\alpha+2} - s^{\alpha+2}}{2(\alpha + 1)ts}
\]

and we have \(Y^{(4)} \in \mathbb{S}(\alpha, 2, 1)\).

\diamond \textit{Dual fractional Brownian motion} \{Y^{(5)}(t) : t \geq 0\} with parameter \(\alpha \in (0, 2]\) is a centered Gaussian process defined as

\[
Y^{(5)}(t) = t^{\alpha+1} \sqrt{\frac{2}{\Gamma(\alpha+1)}} \int_0^\infty B_\alpha(s)e^{-st} \, ds
\]

(see [23]). We have

\[
R_{Y^{(5)}}(t, s) = \frac{t^{\alpha}s + s^{\alpha}t}{t + s}
\]

and \(Y^{(5)} \in \mathbb{S}(\alpha, 2, \alpha/2)\).

In the rest of the paper, \(X(s) := X(s)/\sigma_X(s)\), and \(\Psi(\cdot)\) denotes the tail distribution function of the standard normal random variable.

The following proposition plays a key role in the proofs of our main results, confirming also that the functionals \(\mathcal{H}_Y(\cdot)\) and \(\mathcal{H}_{Y,Y}^{(5)}(\cdot)\) for \(Y \in \mathbb{S}(\alpha, \kappa, c_Y)\) appear in the asymptotics of extremes of Gaussian processes.

**Proposition 2.1.** Let \(Y \in \mathbb{S}(\alpha, \kappa, c_Y)\) and let \(\{X(t) : t \geq 0\}\) be a centered Gaussian process with \(R_X(t, s) = \exp(-AV_Y(t, s))\) for \(a > 0\) and \(\sigma_X(t) = \frac{1}{1+bt^\beta}\) for \(b \geq 0, \beta > 0\).
(i) If $\alpha = \beta$, then as $u \to \infty$,
\[
P \left( \sup_{t \in [0, Tu^{-2/\alpha}]} X(t) > u \right) = \mathcal{H}_Y^{h/\alpha} (a^{1/\alpha} T) \Psi(u) (1 + o(1)).
\]

(ii) If $\alpha < \beta$, then as $u \to \infty$,
\[
P \left( \sup_{t \in [0, Tu^{-2/\alpha}]} X(t) > u \right) = \mathcal{H}_Y (a^{1/\alpha} T) \Psi(u) (1 + o(1)).
\]

The proof of Proposition 2.1 is given in Section 4.1.

3. PICKANDS–PITERBARG CONSTANTS FOR SELF-SIMILAR GAUSSIAN PROCESSES

The aim of this section is to find analogs of Pickands and Piterbarg constants for self-similar Gaussian processes $Y \in S(\alpha, \kappa, c_Y)$.

3.1. Piterbarg constants. For $R > 0$ and $Y \in S(\alpha, \kappa, c_Y)$ let us introduce an analog of the Piterbarg constant $H_{B_1}$ as follows:

\[
H_{Y}^{R} := \lim_{T \to \infty} \mathcal{H}_{Y}^{R} (T) = \lim_{T \to \infty} \mathbb{E} \exp \left( \sup_{t \in [0, T]} \left( \sqrt{2} Y(t) - (1 + R) t^{\alpha} \right) \right).
\]

In the next theorem we prove that $H_{Y}^{R}$ is well-defined and we compare it with the classical Piterbarg constants.

**Theorem 3.1.** Let $Y \in S(\alpha, \kappa, c_Y)$. Then, for any $R > 0$,
\[
H_{Y}^{R} \in (0, \infty).
\]

Furthermore
\[
H_{B_1}^{R/c_1} \leq H_{Y}^{R} \leq H_{B_1}^{R/c_2},
\]

where
\[
c_1 = \inf_{x \in (0, 1)} \frac{V_{Y}(1, x^{\kappa/\alpha})}{|1 - x|^\kappa} \quad \text{and} \quad c_2 = \sup_{x \in (0, 1)} \frac{V_{Y}(1, x^{\kappa/\alpha})}{|1 - x|^\kappa}.
\]

The proof of Theorem 3.1 is given in Section 4.2.

**Proposition 3.2.** Let $Y \in S(\alpha, \kappa, c_Y)$. Then
\[
H_{Y}^{R} \geq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{R}} \right).
\]

The proof of Proposition 3.2 is postponed to Section 4.3.

The following corollary follows from Theorem 3.1 combined with the fact that $H_{B_1}^{R} = 1 + 1/R$ (see, e.g., [13]) and $H_{B_2}^{R} = \frac{1}{2} (1 + \sqrt{1 + 1/R})$ (see, e.g., [20]).
**Corollary 3.3.** Let $Y \in \mathbb{S}(\alpha, \kappa, c_Y)$.

(i) If $\kappa = 1$, then

$$1 + \frac{1}{R} \left( \inf_{x \in [0,1]} \frac{V_Y(1, x^{1/\alpha})}{1 - x} \right) \leq \mathcal{H}_Y^R \leq 1 + \frac{1}{R} \left( \sup_{x \in [0,1]} \frac{V_Y(1, x^{1/\alpha})}{1 - x} \right).$$

(ii) If $\kappa = 2$, then

$$\frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{R}} \right) \leq \mathcal{H}_Y^R \leq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{4k(\alpha + k - 1)}{R(\alpha + 2k)(\alpha + 2k - 2)}} \right).$$

In the following example we specify Corollary 3.3 for some particular self-similar processes introduced in Section 2.

