AN EQUIVALENT CHARACTERIZATION OF WEAK BMO MARTINGALE SPACES

BY

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Abstract. In this paper, we give an equivalent characterization of weak BMO martingale spaces due to F. Weisz [Probab. Math. Statist. 18, 133-148, 1998].

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1. INTRODUCTION

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, and \(\{\mathcal{F}_n\}_{n \geq 0}\) be an increasing sequence of sub-\(\sigma\)-algebras of \(\mathcal{F}\) such that \(\mathcal{F} = \sigma(\cup_{n \geq 0} \mathcal{F}_n)\). The expectation operator and the conditional expectation operator relative to \(\mathcal{F}_n\) are denoted by \(\mathbb{E}\) and \(\mathbb{E}_n\), respectively. A sequence \(f = (f_n)_{n \geq 0}\) of random variables such that \(f_n\) is \(\mathcal{F}_n\)-measurable is said to be a martingale if \(\mathbb{E}(|f_n|) < \infty\) and \(\mathbb{E}_n(f_{n+1}) = f_n\) for every \(n \geq 0\).

The study of the space BMO (Bounded Mean Oscillation) began with the establishment of so-called John-Nirenberg theorem in [9]. Basing mainly on the duality and something else, the space BMO plays a remarkable role both in classical analysis and martingale theory. For example, BMO is a good space in operator actions (see e.g. [12, Chapter 4]). And the martingale space \(BMO_r(\alpha)\) was first introduced by Herz in [3] as the dual of \(H^p_\alpha\) \((0 < p \leq 1)\) associated with the dyadic filtration (see Example 2.1 below). With the help of atomic decomposition, Weisz in [13] extended this result to general case. Let \(T\) be the set of all stopping times with respect to \(\{\mathcal{F}_n\}_{n \geq 0}\). The martingale space \(BMO_r(\alpha)\) \(([14, p. 8] or [13])\) for \(1 \leq r < \infty\) and \(\alpha \geq 0\) is defined as

\[
BMO_r(\alpha) = \{f = (f_n)_{n \geq 0} : \|f\|_{BMO_r(\alpha)} < \infty\},
\]

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where
\[ \|f\|_{BMO_\alpha} = \sup_{\nu \in T} \mathbb{P}(\nu < \infty)^{-\frac{1}{r}} \|f - f''\|_r. \]

We present two well-known results (see [14] or [13]). If \(0 < p \leq 1\) and \(\alpha = \frac{1}{p} - 1\), then \(BMO_2(\alpha)\) is the dual space of the Hardy space \(H^p_2\). If the stochastic basis \(\{F_n\}_{n \geq 1}\) is regular, then \(BMO_\alpha = BMO_1(\alpha)\). And recently, Yi et al. in [15] proved that \(BMO_E(\alpha) = BMO_1(\alpha)\), where \(\alpha = 0\) and \(E\) is a rearrangement invariant Banach function space.

We consider weak BMO martingale space in the present paper. In order to characterize the dual of weak Hardy martingale space \(L^{s,p}_{\infty}\), Weisz in [15] first introduced and studied the weak BMO martingale space. Let us recall the definition. We also refer the reader to [10], [11] for some new results related to weak BMO martingales spaces.

**Definition 1.1.** Let \(1 \leq r < \infty\), \(\alpha r + 1 > 0\). The space \(wBMO_\alpha\) is defined as the set of all martingales \(f \in L_r\) with the norm
\[ \|f\|_{wBMO_\alpha} = \int_0^\infty t_\alpha(x) dx < \infty, \]
where
\[ t_\alpha(x) = x^{-\frac{1}{r} - \alpha} \sup_{\nu \in T, \mathbb{P}(\nu < \infty) \leq x} \|f - f''\|_r. \]

In the very recent paper [8], the generalized BMO martingale space is introduced to be as the dual of Hardy-Lorentz martingale space. Strongly motivated by [8, Definition 1.1], we introduce the following new weak BMO martingale space by stopping time sequences.

