Abstract. The aim of this paper is to study the asymptotic behavior of aggregated Weyl multifractional Ornstein–Uhlenbeck processes mixed with Gamma random variables. This allows us to introduce a new class of processes, Gamma-mixed Weyl multifractional Ornstein–Uhlenbeck processes (GWmOU), and study their elementary properties such as Hausdorff dimension, local self-similarity and short-range dependence. We also prove that these processes approach the multifractional Brownian motion.

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1. INTRODUCTION

Fractional Ornstein–Uhlenbeck (fOU) processes are one of the most well studied and widely applied classes of stochastic processes [8]. Recently, in [10], an interesting class of processes, of interest for various applications, has been introduced employing sequences of fOU processes with random coefficients.

Let us first present a brief summary of their construction. Let $B^H = \{B^H(t), t \in \mathbb{R}\}$ be a fractional Brownian motion (fBm) with Hurst index $H > 1/2$, defined on a probability space $(\Omega_{B^H}, \mathcal{F}_{B^H}, \mathcal{P}_{B^H})$. Consider a sequence of stationary fOU processes $X_k^t$, $k \geq 1$, with random coefficients defined by the stochastic integral

\begin{equation}
X_t^k = \int_{-\infty}^t e^{\gamma_k(t-s)} dB_s^H, \quad t \in \mathbb{R},
\end{equation}

with initial condition $X_0^k = \int_{-\infty}^0 e^{\gamma_k(t-s)} dB_s^H$. The random coefficients
\( \gamma_k, k \geq 1, \) are independent random variables on a probability space \((\Omega_\gamma, \mathcal{F}_\gamma, P_\gamma)\) and for any \( k \geq 1, -\gamma_k \sim \Gamma(1 - h, \lambda)\) with \( 0 < h < 1 - H \) and \( \lambda > 0.\)

Assume that the family \( \{\gamma_k, k \geq 1\} \) is independent of \( B^H.\) The processes \( X_k, k \geq 1, \) defined above are \( P_\gamma\)-almost surely fOU processes (see [8]). Let

\[
Y_n(t) = \frac{1}{n} \sum_{k=1}^{n} X_k(t), \quad t \in \mathbb{R},
\]

denote the so-called aggregated process. It has been proven that as \( n \to \infty,\) \( (Y_n)_{n \geq 1} \) converges weakly and in \( L^2(\Omega_{B^H}) \) for fixed time, \( P_\gamma\)-almost surely to a stochastic process denoted by \( Y^\lambda := \{Y^\lambda(t), t \in \mathbb{R}\}, \) given by the stochastic integral

\[
Y^\lambda(t) = \int_{-\infty}^{t} \left( \frac{\lambda}{\lambda + t - s} \right)^{1-h} dB^H_s, \quad t \in \mathbb{R}.
\]

The limiting process \( Y^\lambda \) is stationary, almost self-similar and exhibits long-range dependence (see [13] or [10]). The asymptotic behavior of \( Y^\lambda \) with respect to \( \lambda \) has also been studied, as \( \lambda \) varies between \( \infty \) and \( 0. \) The process \( Y^\lambda \) ranges from a fBm with index \( H \) to a fBm with index \( h + H.\)

When \( B^H \) is a standard Brownian motion (i.e. \( H = 1/2, \)) Gamma-mixed Ornstein–Uhlenbeck processes have been studied in [13].

Our goal is to construct a new kind of processes, called Gamma-mixed Weyl multifractional Ornstein–Uhlenbeck (GWmOU) processes, in analogy to the limiting procedure that leads to the process defined in (1.3). In our construction we replace the processes \( X_k, 1 \leq k \leq n, \) in the aggregated process (1.2) by Weyl multifractional Ornstein–Uhlenbeck (WmOU) processes mixed with Gamma random variables defined by the Wiener integral

\[
X^k_{\alpha(t)}(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} (t-s)^{\alpha(t)-1} e^{\gamma_k(t-s)} dB_s, \quad t \in \mathbb{R},
\]

where \( B = \{B(s), s \in \mathbb{R}\} \) is a Brownian motion on \((\Omega_B, \mathcal{F}_B, P_B),\) and \( \gamma_k, k \geq 1, \) are independent random variables on \((\Omega_\gamma, \mathcal{F}_\gamma, P_\gamma),\) also independent of \( B,\) and for any \( k \geq 1, -\gamma_k \sim \Gamma(1 - h, \lambda)\) with \( 0 < h < 1 \) and \( \lambda > 0.\) Moreover, \( \alpha \) is a Hölder continuous function with exponent \( 0 < \beta \leq 1.\) The processes \( X^k, k \geq 1, \) are \( P_\gamma\)-almost surely WmOU processes (see Section 2).

We define a GWmOU, denoted \( Y^\lambda_{\alpha}, \) by

\[
Y^\lambda_{\alpha(t)}(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} \left( \frac{\lambda}{\lambda + t - s} \right)^{1-h} (t-s)^{\alpha(t)-1} dB_s, \quad t \in \mathbb{R}.
\]

It is non-stationary, locally asymptotically self-similar and exhibits short-range dependence. We will also study the Hölder exponent and the box and Hausdorff dimension of the process \( Y^\lambda_{\alpha}.\) In addition, we will investigate the asymptotic behavior of \( Y^\lambda_{\alpha} \) with respect to \( \lambda; \) we will prove that \( Y^\lambda_{\alpha} \) approaches the multifractional
Brownian motion (see [17]) as $\lambda \to \infty$, while its integrated renormalized process

$$\hat{Y}_{\alpha}^\lambda(t) = \lambda^{h-1} \int_0^t Y_{\alpha}^\lambda(s) \, ds, \quad t \geq 0,$$

(here we suppose that the function $\alpha$ is constant) converges to a fractional Brownian motion modulo a constant as $\lambda \to 0$.

The motivation of this work comes from two facts. On the one hand, Gamma-mixed processes are good models for various applications; for example, the limiting process $Y^\lambda$ defined by (1.3) is a successful model of heart rate variability and could also be a good model of a lot of Gaussian stationary data with long-range dependence (see [10], [13] for more details). Moreover, the so-called Gamma-mixed Poisson processes (also named Pólya processes) have many practical applications, one of them being the study of reliability of engineering systems [9]. On the other hand, multifractional Ornstein–Uhlenbeck processes are omnipresent in physics. For further details and references, we refer the reader to [15]. Also, for more details about the construction and study of several classes of multifractional processes, see e.g. [3], [2], [5], [4], [17], [19]. The above motivate mixing multifractional Ornstein–Uhlenbeck processes (Weyl version) with Gamma random variables, in order to introduce GWmOU processes, as a counterpart of the limiting process $Y^\lambda$, a new candidate to model several short range, variable fractal dimension and non-stationary physical phenomena.

The paper is structured as follows. Section 2 presents a short summary of results on WmOU processes. In Section 3 we introduce GWmOU processes as limits of aggregated Weyl multifractional Ornstein–Uhlenbeck processes mixed with Gamma-distributed random variables. Finally, Section 4 contains some interesting properties of GWmOU processes including their asymptotic behavior.

2. PRELIMINARIES

WmOU processes have been introduced as a multifractional generalization of Weyl fractional Ornstein–Uhlenbeck processes (WfOU).

Let us begin with a brief review of WfOU processes (see [14]). First, we recall some elementary definitions of fractional calculus (see [16], [18]). The Weyl fractional derivative of order $\alpha > 0$, denoted by $aD_t^\alpha$, for $a = -\infty$, can be defined by its inverse using the Weyl fractional integral,

$$aD_t^{-\alpha} f(t) = aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds, \quad t \geq a.$$  

For $n - 1 \leq \alpha < n$, $aD_t^\alpha$ is defined as the ordinary derivative of order $n$ of the Weyl fractional integral of order $n - \alpha$,

$$aD_t^\alpha = (d/dt)^n aD_t^{\alpha-n}.$$
WfOUs are stochastic processes obtained as solutions of the fractional Langevin equation
$$(aD_t + w)^\alpha X(t) = W(t), \quad \alpha > 0, \ w > 0,$$
where $W(t)$ is a Gaussian white noise. They are defined explicitly by the stochastic integral
$$X_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} e^{-w(t-s)} dB_s, \quad t \in \mathbb{R},$$
where $B = \{B(s), s \in \mathbb{R}\}$ is the standard Brownian motion and $\alpha > 1/2$ to ensure that $X_\alpha(t)$ has finite variance.

Similarly to the generalization of fractional Brownian motion to multifractional Brownian motion (see [17]), an extension of WfOU processes is obtained by replacing the parameter $\alpha$ by a Hölder continuous function with exponent $0 < \beta \leq 1$, i.e. there exists a constant $K$ such that
$$|\alpha(t) - \alpha(s)| \leq K|t - s|^\beta \quad \forall s, t,$$
and $\alpha(t) > 1/2$ for all $t$.

Let us recall WmOU processes and their properties needed in what follows. For more details we refer the reader to [15].