**Example 3.1.** The following bounds hold:

- $k$-fold integrated fractional Brownian motion $Y^{(3),k}$:

$$\frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{R}} \right) \leq \mathcal{H}_Y^{R^{(3),k}} \leq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{4k(\alpha + k - 1)}{R(\alpha + 2k)(\alpha + 2k - 2)}} \right).$$

- Time-average of fractional Brownian motion $Y^{(4)}$ with parameter $\alpha \in (0, 2)$:

$$\frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{R}} \right) \leq \mathcal{H}_Y^{R^{(4)}} \leq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{4}{R(\alpha + 2)}} \right).$$

The above bounds improve the results obtained in [15] for the constants

$$F_{\alpha} = \lim_{T \to \infty} \mathbb{E} \exp \left( \sup_{t \in (0, T]} \int_0^t \left( \sqrt{2B_\alpha(s) - s^\alpha} \right) ds \right) = \lim_{T \to \infty} \mathbb{E} \exp \left( \sup_{t \in (0, T]} \left( \frac{Y^{(4)}(t) - \alpha + 2}{\alpha + 1} t^\alpha \right) \right) = \mathcal{H}_Y^{1/(\alpha+1)},$$

leading to

$$\frac{1}{2} \left( 1 + \sqrt{2 + \alpha} \right) \leq F_{\alpha} \leq \frac{1}{2} \left( 1 + \sqrt{1 + 4(\alpha + 1)/\alpha^2} \right),$$

while in [15] it was proved that $F_{\alpha} \leq 2 + \alpha$ for $\alpha \in [1, 2)$.

- Dual fractional Brownian motion $Y^{(5)}$ with parameter $\alpha \in (0, 2)$:

$$\frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{R}} \right) \leq \mathcal{H}_Y^{R^{(5)}} \leq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{2}{R^2}} \right).$$
3.2. Pickands constants. In this section we focus on an analog of Pickands constants for $Y \in S(\alpha, \kappa, c_Y)$. Let

$$H_Y := \lim_{T \to \infty} \frac{\mathcal{H}_Y(T)}{T^{\alpha/\kappa}} = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0, T]} (\sqrt{2} Y(t) - t^\alpha))}{T^{\alpha/\kappa}}.$$ 

We observe that for $Y(t) = B_{\kappa}(t)$ the above definition agrees with the notion of the classical Pickands constant $H_{B_{\kappa}}$, since $\alpha = \kappa$ in this case.

In the following theorem we show that $H_Y$ is well-defined and find a surprising relation between $H_Y$ and $H_{B_{\kappa}}$.

**Theorem 3.4.** Let $Y \in S(\alpha, \kappa, c_Y)$. Then $H_Y \in (0, \infty)$ and

$$H_Y = \frac{\kappa}{\alpha} (c_Y)^{1/\kappa} H_{B_{\kappa}}.$$ 

A complete proof of Theorem 3.4 is presented in Section 4.4. The following corollary is an immediate consequence of Theorem 3.4 and the fact that $H_{B_1} = 1$ and $H_{B_2} = 1/\sqrt{\pi}$.

**Corollary 3.5.** Let $Y \in S(\alpha, \kappa, c_Y)$.

(i) If $\kappa = 1$, then

$$H_Y = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0, T]} (\sqrt{2} Y(t) - t^\alpha))}{T^{\alpha/2}} = \frac{c_Y}{\alpha}.$$ 

(ii) If $\kappa = 2$, then

$$H_Y = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0, T]} (\sqrt{2} Y(t) - t^\alpha))}{T^{\alpha/2}} = \frac{2}{\alpha} \sqrt{\frac{c_Y}{\pi}}.$$ 

In the following example we specify the findings of this section for self-similar Gaussian processes introduced in Section 2.

**Example 3.2.** The following equalities hold:

- □ Bifractional Brownian motion with parameters $\alpha \in (0, 2)$ and $K \in (0, 1]$:

$$H_{Y(1)} = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0, T]} (\sqrt{2} Y^{(1)}(t) - t^{\alpha K}))}{T} = 2^{\frac{1-K}{\alpha K}} H_{B_{\alpha K}}.$$ 

- □ Sub-fractional Brownian motion with parameter $\alpha \in (0, 2)$:

$$H_{Y(2)} = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0, T]} (\sqrt{2} Y^{(2)}(t) - t^\alpha))}{T} = (2 - 2^{\alpha-1})^{-1/\alpha} H_{B_{\alpha}}.$$
\( k \)-fold integrated fractional Brownian motion with parameters \( k \in \mathbb{N} \) and \( \alpha \in (0, 2] \):

\[
H_{Y^{(3)},k} = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0,T]}(\sqrt{2} Y^{(3)}(t) - t^{k+\alpha/2}))}{T^{k+\alpha/2}} = \sqrt{\frac{4k(\alpha + k - 1)}{\pi(\alpha + 2k)(\alpha + 2k - 2)}}.
\]

Time-average of fractional Brownian motion with parameter \( \alpha \in (0, 2] \):

\[
H_{Y^{(4)}} = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0,T]}(\sqrt{2} Y^{(4)}(t) - t^\alpha))}{T^{\alpha/2}} = \frac{2}{\sqrt{\pi \alpha}}.
\]

Dual fractional Brownian motion with parameter \( \alpha \in (0, 2] \):

\[
H_{Y^{(5)}} = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0,T]}(\sqrt{2} Y^{(5)}(t) - t^\alpha))}{T^{\alpha/2}} = \sqrt{\frac{2}{\pi \alpha}}.
\]

4. PROOFS

In the rest of the paper we use the notation \( v_Y(t) := V_Y(1, t) \). We begin with the following lemma, skipping its straightforward proof.

**Lemma 4.1.** Let \( Y \in S(\alpha, \kappa, c_Y) \) and \( \hat{Y}_{\alpha_1} = Y(t^{\alpha_1}) \) for some \( \alpha_1 > 0 \). Then, for any \( R \geq 0 \) and \( T > 0 \),

(i) \( H_{cY}^R(T) = H_{Y}^R(c^{2/\alpha}T) \) for any \( c > 0 \);

(ii) \( H_{Y_{\alpha_1}}^R(T) = H_{Y}^R(T^{\alpha_1}) \).

