A CONSISTENT ESTIMATOR FOR SPECTRAL DENSITY MATRIX OF A DISCRETE TIME PERIODICALLY CORRELATED PROCESS

BY

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Abstract. In this article, we introduce a weighted periodogram in the class of smoothed periodograms as a consistent estimator for the spectral density matrix of a periodically correlated process. We derive its limiting distribution that appears to be a certain finite linear combination of Wishart distribution. We also provide numerical derivations for our smoothed periodogram and exhibit its asymptotic consistency using simulated data.

2010 AMS Mathematics Subject Classification: Primary: 62M10 ; Secondary: 62M15.

Key words and phrases: Periodically correlated processes, spectral density matrix, Kullback–Liebler distance, smoothed periodogram, Wishart distribution, limiting distribution.

1. INTRODUCTION

A univariate zero mean second order process \( \{X_t; \ t \in \mathbb{Z}\} \) with covariance function \( R(t, s) \) is called periodically correlated, PC in short, if

\[
R(t, s) = R(t + T, s + T) \quad t, \ s \in \mathbb{Z},
\]

for some integer \( T > 0 \). The smallest such \( T \) is defined to be period of the process. The spectral density matrix of this process \( f(\theta) = \{f_{k-j}(\theta + \frac{2\pi j}{T})\}_{j,k = 0, \ldots, T-1} \), if exists, is a non-negative definite matrix-valued function. In this case,

\[
R(t, s) = \sum_{d=-T+1}^{T-1} \int_{0}^{2\pi} e^{-it\delta + is(\theta + \frac{2\pi d}{T})} f_d(\theta) \mu(d\theta),
\]

where \( f_d(\theta), \ d = -T + 1, \ldots, T - 1 \) are called the spectral components of \( f \). Gladyshev (1961). The literature on PC or cyclostationary processes is quite rich.

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Therefore we put light on what is done on spectral density estimation, which we discuss in this article. Nematollahi and Rao (2005) derive a consistent estimator for the spectral density matrix of a PC process using the eigenvalue decomposition of block Toeplitz matrices. Hurd and Miamee (2007) give a bivariate periodogram for a PC sequence to estimate spectral components of the spectral density matrix. Soltani and Azimmohseni (2007), classically, introduce a periodogram matrix, then by using the Cholesky decomposition of spectral density matrix obtain the asymptotic distribution of the periodogram.

In the classical time series it is acknowledged that weighted periodograms for stationary processes are asymptotically consistent, and their variance goes to zero relatively fast. The rate of convergence is also specified. In this article we are concerned with weighted periodograms for periodically correlated processes that form a class of nonstationary processes rich in theory and practice. We introduce a class of weighted periodograms, derive rate of convergence for the variances and give their limiting distributions. This paper is organized as follows.

In Section 2, we provide some important spectral characterizations of PC processes including their spectral representations. We give the periodogram and its asymptotic properties. In Section 3, We derive interesting formulas for the covariance of the periodogram at different Fourier frequencies. In Section 4, we present a smoothed periodogram as an asymptotically consistent estimator for spectral density matrix of a PC process, and give the rate of the convergence. Finally we establish the limiting distribution of the smoothed periodogram.

2. SPECTRAL CHARACTERIZATIONS OF PC PROCESSES

There are various spectral representation for PC processes. Gladyshev (1961) views a PC processes as the Fourier transform of a random measure with certain values dependencies, see also Hurd and Miamee (2007). Soltani and Shishebor (1999) suggested an alternative representation: to view a PC process as a process with time dependent spectrum, namely

\[ X_t = \int_0^{2\pi} e^{-it\theta} V_t(\theta) \Phi(d\theta), \]  

where \( V_t(\theta), t \in \mathbb{Z}, \theta \in [0, 2\pi), \) is a kernel which is T-periodic in \( t; \) \( V_{t+T}(\theta) = V_t(\theta), \theta \in [0, 2\pi), t \in \mathbb{Z}; \) and \( \Phi \) is a random measure with orthogonal increment possessing finite spectral measure \( \mu(d\theta) = E|\Phi(d\theta)|^2 \) on \([0, 2\pi)\).