**Definition 1.2.** Let \(1 \leq r < \infty\) and \(\alpha \geq 0\). The weak BMO martingale space \(wBMO_\alpha\) is defined by
\[ wBMO_\alpha = \{ f \in L_r : \|f\|_{wBMO_\alpha} < \infty \}, \]
where
\[ \|f\|_{wBMO_\alpha} = \sum_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k < \infty)^{1 - \frac{1}{r}} \|f - f''_k\|_r \]
\[ \leq \sup_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k < \infty)^{1 + \alpha} \]
and the supremum is taken over all stopping time sequences \(\{\nu_k\}_{k \in \mathbb{Z}}\) such that \(2^k \mathbb{P}(\nu_k < \infty)^{1 + \alpha} \in \ell_\infty\).

It is a very natural question that what is the relationship between \(wBMO_\alpha\) and \(wBMO_\alpha\). The present paper fully answers this question. Our main result can be described as follows. We simply denote \(wBMO = wBMO(0)\) and \(wBMO = wBMO(0)\).
THEOREM 1.1. Let \( 1 \leq r < \infty \) and \( \alpha \geq 0 \). If the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular, then
\[
\text{wBMO}_r(\alpha) = \text{wBMO}_r(\alpha)
\]
with equivalent norms. In particular,
\[
\text{wBMO}_r = \text{wBMO}_r
\]
with equivalent norms.

In this paper, the set of integers and the set of non-negative integers are always denoted by \( \mathbb{Z} \) and \( \mathbb{N} \), respectively. We use \( C \) to denote a positive constant which may vary from line to line. The symbol \( \subset \) means the continuous embedding.

2. PRELIMINARIES

Firstly, we give the definition of Lorentz spaces. We denote by \( L_{0;r}(\Omega) \), or simply \( L_0(\Omega) \), the space of all measurable functions on \( (\Omega, \mathcal{F}, \mathbb{P}) \). For any \( f \in L_0(\Omega) \), we define the distribution function of \( f \) by
\[
\lambda_s(f) = \mathbb{P}\{\omega \in \Omega : |f(\omega)| > s\}, \quad s \geq 0.
\]
And denote by \( \mu_t(f) \) the decreasing rearrangement of \( f \), defined by
\[
\mu_t(f) = \inf\{s > 0 : \lambda_s(f) \leq t\}, \quad t > 0,
\]
with the convention that \( \inf \emptyset = \infty \).

DEFINITION 2.1. Let \( 0 < p < \infty \) and \( 0 < q \leq \infty \). Then Lorentz space \( L_{p,q}(\Omega) \) consists of those measurable functions such that \( \|f\|_{p,q} < \infty \), where
\[
\|f\|_{p,q} = \left\| \int_0^\infty \left( \frac{1}{t^{1/p}} \mu_t(f) \right)^q \frac{dt}{t} \right\|^{1/q}, \quad 0 < q < \infty,
\]
and
\[
\|f\|_{p,\infty} = \sup_{0 \leq t < \infty} t^{1/p} \mu_t(f), \quad q = \infty.
\]

REMARK 2.1. We refer the reader to \( [3] \) for the following basic properties.

(1) If \( p = q \), then \( L_{p,p}(\Omega) \) becomes \( L_p(\Omega) \).

(2) If \( 0 < p_1 \leq p_2 < \infty \) and \( 0 < q \leq \infty \), then \( \|f\|_{p_1,q} \leq C \|f\|_{p_2,q} \), where \( C \) depends on \( p_1, p_2 \) and \( q \). This is due to \( \mathbb{P}(\Omega) = 1 \).

(3) If \( 0 < p < \infty \) and \( 0 < q_1 \leq q_2 \leq \infty \), then \( \|f\|_{p,q_2} \leq C \|f\|_{p,q_1} \), where \( C \) depends on \( q_1, q_2 \) and \( p \).

Denote by \( M \) the set of all martingales \( f = (f_n)_{n \geq 0} \) relative to \( \{F_n\}_{n \geq 0} \) such that \( f_0 = 0 \). For \( f \in M \), denote its martingale difference by \( d_n f = f_n - f_{n-1} \).
(\( n \geq 0 \), with convention \( f_{-1} = 0 \)). Then the maximal function and the conditional quadratic variation of a martingale \( f \) are respectively defined by

\[
f_n^* = \sup_{0 \leq i \leq n} |f_i|, \quad f^* = \sup_{n \geq 0} |f_n|,
\]

\[
s_n(f) = \left( \sum_{i=1}^{n} E_{i-1} d_i f^2 \right)^{\frac{1}{2}}, \quad s(f) = \left( \sum_{i=1}^{\infty} E_{i-1} d_i f^2 \right)^{\frac{1}{2}}.
\]

Then we define martingale Hardy-Lorentz spaces as follows.