**4.1. Proof of Proposition 2.1.** In the next lemma we present a useful bound on \( V_Y(\cdot, \cdot) \) for \( Y \in S(\alpha, \kappa, c_Y) \).

**Lemma 4.2.** Let \( Y \in S(\alpha, \kappa, c_Y) \). Then there exists a positive constant \( C \) such that for \( \gamma = \min(\alpha, \kappa) \), \( T > 0 \) and all \( t, s \in [0, T] \),

\[
V_Y(t, s) \leq CT^{\alpha-\gamma}|t - s|^{\gamma}.
\]

**Proof.** For \( t = s \) the conclusion is obvious. Suppose that \( 0 \leq s < t \leq T \) and let \( \epsilon \in (0, 1) \) be such that for \( \delta \in (0, 1) \),

\[
(1 - \epsilon)c_Y|1 - x|^\kappa \leq V_Y(1, x) \leq (1 + \epsilon)c_Y|1 - x|^\kappa.
\]
for all $x \in [\delta, 1]$ (due to S2). For $s/t \geq \delta$ we have
\[
V_{Y}(t, s) = t^\alpha V_{Y}(1, s/t) \leq t^\alpha (1 + \epsilon) c_{Y} |1 - s/t|^\kappa
\leq t^\alpha (1 + \epsilon) c_{Y} |1 - s/t|^{\min(\alpha, \kappa)}
= t^{\alpha - \gamma} (1 + \epsilon) c_{Y} |t - s|^\gamma \leq T^{\alpha - \gamma} (1 + \epsilon) c_{Y} |t - s|^\gamma.
\]

For $s/t \leq \delta$ we have $|1 - \delta|^\gamma \leq |1 - s/t|^\gamma$. Hence, $t^\gamma |1 - \delta|^\gamma \leq |t - s|^\gamma$. Then
\[
V_{Y}(t, s) = t^\alpha V_{Y}(1, s/t)
\leq t^\gamma t^{\alpha - \gamma} |1 - \delta|^\gamma \max_{x \in [0, \delta]} V_{Y}(1, x) \leq T^{\alpha - \gamma} \max_{x \in [0, \delta]} V_{Y}(1, x)|1 - \delta|^\gamma.
\]

Hence the proof is completed with $C = \max ((1 + \epsilon) c_{Y}, \max_{x \in [0, \delta]} V_{Y}(1, x)|1 - \delta|^\gamma).$ ■

**Proof of Proposition 2.1.** Since for any Gaussian process $Y(\cdot, \cdot)$, the variogram function $V_{Y}(\cdot, \cdot)$ is negative definite, by the Schoenberg theorem the function $\exp(-V_{Y}(\cdot, \cdot))$ is positive definite. Thus there exists a Gaussian process $\{X(t) : t \geq 0\}$ with $R_{X}(t, s) = \exp(-V_{Y}(t, s)).$

The rest of the proof follows straightforwardly from [12, Theorem 2.1] and Lemma 4.2 applied to $X_{u}(t) = X(tu^{-2/\alpha}).$ ■

4.2. Proof of Theorem 3.1

**Lemma 4.3.** Let $Y \in S(\alpha, \kappa, c_{Y})$ and define $\hat{Y}(t) = Y(t^{\kappa/\alpha}).$ Then $\hat{Y} \in S(\kappa, \kappa, c_{Y}(\kappa/\alpha)\kappa)$ and there exist finite and positive constants
\[
c_{1} = \inf_{x \in [0, 1]} \frac{V_{Y}(1, x^{\kappa/\alpha})}{|1 - x|^\kappa} = \inf_{x \in [0, 1]} \frac{V_{Y}(1, x)}{|1 - x|^\kappa},
c_{2} = \sup_{x \in [0, 1]} \frac{V_{Y}(1, x^{\kappa/\alpha})}{|1 - x|^\kappa} = \sup_{x \in [0, 1]} \frac{V_{Y}(1, x)}{|1 - x|^\kappa}.
\]

Moreover for all $t, s \geq 0$,
\[
c_{1} |t - s|^\kappa \leq V_{Y}(t^{\kappa/\alpha}, s^{\kappa/\alpha}) = V_{\hat{Y}}(t, s) \leq c_{2} |t - s|^\kappa.
\]

**Proof.** Observe that $\hat{Y} \in S(\kappa, \kappa, c_{Y}(\kappa/\alpha)\kappa)$ with $V_{\hat{Y}}(t, s) = V_{Y}(t^{\kappa/\alpha}, s^{\kappa/\alpha})$. Consider the function $f(x) = \frac{V_{Y}(1, x)}{|1 - x|^\kappa}$ for $x \in [0, 1)$. Due to S2, $\lim_{x \to 1^{-}} f(x) = c_{\hat{Y}} > 0,$ $f(0) = 1$ and $f(x) = 0$ only for $x = 1$. Hence, $c_{1}, c_{2} > 0$ exist.