Let us define the process \( Y_t \) by

\[ Y_t = \int_0^{2\pi} e^{-it\theta} \Phi(d\theta), \quad t \in \mathbb{Z}, \]  

then \( Y_t \) is a purely random process on \( \mathbb{Z}; \) \( EY_tY_s = 0, \ t \neq s, \ E|Y_t|^2 = 1. \)
We let
\[ f(\theta) = A(\theta)A^*(\theta), \]
where
\[ A(\theta) = \left\| a_{j-k} \left( \theta + \frac{2\pi j}{T} \right) \right\|_{j \geq k, j, k = 0, \ldots, T-1} \]
denote the Cholesky factor of the spectral density \( f(\theta) \), where \( a_k \) are complex-valued function on \([0, 2\pi]\), subject to
\[ a_0(\theta) > 0, \quad a_k(\theta) = 0, \quad \theta \in [0, 2\pi k/T], \quad k = 0, \ldots, T-1, \ a.e. \theta, \]
so that
\[ T^{-1-d} \sum_{k=0}^{T-1} a_k(\theta) a_{k+d} \left( \theta + \frac{2\pi d}{T} \right) = f_\theta(\theta), \quad d = 1, \ldots, T-1, \]
\[ T^{-1} \sum_{k=0}^{T-1} |a_k(\theta)|^2 = f_0(\theta), \]
\[ T^{-1} \sum_{k=-d}^{T-1} a_k(\theta) a_{k+d} \left( \theta + \frac{2\pi d}{T} \right) = f_\theta(\theta), \quad d = -T + 1, \ldots, -1. \]

A method to construct the kernel \( V_t(\theta) \) in (1) is through the Cholesky factor of the spectral density, as follows:
\[ V_t(\theta) = \sum_{k=0}^{T-1} e^{-\frac{2\pi k}{T}} a_k(\theta + \frac{2\pi k}{T}), \quad \theta \in [0, 2\pi). \]

For the inference on periodogram, we let \( X_0, \ldots, X_{N-1} \) to be a finite segment of the PC process \( \{X_t, t \in \mathbb{Z}\} \). The \( T \)-variate Fourier transform at frequency \( \lambda \in [0, \frac{2\pi}{T}] \) is defined as
\[ d_X(\lambda) = (d_X(\lambda), d_X(\lambda + \frac{2\pi}{T}), \ldots, d_X(\lambda + \frac{2\pi(T-1)}{T})), \]
where
\[ d_X(\lambda + \frac{2\pi j}{T}) = N^{-1/2} \sum_{t=0}^{N-1} X_t e^{i t(\lambda + \frac{2\pi j}{T})}, \quad j = 0, \ldots, T-1. \]
It will be more convenient to denote the periodogram in matrix form:
\[ I_X(\lambda) = d_X(\lambda)d_X^*(\lambda). \]

It is common to define discrete Fourier transform and periodogram with respect to Fourier frequencies \( \lambda_k = \frac{2\pi k}{N}, \quad k = 0, \ldots, N-1 \) as follows:
\[ d_X(\lambda) = d_X(\lambda_k), \text{ for } \lambda_k \leq \lambda < \lambda_{k+1}, \quad k = 0, \ldots, N-1, \]
and similarly

\[ I_X(\lambda) = I_X(\lambda_k), \text{ for } \lambda_k \leq \lambda < \lambda_{k+1}, \ k = 0, \ldots, N-1, \]

Brockwell and Davis (1991). Note that, for a PC process the Fourier frequencies belong to the interval \([0, \frac{2\pi}{T})\). Indeed

(i) For \(\lambda_j \in \left[0, \frac{2\pi}{T}\right)\), the set of \(\left\{ \lambda_j, \lambda_j + \frac{2\pi}{T}, \ldots, \lambda_j + \frac{2\pi(T-1)}{T} \right\}\) are Fourier frequencies in \([0, 2\pi)\).

(ii) Every Fourier frequency \(\lambda_k \in \left[0, \frac{2\pi}{T}\right)\) is uniquely represented as \(\lambda_k = \lambda_j + \frac{2\pi u(k,j)}{T}\), where \(\lambda_j \in \left[0, \frac{2\pi}{T}\right)\).

(iii) If \(\lambda_i, \lambda_j\) are two different Fourier frequencies in \([0, \frac{2\pi}{T})\) then \(\lambda_i + \frac{2\pi k}{T} \neq \lambda_j + \frac{2\pi s}{T}\), for all \(k, s = 0, \ldots, T-1\).

(iv) Every Fourier frequency in \([0, \frac{2\pi}{T})\) is represented as

\[ \lambda_k = \frac{2\pi k}{N} \quad k = 0, \ldots, m-1. \]

(v) If \(\lambda_i + \lambda_j = \frac{2\pi}{T}\), then \(\lambda_i + \frac{2\pi p}{T} + \lambda_j + \frac{2\pi s}{T} = 2\pi\) for \(p + s = T - 1\). If \(\lambda_i + \lambda_j \neq \frac{2\pi}{T}\) then \(\lambda_i + \frac{2\pi p}{T} + \lambda_j + \frac{2\pi s}{T} \neq 2\pi\).