**Definition 2.2.** Let \( 0 < p < \infty \), \( 0 < q \leq \infty \). Define

\[
H_{p,q}^* = \left\{ f \in \mathcal{M} : \| f \|_{H_{p,q}^*} = \| f^* \|_{p,q} < \infty \right\},
\]

\[
H_{p,q}^s = \left\{ f \in \mathcal{M} : \| f \|_{H_{p,q}^s} = \| s(f) \|_{p,q} < \infty \right\}.
\]

If \( p = q \), then the martingale Hardy-Lorentz spaces recover the martingale Hardy spaces \( H_{p}^* \) and \( H_{p}^s \) (see [14]).

Recall that the stochastic basis \( \{F_n\}_{n \geq 0} \) is said to be regular, if there exists a positive constant \( R > 0 \) such that

\[
(2.1) \quad f_n \leq R f_{n-1}, \quad \forall n > 0,
\]

holds for all nonnegative martingales \( f = (f_n)_{n \geq 0} \). Condition (2.1) can be replaced by several other equivalent conditions (see [12, Chapter 7]). We refer the reader to [12, p. 265] for examples for regular stochastic basis. Here, we give a special case.

**Example 2.1.** Let \( ((0,1], \mathcal{F}, \mu) \) be a probability space such that \( \mu \) is the Lebesgue measure and subalgebras \( \{F_n\}_{n \geq 0} \) generated as follows:

\[
F_n = \sigma\text{-algebra generated by atoms } \left( \frac{j}{2^n}, \frac{j+1}{2^n} \right], j = 0, \ldots, 2^n - 1.
\]

Then \( \{F_n\}_{n \geq 0} \) is regular. And all martingales with respect to such \( \{F_n\}_{n \geq 0} \) are called dyadic martingales.

The method of atomic decompositions plays an important role in martingale theory (see for example [3], [4], [5], [14], [15]). The atomic decompositions of Hardy-Lorentz martingale spaces \( H_{p,q}^* \) and martingale inequalities were given in [6], [8]. Let us first introduce the concept of an atom (see [14, p. 14]).

**Definition 2.3.** Let \( 0 < p < \infty \) and \( p < r \leq \infty \). A measurable function \( a \) is called a \((1, p, r)\)-atom (or \((3, p, r)\)-atom) if there exist a stopping time \( \nu \in \mathcal{T} \) such that \( a_n = E_n(a) = 0 \) if \( \nu \geq n \), and

\[
(1) \quad \| s(a) \|_r, \| a^* \|_r \leq \mathbb{P}(\nu < \infty)^{\frac{1}{r} - \frac{1}{p}}.
\]
Let \( 0 < p < r \leq \infty \) and \( 0 < q \leq r \). If \( a \) is a \((1, p, r)\)-atom, then we have \( \|a\|_{H_{p,q}^a} \leq C \). Choose \( p_1, p_2 \) such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} \). By Hölder’s inequality, we have (\( \nu \) is the stopping time corresponding to the atom \( a \))

\[
\|a\|_{H_{p,q}^a} = \|s(a)\chi_{\{\nu < \infty\}}\|_{p,q} \leq C\|s(a)\|_{r,r}\|\chi_{\{\nu < \infty\}}\|_{p_1,q_1} \leq C\|\nu\|_{p,q}^{\frac{1}{p} - \frac{1}{p_1}} \left( \int_0^{\infty} t^{p_1 - 1}\chi_{(0,\infty)}(dt) \right)^{\frac{1}{q}} \leq C.
\]

Similarly, we have \( \|a\|_{H_{p,q}^a} \leq C \) for a \((3, p, r)\)-atom \( a \). If \( p = q \), then \( C = 1 \).