Moreover, for all $t \geq s \geq 0$,
\[
c_{1} |t - s|^\kappa = c_{1} t^\kappa |1 - s/t|^\kappa \leq t^\kappa V_{Y}(1, s/t) \leq c_{2} t^\kappa |1 - s/t|^\kappa = c_{2} |t - s|^\kappa.
\]

This completes the proof. ■
Proof of Theorem 3.1 Let $R, T > 0$. Define $\hat{Y}(t) = Y(t^{\kappa/\alpha})$. Then $\hat{Y} \in S(\kappa, \kappa, c_{2}\kappa/\alpha)$ with $V_{\hat{Y}}(1) = V_{Y}(1, x^{\kappa/\alpha})$ and $c_{1}, c_{2} > 0$ exist. Let $\{X(t) : t \geq 0\}, \{X_{2}(t) : t \geq 0\}$ be centered Gaussian processes with $R_{X}(t, s) = \exp(-V_{Y}(t, s)), R_{X_{2}}(t, s) = \exp(-c_{1}V_{B_{\kappa}}(t, s))$ and $\sigma_{X}(t) = \sigma_{X_{2}}(t) = \frac{1}{1 + R_{x_{\kappa}}^2}$. Then, for all $t, s > 0$,

$$R_{X_{1}}(t, s) = \exp(-c_{1}V_{B_{\kappa}}(t, s)) \geq R_{X}^{0}(t, s) = \exp(-V_{Y}(t, s)) \geq \exp(-c_{2}V_{B_{\kappa}}(t, s)) = R_{X_{2}}(t, s)$$

and hence, due to Slepian’s inequality (see, e.g., [1, Corollary 2.4]) we find that for all $u > 0$,

$$P\left(\sup_{t \in [0, T^{u^{-2/\kappa}}]} X_{1}(t) > u\right) \leq P\left(\sup_{t \in [0, T^{u^{-2/\kappa}}]} X(t) > u\right)$$

$$\leq P\left(\sup_{t \in [0, T^{u^{-2/\kappa}}]} X_{2}(t) > u\right) .$$

Application of Proposition 2.1(i) to the inequalities above gives

$$H_{B_{\kappa}}^{R/c_{1}}(c_{1}^{1/\kappa}T) \leq H_{Y}^{R}(T) \leq H_{Y}^{R}(T^{\kappa/\alpha}) \leq H_{B_{\kappa}}^{R/c_{2}}(c_{2}^{1/\kappa}T) ,$$

where the equality follows from Lemma 4.1(ii). Note that all functions in (4.1) are increasing in $T$, and hence letting $T \to \infty$ in (4.1) completes the proof. ■

4.3. Proof of Proposition 3.2 Since $\sigma_{Y}^{2}(t) = t^{\alpha}$, the Schwarz inequality implies that $R_{Y}(t, s) \leq (ts)^{\alpha/2}$ for all $t, s > 0$. Therefore for $t, s > 0$,

$$V_{Y}(t, s) \geq t^{\alpha} + s^{\alpha} - 2(ts)^{\alpha/2} = |t^{\alpha/2} - s^{\alpha/2}|^{2} = V_{B_{2}}(t, s) ,$$

where $B_{2}(t) = B_{2}(t^{\alpha/2})$. Thus, by Slepian’s inequality,

$$H_{Y}^{R}(T) = \int \mathbb{E}^{x}P\left(\sup_{t \in [0, T]} (\sqrt{2} Y(t) - (1 + R)t^{\alpha} > x)\right) dx$$

$$\geq \int \mathbb{E}^{x}P\left(\sup_{t \in [0, T]} (\sqrt{2} \hat{B}_{2}(t) - (1 + R)t^{\alpha} > x)\right) dx$$

$$= H_{B_{2}}^{R}(T) = H_{B_{2}}^{R}(T^{\alpha/2}) ,$$

where the last equality follows from Lemma 4.1(ii). Letting $T \to \infty$ yields the conclusion. ■

4.4. Proof of Theorem 3.4 To prove Theorem 3.4 we need some technical lemmas.

Lemma 4.4. Let $\hat{Y} \in S(\kappa, \kappa, c_{\hat{Y}})$. For any $\epsilon \to 0^{+}$ there exists $\delta_{\epsilon} \to 0^{+}$ such that for any $T > 0$ and $A \geq T/\delta_{\epsilon}$,

$$(1 - \epsilon)c_{\hat{Y}}|t - s|^{\kappa} \leq V_{\hat{Y}}(A + t, A + s) \leq (1 + \epsilon)c_{\hat{Y}}|t - s|^{\kappa}$$

for all $t, s \in [0, T]$. 


Proof. Let \( \epsilon \in (0, 1) \) be sufficiently small such that

\[
(4.2) \quad (1 - \epsilon)c_Y|h|^{\kappa} \leq V_Y(1, 1 - h) \leq (1 + \epsilon)c_Y|h|^{\kappa}
\]

for all \( h \in [0, \delta_\epsilon] \) and \( \delta_\epsilon \in (0, 1) \) (due to S2). Then, for any \( T > 0 \), \( A > T/\delta_\epsilon \) and \( 0 \leq s \leq t \leq T \) we have \( t-s \leq T/A \leq \delta_\epsilon \). Combining the fact that

\[
V_Y(A + s, A + t) = (A + t)^\kappa V_Y(1, 1/A + t) = (A + t)^\kappa V_Y(1, 1 - t - s / A + t)
\]

with (4.2) for \( h = \frac{t-s}{A+t} \leq \delta_\epsilon \), we obtain the assertion. \( \blacksquare \)

Lemma 4.5. Let \( \hat{Y} \in S(\kappa, \kappa, c_Y) \) and let \( c_1, c_2 \) be as in Lemma 4.3. Consider a centered Gaussian process \( \{X(t) : t \geq 0\} \) with \( R_X(t, s) = \exp(-aV_Y(t, s)) \) with \( a > 0 \).