We apply

\[ \hat{X}_t^N = N^{-1/2} \sum_{p=0}^{N-1} e^{i\lambda_p}V_t(\lambda_p)d_Y(\lambda_p). \]

(5)

to approximate the random integral in (1). We form the finite segment \(\hat{X}_0, \ldots, \hat{X}_{N-1}\) using the observed segment \(X_0, \ldots, X_{N-1}\). Let \(d_X(\cdot)\) be the \(T\)-variate Fourier transform of \(\hat{X}_0, \ldots, \hat{X}_{N-1}\), then we deduce that

\[ d_X(\lambda_j) = A(\lambda_j)d_Y(\lambda_j), \]

(6)

where \(A(\cdot)\) is the Cholesky factor of the spectral density \(f(\cdot)\) and the \(T\)-variate \(d_Y(\lambda_j)\) is similarly defined as in (3) for the purely random series given in (2), note that \(E(d_Y(\lambda_j)d_Y^*(\lambda_j)) = I_T\).

For the Fourier frequency \(\lambda_j \in \left[0, \frac{2\pi}{T}\right)\), the components of vector \(d_X(\lambda_j)\) are given by

\[ d_X\left(\lambda_j + \frac{2\pi s}{T}\right) = \sum_{k=0}^{T-1} a_k \left(\lambda_j + \frac{2\pi s}{T}\right) d_Y\left(\lambda_{p(k,j,s)}\right), \]

(7)

where \(\lambda_{p(k,j,s)}\) is uniquely determined by \(\lambda_{p(k,j,s)} = \lambda_j + \frac{2\pi(s-k)}{T}\) for \(s = 0, \ldots, T - 1\). Moreover the periodogram in terms of \(X_0, \ldots, \hat{X}_{N-1}\) is defined by:

\[ I_{\hat{X}}(\lambda_j) = d_X^*(\lambda_j)d_X(\lambda_j), \]

\[ = \left[I_{pq}(\lambda_j)\right]_{p,q=0,\ldots,T-1} \]

(8)
Estimation of spectral density matrix of PC processes

where

\[ \hat{I}_{pq}(\lambda_j) = d_X(\lambda_j + \frac{2\pi p}{T})d_X(\lambda_j + \frac{2\pi q}{T}). \]

It is clear that

\[ E(\hat{I}_X(\lambda_j)) = f(\lambda_j). \]

Moreover, If \( \lambda_1 < \ldots < \lambda_J \) are arbitrary Fourier frequencies in \((0, \frac{2\pi}{T})\) then \( d_X(\lambda_1), \ldots, d_X(\lambda_J) \) are uncorrelated. If white noise \( \{Y_t, t \in \mathbb{Z}\} \) is Gaussian, then these vectors are independent, and consequently, \( I_X(\lambda_j), j = 1, \ldots, J \) are independent and distributed as \( \mathbb{W}_F(1, f(\lambda_j)) \), where \( \mathbb{W}_F(1, f) \) stands for complex Wishart distribution.

In order to investigate the asymptotic properties of the discrete Fourier transform and the periodogram of PC processes, we apply \( \{X_t, t \in \mathbb{Z}\} \) as an auxiliary operator, as in Soltani and Azimmohseni (2007), where it is proved that under the condition that the Cholesky factor \( A(\cdot) \) of the spectral density matrix \( f(\cdot) \) is continuous, for arbitrary frequencies \( \lambda_1 < \ldots < \lambda_J \) in \((0, \frac{2\pi}{T})\), \( d_X(\lambda_1), \ldots, d_X(\lambda_J) \) are asymptotically independent and distributed as \( \mathbb{N}_F(0, f(\lambda_j)), j = 1, \ldots, J \). Moreover \( I_X(\lambda_j) \) are asymptotically independent \( \mathbb{W}_F(1, f(\lambda_j)) \), for \( j = 1, \ldots, J \). Using the auxiliary operator is somewhat new and is a short cut to the classical lengthy procedure of derivations of the periodogram asymptotic distribution given by Brockwell and Davis (1999).

3. PERIODOGRAM COVARIANCES

This section is devoted to the formulations for covariances between the periodogram at Fourier frequencies. The following kernels are basic tools, namely the Dirichlet kernel, the Fejer kernel and the generalized spectral kernel \( S_N(\cdot; \ldots) \) introduced by Soltani and Azimmohseni (2007), given below.

Suppose \( D_N(\theta) = \sum_{t=0}^{N-1} e^{it\theta}, \theta \in [0, 2\pi) \) is Dirichlet kernel, and let

\[ S_N(\theta; \eta, \eta') = \frac{D_N(\theta - \eta)D_N(\theta - \eta')}{N}, \theta \in [0, 2\pi], \eta, \eta' \in [0, 2\pi), \] (9)

then \( S_N(\theta, \eta, \eta') \), as a function of \( \theta \), possesses the following properties:

(i) \( S_N(\theta; \eta, \eta') = K_N(\theta - \eta) \), where \( K_N \) is the Fejer kernel.

(ii) \( S_N(\theta; \eta, \eta') \to 0, N \to \infty, \) for \( \eta \neq \eta', \theta \in [0, 2\pi); \theta \neq \eta, \theta \neq \eta' \).