The following result is from [8]. And the result about Hardy space \( H_{p,q}^a \) follows from the combing of [3, Theorem 3.3] and [8, Lemma 5.1].

**Theorem 2.1.** If \( f = \{f_n\}_{n \geq 0} \in H_{p,q}^a \) for \( 0 < p < \infty \), \( 0 < q \leq r \), there exist a sequence \( \{a^k\}_{k \in \mathbb{Z}} \) of \((1, p, \infty)\)-atoms and a positive number \( A \) satisfying \( \mu_k = A \cdot 2^k \mathbb{P}(\nu_k < \infty) \) (where \( \nu_k \) is the stopping time corresponding to \( a^k \)) such that

\[
(2.2) \quad f_n = \sum_{k \in \mathbb{Z}} \mu_k a^k_n, \quad \text{a.e.,} \quad n \in \mathbb{N},
\]

and

\[
\|\{\mu_k\}\|_{l_q} \leq C\|f\|_{H_{p,q}^a}.
\]

Conversely, if the martingale \( f \) has the above decomposition, then \( f \in H_{p,q}^a \) and \( \|f\|_{H_{p,q}^a} \approx \inf \|\{\mu_k\}\|_{l_q} \), where the infimum is taken over all the above decompositions.

Moreover, if the stochastic basis \( \{\mathcal{F}_n\}_{n \geq 0} \) is regular and if we replace \( H_{p,q}^a \) by \( H_{p,q}^a \), then the conclusions above still hold.

**Lemma 2.1.** ([3, Lemma 1.2]) Let \( 0 < p < \infty \), and let the nonnegative sequence \( \{\mu_k\} \) be such that \( \{2^k \mu_k\} \in \ell^q \), \( 0 < q \leq \infty \). Further, suppose that the nonnegative function \( \varphi \) satisfies the following property: there exists \( 0 < \varepsilon \leq \min(1, q/p) \) such that, given an arbitrary integer \( k_0 \), we have \( \varphi \leq \psi_{k_0} + \eta_{k_0} \), where \( \psi_{k_0} \) and \( \eta_{k_0} \) satisfy

\[
2^{k_0 p} \mathbb{P}(\psi_{k_0} > 2^{k_0})^\varepsilon \leq C \sum_{k=-\infty}^{k_0-1} \left(2^k \mu_k^\varepsilon\right)^p,
\]

\[
2^{k_0 p} \mathbb{P}(\eta_{k_0} > 2^{k_0}) \leq C \sum_{k=k_0}^{\infty} (2^k \mu_k)^p.
\]

Then \( \varphi \in L_{p,q} \) and \( \|\varphi\|_{p,q} \leq C\|\{2^k \mu_k\}\|_{l_q} \).
3. A JOHN-NIRENBERG THEOREM

In this section, we prove a John-Nirenberg theorem when the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular. The main idea and method are similar to those of \([8]\). The following lemma can be found in \([5]\), \([14]\). In fact, it follows from \([5\), Theorem 7.14] and \([14\), Corollary 5.13].

**Lemma 3.1.** Suppose that \( 0 < q \leq \infty \) and the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular.
- If \( 0 < p < \infty \), then \( H_{p,q}^s \) and \( H_{p,q}^s \) are equivalent.
- If \( 1 < p < \infty \), then \( H_{p,q}^s \), \( H_{p,q}^s \) and \( L_{p,q} \) are all equivalent.

\( L_p \) is not dense in \( L_{p,\infty} \). This fact is mentioned in \([15\), p. 143] (see also \([2\), Remark 1.4.14]). Hence, in order to describe the duality, we need the following definition from \([7\), Remark 1.7].