(i) Let \( \{X_i(t) : t \geq 0\} \), \( i = 1, 2 \), be centered stationary Gaussian processes with \( R_X_i(t, s) = \exp(-ac_1V_B(t, s)) \). Then for all \( u > 0 \) and any \( T, A = A(u) > 0 \),

\[
P\left( \sup_{t \in [0,T]} X_1(t) > u \right) \leq P\left( \sup_{t \in [A,A+T]} X(t) > u \right) \leq P\left( \sup_{t \in [0,T]} X_2(t) > u \right).
\]

(ii) For any \( \epsilon > 0 \), let \( \{X_i(t) : t \geq 0\} \), \( i = 1, 2 \), be centered stationary Gaussian processes with \( R_X_i(t, s) = \exp(-a(1 + (-1)^i\epsilon)c_YV_B(t, s)) \). Then for any \( \epsilon \to 0^+ \) there exists \( \delta_\epsilon \to 0^+ \) such that for any \( T > 0 \) and \( A = A(u) > T/\delta \),

\[
P\left( \sup_{t \in [0,T]} X_1(t) > u \right) \leq P\left( \sup_{t \in [A,A+T]} X(t) > u \right) \leq P\left( \sup_{t \in [0,T]} X_2(t) > u \right).
\]

Proof. (i) The argument is the same as for Theorem 3.1. From Lemma 4.3 we know that for all \( t, s \geq 0 \),

\[
V_{X_1}(t, s) \leq V_X(t, s) \leq V_{X_2}(t, s)
\]

and hence, due to Slepian’s inequality, for all \( u > 0 \),

\[
P\left( \sup_{t \in [A,A+T]} X_2(t) > u \right) \leq P\left( \sup_{t \in [A,A+T]} X(t) > u \right) \leq P\left( \sup_{t \in [A,A+T]} X_2(t) > u \right).
\]

Due to stationarity of \( X_i(\cdot) \) we obtain the assertion.
(ii) From Lemma 4.4 for any $\epsilon \to 0^+$ there exists $\delta_\epsilon \to 0^+$ such that for any $T > 0$ and $A \geq T/\delta_\epsilon$,

$$ (1 - \epsilon)c_\gamma |t - s|^\kappa \leq V_\gamma(t, s) \leq (1 + \epsilon)c_\gamma |t - s|^\kappa $$

for all $t, s \in [A, A + T]$. The same argument as in the proof of (i) completes the proof.

**Lemma 4.6.** Suppose that $\lim_{u \to \infty} f(u)/u = c$ for some $c > 0$. Under the notation of Lemma 4.5 there exist absolute constants $F, G > 0$ such that

$$ P\left( \sup_{t \in [A, A + T]} X(t) > f(u), \sup_{t \in [t_0, t_0 + T]} X(t) > f(u) \right) $$

$$ \leq FT^2 \exp(-G(t_0 - (A + T))^{\kappa}) \Psi(f(u)) $$

for all $t_0 > A + T > 0$, $T \geq 1$ and any $u \geq u_0 = (2ac_2)^2(t_0 + T)^{\kappa/2}$.

**Proof.** The argument is similar to the one given in, e.g., [6, Lemma 6.2] or [12, Theorem 2.1]. Thus we only present the main steps.

Let $u_0 = (2ac_2)^2(t_0 + T)^{\kappa/2}$ and $\{Z_u(t_1, t_2) : (t_1, t_2) \in [A, A + T] \times [t_0, t_0 + T]\}$, where $Z_u(t_1, t_2) = X(t_1u^{-2/\kappa}) + X(t_2u^{-2/\kappa})$. Note that

$$ P\left( \sup_{t \in [A, A + T]} X(t) > f(u), \sup_{t \in [t_0, t_0 + T]} X(t) > f(u) \right) $$

$$ \leq P\left( \sup_{(t_1, t_2) \in [A, A + T] \times [t_0, t_0 + T]} Z_u(t_1, t_2) > 2f(u) \right). $$

Since $(t_0 + T)u^{-2/\kappa} \leq (2ac_2)^{-1/\kappa}$, by Lemma 4.3 for all $t_1, t_2 \leq t_0 + T$.

$$ ac_1u^{-2}|t_2 - t_1|^\kappa \leq aV_\gamma(t_1u^{-2/\kappa}, t_2u^{-2/\kappa}) $$

$$ \leq ac_2u^{-2}|t_2 - t_1|^\kappa \leq ac_2|(t_0 + T)u^{-2/\kappa}|^\kappa \leq 1/2. $$

Hence, as $x \leq 2(1 - e^{-x}) \leq (1 - e^{-4x})$ for $x \in [0, 1/2]$, we obtain

$$ V_\gamma(t_1u^{-2/\kappa}, t_2u^{-2/\kappa}) = 2(1 - \exp(-aV_\gamma(t_1u^{-2/\kappa}, t_2u^{-2/\kappa}))) $$

$$ \geq aV_\gamma(t_1u^{-2/\kappa}, t_2u^{-2/\kappa}) \geq ac_1u^{-2}|t_2 - t_1|^\kappa, $$

$$ V_\gamma(t_1u^{-2/\kappa}, t_2u^{-2/\kappa}) \leq 2(1 - \exp(-ac_2u^{-2}|t_2 - t_1|^\kappa)) $$

$$ \leq (1 - \exp(-4ac_2u^{-2}|t_2 - t_1|^\kappa)) $$

for all $t_1, t_2 \leq t_0 + T$. Since

$$ \sigma^2_{Z_u}(t_1, t_2) = 2 + 2\exp(-aV_\gamma(t_1u^{-2/\kappa}, t_2u^{-2/\kappa})) $$

$$ = 4 - 2(1 - \exp(-aV_\gamma(t_1u^{-2/\kappa}, t_2u^{-2/\kappa}))), $$
from (4.5), for any \((t_1, t_2) \in [A, A + T] \times [t_0, t_0 + T]\),

(4.7) \[ 2 \leq \sigma_{Z_u}^2(t_1, t_2) \leq 4 - ac_1u^{-2}(t_0 - (A + T))^{\kappa}. \]