A WmOU process is a Gaussian process defined by the Wiener integral
$$X_{\alpha(t)}(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} (t-s)^{\alpha(t)-1} e^{-w(t-s)} dB_s, \quad t \in \mathbb{R}.$$

We have
\begin{equation}
E_B[(X_{\alpha(t)}(t + s) - X_{\alpha(t)}(t))^2] = \frac{-|s|^{2\alpha(t)-1}}{\Gamma(2\alpha(t)) \cos(\pi \alpha(t))} - 2|s|^2 w^{3-2\alpha(t)} S_{\alpha(t)}(w|s|),
\end{equation}
where $S_\vartheta(x)$ is a continuous function given explicitly by
$$S_\vartheta(x) = -\frac{\sqrt{\pi}}{8\Gamma(\vartheta) \cos(\pi \vartheta)} \left[ \sum_{m=0}^{\infty} \frac{x^{2m}}{2^{2m}(m+1)! \Gamma(m+5/2-\vartheta)} \right. \left. - \left( \frac{x}{2} \right)^{2\vartheta-1} \sum_{m=0}^{\infty} \frac{x^{2m}}{2^{2m}(m+1)! \Gamma(m+3/2+\vartheta)} \right]$$
for every $x > 0$ and $1/2 < \vartheta < 3/2$. The relevant variance is equal to
$$E[X_{\alpha(t)}(t)^2] = \frac{(2w)^{1-2\alpha(t)} \Gamma(2\alpha(t) - 1)}{\Gamma(\alpha(t))^2}.$$
On the other hand, for $s < t$ the covariance of the WmOU is given by

$$E[X_{\alpha(t)}(t) X_{\alpha(s)}(s)] = \frac{e^{-w(t-s)}(t-s)^{(\alpha(t)+\alpha(s)-1)}}{\Gamma(\alpha(t))} \psi(\alpha(s), \alpha(s) + \alpha(t); 2w(t-s)),$$

where $\psi(\alpha, \gamma; z)$ is the confluent hypergeometric function. The variance and the covariance functions are divergent when $w \to 0$. However, if we set $Z_{\alpha}(t) = X_{\alpha(t)}(t) - X_{\alpha(t)}(0)$, it has been proven in [15] that for $\alpha(t) \in (1/2, 3/2)$ and by identifying $\alpha(t)$ with $H(t)+1/2$, when $w \to 0$ the process $Z_{\alpha}(t)$ approaches (in the sense of finite-dimensional distributions) $B_{H(t)}(t)$, the multifractional Brownian motion (with moving average definition) defined in [17] by

$$B_{H(t)}(t) = \frac{1}{\Gamma(H(t) + 1/2)} \times \left( \int_{-\infty}^{0} [(t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2}] dB_s + \int_{0}^{t} (t-s)^{H(t)-1/2} dB_s \right).$$

For the basic properties of WmOU processes such as short-range dependence, local self-similarity and Hausdorff dimension, we refer the reader to [15].

Let us now recall a sufficient criterion for weak convergence, which will be needed in what follows. By Prokhorov’s theorem, the convergence of finite-dimensional distributions and tightness yield weak convergence. For processes $X_n$, $X$, $n \geq 1$, with paths in $C([a,b], \mathbb{R})$, one has the following sufficient criterion (Billingsley [6, Theorem 12.3], or [7]).

**THEOREM 2.1.** Suppose that the finite-dimensional distributions of the family $(X_n)_{n \geq 1}$ converge to those of $X$. If, in addition, there exist constants $\zeta > 0$, $\theta > 1$ and $c_{\zeta, \theta}$, depending only on $\zeta$ and $\theta$, such that for all $s, t \in [a, b]$ with $a, b \in \mathbb{R}$, $a < b$,

$$E[|X_n(t) - X_n(s)|^\zeta] \leq c_{\zeta, \theta}|t - s|^\theta$$

for all $n \geq 1$, then the family $(X_n)_{n \geq 1}$ is tight and consequently

$$X_n \to X \quad \text{weakly in } C[a, b] \text{ as } n \to \infty.$$

3. AGGREGATED WEYL MULTIFRACTIONAL ORNSTEIN–UHLENBECK PROCESSES MIXED WITH GAMMA DISTRIBUTION

Let us now consider a sequence of WmOU processes mixed with Gamma distribution random variables $X_{\alpha(t)} := \{X_{\alpha(t)}^k(t), t \in \mathbb{R}\}$ defined by the following Wiener integral:

$$X_{\alpha(t)}^k(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} (t-s)^{(\alpha(t)-1)} e^{\gamma_k(t-s)} dB_s, \quad t \in \mathbb{R},$$
where $B = \{B(s), s \in \mathbb{R}\}$ is a Brownian motion defined on a probability space $(\Omega_B, \mathcal{F}_B, P_B)$ and for any $k \geq 1$, $-\gamma_k \sim \Gamma(1-h, \lambda)$ with $0 < h < 1$ and $\lambda > 0$ are independent random variables, also independent of $B$, defined on a probability space $(\Omega_\gamma, \mathcal{F}_\gamma, P_\gamma)$.

The processes $X^k_\alpha$, $k \geq 1$, are $P_\gamma$-almost surely WmOU processes defined on $(\Omega_B, \mathcal{F}_B, P_B)$. We define their empirical mean by

$$Y_n^{\alpha(t)}(t) = \frac{1}{n} \sum_{k=1}^{n} X^k_\alpha(t)$$

for every $t \in \mathbb{R}$ and $n \geq 1$.

Throughout the paper we assume that

$$1/2 < \alpha_{\inf} \leq \alpha_{\sup} < 3/2,$$

where $\alpha_{\inf} := \inf_{t \in \mathbb{R}} \alpha(t)$ and $\alpha_{\sup} := \sup_{t \in \mathbb{R}} \alpha(t)$.

We will also need the following notations:

- $m_\alpha[a, b] = \min\{\alpha(t) : t \in [a, b]\}$ and $M_\alpha[a, b] = \max\{\alpha(t) : t \in [a, b]\}$ for all real $a < b$. $E_B$ and $E_\gamma$ denote the expectations with respect to $P_B$ and $P_\gamma$ respectively.

- $C$ denotes a generic constant depending only on $[a, b]$, $\lambda$ and $h$.

- $C^{x,y}$ denotes a generic constant depending on $[a, b]$, $\lambda$, $h$, $x$ and $y$ such that $0 < x < 2m_\alpha[a, b] - 1$ and $0 < y < 3/2 - h - M_\alpha[a, b]$.

- $C^{x,y}_\gamma$ denotes a generic constant depending on $[a, b]$, $\lambda$, $h$, $x$ and $y$ such that $0 < x < 2m_\alpha[a, b] - 1, \ 0 < y < 3/2 - h - M_\alpha[a, b] \text{ and } 0 \leq \eta < m_\alpha[a, b] - 1/2$.

3.1. The limit of aggregated processes. If $0 < h < 3/2 - \alpha_{\sup}$, we define a zero mean Gaussian process $Y_\alpha^\lambda := \{Y_\alpha^\lambda(t), t \in \mathbb{R}\}$ by

$$Y_\alpha^\lambda(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} \left( \frac{\lambda}{\lambda + t - s} \right)^{1-h} (t-s)^{\alpha(t)-1} dB_s, \quad t \in \mathbb{R}. $$

It is easy to see that the Wiener integral in (3.3) is well-defined. The process $Y_\alpha^\lambda$ will be called a Gamma-mixed Weyl multifractional Ornstein–Uhlenbeck process, abbreviated as GWMOU.

Given a compact interval $[a, b] \subset \mathbb{R}$, the following result proves that $P_\gamma$-a.s., as $n \rightarrow \infty$, $Y_n^{\alpha(t)}(t)$ converges to $Y_\alpha^\lambda(t)$ in $L^2(\Omega_B)$, uniformly in $t \in [a, b]$.

**Theorem 3.1.** Fix real numbers $a, b$ such that $a < b$. If $0 < h < 3/2 - M_\alpha[a, b]$, then $P_\gamma$-a.s.,

$$Y_n^{\alpha(t)}(t) \xrightarrow{n \rightarrow \infty} Y_\alpha^\lambda(t) \quad \text{in } L^2(\Omega_B)$$
uniformly in $t \in [a, b]$. In particular, if $0 < h < 3/2 - \alpha_{\text{sup}}$, then $P_{\gamma}$-a.s., for every $t \in \mathbb{R}$,

\begin{equation}
Y^n_{\alpha(t)}(t) \xrightarrow{n \to \infty} Y^\lambda_{\alpha(t)}(t) \quad \text{in } L^2(\Omega_B).
\end{equation}

**Proof.** We prove (3.4). For every $x > 0$, $n \geq 1$, set

\[ f_n(x) := \frac{1}{n} \sum_{k=1}^{n} e^{\gamma_k x}, \quad c(x) := E_{\gamma}[e^{\gamma_1 x}] = \left( \frac{\lambda}{\lambda + x} \right)^{1-h}. \]

By the law of large numbers, we have $P_{\gamma}$-a.s., for every $x > 0$,

\begin{equation}
\frac{1}{n} \sum_{k=1}^{n} e^{\gamma_k x} \xrightarrow{n \to \infty} c(x),
\end{equation}

and for every $c > 0$ and $d < 3/2 - h$,

\begin{equation}
\frac{1}{n} \sum_{k=1}^{n} \frac{e^{\gamma_k c}}{(-\gamma_k)^d - 1/2} \xrightarrow{n \to \infty} E_{\gamma} \left[ \frac{e^{\gamma_1 c}}{(-\gamma_1)^d - 1/2} \right] = \frac{\lambda^{1-h} \Gamma(3/2 - d - h)}{\Gamma(1-h)(\lambda + c)^{3/2 - d - h}}.
\end{equation}

Using the change of variable $u = t - s$, we can write

\[ E_B[(Y^n_{\alpha(t)}(t) - Y^\lambda_{\alpha(t)}(t))^2] \]

\[ = \frac{1}{\Gamma(\alpha(t))^2} E_B \left[ \left( \int_{-\infty}^{t} (t-s)^{\alpha(t)-1} (f_n(t-s) - c(t-s)) dB_s \right)^2 \right] \]

\[ = \frac{1}{\Gamma(\alpha(t))^2} \int_{-\infty}^{t} (t-s)^{2\alpha(t)-2} (f_n(t-s) - c(t-s))^2 \, ds \]

\[ = \frac{1}{\Gamma(\alpha(t))^2} \int_{0}^{\infty} u^{2\alpha(t)-2} (f_n(u) - c(u))^2 \, du. \]

Hence, for every $m \geq 2$ and $t \in [a, b]$,

\begin{equation}
E_B[(Y^n_{\alpha(t)}(t) - Y^\lambda_{\alpha(t)}(t))^2] \leq K \left[ \int_{0}^{1} u^{2m_{a,b}-2} (f_n(u) - c(u))^2 \, du + \int_{1}^{m} u^{2\alpha(t)-2} (f_n(u) - c(u))^2 \, du \right. \]

\[ \left. + \int_{m}^{\infty} u^{2\alpha(t)-2} (f_n(u) - c(u))^2 \, du \right] \leq K \left[ \int_{0}^{1} u^{2m_{a,b}-2} (f_n(u) - c(u))^2 \, du + \int_{1}^{m} u^{2M_{a,b}-2} (f_n(u) - c(u))^2 \, du \right. \]

\[ \left. + \int_{m}^{\infty} u^{2M_{a,b}-2} (f_n(u) - c(u))^2 \, du \right] := K[A(n, m) + B(n, m) + C(n, m)], \]
where \( K \) is the maximum of the continuous function \( z \mapsto 1/\Gamma(z) \) on the interval \([m_\alpha[a, b], M_\alpha[a, b]]\).