**Definition 3.1.** Let measurable set \( A_k \subset \Omega \) satisfy \( P(A_k) \to 0 \) as \( k \to \infty \).
Define \( \mathcal{L}_{p,\infty} \) as the set of all \( f \in L_{p,\infty} \) having the absolute continuous quasi-norm defined by
\[
\mathcal{L}_{p,\infty} = \{ f \in L_{p,\infty} : \lim_{k \to \infty} \| f \chi_{A_k} \|_{p,\infty} = 0 \}.
\]
\( \mathcal{L}_{p,\infty} \) is a closed subspace of \( L_{p,\infty} \) and \( L_p \subset \mathcal{L}_{p,\infty} \subset L_{p,\infty} \) (see \([7]\)). Now we define
\[
\mathcal{H}_{p,\infty}^s = \{ f = (f_n)_{n \geq 0} : s(f) \in L_{p,\infty} \},
\]
which is closed subspace of \( H_{p,\infty}^s \). Similarly, we define \( \mathcal{H}_{p,\infty}^s \).

**Remark 3.1.** (1) According to \([7\), Remark 2.2], we can conclude that \( H_2^s = L_2 \) is dense in \( H_{p,\infty}^s \).
(2) If the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular, then, by the same argument of \([7\), Remark 2.2], \( L_\infty \) is dense in \( H_{p,\infty}^s \).

**Lemma 3.2.** Let \( 0 < p \leq 1 \). If the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular, then
\[
(\mathcal{H}_{p,\infty}^s)' = wBMO_1(\alpha), \quad \alpha = \frac{1}{p} - 1.
\]

**Proof.** Let \( g \in wBMO_1(\alpha) \). Define
\[
\phi_g(f) = \mathbb{E}(fg), \quad f \in L_\infty.
\]
Then, by Theorem \([5\), we find that \( \nu_k \) is the stopping time corresponding the atom.
\( a^k \) for every \( k \in \mathbb{Z} \)

\[
|\phi_g(f)| \leq \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{E}(a^k(g - g'\nu_k)) \leq \sum_{k \in \mathbb{Z}} |\mu_k| \|a^k\|_\infty \|g - g'\nu_k\|_1 \\
\leq C \sum_{k \in \mathbb{Z}} |\mu_k| \|a^k\|_\infty \|g - g'\nu_k\|_1 \\
\leq C \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}} \|g - g'\nu_k\|_1 \\
= C \cdot A \sum_{k \in \mathbb{Z}} 2^k \|g - g'\nu_k\|_1.
\]

By the definition of \( \| \cdot \|_{wBMO_r(\alpha)} \), we obtain

\[
|\phi_g(f)| \leq C \cdot A \sup_k 2^k \mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}} \|g\|_{wBMO_1(\alpha)} \\
\leq C \|f\|_{L^p_{\alpha,\infty}} \|g\|_{wBMO_1(\alpha)}.
\]

Since the stochastic basis \( \{\mathcal{F}_n\}_{n \geq 0} \) is regular, \( L_\infty \) is dense in \( H^*_{p,\infty} \) (see Remark 3.1(2)). Then \( \phi_g \) can be uniquely extended to be a continuous linear functional on \( H^*_{p,\infty} \).

Conversely, let \( \phi \in (H^*_{p,\infty})^* \). Since \( L_2 \) is dense in \( H^*_{p,\infty} \) (see Remark 5.1(2)), there exists \( g \in L_2 \subset L_1 \) such that

\[
\phi(f) = \mathbb{E}(fg), \quad f \in L_\infty.
\]

Let \( \{\nu_k\}_{k \in \mathbb{Z}} \) be a stopping time sequence satisfying \( \{2^k \mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}}\}_{k \in \mathbb{Z}} \in l_\infty \). Let

\[
h_k = \text{sign}(g - g'\nu_k), \quad a^k = \frac{1}{2}(h_k - h_k'\nu_k)\mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}},
\]

then \( a^k \) is a \((3, p, \infty)\)-atom. Let \( f^N = \sum_{k=-N}^N 2^{k+1} \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} a^k \), where \( N \) is an arbitrary nonnegative integer. By Theorem 2.1, we have \( f^N \in H^*_{p,\infty} \) and

\[
\|f^N\|_{H^*_{p,\infty}} \leq C \sup_k 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}.
\]

Consequently,

\[
\sum_{k=-N}^N 2^k \|g - g'\nu_k\|_1 = \sum_{k=-N}^N 2^k \mathbb{E}(h_k(g - g'\nu_k)) = \sum_{k=-N}^N 2^k \mathbb{E}((h_k - h_k'\nu_k)g) \\
= \mathbb{E}(f^Ng) = \phi(f^N) \leq \|f^N\|_{H^*_{p,\infty}} \|\phi\| \\
\leq C \sup_k 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} \|\phi\|.
\]
Thus we have
\[ \frac{1}{2^k} \| g - g^{\nu_k} \|_1 \leq C \| \phi \|. \]
This implies \( \| g \|_{wBMO(\alpha)} \leq C \| \phi \|. \) The proof is complete.