Now observe that

\[
P\left( \sup_{(t,s)\in[A,A+T]\times[t_0,t_0+T]} Z_u(t_1, t_2) > 2f(u) \right)
\leq P\left( \sup_{(t_1,t_2)\in[A,A+T]\times[t_0,t_0+T]} Z_u(t_1, t_2) > \frac{2f(u)}{\sqrt{4 - ac_1u^{-2}(t_0 - (A + T))^{\kappa}}} \right).
\]

Note that for any \((t_1, t_2), (s_1, s_2) \in [A, A + T] \times [t_0, t_0 + T]\), we have

(4.8) \[
\mathbf{Var}(\overline{Z}_u(t_1, t_2) - \overline{Z}_u(s_1, s_2)) \leq \frac{\mathbf{Var}(Z_u(t_1, t_2) - Z_u(s_1, s_2))}{\sigma_{Z_u}(t_1, t_2)^2} \tag{6}
\]

\[
\leq \frac{1}{2} E\left( (\overline{X}(t_1u^{-2/\kappa}) - \overline{X}(s_1u^{-2/\kappa})) + (\overline{X}(t_2u^{-2/\kappa}) - \overline{X}(s_2u^{-2/\kappa})) \right)^2
\leq V_{\overline{X}}(t_1u^{-2/\kappa}, s_1u^{-2/\kappa}) + V_{\overline{X}}(t_2u^{-2/\kappa}, s_2u^{-2/\kappa})
\leq (1 - \exp(-4ac_2u^{-2}|t_1 - s_1|^\kappa)) + (1 - \exp(-4ac_2u^{-2}|t_2 - s_2|^\kappa)),
\]

where the next-to-last inequality follows from \((x + y)^2 \leq 2(x^2 + y^2)\), and the last one follows from (4.6).

Denote \(u^* = \frac{2f(u)}{\sqrt{4 - ac_1u^{-2}(t_0 - (A + T))^{\kappa}}} \) and let \(c, \overline{c} > 0\) be constants such that \(c \leq f(u)/u \leq \overline{c}\) for all \(u \geq u_0\). Note by (4.4) that \(f(u) \leq u^* \leq \sqrt{8/\overline{c}} f(u)\) for \(u \geq u_0\). Hence, \(cu \leq u^* \leq \sqrt{8/\overline{c}}cu\) for \(u \geq u_0\), and therefore \(u^{-2} \leq \frac{8}{\overline{c}}u^* - 2\) for \(u \geq u_0\).

Consider two independent, identically distributed centered stationary Gaussian processes \(\{Z_{1,u^*}(t_1) : t_1 \geq 0\}, \{Z_{2,u^*}(t_2) : t_2 \geq 0\}\) with \(R_{Z_{1,u^*}}(t_1, s_1) = \exp(-\frac{32}{7}a\overline{c}^2c_2(u^*)^{-2}|t_1 - s_1|^\kappa)\) and let \(Z_{u^*}(t_1, t_2) = \frac{1}{\sqrt{2}}(Z_{1,u^*}(t_1) + Z_{2,u^*}(t_2))\). Hence, by (4.8), for any \((t_1, t_2), (s_1, s_2) \in [A, A + T] \times [t_0, t_0 + T]\),

\[
\mathbf{Var}(Z_u(t_1, t_2) - Z_u(s_1, s_2))
\leq (1 - \exp(-4ac_2u^{-2}|t_1 - s_1|^\kappa)) + (1 - \exp(-4ac_2u^{-2}|t_2 - s_2|^\kappa))
\leq (1 - \exp(-\frac{32}{7}a\overline{c}^2c_2(u^*)^{-2}|t_1 - s_1|^\kappa))
+ (1 - \exp(-\frac{32}{7}a\overline{c}^2c_2(u^*)^{-2}|t_2 - s_2|^\kappa))
= \mathbf{Var}(Z_{u^*}(t_1, t_2) - Z_{u^*}(s_1, s_2)),
\]
and due to Slepian’s inequality, we obtain

\[
\mathbb{P}
\left( \sup_{(t_1, t_2) \in [A, A+T] \times [t_0, t_0+T]} Z_u(t_1, t_2) > u^* \right)
\leq \mathbb{P}
\left( \sup_{(t_1, t_2) \in [A, A+T] \times [t_0, t_0+T]} Z_{u^*}(t_1, t_2) > u^* \right)
= \mathbb{P}
\left( \sup_{(t_1, t_2) \in [0, T]^2} Z_{u^*}(t_1, t_2) > u^* \right)
\]

as \( u \to \infty \), where the equality follows from stationarity of \( Z_{u^*}(. , .) \). Now

\[
\lim_{u^* \to \infty} \frac{\mathbb{P}\left( \sup_{(t_1, t_2) \in [0, T]^2} Z_{u^*}(t_1, t_2) > u^* \right)}{\Psi(u^*)} = \left( \mathcal{H}_{B_c}(\frac{16ac^2c_2/7}{1/\kappa}T) \right)^2
\leq (\mathcal{H}_{B_c}(1))^2 \max(1, (16ac^2c_2/7)^{2/\kappa})T^2,
\]

where the equality follows from, e.g., [6, Theorem 2.1] (see also [12, Theorem 3.1]) and the inequality follows from the fact that \( \mathcal{H}_{B_c}(AT) \leq T \max(1, A) \mathcal{H}_{B_c}(1) \) for any \( T > 1 \) and \( A > 0 \) [26, Corollary D.1]. Hence, there exists a constant \( F' \) (which does not depend on \( t_0, A, T \)) such that

\[
\mathbb{P}
\left( \sup_{(t_1, t_2) \in [0, T]^2} Z_{u^*}(t_1, t_2) > u^* \right) \leq F'T^2 \Psi(u^*)
\]

for all \( u^* \geq u_0^* = c u_0 \) (i.e. \( u \geq u_0 \)). Thus

\[
\mathbb{P}
\left( \sup_{t \in [A, A+T]u^{-2/\kappa}} \overline{X}(t) > f(u), \sup_{t \in [t_0, t_0+T]u^{-2/\kappa}} \overline{X}(t) > f(u) \right)
\leq F'T^2 \Psi(u^*)
\]

for \( u \geq u_0 \).