Combining (3.6), \( f_n(u) \leq 1, c(u) \leq 1 \) and (3.2) with Lebesgue’s dominated convergence theorem, we conclude that \( P_{\gamma}\text{-a.s.} \), for every \( m \geq 2 \),

\[
(3.9) \quad A(n, m) \xrightarrow{n \to \infty} 0, \quad B(n, m) \xrightarrow{n \to \infty} 0.
\]

Now we will estimate \( C(n, m) \) for all \( m \geq 2 \). We have

\[
C(n, m) = \int_m^\infty (f_n(u) - c(u))^2 u^{2M_\alpha[a, b]-2} du
\]

\[
\leq 2 \int_m^\infty f_n(u)^2 u^{2M_\alpha[a, b]-2} du + 2 \int_m^\infty c(u)^2 u^{2M_\alpha[a, b]-2} du.
\]

Moreover, by the change of variable \( v = (-\gamma_j - \gamma_k)u \) and \( 2\sqrt{(-\gamma_j)(-\gamma_k)} \leq -\gamma_j - \gamma_k \),

\[
\int_m^\infty f_n(u)^2 u^{2M_\alpha[a, b]-2} du = \frac{1}{n^2} \sum_{k,j=1}^n \int_m^\infty e^{\gamma_j u} e^{\gamma_k u} u^{2M_\alpha[a, b]-2} du
\]

\[
= \frac{1}{n^2} \sum_{k,j=1}^n \frac{1}{(-\gamma_j - \gamma_k)^{2M_\alpha[a, b]-1}} \int_m^\infty v^{2M_\alpha[a, b]-2} e^{-v} dv
\]

\[
\leq \frac{2^{1-2M_\alpha[a, b]}}{n^2} \sum_{k,j=1}^n \frac{e^{-\frac{m}{2}(-\gamma_j - \gamma_k)}}{((-\gamma_j)(-\gamma_k))^{M_\alpha[a, b]-1/2}} \int_m^\infty v^{2M_\alpha[a, b]-2} e^{-v/2} dv
\]

\[
\leq \Gamma(2M_\alpha[a, b] - 1) \left( \frac{1}{n} \sum_{j=1}^n \frac{e^{-\frac{m}{2}(-\gamma_j)}}{((-\gamma_j)(-\gamma_k))^{M_\alpha[a, b]-1/2}} \right)^2.
\]

Combining this with (3.7) we get, \( P_{\gamma}\text{-a.s.} \),

\[
\limsup_{n \to \infty} \int_m^\infty f_n(u)^2 u^{2M_\alpha[a, b]-2} du
\]

\[
\leq \Gamma(2M_\alpha[a, b] - 1) \left( \frac{\lambda^{1-h} \Gamma(3/2 - M_\alpha[a, b])/(-h)}{\Gamma(1-h)(\lambda + m/2)^{3/2-M_\alpha[a, b]}} \right)^2 \xrightarrow{m \to \infty} 0.
\]

On the other hand, since

\[
\int_0^\infty \left( \frac{\lambda}{\lambda + u} \right)^{2-2h} u^{2M_\alpha[a, b]-2} du
\]

\[
= \lambda^{2M_\alpha[a, b]-1} \beta(3 - 2M_\alpha[a, b] - 2h, 2M_\alpha[a, b] - 1) < \infty,
\]
we have
\[ \int c(u)^2 u^{2M_{[a, b]} - 2} \, du = \int \left( \frac{\lambda}{\lambda + u} \right)^{2-2h} u^{2M_{[a, b]} - 2} \, du \xrightarrow{m \to \infty} 0. \]
which implies that \( P_{\gamma}\)-a.s.,
\[ \limsup_{n \to \infty} C(n, m) \xrightarrow{m \to \infty} 0. \tag{3.10} \]
Therefore, by applying the convergences (3.9) and (3.10) in (3.8) we deduce that \( P_{\gamma}\)-a.s.,
\[ \limsup_{n \to \infty} \sup_{t \in [a, b]} E_B[(Y_{\alpha(t)}^n(t) - Y_{\lambda,\alpha(t)}(t))^2] = 0, \]
which finishes the proof of (3.4).

Finally, the convergence (3.5) is a direct consequence of (3.4), (3.2) and \( 0 < h < 3/2 - \alpha_{\sup}. \)

The weak convergence of the sequence \((Y_{\alpha}^n)_{n \geq 1}\) is established in our next theorem.

**Theorem 3.2.** Fix real \( a < b \). Suppose that \( 0 < h < 3/2 - M_{[a, b]} \) and \( \min\{2m_{[a, b]} - 1, 2\beta\} < 1. \) Then \( P_{\gamma}\)-a.s.,
\[ Y_{\alpha}^n \xrightarrow{n \to \infty} Y_{\alpha}^\lambda \quad \text{in} \, C[a, b], \tag{3.11} \]
where \( C[a, b] \) is the space of continuous functions on \([a, b]\).

**Proof.** First, since \( P_{\gamma}\)-almost surely, \( Y_{\alpha}^n \) and \( Y_{\alpha}^\lambda \) are zero mean Gaussian processes whose finite-dimensional distributions are determined by their covariances, (3.4) implies the convergence \( P_{\gamma}\)-almost surely of the finite-dimensional distributions of \((Y_{\alpha}^n)_{n \geq 1}\) to those of \( Y_{\alpha}^\lambda \). Thus, in order to prove (3.11) it remains to prove the \( P_{\gamma}\)-a.s. tightness of \((Y_{\alpha}^n)_{n \geq 1}\) by using Theorem 2.1.

Throughout the proof all the results are given \( P_{\gamma}\)-almost surely.

Let \( t, t + \tau \in [a, b] \) be such that \(|\tau| < \min(\lambda/2, 1)\). Then
\[ E_B[(Y_{\alpha(t+\tau)}^n(t + \tau) - Y_{\alpha(t)}^n(t))^2] \]
\[ = E_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} (X_{\alpha(t+\tau)}^k(t + \tau) - X_{\alpha(t)}^k(t)) \right)^2 \right] \]
\[ \leq 2E_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} U_{t}^k(\tau) \right)^2 \right] + 2E_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} V_{t}^k(\tau) \right)^2 \right], \tag{3.12} \]
where
\[ U_{t}^k(\tau) := X_{\alpha(t)}^k(t + \tau) - X_{\alpha(t)}^k(t), \quad V_{t}^k(\tau) := X_{\alpha(t+\tau)}^k(t + \tau) - X_{\alpha(t)}^k(t + \tau). \]
We will first prove that for every \( n \geq 1 \),

\[
(3.13) \quad E_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} U_i^k(\tau) \right)^2 \right] \leq C|\tau|^{2m_\alpha[a,b]-1}. 
\]

To this end, by using Hölder’s inequality and (2.1), we can write

\[
E_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} U_i^k(\tau) \right)^2 \right] \leq \frac{1}{n} \sum_{k=1}^{n} E_B[U_i^k(\tau)^2] = \frac{-|\tau|^{2\alpha(t)-1}}{\Gamma(2\alpha(t)) \cos(\pi \alpha(t))} - 2|\tau|^{2} \frac{1}{n} \sum_{k=1}^{n} (-\gamma_k)^{3-2\alpha(t)} S_{\alpha(t)}(-\gamma_k|\tau|). 
\]

Since \( 1/2 < \alpha(t) < 3/2 \) and \( \cos(\pi \alpha(t)) < 0 \), we get

\[
-S_{\alpha(t)}(-\gamma_k|\tau|) 
= \frac{\sqrt{\pi}}{8\Gamma(\alpha(t)) \cos(\pi \alpha(t))} \sum_{m=0}^{\infty} \frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}(m+1)! \Gamma(m+5/2-\alpha(t))} 
- \frac{\sqrt{\pi}}{8\Gamma(\alpha(t)) \cos(\pi \alpha(t))} \left( \frac{-\gamma_k|\tau|}{2} \right)^{2\alpha(t)-1} \sum_{m=0}^{\infty} \frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}(m+1)! \Gamma(m+3/2+\alpha(t))} 
\leq C(-\gamma_k|\tau|)^{2\alpha(t)-1} \sum_{m=0}^{\infty} \frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}(m+1)!^2},
\]

where the last inequality comes from \( \Gamma(m+3/2+\alpha(t)) \geq (m+1)! \) and the fact that the functions \( \Gamma(x) \) and \( \cos(\pi x) \) are continuous at every \( x \) with \( 1/2 < m_\alpha[a,b] \leq x \leq M_\alpha[a,b] < 3/2 \).

As a consequence,

\[
(3.14) \quad E_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} U_i^k(\tau) \right)^2 \right] \leq C|\tau|^{2\alpha(t)-1} \left( 1 + \frac{1}{n} \sum_{k=1}^{n} (-\gamma_k)^{2} \sum_{m=0}^{\infty} \frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}(m+1)!^2} \right). 
\]

Moreover, by the law of large numbers, we obtain

\[
(3.15) \quad \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (-\gamma_k)^{2} \sum_{m=0}^{\infty} \frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}(m+1)!^2} 
= E_\gamma \left[ (-\gamma_1)^{2} \sum_{m=0}^{\infty} \frac{(-\gamma|\tau|)^{2m}}{2^{2m}((m+1)!)^2} \right] \leq \frac{1}{\Gamma(1-h)\lambda^2} \sum_{m=0}^{\infty} \frac{\Gamma(2m+3-h)}{2^{2m}((m+1)!)^2} \left( \frac{|\tau|}{\lambda} \right)^{2m} 
\leq C \sum_{m=0}^{\infty} \frac{\Gamma(2m+3-h)}{2^{2m}((m+1)!)^2} \left( \frac{1}{2} \right)^{2m} < \infty,
\]
where we have used the fact that the radius of convergence of the power series \( \sum_{m=0}^{\infty} \frac{\Gamma(2m+3-h)}{2^{2m}(m+1)!} x^m \) is 1. By combining (3.14) and (3.15), we obtain (3.13).