**Lemma 3.3.** Let \( 0 < p \leq 1, 1 < r < \infty \). If the stochastic basis \( \{ \mathcal{F}_n \}_{n \geq 0} \) is regular, then
\[ (\mathcal{H}_{p,\infty})^* = wBMO_r(\alpha), \quad \alpha = \frac{1}{p} - 1. \]

**Proof.** By Hölder’s inequality, we have \( \| f \|_{wBMO(\alpha)} \leq \| f \|_{wBMO(\alpha)} \) for any \( f \in wBMO_r(\alpha) \). Let \( g \in wBMO_r(\alpha) \subset L_r \). We define
\[ \phi_g(f) = \mathbb{E}(fg), \quad \forall f \in L_r. \]
Then, by Lemma 3.2, we have
\[ |\phi_g(f)| \leq C \| f \|_{H_{p,\infty}^*} \| g \|_{wBMO(\alpha)} \leq C \| f \|_{H_{p,\infty}^*} \| g \|_{wBMO(\alpha)}. \]
It follows from Remark 3.1 (2) that \( L_r \) is dense in \( \mathcal{H}_{p,\infty}^* \). Thus \( \phi_g \) can be uniquely extended to be a continuous linear functional on \( \mathcal{H}_{p,\infty}^* \).

Conversely, if \( \phi \in (\mathcal{H}_{p,\infty}^*)^* \). By Doob’s maximal inequality, we have \( L_r = H_{p,r}^* \subset H_{p,\infty}^* \). Then \( (\mathcal{H}_{p,\infty}^*)^* = L_r \). Thus there exists \( g \in L_r \) such that
\[ \phi(f) = \phi_g(f) = \mathbb{E}(fg), \quad \forall f \in L_r. \]
Let \( \{ \nu_k \}_{k \in \mathbb{Z}} \) be a stopping time sequence such that \( \{ 2^k \mathbb{P}(\nu_k < \alpha) \}_{k \in \mathbb{Z}} \in l_\infty \) and \( N \) be an arbitrary nonnegative integer. Let
\[ h_k = \frac{|g - g^{\nu_k}|^{r-1} \text{sign}(g - g^{\nu_k})}{|g - g^{\nu_k}|_{r-1}}, \quad f = \sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \alpha) \frac{1}{r} (h_k - h_k^{\nu_k}). \]
For an arbitrary integer \( k_0 \) which satisfies \( -N \leq k_0 \leq N \) (for \( k_0 \leq -N \), let \( G = 0 \) and \( H = f \), for \( k_0 > N \), let \( H = 0 \) and \( G = f \)), let
\[ f = G + H, \]
where \( G = \sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \alpha) \frac{1}{r} (h_k - h_k^{\nu_k}) \) and \( H = \sum_{k=k_0}^{N} 2^k \mathbb{P}(\nu_k < \alpha) \frac{1}{r} (h_k - h_k^{\nu_k}) \). Obviously \( \| h_k \|_{r'} = 1 \) and \( \| G \|_{r'} \leq 2 \sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \alpha) \frac{1}{r}. \) By the sublinearity of the maximal operator *, we have \( f^* \leq G^* + H^* \). Let \( \varepsilon = \frac{2}{r} \) (0 < \( \varepsilon < 1 \).
By Doob’s maximal inequality, we have

\[ P(G^* > 2^{k_0}) \leq \frac{1}{2^{k_0 r'}} \|G^*\|_{r'}' \leq C \frac{1}{2^{k_0 r'}} \|G\|_{r'}' \]

\[ \leq 2^{k_0 r'} \left( \sum_{k=-N}^{k_0-1} 2^k P(\nu_k < \infty) \frac{1}{r'} \right)^{r'}' \]