(iii) For any \( 0 < \delta < \frac{1}{2} |\eta - \eta'|, |S_N(\theta; \eta, \eta')| < \frac{1}{\sin^2(\frac{\delta}{2})}, N \geq 1, \theta \in [0, 2\pi); \eta, \eta' \in [0, 2\pi), \eta \neq \eta'. \)

(iv) If either \( \theta - \eta \) or \( \theta - \eta' \) be a Fourier frequency in \((0, 2\pi)\) then \( S_N(\theta, \eta, \eta') = 0 \), also if both \( \theta - \eta, \theta - \eta' \in (0, 2\pi) \) then \( S_N(\theta, \eta, \eta') = N \).
The following Lemma is a version of the classic result for periodogram of white noise processes given by Brackwell and Davis (1991). For notational convenience we let
\[
\sigma_{ij}^{(Y)}(p, q; s, r) = \text{Cov}(dy_i(\lambda_i + \frac{2\pi p}{T}), dy_j(\lambda_j + \frac{2\pi q}{T}))
\]
to denote the periodogram covariance of a process \( \{Y_t, t \in \mathbb{Z}\} \).

**Lemma 3.1.** Let \( Y_t \) be a white noise process and \( dy(.) \) denoted the finite Fourier transform of the finite segment \( Y_0, \ldots, Y_{N-1} \) and assume \( \lambda_i, \lambda_j \in (0, \frac{2\pi}{T}) \), then:

(i) For \( \lambda_i \neq \lambda_j \) and \( \lambda_i + \lambda_j \neq \frac{2\pi}{T} \),
\[
\sigma_{ij}^{(Y)}(p, q; s, r) = O\left(\frac{1}{N}\right).
\]

(ii) For \( \lambda_i \neq \lambda_j \) and \( \lambda_i + \lambda_j = \frac{2\pi}{T} \),
\[
\sigma_{ij}^{(Y)}(p, q; s, r) = \begin{cases} 
1 - O\left(\frac{1}{N}\right) & \text{if } p + s = q + r = T - 1 \\
O\left(\frac{1}{N}\right) & \text{o.w.}
\end{cases}
\]

(iii) For \( \lambda_i = \lambda_j \),
\[
\sigma_{ii}^{(Y)}(p, q; s, r) = \begin{cases} 
1 + O\left(\frac{1}{N}\right) & \text{if } p = q = s = r \\
1 + O\left(\frac{1}{N}\right) & \text{if } (p = r) \neq (s = q) \\
O\left(\frac{1}{N}\right) & \text{o.w.}
\end{cases}
\]

In the following we bring our formulations for the covariances of the periodogram of \( X_t \) at Fourier frequencies.

**Theorem 3.1.** Assume the spectral density \( f(.) \) is positive definite and continuous. Then for \( \lambda_i, \lambda_j \in [0, \frac{2\pi}{T}) \) such that \( \lambda_i + \lambda_j \neq \frac{2\pi}{T} \) and for all \( p, q, s, r = 0, 1, \ldots, T - 1 \),
\[
\sigma_{ij}^{(X)}(p, q; s, r) = \begin{cases} 
O(1/N) & \lambda_i \neq \lambda_j, \\
fr-p(\lambda_i + 2\pi q/T)f_{s-q}(\lambda_i + 2\pi p/T) + O(1/N) & \lambda_i = \lambda_j, 
\end{cases}
\]
for all \( p, q, s, r \in \{0, 1, \ldots, T - 1\} \).

**Corollary 3.1.** Assume \( I_X(.) = [I_{pq}(.)]_{p, q=0, \ldots, T-1} \) be the periodogram of \( \{X_0, \ldots, X_{N-1}\} \). Then for all \( \lambda_i, \lambda_j \in (0, \frac{2\pi}{T}) \) such that \( \lambda_i + \lambda_j \neq \frac{2\pi}{T} \) and for \( p, q, r, s \in \{0, \ldots, T - 1\} \),
\[
\text{Cov}(I_{pq}(\lambda_i), I_{sr}(\lambda_j)) = \begin{cases} 
O(1/N) & \lambda_i \neq \lambda_j, \\
fr-p(\lambda_i + 2\pi q/T)f_{s-q}(\lambda_i + 2\pi p/T) + O(1/N) & \lambda_i = \lambda_j.
\end{cases}
\]
Corollary 3.2. Special cases of corollary 3.1 can be considered as follows:

\[
\text{Var}(\hat{I}_{pp}(\lambda_i)) = f_0 \left( \lambda_i + \frac{2\pi p}{T} \right) f_0 \left( \lambda_i + \frac{2\pi p}{T} \right) + O(N^{-1})
\]

\[
\text{Var}(\hat{I}_{pq}(\lambda_i)) = f_{q-p} \left( \lambda_i + \frac{2\pi p}{T} \right) f_{p-q} \left( \lambda_i + \frac{2\pi q}{T} \right) + O(N^{-1}).
\]

According to corollaries 3.1 and 3.2, the periodogram is not a consistent estimator for spectral density of a PC process.