Let us now turn to the second term in (3.12). It remains to prove that for every \( n \geq 1 \),

\[
(3.16) \quad E_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} V_{t,1}^k(\tau) \right)^2 \right] \leq C \delta |\tau|^{2\beta}.
\]

To this end, from (3.1) we can write

\[
(3.17) \quad V_t^k(\tau) = V_{t,1}^k(\tau) + V_{t,2}^k(\tau),
\]

where

\[
V_{t,1}^k(\tau) = \left( \frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))} \right) \int_{-\infty}^{t+\tau} (t+\tau-u)^{\alpha(t+\tau)-1} e^{\gamma_k(t+\tau-u)} dB_u,
\]

\[
V_{t,2}^k(\tau) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t+\tau} \left( (t+\tau-u)^{\alpha(t+\tau)-1} - (t+\tau-u)^{\alpha(t)-1} \right) e^{\gamma_k(t+\tau-u)} dB_u.
\]

Then

\[
E_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} V_t^k(\tau) \right)^2 \right] \leq 2E_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} V_{t,1}^k(\tau) \right)^2 \right] + 2E_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} V_{t,2}^k(\tau) \right)^2 \right].
\]

Combining the mean value theorem and the fact that any continuous function has a maximum on any compact interval, we get

\[
\left| \frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))} \right|^2 \Gamma(2\alpha(t+\tau) - 1) \leq C |\alpha(t+\tau) - \alpha(t)|^2.
\]

Moreover, since \( \alpha \) is \( \beta \)-Hölder continuous, and since \( 2\sqrt{(-\gamma_j)(-\gamma_k)} \leq -\gamma_j - \gamma_k \) and \( 1 - 2\alpha(t+\tau) < 0 \), we have

\[
E_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} V_{t,1}^k(\tau) \right)^2 \right] = \left( \frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))} \right)^2 \frac{1}{n^2} \sum_{j,k=1}^{n} \int_{-\infty}^{t+\tau} (t+\tau-u)^{2\alpha(t+\tau)-2} e^{(\gamma_j+\gamma_k)(t+\tau-u)} du
\]

\[
= \left( \frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))} \right)^2 \Gamma(2\alpha(t+\tau) - 1) \frac{1}{n^2} \sum_{j,k=1}^{n} (-\gamma_j - \gamma_k)^{1-2\alpha(t+\tau)}
\]

\[
\leq C |\tau|^{2\beta} \left[ \frac{1}{n} \sum_{k=1}^{n} \left( -\gamma_k \right)^{1/2-\alpha(t+\tau)} \right]^2.
\]
Moreover,
\[
\frac{1}{n} \sum_{k=1}^{n} (-\gamma k)^{1/2-\alpha(t+\tau)} \xrightarrow{n \to \infty} \frac{\lambda^\alpha(t+\tau)-1/2}{\Gamma(1-h)} \Gamma(3/2 - \alpha(t + \tau) - h) < \infty.
\]

Thus, we conclude that for every \( n \geq 1, \)
\[
(3.18) \quad \mathbb{E}_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} V_{t,1}^k(\tau) \right)^2 \right] \leq C|\tau|^{2\beta}.
\]

On the other hand, by the change of variable \( t + \tau - u = x, \) we have
\[
\mathbb{E}_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} V_{t,2}^k(\tau) \right)^2 \right]
\]
\[
\quad = \frac{1}{\Gamma(\alpha(t))^2} \frac{1}{n^2} \sum_{j,k=1}^{n} \int_{-\infty}^{t+\tau} \left[ (t + \tau - u)^{\alpha(t+\tau)-1} - (t + \tau - u)^{\alpha(t)-1} \right]^2 e^{(\gamma_j + \gamma_k)(t+\tau-u)} du
\]
\[
\quad = \frac{1}{\Gamma(\alpha(t))^2} \frac{1}{n^2} \sum_{j,k=1}^{n} \int_{0}^{\infty} \left[ x^{\alpha(t+\tau)-1} - x^{\alpha(t)-1} \right]^2 e^{(\gamma_j + \gamma_k)x} dx
\]
\[
\quad = \frac{[\alpha(t + \tau) - \alpha(t)]^2}{\Gamma(\alpha(t))^2} \frac{1}{n^2} \sum_{j,k=1}^{n} \int_{0}^{\infty} \log(x) x^{\alpha(t+\tau)-1} e^{(\gamma_j + \gamma_k)x} dx
\]
\[
\quad = \frac{\alpha(t + \tau) - \alpha(t)}{\Gamma(\alpha(t))^2} \frac{1}{n^2} \sum_{j,k=1}^{n} \left( \int_{0}^{1} \log(x) x^{\alpha(t+\tau)-1} e^{(\gamma_j + \gamma_k)x} dx + \int_{1}^{\infty} \log(x) x^{\alpha(t+\tau)-1} e^{(\gamma_j + \gamma_k)x} dx \right)
\]
\[
\quad \leq C|\tau|^{2\beta} \left( \int_{0}^{1} x^{2\alpha \tau} e^{(\gamma_j + \gamma_k)x} dx + \frac{1}{n^2} \sum_{j,k=1}^{n} \int_{0}^{\infty} x^{2\alpha \tau + 2} e^{(\gamma_j + \gamma_k)x} dx \right)
\]
\[
\quad \leq C|\tau|^{2\beta} \left( \mu + \frac{1}{n^2} \sum_{j,k=1}^{n} \int_{0}^{\infty} x^{2\alpha \tau + 2} e^{(\gamma_j + \gamma_k)x} dx \right)
\]
for some \( c_{\tau,1} \in (m_{\alpha[a,b]}, M_{\alpha[a,b]}), \) where the last equality comes from the mean value theorem.

Let \( 0 < \delta < 2\alpha \tau + 1, \) \( 0 < \rho < 3/2 - \alpha \tau - h \) and define \( \mu = 1/(2\alpha \tau + 1 - \delta). \) Since \( \alpha \) is \( \beta \)-Hölder continuous, we can write
\[
\mathbb{E}_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} V_{t,2}^k(\tau) \right)^2 \right]
\]
\[
\quad = \frac{[\alpha(t + \tau) - \alpha(t)]^2}{\Gamma(\alpha(t))^2} \frac{1}{n^2} \sum_{j,k=1}^{n} \int_{0}^{\infty} \log(x) x^{\alpha(t+\tau)-1} e^{(\gamma_j + \gamma_k)x} dx
\]
\[
\quad = \frac{\alpha(t + \tau) - \alpha(t)}{\Gamma(\alpha(t))^2} \frac{1}{n^2} \sum_{j,k=1}^{n} \left( \int_{0}^{1} \log(x) x^{\alpha(t+\tau)-1} e^{(\gamma_j + \gamma_k)x} dx + \int_{1}^{\infty} \log(x) x^{\alpha(t+\tau)-1} e^{(\gamma_j + \gamma_k)x} dx \right)
\]
\[
\quad \leq C|\tau|^{2\beta} \left( \int_{0}^{1} x^{2\alpha \tau} e^{(\gamma_j + \gamma_k)x} dx + \frac{1}{n^2} \sum_{j,k=1}^{n} \int_{0}^{\infty} x^{2\alpha \tau + 2} e^{(\gamma_j + \gamma_k)x} dx \right)
\]
\[
\quad \leq C|\tau|^{2\beta} \left( \mu + \frac{1}{n^2} \sum_{j,k=1}^{n} \int_{0}^{\infty} x^{2\alpha \tau + 2} e^{(\gamma_j + \gamma_k)x} dx \right)
\]
Theorem 2.1, implies that the family for all

Consequently, given

Let

\[ a < b \]

\[ \min \]

\[\begin{align*}
    \leq C|\tau|^{2\beta} & \left( \mu + \frac{1}{n^2} \sum_{j,k=1}^{n} \frac{\Gamma(2M_\alpha[a,b]-1+2\rho)}{(-\gamma_j-\gamma_k)^{2M_\alpha[a,b]-1+2\rho}} \right) \\
    \leq C|\tau|^{2\beta} & \left( \mu + \frac{2^{1-2\rho-2M_\alpha[a,b]}}{n^2} \sum_{j,k=1}^{n} \frac{\Gamma(2M_\alpha[a,b]-1+2\rho)}{\sqrt{(-\gamma_j)(-\gamma_k)}^{2M_\alpha[a,b]-1+2\rho}} \right) \\
    = C|\tau|^{2\beta} & \times \left( \mu + 2^{1-2\rho-2M_\alpha[a,b]} \frac{\Gamma(2M_\alpha[a,b]-1+2\rho)}{\Gamma(1-h)} \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{(-\gamma_k)^{M_\alpha[a,b]-1/2+\rho}} \right)^2 \right). 
\end{align*}\]

Combining this with

\[ \frac{1}{n} \sum_{k=1}^{n} \frac{1}{(-\gamma_k)^{M_\alpha[a,b]-1/2+\rho}} \xrightarrow[n \to \infty]{} \frac{\lambda^{M_\alpha[a,b]-1/2+\rho}}{\Gamma(1-h)} \Gamma(3/2-M_\alpha[a,b]-h-\rho) < \infty, \]

we deduce that for every \( n \geq 1 \),

\[ E_B \left[ \left( \frac{1}{n} \sum_{k=1}^{n} V_{k,2}(\tau) \right)^2 \right] \leq C^{\delta,\rho}|\tau|^{2\beta}. \]

Thus, combining (3.18) and (3.19), we get (3.16).