Since (in view of \( \frac{1}{1-x} \geq 1 + x \) for \( x \geq 0 \))

\[
(u^*)^2 = \frac{4f^2(u)}{4 - ac_1u^{-2}(t_0 - (A + T))^{\kappa}} \geq f^2(u) + \frac{ac_1}{4} \left( \frac{f(u)}{u} \right)^2 (t_0 - (A + T))^{\kappa}
\geq f^2(u) + \frac{ac_1c_2^2}{4} (t_0 - (A + T))^{\kappa},
\]

we have

\[
\Psi(u^*) \leq \exp\left( -\frac{1}{2} \left( f^2(u) + \frac{ac_1c_2^2}{4} (t_0 - (A + T))^{\kappa} \right) \right)
\leq \frac{\sqrt{2\pi}}{\sqrt{f^2(u)}} \exp\left( -\frac{ac_1c_2^2}{8} (t_0 - (A + T))^{\kappa} \right).
\]
Combination of (4.10) with (4.11) gives

\[
P \left( \sup_{t \in [A, A + T]} \mathcal{X}(t) > f(u), \quad \sup_{t \in [0, t_0 + T]} \mathcal{X}(t) > f(u) \right) 
\leq F_1 F'T^2 \exp \left( - \frac{ac_1 \xi^2}{8} (t_0 - (A + T))^\kappa \right) \Psi(f(u))
\]

for any \( u \geq u_0 \) and some positive constant \( F_1 \) such that \( \exp(\frac{-1}{2} f^2(u)) \leq F_1 \Psi(f(u)) \) for \( u > u_0 \). This completes the proof with \( F = F_1 F' \) and \( G = ac_1 \xi^2 / 8 \).

**Lemma 4.7.** With the notation of Lemma 4.5, there exist absolute constants \( F, G > 0 \) such that

\[
P \left( \sup_{t \in [A, A + T]} \mathcal{X}(t) > u, \quad \sup_{t \in [A + T, A + 2T]} \mathcal{X}(t) > u \right) 
\leq F(T^2 \exp(-G\sqrt{T^\kappa}) + \sqrt{T}) \Psi(u)
\]

for all \( A > 0, T > 1 \) and any \( u \geq u_0 = (2ac_2)^2(A + 2T)^{\kappa/2} \), i.e. \( (A + 2T)u^{-2/\kappa} \leq (2ac_2)^{-1/\kappa} \).

**Proof.** Let \( u_0 = (2ac_2)^2(A + 2T)^{\kappa/2} \) and \( \mathcal{X}_u(t) = \mathcal{X}(tu^{-2/\kappa}) \). We have

\[
P \left( \sup_{t \in [A, A + T]} \mathcal{X}_u(t) > u, \quad \sup_{t \in [A + T, A + 2T]} \mathcal{X}_u(t) > u \right) 
= P \left( \sup_{t \in [A, A + T]} \mathcal{X}_u(t) > u, \quad \sup_{t \in [A + T, A + 2T]} \mathcal{X}_u(t) > u \right) 
\leq P \left( \sup_{t \in [A, A + T]} \mathcal{X}_u(t) > u, \quad \sup_{t \in [A + T, A + \sqrt{T}]} \mathcal{X}_u(t) > u \right) 
\quad + P \left( \sup_{t \in [A + T, A + T + \sqrt{T}]} \mathcal{X}_u(t) > u \right) 
\leq F_1 T^2 \exp(-G\sqrt{T^{\kappa}}) \Psi(u) + P \left( \sup_{t \in [A + T, A + T + \sqrt{T}]} \mathcal{X}_u(t) > u \right),
\]

where the last inequality follows from Lemma 4.6 with \( t_0 = A + T + \sqrt{T} \). Applying Lemma 4.5(i) and Proposition 2.1 (choose \( b = 0 \)) to \( P(\sup_{t \in [A + T, A + T + \sqrt{T}]} \mathcal{X}_u(t) > u) \), we finally obtain, for sufficiently large \( u \geq u_0 \),
for some constant $F > 0$, where the next-to-last inequality follows from subadditivity of $H_{B_\kappa}(\cdot) [26, Corollary D.1].$  ■

Proof of Theorem 3.4. First, we prove the conclusion of the theorem for $\hat{Y} \in \mathcal{S}(\kappa, \kappa, c_Y(\kappa/\alpha)\kappa)$, where $\hat{Y}(t) = Y(t^\kappa/\alpha)$. Let $c_1, c_2$ be the constants of Lemma 4.3. Consider a centered Gaussian process $\{X(t) : t \geq 0\}$ with $R_X(t, s) = \exp(-V_{\hat{Y}}(t, s))$ and let $\bar{X}_u(t) = X(tu^{-2/\kappa})$. Let $n \in \mathbb{N}$ and choose $\epsilon_n \in (0, 1)$, $\delta_{\epsilon_n} = 1/n$ so that the conclusion of Lemma 4.5(ii) holds. We find a lower bound and an upper bound separately.