In the next section, we present a consistent estimator for spectral density by smoothing the periodogram matrix.

4. ASYMPTOTICALLY CONSISTENT ESTIMATORS

In order to construct a consistent estimator for the spectral density, we propose the following weighted (smoothed) periodogram:

\[
I_W(\lambda) = \sum_{|k| \leq u_n} W_n(k) I(\lambda + \lambda_k),
\]

where \(\{u_n\}\) and \(\{W_n(.)\}\) are sequences of band-widths and weight functions respectively. We use similar weight functions for all the elements of periodogram matrix. Technically, we need to impose the following assumptions on the weight functions \(\{W_n(.)\}\) and the band-width sequence \(\{u_n, n \in \mathbb{Z}\}\).

(i) \(u_n \to \infty\) and \(\frac{u_n}{N} \to 0\) as \(N \to \infty\).

Therefore for all \(k \leq u_n\) and for a fixed frequency \(\lambda, \lambda + \lambda_k(u_n) \to \lambda\) as \(N \to \infty\).

(ii) \(W_n(k) = W_n(-k)\) and \(W_n(k) \geq 0\) for all \(k\).

(iii) \(\sum_{|k| \leq u_n} W_n(k) = 1\).

(iv) \(\sum_{|k| \leq u_n} W^2_n(k) \to 0\) as \(N \to \infty\),

see Brockwell and Davis (1991). In order to show the consistency of the weighted periodogram, we apply the weighted periodogram \(I_W^X(\lambda) = \left[ I_{pq}^W(\lambda) \right]_{p, q=0, \ldots, T-1}\) in terms of \(\{\tilde{X}_0, \ldots, \tilde{X}_{N-1}\}\) as follows

\[
I_W^X(\lambda) = \sum_{|k| \leq u_n} W_n(k) I_X(\lambda + \lambda_k).
\]
In particular, put \( u_n = n \) and \( W_n(k) = \frac{1}{2|n + 1|} \) for \(|k| \leq n\) where \( \frac{n}{N} \to 0\) as \( N \to +\infty \) then it is easy to see that

\[
I_X^W(\lambda_j) = \frac{1}{2n + 1} \sum_{|k| \leq n} I_X(\lambda_j + \lambda_k)
\]

is distributed as \( W_{2n + 1}^{(\lambda_j)}(2n + 1, \frac{f(\lambda_j)}{2n + 1}) \). This fact shows that the weighted periodogram is a consistent estimator for spectral density.

The following theorem is for the general case.

**Theorem 4.1.** Suppose the spectral density \( f(.) = [f_{pq}(.)]_{p=q=0,\ldots,T-1} \) on \([0, \frac{2\pi}{T}]\) is continuous then \( \hat{f}(.) = I_X^W(.) \) possesses the following properties

(a) \( E(\hat{f}(\lambda_j)) = f(\lambda_j) \) for \( \lambda_j \in [0, \frac{2\pi}{T}] \).

(b) For all \( \lambda_i \neq \lambda_j \in [0, \frac{2\pi}{T}] \) and \( p, q, s, r = 0, \ldots, T - 1 \), \( \text{cov}(\hat{f}_{pq}(\lambda_i), \hat{f}_{rs}(\lambda_j)) \to 0 \), as \( N \to \infty \).

(c) For all \( \lambda_i \in [0, \frac{2\pi}{T}] \) and \( p, q, s, r = 0, \ldots, T - 1 \), \( \text{cov}(\hat{f}_{pq}(\lambda_i), \hat{f}_{rs}(\lambda_i)) \to 0 \) as \( N \to \infty \). Specially \( \text{var}(\hat{f}_{pq}(\lambda_i)) \to 0 \), and \( \text{var}(\hat{f}_{pq}(\lambda_i)) \to 0 \) as \( N \to \infty \).

(d) For \( \lambda_i, \lambda_j \in [0, \frac{2\pi}{T}] \),

\[
\lim_{N \to \infty} \left( \sum_{|k| \leq u_n} W_n^2(k) \right)^{-1} \text{cov}(\hat{f}_{pq}(\lambda_i), \hat{f}_{rs}(\lambda_j)) = \begin{cases} 0 & \lambda_i \neq \lambda_j \\ \frac{f_{pq}(\lambda_i + \frac{2\pi}{T})f_{rs}(\lambda_j + \frac{2\pi}{T})}{\lambda_i \neq \lambda_j} & \lambda_i = \lambda_j. \end{cases}
\]