Therefore, from (3.12), (3.13) and (3.16) we obtain, for every \( n \geq 1 \),

\[ E_B [(Y_{\alpha(t+\tau)}^n(t + \tau) - Y_{\alpha(t)}^n(t))^2] \leq C^{\delta,\rho}|\tau|^{\min\{2m_\alpha[a,b]-1,2\beta\}}. \]

Let \( a < b \). For \( s < t \in [a,b] \), we can find \( 2k+2 \) points \( u_1, \ldots, u_{2k+2} \in [s,t] \) with \( b-a = k \min \{ \lambda/2, 1 \} + c, 0 \leq c < \min \{ \lambda/2, 1 \} \) and \( 0 < u_{i+1} - u_i < \min \{ \lambda/2, 1 \} \) such that \( [t,s] = \bigcup_{i=1}^{2k+2} [u_i, u_{i+1}] \).

Using Minkowski’s inequality, (3.20) and Proposition 4.1 (because \( 0 < \min \{2m_\alpha[a,b] - 1, 2\beta\} < 1 \)) we conclude that for every \( n \geq 1 \) and \( s,t \in [a,b] \),

\[ E_B [(Y_{\alpha(t)}^n(t) - Y_{\alpha(s)}^n(s))^2] \leq C^{\delta,\rho}|t-s|^{\min\{2m_\alpha[a,b]-1,2\beta\}}. \]

Consequently, given \( r > 0 \) and using again the fact that \( Y_{\alpha}^n \) is \( P_{\gamma} \)-almost surely Gaussian, there exists a constant \( C_r \) depending only on \( r \) such that

\[ E_B [\left| Y_{\alpha(t)}^n(t) - Y_{\alpha(s)}^n(s) \right|^r] = C_r (E_B [(Y_{\alpha(t)}^n(t) - Y_{\alpha(s)}^n(s))^2])^{r/2} \leq C_r (C^{\delta,\rho})^{r/2}|t-s|^{r\min\{2m_\alpha[a,b]-1,2\beta\}} \]

for all \( n \geq 1 \) and \( s,t \in [a,b] \). If we choose so that \( r \min\{m_\alpha[a,b] - 1/2, \beta\} > 1 \), Theorem 2.1 implies that the family \( (Y_{\alpha}^n)_{n \geq 1} \) is tight, as desired.
3.2. Properties of GWmOU processes and asymptotic behavior with respect to \( \lambda \).

In this section we study several interesting properties of the GWmOU process \( Y_{\alpha}^\lambda \), such as the Hölder exponent and short-range dependence. In addition, we investigate the asymptotic behavior of \( Y_{\alpha}^\lambda \) when \( \lambda \to \infty \) and when \( \lambda \to 0 \).

Let us first compute the variance and the covariance of \( Y_{\alpha}^\lambda \). An easy computation shows that for all \( t \in \mathbb{R} \) the variance is given by

\[
E_B[Y_{\alpha(t)}^\lambda(t)^2] = \frac{\lambda^{2-2h}}{\Gamma(\alpha(t))^2} \int_{-\infty}^{t} (\lambda + t - s)^{2h-2} (t - s)^{2\alpha(t)-2} \, ds
\]

\[
= \frac{\lambda^{2\alpha(t)-1}}{\Gamma(\alpha(t))^2} \beta(3 - 2h - 2\alpha(t), 2\alpha(t) - 1),
\]

where \( \beta \) is the beta function defined by \( \beta(x, y) = \int_{0}^{1} u^{x-1} (1 - u)^{y-1} \, du \) for \( x, y > 0 \). Hence a GWmOU process is in general not stationary.

In addition, for \( s < t \), using the change of variable \( z = \lambda/(\lambda + s - u) \), the covariance of \( Y_{\alpha}^\lambda \) is given by

\[
E_B[ Y_{\alpha(t)}^\lambda(t) Y_{\alpha(s)}^\lambda(s) ] = \frac{1}{\Gamma(\alpha(t))\Gamma(\alpha(s))} \times \int_{-\infty}^{s} \left( \frac{\lambda}{\lambda + t - u} \right)^{1-h} \left( \frac{\lambda}{\lambda + s - u} \right)^{1-h} (t - u)^{\alpha(t)-1} (s - u)^{\alpha(s)-1} \, du
\]

\[
= \frac{\lambda^{\alpha(t)+\alpha(s)-1}}{\Gamma(\alpha(t))\Gamma(\alpha(s))} G(\alpha(t), \alpha(s), h, t - s/\lambda),
\]

with

\[
G(a, b, c, d) = \frac{1}{\Gamma(a)} \frac{(1 + dz)^{c-1}}{(1 + [d-1]z)^{1-a}} (1 - z)^{b-1} z^{-[a+b]-2c} \, dz.
\]

In order to study the local properties of GWmOU processes we will need the following result.

**Proposition 3.1.** Fix a compact interval \( [a, b] \subset \mathbb{R} \).

1. If \( 0 < h < 3/2 - M_\alpha[a, b] \), then there exists a constant \( C^{\delta, \rho} \) such that

\[
E_B[(Y_{\alpha(t+\tau)}^\lambda(t+\tau) - Y_{\alpha(t)}^\lambda(t))^2] \leq C^{\delta, \rho} \tau^{\min\{2M_\alpha[a, b]^{-1}, 2\beta\}}
\]

for all \( t, t + \tau \in [a, b] \) with \( |\tau| < \min\{\lambda/2, 1\} \).

2. If \( 0 < h < 3/2 - M_\alpha[a, b] \), \( M_\alpha[a, b] < 1 \) and \( \alpha(t) - 1/2 < \beta \) for all \( t \), then

   a. there exist constants \( C_2 \) and \( \epsilon < 1 \) such that

\[
E_B[(Y_{\alpha(t+\tau)}^\lambda(t+\tau) - Y_{\alpha(t)}^\lambda(t))^2] \geq (C_2/2) \tau^{2M_\alpha[a, b]^{-1}}
\]

   for all \( t, t + \tau \in [a, b] \) with \( |\tau| < \epsilon \).
(b) as $\tau \to 0$,

\begin{equation}
E_B[(Y_{\alpha(t+\tau)}^\lambda(t+\tau) - Y_{\alpha(t)}^\lambda(t))^2] = C_2 |\tau|^{\alpha(t) - 1/2} + O(|\tau|^{2\alpha(t) - 1}).
\end{equation}

**Proof.** The inequality (3.23) is a direct consequence of (3.20) and (3.4).

Let us now prove (3.24). For convenience, for all $t, t + \tau \in [a, b]$ with $|\tau| < 1$, we set

\begin{align*}
U_t^\lambda(\tau) &= Y_{\alpha(t)}^\lambda(t + \tau) - Y_{\alpha(t)}^\lambda(t), \\
V_t^\lambda(\tau) &= Y_{\alpha(t+\tau)}^\lambda(t+\tau) - Y_{\alpha(t)}^\lambda(t+\tau) \\
&= V_{t,1}^\lambda(\tau) + V_{t,2}^\lambda(\tau),
\end{align*}

where

\begin{align*}
V_{t,1}^\lambda(\tau) &= \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right) t + \tau - u \frac{\lambda^{1-h}}{\lambda^{1+h}} dB_u, \\
V_{t,2}^\lambda(\tau) &= \frac{1}{\Gamma(\alpha(t))} \\
&\times \int_{-\infty}^{t + \tau} [(t + \tau - u)^{\alpha(t+\tau) - 1} - (t + \tau - u)^{\alpha(t) - 1}] \frac{\lambda^{1-h}}{\lambda^{1+h}} dB_u.
\end{align*}

Hence

\begin{equation}
E_B[(Y_{\alpha(t+\tau)}^\lambda(t+\tau) - Y_{\alpha(t)}^\lambda(t))^2] \geq E_B[U_t^\lambda(\tau)^2] + 2E_B[U_t^\lambda(\tau) V_t^\lambda(\tau)] \\
\geq E_B[U_t^\lambda(\tau)^2] - 2E_B[U_t^\lambda(\tau)^2]^{1/2}E[V_t^\lambda(\tau)^2]^{1/2}.
\end{equation}

The last inequality follows from the Cauchy–Schwarz inequality. By Lemma 4.1 below and the inequality (4.10), there exist constants $C_1$ and $C_2$ depending only on $[a, b]$, $\lambda$ and $h$ such that

\begin{equation}
C_2 |\tau|^{2\alpha(t) - 1} \leq E[(U_t^\lambda)^2] \leq C_1 |\tau|^{2\alpha(t) - 1}.
\end{equation}

On the other hand,

\begin{equation*}
E_B[V_t^\lambda(\tau)^2] \leq 2(E_B[V_{t,1}^\lambda(\tau)^2] + E_B[V_{t,2}^\lambda(\tau)^2]).
\end{equation*}