On the other hand, \( \{ H^* > 0 \} \subset \bigcup_{k=k_0}^{N} \{ \nu_k < \infty \} \). Then for each \( 0 < \varepsilon < 1 \), we have

\[ 2^{k_0 r'} P(H^* > 2^{k_0}) \leq \sum_{k=k_0}^{N} 2^{k_0 r'} P(\nu_k < \infty) \leq \sum_{k=k_0}^{N} \left( 2^{k_0 r'} P(\nu_k < \infty) \frac{1}{r'} \right)^{r'} \]

By Lemma 7.11 we have \( f^* \in L_{p,\infty} \) and \( \| f^* \|_{p,\infty} \leq C \| \{ 2^{k_0 r'} P(\nu_k < \infty) \frac{1}{r'} \} \}_{k \in \mathbb{Z}} \| l_{\infty} \).

Thus \( f \in H^*_{p,\infty} \) and

\[ \| f \|_{H^*_{p,\infty}} \leq C \sup_k 2^{k_0 r'} P(\nu_k < \infty) \frac{1}{r'} \]

Consequently,

\[ \sum_{k=-N}^{N} 2^{k} P(\nu_k < \infty)^{1 - \frac{1}{r'}} \| g - g^{\nu_k} \|_r = \sum_{k=-N}^{N} 2^{k} P(\nu_k < \infty) \frac{1}{r'} E(h_k (g - g^{\nu_k})) \]

\[ = \sum_{k=-N}^{N} 2^{k} P(\nu_k < \infty) \frac{1}{r'} E((h_k - h_k^{\nu_k}) g) \]

\[ = E(f g) = \varphi(f) \leq \| f \|_{H^*_{p,q}} \| \varphi \| \]

\[ \leq C \sup_k 2^{k_0 r'} P(\nu_k < \infty)^{\frac{1}{r'}} \]

Thus we obtain

\[ \frac{\sum_{k=-N}^{N} 2^{k} P(\nu_k < \infty)^{1 - \frac{1}{r'}} \| g - g^{\nu_k} \|_r}{\sup_k 2^{k_0 r'} P(\nu_k < \infty)^{\frac{1}{r'}}} \leq C \| \varphi \|. \]

Taking \( N \to \infty \) and the supremum over all of such stopping time sequences such that \( \{ 2^{k_0 r'} P(\nu_k < \infty) \frac{1}{r'} \} \}_{k \in \mathbb{Z}} \in l_{\infty} \), we get \( \| g \|_{wBMO_{r,\alpha}} \leq C \| \varphi \|. \) ■
Now we formulate the weak version of the John-Nirenberg theorem, which directly follows from Lemma 3.2 and Lemma 3.3.

**Theorem 3.1.** Let \( \alpha \geq 0 \) and \( 1 \leq r < \infty \). If the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular, then
\[
\text{wBMO}_r(\alpha) = \text{wBMO}_1(\alpha)
\]
with equivalent norms.

According to Lemma 3.1, Lemma 3.3 holds if we replace \( H^s_{p,\infty} \) by \( H^s_{r,\infty} \).

Without the regularity of stochastic basis \( \{F_n\}_{n \geq 0} \), we also get a duality result.

**Proposition 3.1.** Let \( 0 < p \leq 1 \). Then \( (H^s_{p,\infty})^* = \text{wBMO}_2(\alpha) \) with \( \alpha = 1/p - 1 \).

**Proof.** Note that \( H^s_2 = L_2 \) is dense in \( H^s_{p,\infty} \) by Remark 3.1 (1). The first part of the proof is similar to that of Lemma 3.2. And the converse part of the proof is similar to that of Lemma 3.3 with \( r = 2 \). We omit the proof. ■

4. PROOF OF THE MAIN THEOREM

In this section we complete the proof of Theorem 1.1.

Let \( \overline{H}^s_{p,\infty} \) be the \( H^s_{p,\infty} \) closure of \( H^s_{\infty} \). Since \( H^s_{\infty} \subset H^s_2 = L_2 \) and Remark 3.1 (1), \( \overline{H}^s_{p,\infty} \subset H^s_{p,\infty} \). Then \( (H^s_{p,\infty})^* \subset (\overline{H}^s_{p,\infty})^* \).