(e) For \( \lambda_i \in [0, \frac{2\pi}{T}] \), \( \hat{f}(\lambda_i) \) has the same asymptotic distribution as a linear combination of independent complex Wishart matrices:

\[
\tilde{f}(\lambda_i) \overset{d}{=} \sum W_n(k)U_k,
\]

where \( U_k \) is distributed as \( W_{2n+1}^{(\lambda)}(1, f(\lambda_i+k)) \)

Note that a linear combination of Wishart matrices with positive coefficients can be approximated with a Wishart distribution, Tan and Gupta (1983) and Khuri et. al (1994). Although the method have been obtained for real Wishart matrices, but it can be effectively used for complex Wishart matrices with real positive coefficients. Therefore, the distribution of weighted periodogram in theorem 4.1(e), for fixed \( \lambda_i \in [0, \frac{2\pi}{T}] \), can be approximated as \( W_{2n+1}^{(\lambda_i)}(u, g(\lambda_i)) \) where \( u \) and \( g(\lambda_i) \) can be expressed in terms of \( f(\lambda_i+k), |k| \leq u_n \). Using similar notation as Khuri.
et. al (1994), let \( d_t^{(pg, st)} = \sum_{|k| \leq u_n} W_n^2(k) f_{pq}(\lambda_{i+k}) f_{st}(\lambda_{i+k}) \). Also, let \( f^*(\lambda_i) \) denote the \( T^* \times T^* \) matrix, \( T^* = \frac{1}{2} T(T + 1) \), whose elements are \( d_t^{(pg, st)} \) arranged in lexicographic order; that is, \( d_t^{(p_1 q_1, s_1 t_1)} \) appears before \( d_t^{(p_2 q_2, s_2 t_2)} \) in a row if \( q_2 > q_1 \) or, \( q_2 = q_1 \) and \( p_2 > p_1 \). Similarly, \( d_t^{(p_1 q_1, s_1 t_1)} \) before \( d_t^{(p_2 q_2, s_2 t_2)} \) in a column if \( t_2 > t_1 \) or, \( t_2 = t_1 \) and \( s_2 > s_1 \). Now, \( u \) and \( g(\lambda_i) \) are computed as follows:

\[
\begin{align*}
  u &= \left( \frac{\sum_{|k| \leq u_n} W_n(k) f(\lambda_{i+k})^{|T+1}}{|f^*(\lambda_i)|} \right)^{\frac{1}{2}}, \\
  g(\lambda_i) &= \frac{1}{u} \sum_{|k| \leq u_n} W_n(k) f(\lambda_{i+k}).
\end{align*}
\]

Let us now present the asymptotic properties of the actual estimator of spectral density matrix, \( \hat{f}(\lambda) \) i.e.,

\[
\hat{f}(\lambda) = I_N^W(\lambda) = \sum_{|k| \leq u_n} W_n(k) I_X(\lambda + \lambda_k). \tag{15}
\]

Since the periodograms of the actual series \( X_1, \ldots, X_N \) and the auxiliary series \( \hat{X}_1, \ldots, \hat{X}_N \) have the same asymptotic distribution, the same results as in theorem 4.1 can be achieved for the estimator (15).

**Theorem 4.2.** Suppose the spectral density \( f(.) = [f_{ij}(\lambda)]_{i,j=0,\ldots,T-1} \) is continuous on \([0, \frac{2\pi}{T}] \), then \( \hat{f}(\lambda) = I_N^W(\lambda) \) possesses the following properties

**a)** \( E(\hat{f}(\lambda_j)) = f(\lambda_j) \) for \( \lambda_j \in [0, \frac{2\pi}{T}] \).

**b)** For all \( \lambda_1 \neq \lambda_2 \in [0, \frac{2\pi}{T}] \) and \( p, q, s, r = 0, \ldots, T - 1 \),\( \text{cov} \left( \hat{f}_{pq}(\lambda_i), \hat{f}_{rs}(\lambda_j) \right) \to 0 \), as \( N \to \infty \).

**c)** For all \( \lambda_i \in [0, \frac{2\pi}{T}] \) and \( p, q, s, r = 0, \ldots, T - 1 \),\( \text{cov} \left( \hat{f}_{pq}(\lambda_i), \hat{f}_{rs}(\lambda_i) \right) \to 0 \) as \( N \to \infty \). Specially\( \text{var} \left( \hat{f}_{pq}(\lambda_i) \right) \to 0 \), and \( \text{var} \left( \hat{f}_{pp}(\lambda_i) \right) \to 0 \) as \( N \to \infty \).