A standard computation combined with the mean value theorem and the fact that any continuous function has a maximum on any compact interval, we obtain

\begin{align*}
E_B[V_{t,1}^\lambda(\tau)^2] \\
&= \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right)^2 \lambda^{2\alpha(t+\tau) - 1}\beta(3 - 2\alpha(t + \tau) - 2h, 2\alpha(t + \tau) - 1) \\
&\leq C|\alpha(t + \tau) - \alpha(t)| \leq C|\tau|^{2\beta}.
\end{align*}
Moreover, by the change of variable \( x = t + \tau - u \), we have

\[
E_B[V_{t,2}^\lambda(\tau)^2] = \frac{\lambda^{2-2h}}{\Gamma(\alpha(t))^2} \int_{-\infty}^{t+\tau} [(t + \tau - u)^\alpha(t+\tau)-1 - (t + \tau - u)^\alpha(t)-1]^2 (\lambda + t + \tau - u)^{2h-2} du
\]

\[
= \frac{\lambda^{2-2h}}{\Gamma(\alpha(t))^2} \int_0^{\infty} [x^{\alpha(t+\tau)-1} - x^{\alpha(t)-1}]^2 (\lambda + x)^{2h-2} dx
\]

\[
= \frac{\lambda^{2-2h}}{\Gamma(\alpha(t))^2} \int_0^{\infty} \log(x)^2 x^{2\alpha(t),\tau-2} (\lambda + x)^{2h-2} du
\]

for some \( a_{t,\tau}^x \in (m_\alpha[a,b], M_\alpha[a,b]) \), the last equality following from the mean value theorem. Let \( 0 < \sigma < 2m_\alpha[a,b] - 1 \) and \( 0 < \varsigma < 3/2 - M_\alpha[a,b] - h \). Since \( \alpha \) is \( \beta \)-Hölder continuous, for \( c = 3 - 2h - 2M_\alpha[a,b] - 2\varsigma \) and \( d = 2M_\alpha[a,b] - 1 + 2\varsigma \) we have

\[
E_B[V_{t,2}^\lambda(\tau)^2] = \frac{\lambda^{2-2h}[\alpha(t + \tau) - \alpha(t)]}{\Gamma(\alpha(t))^2} \left( \int_1^\infty \log(x)^2 x^{2\alpha(t),\tau-2} (\lambda + x)^{2h-2} dx \right)
\]

\[
\leq C|\tau|^{2\beta} \left( \int_0^1 x^{2m_\alpha[a,b]-2-\sigma} dx + \int_1^\infty x^{2M_\alpha[a,b]-2+2\varsigma} (\lambda + x)^{2h-2} dx \right)
\]

\[
\leq C|\tau|^{2\beta} \left( 1/(2m_\alpha[a,b] - 1 - \sigma) + \beta(c,d) \right) \leq C^{\sigma,\varsigma}|\tau|^{2\beta}.
\]

We then deduce that

\[(3.28) \quad E_B[V_{t}^\lambda(\tau)^2] \leq C^{\sigma,\varsigma}|\tau|^{2\beta}.\]

Combining (3.27), (3.28) and the Cauchy–Schwarz inequality yields

\[(3.29) \quad |E[U_{t}^\lambda(\tau)V_{t}^\lambda(\tau)]| \leq E[U_{t}^\lambda(\tau)^{2}]^{1/2}E[V_{t}^\lambda(\tau)^{2}]^{1/2} \leq C^{\sigma,\varsigma}|\tau|^{\beta+\alpha(t)-1/2}.
\]

Thus, by plugging (3.27) and (3.29) in (3.26), we get

\[
E_B[(Y_{\alpha(t+\tau)}^\lambda(t + \tau) - Y_{\alpha(t)}^\lambda(t))^2] \geq C_2|\tau|^{2\alpha(t)-1} - C^{\sigma,\varsigma}|\tau|^{\alpha(t)-1/2+\beta}
\]

\[\geq |\tau|^{2M_\alpha[a,b]-1} (C_2 - C^{\sigma,\varsigma}|\tau|^{\beta-M_\alpha[a,b]+1/2}).\]

By assuming that \( \alpha(t) - 1/2 \leq M_\alpha[a,b] - 1/2 < \beta \), the function

\[g : \tau \mapsto C_2 - C^{\sigma,\varsigma}|\tau|^{\beta-M_\alpha[a,b]+1/2}\]

is continuous in \( \tau \) and converges to \( C_2 \) when \( \tau \to 0 \). So there exists \( \epsilon > 0 \) such that \( g(\tau) > C = C_2 \) for \( |\tau| < \epsilon \), which gives the inequality (3.24).
On the other hand, by the assumption $\alpha(t) - 1/2 \leq M_\alpha[a, b] - 1/2 < \beta$ and using the equivalence (4.11), (3.28) and (3.29), we immediately obtain (3.25).  

In the following, we state interesting properties of GWmOU processes such as continuity, Hölder exponent at $t$, Hausdorff dimension and local asymptotic self-similarity. The same properties hold for WmOU processes, the proofs of which are based on [15, Lemma 3.1], of which Proposition 3.1 is the counterpart for GWmOU processes. Having Proposition 3.1 at hand, the proofs for GWmOU processes proceed analogously to those in [15]. Therefore, we omit them.

3.2.1. Continuity

**Proposition 3.2.** The process $\{Y_\lambda^{\alpha(t)}(t), t \in \mathbb{R}\}$ admits a continuous modification.

In the following properties: Hölder exponent, Hausdorff dimension and local asymptotic self-similarity, we make the additional assumptions that $\alpha(t) - 1/2 < \beta$ for all $t$ in the domain of $\alpha$ and $M_\alpha[a, b] < 1$.

3.2.2. Hölder exponent

**Proposition 3.3.** Let $[a, b] \subset \mathbb{R}$ be an interval. For any $0 \leq \eta < m_\alpha[a, b] - 1/2$, with probability 1, there exists a constant $C_\eta^{\delta, \rho}$ such that

$$|Y_\lambda^{\alpha(t)}(t) - Y_\lambda^{\alpha(s)}(s)| \leq C_\eta^{\delta, \rho}|t - s|^{\eta} \quad \forall t, s \in [a, b].$$

We now turn to the Hölder continuity of GWmOU processes. Let us first recall the following definition.

**Definition 3.1.** A real-valued function is said to have Hölder exponent $\beta$ at a point $t_0$ if

$$\lim_{h \to 0} \frac{|f(t_0 + h) - f(t_0)|}{|h|^\gamma} = 0 \quad \text{for any } \gamma < \beta,$$

$$\limsup_{h \to 0} \frac{|f(t_0 + h) - f(t_0)|}{|h|^\gamma} = \infty \quad \text{for any } \gamma > \beta.$$

**Proposition 3.4.** With probability 1, the Hölder exponent of $Y_\lambda^{\alpha(t)}(t)$ at a point $t_0$ in the domain is $\alpha(t_0) - 1/2$.

3.2.3. Hausdorff dimension. Let $\dim_H A$, $\dim_B A$, and $\overline{\dim}_B A$ denote the Hausdorff dimension, the lower box dimension, and the upper box dimension of a set $A$ in $\mathbb{R}^n$, respectively. Given a compact interval $[a, b] \subset \mathbb{R}$, $G_\alpha[a, b] = \{(t, Y_\lambda^{\alpha(t)}(t)) : t \in [a, b]\}$ stands for the graph of the process $Y_\lambda^{\alpha(t)}(t)$ restricted to $[a, b]$. For more information on these notions see [11]. We now formulate our result.
**Proposition 3.5.** Let \([a, b]\) be an interval in the domain of definition of \(\alpha\). With probability 1, \(\dim_H G_\alpha[a, b] = \dim_B G_\alpha[a, b] = \dim_B G_\alpha[a, b] = 5/2 - m_\alpha[a, b].\)

### 3.2.4. Local asymptotic self-similarity. WmOU processes are locally asymptotically self-similar, in the following sense defined in \([4]\).

**Definition 3.2.** Let \(X(t)\) be a Gaussian process. We say that \(X(t)\) is **locally asymptotically self-similar** with parameter \(H\) at a point \(t_0\) if the limit process

\[
\left\{ \lim_{h \to 0^+} \frac{X(t_0 + hu) - X(t_0)}{h^H}, u \in \mathbb{R} \right\}
\]

exists and is nontrivial for every \(t_0\).

This property holds true for GWmOU processes. Before stating this result, let us first recall that a **fractional Brownian motion** with Hurst index \(H\) is a centered Gaussian process with covariance

\[
E[B^H(t)B^H(s)] = \frac{1}{2} [|t|^{2H} + |s|^{2H} - |t - s|^{2H}].
\]

**Proposition 3.6.** For any \(t_0\) the stochastic process

\[
\left\{ \lim_{h \to 0^+} \frac{Y^\lambda_{\alpha(t_0 + hu)}(t_0 + hu) - Y^\lambda_{\alpha(t_0)}(t_0)}{h^{\alpha(t_0)/2 - 1/4}}, u \in \mathbb{R} \right\}
\]

is, modulo a constant, a fractional Brownian motion with Hurst index \(\alpha(t_0)/2 - 1/4\).

### 3.2.5. Short-range dependence.** We are now interested in the strength of the dependence of GWmOU processes.

**Definition 3.3** (\([12]\)). Let \(X(t)\) be a Gaussian process with covariance denoted by \(c(s, t) = \text{cov}(X(s), X(t))\) and correlation \(\rho(s, t)\) defined by

\[
\rho(s, t) = \frac{c(s, t)}{\sqrt{c(t, t)c(s, s)}}.
\]

We say that \(X(t)\) is **long-range dependent** if

\[
\int_0^\infty |\rho(t, t + \tau)| d\tau = \infty,
\]

and it is **short-range dependent** if the integral is finite.
The following lemma provides an upper bound for the inverse of the variance of the process $Y^\lambda_{\alpha}$ with $0 < h < 3/2 - \alpha_{\text{sup}}$ and $1/2 < \alpha(t)$ for all $t$.

**Lemma 3.1.** For all $t$ the function $t \mapsto 1/E_B[Y^\lambda_{\alpha(t)}(t)^2]$ is upper bounded.

**Proof.** From (3.21), we find that

$$1/E_B[Y^\lambda_{\alpha(t)}(t)^2] = \frac{\lambda^{-2\alpha(t)}[2\alpha(t) - 1]\Gamma(\alpha(t))^2\Gamma(2 - 2h)}{\Gamma(2\alpha(t))\Gamma(3 - 2h - 2\alpha(t))}.$$  

The functions $z \mapsto \lambda^{1-2z}$, $z \mapsto 2z - 1$, $z \mapsto \Gamma(z)^2$, $z \mapsto \Gamma(2z)$ and $z \mapsto \Gamma(3 - 2h - 2z)$ are continuous for $z \in [1/2, \alpha_{\text{sup}}]$. As a consequence,

$$1/E_B[Y^\lambda_{\alpha(t)}(t)^2] \leq C.$$  

We are thus led to the following short-range dependence property of GWmOU processes.