**Lemma 4.1.** ([3], Corollary 6) Let \( 0 < p < 2 \). Then the dual space of \( \overline{H}^s_{p,\infty} \) is \( \text{wBMO}_2(\alpha) \) with \( \alpha = 1/p - 1 \).

**Lemma 4.2.** ([3], Corollary 8) Suppose that the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular and \( 1 \leq r < \infty \). If \( \alpha + 1 > 0 \) for a fixed \( \alpha \), then
\[
\text{wBMO}_r(\alpha) = \text{wBMO}_2(\alpha)
\]
with equivalent norms.

**Theorem 4.1.** Suppose that \( \alpha \geq 0 \). Then
\[
\text{wBMO}_2(\alpha) = \text{wBMO}_2(\alpha)
\]
with equivalent norms.

**Proof.** Let \( p = \frac{1}{1+\alpha} \). Since that \( (H^s_{p,\infty})^* \subset (\overline{H}^s_{p,\infty})^* \), it follows from Proposition 3.1 and Lemma 4.1 that
\[
\text{wBMO}_2(\alpha) \supset \text{wBMO}_2(\alpha).
\]
To obtain
\[
\text{wBMO}_2(\alpha) \supset \text{wBMO}_2(\alpha),
\]
we shall show that
\[ C \| f \|_{wBMO_2(\alpha)} \geq \| f \|_{wBMO_2(\alpha)} \]
for any \( f \in wBMO_2(\alpha) \). Suppose that \( \{\nu_k\}_{k \in \mathbb{Z}} \) is an arbitrary stopping time sequence such that \( \{2^k \mathbb{P}(\nu_k < \infty)^{1/p}\}_{k \in \mathbb{Z}} \in \ell_\infty \). Let
\[ B = \sup_k 2^k \mathbb{P}(\nu_k < \infty)^{1/p}. \]

We can claim that
\[ \sum_{k=-\infty}^{\infty} t_\alpha^2(B^{2^{-k}p}) \leq C \| f \|_{wBMO_2(\alpha)}. \]

To this end, let \( C_k = B^{2^{-k}p} \). Then, for any \( x \in (C_{k+1}, C_k) \), we have
\[ C_k^{1/2+\alpha} t_\alpha(C_k) \leq x^{\frac{1}{2}+\alpha} t_\alpha(x) \leq C_k^{1/2+\alpha} t_\alpha(C_k). \]

We refer to [13, p. 144] for more general case of the inequalities above. Hence,
\[ \int_0^\infty \frac{t_\alpha(x)}{x} \, dx = \sum_{k=-\infty}^{\infty} \frac{C_k}{C_{k+1}} \int \frac{t_\alpha(x)}{x} \, dx \geq (1 - 2^{-p}) 2^{-p(1/2+\alpha)} \sum_{k=-\infty}^{\infty} t_\alpha^2(B^{2^{-k}p}). \]

On the other hand, since \( B^{2^{-k}p} \geq \mathbb{P}(\nu_k < \infty) \) for all \( k \), we have
\[ \sum_{k=-\infty}^{\infty} t_\alpha^2(B^{2^{-k}p}) \geq \sum_{k=-\infty}^{\infty} \frac{2^k(B^{2^{-k}p})^{1/2} \| f - f^{\nu_k} \|_2}{B} \geq \sum_{k=-\infty}^{\infty} \frac{2^k \mathbb{P}(\nu_k < \infty)^{1/2} \| f - f^{\nu_k} \|_2}{B}. \]

By the definition of \( wBMO_2(\alpha) \), we complete the proof. \( \blacksquare \)

**Remark 4.1.** If one proves the dual space of \( H_{p,\infty}^s \) is \( wBMO_0(\alpha) \), then Theorem 4.1 holds. If one shows \( H_{p,\infty}^s = \overline{H}_{p,\infty}^s \), then Proposition 4.1 implies Theorem 4.1. We leave these to the interesting reader.

Now we are ready to prove the main result of the paper.

**Proof of Theorem 1.1.** It directly follows from Theorem 4.1, Theorem 4.1 and Lemma 4.2. \( \blacksquare \)

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