**d)** For \( \lambda_i, \lambda_j \in [0, \frac{2\pi}{T}] \),

\[
\lim_{N \to \infty} \left( \sum_{|k| \leq u_n} W_n^2(k) \right)^{-1} \text{cov}(\hat{f}_{pq}(\lambda_i), \hat{f}_{sr}(\lambda_j)) =
\begin{cases}
0 & \lambda_i \neq \lambda_j \\
\frac{f_{r-p}(\lambda_i + \frac{2\pi}{T}) f_{s-q}(\lambda_j + \frac{2\pi}{T})}{\lambda_i \neq \lambda_j}
\end{cases}
\]
For $\lambda_i \in \left[0, \frac{2\pi}{T}\right)$, $\hat{f}(\lambda_i)$ has the same asymptotic distribution as a linear combination of independent complex Wishart matrices:

$$\hat{f}(\lambda_i) \overset{d}{=} \sum W_n(k) U_k,$$

where $U_k$ is distributed as $W^C_T(1, f(\lambda_{i+k}))$.

5. NUMERICAL RESULTS

In this section we conduct a simulation study to illustrate the efficiency of the estimator (15). In order to make a comparison between the actual spectral density $f(\lambda)$ and its estimator $\hat{f}(\lambda)$, we utilize the following distance measuring function

$$D_H(\hat{f}; f) = \frac{2}{T} \int_0^T H \left( \hat{f}(\lambda) f^{-1}(\lambda) \right) d\lambda,$$

(16)

for some matrix-valued function $H(\cdot)$. In order to have symmetric property for our distance measuring function, we replace the function $H$ in (16) with

$$\tilde{H}(Z) = H(Z) + H(Z^{-1}).$$

There are two commonly used functions $H$:

$$H_1(Z) = \log |Z| - \log |Z| - T,$$

(17)

$$H_\alpha(Z) = \log |\alpha Z + (1 - \alpha) I_T| - \alpha \log |Z|,$$

(18)

where $|\cdot|$ stands for determinant of a matrix. The Distance measuring functions (17) and (18) are called Kullback-Liebler and Chernoff disparity measures respectively, Kakizawa et al. (1998).

In practice, the integral in (16) is approximated by sum over Fourier frequencies at interval $[0, \frac{2\pi}{T})$, i.e., $\lambda_k = \frac{2\pi k}{N}$, $k = 0, ..., m - 1$, where $m = \frac{N T}{T}$. By using $\tilde{H}(\cdot)$ and the estimator of spectral density, we derive the following integral approximation.

$$D_{\tilde{H}}(\hat{f}; f) \approx \sum_{k=0}^{m-1} \tilde{H} \left( \hat{f}(\lambda_k) f^{-1}(\lambda_k) \right).$$

(19)

Let us give three examples to illustrate the asymptotic properties of the estimator (15). For each example we use Daniell kernel to smooth the periodograms. Moreover, to calculate the average distance between the actual spectral density matrices and their corresponding estimators, we replicate the simulation 100 times.

**Example 4.1.** Assume a zero mean PC process $\{X_t, t \in \mathbb{Z}\}$ is an PAR(1), PCAR(1), process:

$$X_{kT+\nu} = \phi \nu X_{kT+\nu-1} + \epsilon_{kT+\nu}, \quad k \in \mathbb{Z}, \quad \nu = 0, \cdots, T - 1,$$

(20)
with the spectral components $f_k(\theta) = g_k(\theta)$, for $\theta \in [0, \frac{2\pi(T-k)}{T})$, $f_{-k}(\theta) = g_{T-k}(\theta)$, for $\theta \in [\frac{2\pi(T-k)}{T}, 2\pi)$, where

$$g_k(\theta) = (1 - Ae^{-iT\theta})^{-2} \sum_{l=0}^{T-1} \hat{G}_l(\theta + \frac{2\pi l}{T}) \bar{G}_{l-k}(\theta + \frac{2\pi l}{T}), \quad k = 0, \ldots, T-1,$$

in which

$$\hat{G}_j(\theta) = \frac{1}{T} \sum_{n=0}^{T-1} G_n(\theta)e^{i2\pi jn/T}, \quad j \in \mathbb{Z},$$

with $G_n(\theta) = \sum_{k=0}^{T-1} A_{n-k+1}^e e^{-ik\theta}$, $A_r^s = \prod_{j=r}^{s} \nu_j$, for $r \leq s$, and $A_r^s = 1$ for $r > s$, and $A = \prod_{j=0}^{T-1} \nu_j$. To evaluate the distance between the spectral density matrix and its estimator in this example, we take exactly the same data as Nematollahi and Rao (2005).

Figure 1 shows the average distance for different choices of sample size.

**Figure 1.** The Kullback-Liebler distance between actual spectral density matrix of the process (20) and its estimate for different choices of sample size.