**Proposition 3.7.** For $0 < h < 1 - \alpha_{\text{sup}}$, the GWmOU process is short-range dependent.

**Proof.** Set $y = \tau/\lambda$. Using (3.22) and (3.30), we have

$$0 \leq \rho_\alpha(t, t + \tau) \leq CG(\alpha(t + \tau), \alpha(t), h, y).$$

Since $0 \leq u \leq 1$ and $1/2 < \alpha(t) < 1$ for all $t$, we obtain

$$G(\alpha(t + \tau), \alpha(t), h, y)$$

$$= \frac{1}{0} u^{2-[\alpha(t)+\alpha(t+\tau)]} - 2h (1 - u)^{\alpha(t)-1}(1 + yu)^{h-1}(yu + 1 - u)^{\alpha(t+\tau)-1} du$$

$$\leq y^{\alpha(t+\tau)-1}(y + 1)^{h-1} \frac{1}{0} u^{-\alpha(t)-h}(1 - u)^{\alpha(t)-1} du.$$  

Therefore,

$$\int_0^\infty |\rho_\alpha(t, t + \tau)| d\tau \leq C \int_0^\infty y^{\alpha(t+\lambda y)-1}(y + 1)^{h-1} dy \int_0^1 u^{-\alpha(t)-h}(1 - u)^{\alpha(t)-1} du \leq C\beta(1 - h - \alpha_{\text{sup}}, 1/2)\beta(1 - h - \alpha(t), \alpha(t)) < \infty,$$

since $0 < h < 1 - \alpha_{\text{sup}}$.

We are now interested in the asymptotic behavior of the process $Y^\lambda_{\alpha}$ when $\lambda \to \infty$. 

**Proposition 3.8.** Let $\{Y^λ_{α(t)}(t), t \geq 0\}$ be a GWmOU process restricted to $t \geq 0$ and set $α(t) = H(t) + 1/2$ with $0 < h < 3/2 - α_{sup}$. Then for fixed $t$ in $\mathbb{R}^+$, 

$$Y^λ_{α(t)}(t) - Y^λ_{α(t)}(0) \xrightarrow{λ \to \infty} B_H(t) \quad \text{in} \quad L^2(Ω_B).$$

**Proof.** For each $s \leq t$ set $c_λ(t - s) = (λ/(λ + t - s))^{1-h}$, for each $t \geq 0$ let $X^λ_{α(t)}(t) = Y^λ_{α(t)}(t) - Y^λ_{α(t)}(0)$, and denote

$$A^λ_{H(t)}(t) = \int_{−∞}^{0} [(c_λ(t - s)(t - s)^H(t)−1/2 - c_λ(-s)(−s)^H(t)−1/2]$$

$$− [(t - s)^H(t)−1/2 - (−s)^H(t)−1/2)] dB_s$$

$$=: \int_{−∞}^{0} [A^λ_{1,H(t)}(t, s) - A_{1,H(t)}(t, s)] dB_s,$$

$$D^λ_{H(t)}(t) = \int_{0}^{t} (t - s)^H(t)−1/2(c_λ(t - s) - 1) dB_s.$$

By substituting $α(t) = H(t) + 1/2$ we get

$$X^λ_{α(t)}(t) - B_H(t) = X^λ_{H(t)+1/2}(t) - B_H(t)$$

$$= \frac{1}{Γ(H(t) + 1/2)}[A^λ_{H(t)}(t) + D^λ_{H(t)}(t)].$$

Thus,

$$E_B[(X^λ_{α(t)}(t) - B_H(t))^2] = \frac{1}{Γ(H(t) + 1/2)^2}E_B[(A^λ_{H(t)}(t) + D^λ_{H(t)}(t))^2]$$

$$\leq \frac{2}{Γ(H(t) + 1/2)^2}(E_B[A^λ_{H(t)}(t)^2] + E_B[D^λ_{H(t)}(t)^2]).$$

Let us first evaluate the asymptotic behavior of $E_B[A^λ_{H(t)}(t)^2]$ when $λ \to \infty$.

For fixed $t \geq 0$, it is easily seen that

$$A^λ_{1,H(t)}(t, s) \xrightarrow{λ \to \infty} A_{1,H(t)}(t, s).$$

Using the elementary inequality, for any $p \geq 0$ and $x, y \in \mathbb{R}$,

$$\|x|^p - |y|^p \leq (p \lor 1)2^{p-2}^+\|(x - y)|^p + |y|^{(p-1)^+} |x - y|^{p^1}|$$

and the fact that $c_λ(x) \leq 1$ for all $x > -λ$, we have, for $s < 0$,

$$|c_λ(t - s)(t - s)^H(t)−1/2 - c_λ(-s)(−s)^H(t)−1/2|$$

$$\leq |c_λ(t - s)(t - s)^H(t)−1/2 - (−s)^H(t)−1/2| + (−s)^H(t)−1/2|c_λ(t - s) - c_λ(-s)|$$

$$\leq |(t - s)^H(t)−1/2 - (−s)^H(t)−1/2| + 2t^{1-h}(−s)^H(t)−1/2(t - s)^{h-1}.$$
Moreover, for fixed $t > 0$ such that $H(t) \neq 1/2$, when $s \to -\infty$ we get

$$
((t - s)^{H(t)-1/2} - (-s)^{H(t)-1/2})^2 \sim (H(t) - 1/2)^2 t^2 (-s)^{2H(t)-3}.
$$

As a result, $s \mapsto ((t - s)^{H(t)-1/2} - (-s)^{H(t)-1/2})^2$ is integrable at $-\infty$, because $2H(t) - 3 < -1$, and as $s \to 0^-$ as well, since $2H(t) - 1 > -1$. Consequently,

$$
0 \int_{-\infty}^{0} ((t - s)^{H(t)-1/2} - (-s)^{H(t)-1/2})^2 ds < \infty.
$$

Also, by the hypothesis $2 - 2H(t) - 2h > 0$,

$$
0 \int_{-\infty}^{0} (-s)^{2H(t)-1} (t - s)^{2h-2} ds = t^{2H(t)+2h-2} \beta(2 - 2H(t) - 2h, 2H(t)) < \infty.
$$

The dominated convergence theorem shows that for fixed $t \geq 0$,

$$
\lim_{\lambda \to \infty} E_B[A_{H(t)}^\lambda(t)^2] = 0.
$$

Similarly, one shows that for fixed $t \geq 0$, \(\lim_{\lambda \to \infty} E_B[D_{H(t)}^\lambda(t)^2] = 0\), which proves the desired result. ■

On the other hand, we now consider the asymptotic behavior of $Y_{\alpha}^\lambda$ when $\lambda \to 0$. In the following result, it is assumed that $\alpha(t) = \alpha$ for all $t$, $1 - \alpha < h < 3/2 - \alpha$ and $1/2 < \alpha < 1$.

**Proposition 3.9.** Let $\{Y_{\alpha}^\lambda, t \geq 0\}$ be the process defined by

$$
\hat{Y}_{\alpha}^\lambda(t) = \lambda^{h-1} \int_0^t Y_{\alpha}^\lambda(s) ds, \quad t \geq 0.
$$

Then

$$
\hat{Y}_{\alpha}^\lambda(t) \xrightarrow{\lambda \to 0} Y_{\alpha}(t) \quad \text{in } L^2(\Omega_B),
$$

where

$$
Y_{\alpha}(t) := \frac{1}{\Gamma(\alpha)(h + \alpha - 1)} \left[ \int_{-\infty}^{0} (t - u)^{h+\alpha-1} - (-u)^{h+\alpha-1} dB_u + \int_0^{t} (t - u)^{h+\alpha-1} dB_u \right].
$$

Moreover, the process $(Y_{\alpha}(t))_{t \geq 0}$ is (modulo a constant) a fractional Brownian motion with Hurst index $h + \alpha - 1/2$. 

Proof. For each $t \geq 0$, we have
\[
\lambda^{h-1} \int_0^t Y_\lambda^\alpha (s) \, ds = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t dB_u \int_u^t (\lambda + s - u)^{h-1} (s-u)^{\alpha-1} \, ds
\]
\[
= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^0 dB_u \int_0^t (\lambda + s - u)^{h-1} (s-u)^{\alpha-1} \, ds
\]
\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t dB_u \int_u^t (\lambda + s - u)^{h-1} (s-u)^{\alpha-1} \, ds.
\]
Using the same computations as in the proof of Proposition 3.8, it is easily checked that for every $t \geq 0$, 
\[
\hat{Y}_\lambda^\alpha (t) \xrightarrow{\lambda \to 0} Y_\alpha (t) := \frac{1}{\Gamma(\alpha)(h+\alpha-1)} \left[ \int_{-\infty}^0 (t-u)^{h+\alpha-1} - (-u)^{h+\alpha-1} \, dB_u + \int_0^t (t-u)^{h+\alpha-1} \, dB_u \right]
\]
in $L^2(\Omega_B)$. Moreover, it is obvious that the process $(Y_\alpha (t))_{t \geq 0}$ is (modulo a constant) a fractional Brownian motion (with moving average definition) with Hurst index $h+\alpha-1/2$. ■

4. APPENDIX

PROPOSITION 4.1. For all $0 < p < 1$ and $k \geq 2$,

\begin{equation}
\sum_{i=1}^k x_i^p \leq 2^{(k-1)(1-p)} \left( \sum_{i=1}^k x_i \right)^p \quad \text{if } x_i \geq 0 \text{ for } i = 1, \ldots, k.
\end{equation}

Proof. For $k \geq 2$, $0 < p < 1$ and $x_i \geq 0$ for all $i = 1, \ldots, k$, we will denote by $A(k)$ the inequality

\[ A(k) : \sum_{i=1}^k x_i^p \leq 2^{(k-1)(1-p)} \left( \sum_{i=1}^k x_i \right)^p. \]

Let $k = 2$. Since the function $x \mapsto x^p$, $x \geq 0$, is concave for every $0 < p < 1$, we get

\[ x^p + y^p \leq 2^{1-p} (x+y)^p, \]

so $A(2)$ holds true.

Let us assume that $A(n-1)$ holds. Using $A(2)$, $A(n-1)$, by easy computations we get $A(n)$. Thus by induction, the proof is complete. ■
Throughout the appendix, it is supposed that $0 < h < 3/2 - M_\alpha[a, b], M_\alpha[a, b] < 1$ and $m_\alpha[a, b] > 1/2$ for any compact interval $[a, b] \subset \mathbb{R}$.