Upper bound. Let $T \in \mathbb{N}$ be such that $T > n$. For any $u > 0$,

\[
P\left(\sup_{t \in [0, T^2]} X_u(t) > u\right) \leq P\left(\sup_{t \in [0, T]} X_u(t) > u\right) + \sum_{k=n}^{T-1} P\left(\sup_{t \in [kT, (k+1)T]} X_u(t) > u\right).
\]

Applying Lemma 4.5 to the right side, by Proposition 2.1 we obtain

\[
\mathcal{H}_{\hat{Y}}(T^2) \leq \mathcal{H}_{B_\kappa}(c_2^{1/\kappa} nT) + (T - n)\mathcal{H}_{B_\kappa}\left(((1 + \epsilon_n)c_{\hat{Y}})^{1/\kappa}T\right),
\]

and hence

\[
\frac{\mathcal{H}_{\hat{Y}}(T^2)}{T^2} \leq \frac{\mathcal{H}_{B_\kappa}(c_2^{1/\kappa} nT)}{T^2} + \frac{((1 + \epsilon_n)c_{\hat{Y}})^{1/\kappa}T^2}{T^2} \frac{\mathcal{H}_{B_\kappa}\left(((1 + \epsilon_n)c_{\hat{Y}})^{1/\kappa}T\right)}{((1 + \epsilon_n)c_{\hat{Y}})^{1/\kappa}T}.
\]

Since $\lim_{S \to \infty} S^{-1}\mathcal{H}_{B_\kappa}(S) = \mathcal{H}_{B_\kappa}$, after letting $T \to \infty$ in (4.12) we get

\[
\limsup_{T \to \infty} \frac{\mathcal{H}_{\hat{Y}}(T)}{T} \leq ((1 + \epsilon_n)c_{\hat{Y}})^{1/\kappa}\mathcal{H}_{B_\kappa}.
\]

Since this bound holds for any $\epsilon_n \to 0^+$, we have

\[
\limsup_{T \to \infty} \frac{\mathcal{H}_{\hat{Y}}(T)}{T} \leq (c_{\hat{Y}})^{1/\kappa}\mathcal{H}_{B_\kappa}.
\]
**Lower bound.** For $T \in \mathbb{N}$ such that $T > n$, with $\Delta_k = [kT, (k+1)T]$, again from Bonferroni’s inequality we deduce that for any $u > 0$,

$$
\text{(4.14)} \quad P\left( \sup_{t \in [0,T^2]} X_u(t) > u \right) \geq P\left( \sup_{t \in [nT,T^2]} X_u(t) > u \right) \\
\geq \sum_{k=n}^{T-1} P\left( \sup_{t \in \Delta_k} X_u(t) > u \right) - \sum_{1 \leq k < l \leq T-1} P\left( \sup_{t \in \Delta_k} X_u(t) > u, \sup_{t \in \Delta_l} X_u(t) > u \right) \\
\geq \sum_{k=n}^{T-1} P\left( \sup_{t \in \Delta_k} X_u(t) > u \right) - \Sigma_1 - \Sigma_2,
$$

where

$$
\Sigma_1 = \sum_{k=1}^{T-2} P\left( \sup_{t \in \Delta_k} X_u(t) > u, \sup_{t \in \Delta_{k+1}} X_u(t) > u \right), \\
\Sigma_2 = \sum_{1 \leq k < l \neq k+1}^{T-1} P\left( \sup_{t \in \Delta_k} X_u(t) > u, \sup_{t \in \Delta_l} X_u(t) > u \right).$$

By Lemma 4.5(ii) and Proposition 2.1 as $u \to \infty$,

$$
\text{(4.15)} \quad \sum_{k=n}^{T-1} P\left( \sup_{t \in \Delta_k} X_u(t) > u \right) \geq (T - n) H_{B_\kappa} \left( \left( (1 - \epsilon_n) \hat{c}_Y \right)^{1/\kappa} T \right) \Psi(u) (1 + o(1)).
$$

From Lemma 4.7 for sufficiently large $u$, we have

$$
\text{(4.16)} \quad \Sigma_1 \leq TF_1 \left( T^2 \exp(-G_1 \sqrt{T^\kappa}) + \sqrt{T} \right) \Psi(u).
$$

From Lemma 4.6 for sufficiently large $u$,

$$
\text{(4.17)} \quad \Sigma_2 \leq T^2 F_2 T^2 \exp(-G_2 T^\kappa) \Psi(u).
$$

Inserting (4.15)–(4.17) in (4.14) (and using Proposition 2.1), we obtain

$$
\frac{H_{\hat{Y}}(T^2)}{T^2} \geq \frac{(T - n) H_{B_\kappa} \left( \left( (1 - \epsilon_n) \hat{c}_Y \right)^{1/\kappa} T \right) T^2}{F_1 \left( T^3 \exp(-G_1 \sqrt{T^\kappa}) + T^{3/2} \right) + F_2 T^4 \exp(-G_2 T^\kappa)}.
$$

Letting $T \to \infty$ and then $\epsilon_n \to 0^+$ yields

$$
\text{(4.18)} \quad \liminf_{T \to \infty} \frac{H_{\hat{Y}}(T)}{T} \geq \left( \hat{c}_Y \right)^{1/\kappa} H_{B_\kappa}.
$$

From the upper bound (4.13) and the lower bound (4.18) we conclude that

$$
\lim_{T \to \infty} \frac{H_{\hat{Y}}(T)}{T} = \left( \hat{c}_Y \right)^{1/\kappa} H_{B_\kappa}.
$$
For $Y \in \mathcal{S}(\alpha, \kappa, c_Y)$, $c_Y = c_Y (\kappa/\alpha) \kappa$ and from Lemma 4.1(ii) it follows that
\[
\frac{\kappa}{\alpha} (c_Y)^{1/\kappa} \mathcal{H}_{B_\kappa} = (c_Y)^{1/\kappa} \mathcal{H}_{B_\kappa} = \lim_{T \to \infty} \frac{\mathcal{H}_Y(T)}{T} = \lim_{T \to \infty} \frac{\mathcal{H}_Y(T)}{T^{\kappa/\alpha}} = \lim_{T \to \infty} \frac{\mathcal{H}_Y(T)}{T^{\alpha/\kappa}}.
\]

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