**Lemma 4.1.** Fix a compact interval $[a, b] \subset \mathbb{R}$. There exists a constant $C$ depending only on $[a, b], \lambda$ and $h$ such that

$$E_B[(Y^\lambda_{\alpha(t)}(t + \tau) - Y^\lambda_{\alpha(t)}(t))^2] \leq C|\tau|^{2\alpha(t) - 1}$$

for all $t, t + \tau \in [a, b]$ with $|\tau| < 1$.

**Proof.** Set $\eta = 3 - 2h - 2\alpha(t), \nu = 2\alpha(t) - 1$ and $y = |\tau|/\lambda$. Using (3.21), we get

$$E_B[(Y^\lambda_{\alpha(t)}(t + \tau) - Y^\lambda_{\alpha(t)}(t))^2] = E_B[(Y^\lambda_{\alpha(t)}(t + \tau))^2] + E_B[Y^\lambda_{\alpha(t)}(t + \tau)Y^\lambda_{\alpha(t)}(t)] - 2E_B[Y^\lambda_{\alpha(t)}(t + \tau)Y^\lambda_{\alpha(t)}(t)] = \frac{2\lambda^\nu}{\Gamma(\alpha(t))^2} \beta(\eta, \nu) - 2E_B[Y^\lambda_{\alpha(t)}(t + \tau)Y^\lambda_{\alpha(t)}(t)].$$

Let us first evaluate the second term on the right hand side. By (3.22) we have

$$-2E_B[Y^\lambda_{\alpha(t)}(t + \tau)Y^\lambda_{\alpha(t)}(t)] = \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2} \int_0^1 u^{\eta-1}(1-u)^{\alpha(t)-1}(1+yu)^{h-1}(yu + 1-u)^{\alpha(t)-1} du.$$

By applying the mean value theorem to the function $t \mapsto (1+yt)^{h-1}$ for $t \in [0, u]$, we obtain

$$-2E_B[Y^\lambda_{\alpha(t)}(t + \tau)Y^\lambda_{\alpha(t)}(t)] = \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2} \int_0^1 u^{\eta-1}(1-u)^{\nu-1} \left(1 + y \frac{u}{1-u}\right)^{\alpha(t)-1} du + \frac{2\lambda^\nu(1-h)}{\Gamma(\alpha(t))^2} y \int_0^1 (1 + yC_u)^{h-2} u^{\eta-1}(1-u)^{\alpha(t)-1}(yu + 1-u)^{\alpha(t)-1} du$$

$$=: A_{\lambda, h}(\alpha(t), y) + B_{\lambda, h}(\alpha(t), y).$$

Let us begin by providing an upper bound for $A_{\lambda, h}$. Using the inequality

$$1 - \frac{yu}{1 - (1-y)u} \leq \left(1 + y \frac{u}{1-u}\right)^{\alpha(t)-1}$$
for $y \neq 0$ we have

\[(4.5) \quad A_{\lambda,h}(\alpha(t), y) \leq \frac{-2\lambda'}{\Gamma(\alpha(t))^2} \int_0^1 u^{\eta-1}(1-u)^{\nu-1} du + \frac{2\lambda' y}{\Gamma(\alpha(t))^2} \int_0^1 u^{\eta}(1-u)^{\nu-1} du \]

where $\frac{-2\lambda'}{\Gamma(\alpha(t))^2} \beta(\eta, \nu) + \frac{2\lambda' y}{\Gamma(\alpha(t))^2} \beta(\nu, \eta)$ $2F_1(1, \eta + 1, 3 - 2h, 1 - y),$ where $2F_1$ is called the hypergeometric function, and the last equality is due to Euler’s representation integral of $2F_1$ (see \cite[Theorem 2.2.1]{1}).

Using Euler’s transformation formula (see \cite[Theorem 2.2.5]{1}), we get

\[(4.6) \quad 2F_1(1, \eta + 1, 3 - 2h, 1 - y) = y^{2\alpha(t) - 2} 2F_1(2 - 2h, \nu, 3 - 2h, 1 - y).\]

Set $a = 2 - 2h, b = 2m_\alpha[a, b] - 1$ and $c = 3 - 2h - 2M_\alpha[a, b] + 2m_\alpha[a, b].$ For $y \neq 0,$ we have

\[(4.7) \quad 2F_1(a, \nu, a + 1, 1 - y) = \frac{\Gamma(a + 1)}{\Gamma(\nu)\Gamma(\eta + 1)} \int_0^1 x^{\nu-1}(1-x)^{\eta}(1 - (1-y)x)^{-a} dx \]

\[\leq C \int_0^1 x^{b-1}(1-x)^{c-b-1}(1 - (1-y)x)^{-a} dx = CF(y) \leq C,\]

the last inequality coming from the fact that the function $F$ is continuous on $[1, 1/\lambda].$ By plugging (4.6) in (4.5) and using (4.7), we infer that

\[(4.8) \quad A_{\lambda,h}(\alpha(t), |\tau|) \leq \frac{-2\lambda'}{\Gamma(\alpha(t))^2} \beta(\eta, \nu) + C|\tau|^\nu.\]

On the other hand, since $\eta, \lambda, C_u > 0, 0 < h < 1,$ and $\alpha(t) < 1,$ we have

\[(4.9) \quad B_{\lambda,h}(\alpha(t), |\tau|) \leq \frac{2\lambda^{a(t)-1}(1-h)}{\Gamma(\alpha(t))^2} \beta(\alpha(t), \alpha(t))|\tau|^{a(t)} \leq M|\tau|^\nu,\]

where $M$ is the maximum of the continuous function

$z \mapsto (2\lambda z^{-1}(1-h)/\Gamma(z)^2)\beta(z, z)$

on $[m_\alpha[a, b], M_\alpha[a, b]].$ Thus, by plugging (4.8) and (4.9) in (4.4), we get

$-2EB[Y_{\alpha(t)}^\lambda(t + \tau) - Y_{\alpha(t)}^\lambda(t)] \leq \frac{-2\lambda'}{\Gamma(\alpha(t))^2} \beta(\eta, \nu) + (M + C)|\tau|^\nu.$

Then

\[EB[(Y_{\alpha(t)}^\lambda(t + \tau) - Y_{\alpha(t)}^\lambda(t))^2] \leq C_1|\tau|^\nu,\]

where $C_1 = M + C,$ which establishes the desired result.  ■
LEMMA 4.2. Fix a compact interval \([a, b] \subset \mathbb{R}\).

(1) There exists a constant \(C_2\) depending only on \([a, b]\), \(\lambda\) and \(h\) such that

\[
E_B[(Y_{\alpha(t)}^\lambda(t + \tau) - Y_{\alpha(t)}^\lambda(t))^2] \geq C_2|\tau|^{2\alpha(t) - 1}
\]

for all \(t, t + \tau \in [a, b]\) with \(|\tau| < 1\).

(2) As \(\tau \to 0\),

\[
E_B[(Y_{\alpha(t)}^\lambda(t + \tau) - Y_{\alpha(t)}^\lambda(t))^2] = C_2|\tau|^{\alpha(t) - 1/2} + O(|\tau|^{2\alpha(t) - 1}).
\]

Proof. With the notation of Lemma 4.1, since \(\alpha(t) < 1\) for all \(t\), we have

\[
B_{\lambda,h}(\alpha(t), y) \leq E_B[(Y_{\alpha(t)}^\lambda(t + \tau) - Y_{\alpha(t)}^\lambda(t))^2].
\]

For \(\tau \neq 0\), using \(C_u \in ]0, 1]\) we get

\[
(1 + |\tau| + 1)\frac{1}{\lambda} \left(1 + |\tau|^{2\alpha(t) - 2} \leq (1 + y)^{h + \alpha(t) - 3}
\]

\[
\leq (1 - u + yu)^{\alpha(t) - 1/2}(1 + yC_u)^{h - 2}.
\]

Set

\[
h(z, x) = (\lambda x)^{3-z-h}(x + \lambda)^{z+h-3},
\]

a continuous function on \([m_\alpha[a, b], M_\alpha[a, b]] \times [0, 1]\), and let \(C_2\) be the minimum of the function

\[
(z, x) \mapsto \frac{2\lambda^{2z-2}(1 - h)}{\Gamma(z)^2} \beta(4 - 2h - 2z, z)h(z, x)
\]

for \((z, x) \in [m_\alpha[a, b], M_\alpha[a, b]] \times [0, 1]\). Then

\[
E_B[(Y_{\alpha(t)}^\lambda(t + \tau) - Y_{\alpha(t)}^\lambda(t))^2] \geq C_2|\tau|^\nu,
\]

which gives (4.10).

Let us now prove (4.11). If instead of (4.12) we use the following inequality for \(y \neq 0:\)

\[
(1 + |\tau| + 1)\frac{1}{\lambda} \left(\alpha(t) + h - 3\right) \leq (1 + y)^{\alpha(t) + h - 3},
\]

then (4.10) becomes

\[
E_B[(Y_{\alpha(t)}^\lambda(t + \tau) - Y_{\alpha(t)}^\lambda(t))^2] \geq C_2|\tau|^\alpha(t) - 1/2.
\]
Combining Lemma 4.1 and the last inequality, we get
\[ C_2|\tau|^{\alpha(t)-1/2} \leq E_B[(Y_{\alpha(t)}^\lambda(t + \tau) - Y_{\alpha(t)}^\lambda(t))^2] \leq C_1|\tau|^{\alpha(t)-1/2}. \]

To prove (4.11), it remains to show that \( C_2 \leq C_1 = M + C. \) Since \( 0 < h < 3/2 - z \) and \( z < 1 \), we obtain
\[ \beta(4 - 2h - 2z, z) \leq \beta(z, z) \quad \text{and} \quad h(z, x) \leq \lambda^{1-z}. \]

Therefore,
\[ C_2 \leq \frac{2\lambda^{2z-2}(1-h)}{\Gamma(z)^2} \beta(4 - 2h - 2z)h(z, x) \leq M \leq C_1, \]
which completes the proof. ■

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