**Example 4.2.** Assume a zero-mean PC process $\{X_t, t \in \mathbb{Z}\}$ to be PCMA(2), with the following structure:

$$X_{kT+\nu} = Z_{kT+\nu} + \cos(\nu)Z_{kT+\nu-1} + \sin(\nu)Z_{kT+\nu-2}, \quad k \in \mathbb{Z}, \quad \nu = 0, \ldots, T-1.$$

For $T = 2$ the spectral density matrix can be expressed as

$$f(\theta) = 2D(2\theta)h(2\theta)D^*(2\theta), \quad \theta \in [0, \pi),$$
where $D(2\theta) = \begin{pmatrix} 0.5 & 0.5e^{-i\theta} \\ 0.5 & -0.5e^{-i\theta} \end{pmatrix}$ and $h(\theta) = \begin{pmatrix} 2(1 + \cos(\theta)) & \sin(1)(e^{i2\theta} + e^{i\theta}) \\ \sin(1)(e^{-i2\theta} + e^{-i\theta}) & 2(1 + \cos(1)\cos(\theta)) \end{pmatrix}$.

Taking $T = 2$, figure 2 depicts average distance between the actual spectral density matrix and its estimator for different choices of sample size.

![Figure 2](image)

**Example 4.3.** Assume a zero mean PC process $\{X_t, t \in \mathbb{Z}\}$ admits the following representation

$$X_t = g(t)Y_t, \quad (22)$$

where $g(t)$ is a periodic function with period $T$ and $Y_t$ is a stationary process. In general, the spectral components of this process are given by:

$$f_k(\theta) = \sum_{p=0}^{T-1} f_Y(\theta - \frac{2\pi p}{T})G_pG_{p-k}, \quad k = 0, \ldots, T - 1,$$

where $G_p = \frac{1}{T} \sum_{t=0}^{T-1} g(t)e^{-i\frac{2\pi pt}{T}}$ and $f_Y(\theta)$ is the spectral density matrix of $Y_t$, Hurd and Miamee (2007).

To perform a numerical study in this example, let us assume $g(t) = a(1 + \cos(\frac{2\pi t}{T}))$ and $Y_t$ to be an AR(1) process, $Y_t = \phi Y_{t-1} + Z_t$. Figure 3 shows the average distance between the actual spectral density matrix and its estimator for different choices of sample size and period $T = 4$.

Figures 1-3 evidently show the consistency of the weighted periodogram (15) to estimate the spectral density matrix of PC processes.
Figure 3. The Chernoff distance ($\alpha = 0.95$) between actual spectral density matrix of the process (22) and its estimate by simulated data for different choices of sample size: $a = 2$ and $\phi = 0.8$.

6. APPENDIX

Proof of Lemma 3.1. (a) Following the proposition 10.1 in Brackwell and Davis (1991), we can conclude that

$$
\sigma_{ij}^{(Y)}(p, q; s, r) = \begin{cases}
\frac{1}{\sqrt{2N}} (2p^2 + s - r) / \sqrt{N} \\
\left\{ K_N(\lambda_i + \lambda_j + 4pT) - \frac{1}{2} \right\} / \sqrt{N} \\
\left\{ K_N(\lambda_i + 2\pi(p + r)/T) / \sqrt{N} \\
\left\{ K_N(\lambda_j - \lambda_j + 2\pi(p - r)/T) / \sqrt{N} \right\}
\end{cases}
$$

Without loss of generality suppose $\lambda_i < \lambda_j$ and also $\lambda_i^{(N)}$, $\lambda_j^{(N)}$ are two sequences of Fourier frequencies so that $\lambda_i^{(N)} < \lambda_i < \lambda_j^{(N)} < \lambda_j$ in which $\lambda_i^{(N)} \to \lambda_i$ and $\lambda_j^{(N)} \to \lambda_j$. By replacing $\lambda_i$ and $\lambda_j$ with $\lambda_i^{(N)}$ and $\lambda_j^{(N)}$ respectively and using the properties (iv) and (v) of $S_N(., .)$ for large $N$, we can deduce that $\sigma_{ij}^{(Y)}(p, q; s, r) = O(1/N)$.

This proves the statement (i) of Lemma 3.1. Other statements can be concluded in similar arguments. \[ \Box \]

Proof of theorem 3.1. Similar to the proof of Lemma 3.1 suppose $\lambda_i^{(N)} < \lambda_j$, $\lambda_i < \lambda_i^{(N)} < \lambda_j$. Let us use the notation $\sigma_{ij}^{(x)}(p, q; s, r || N)$ while $\lambda_i$ and $\lambda_j$ are replaced by $\lambda_i^{(N)}$ and $\lambda_j^{(N)}$. By using (7) we can write $\sigma_{ij}^{(x)}(p, q; s, r || N) = \left\{ \sum_{k = 0}^{T-1} a_k(\lambda_i^{(N)} | 2\pi p/T) \right\} a_k(\lambda_j^{(N)} | 2\pi q/T) a_k(\lambda_j^{(N)} | 2\pi r/T)$$

Since $\lambda_i, \lambda_j \in [0, 2\pi/T]$, we can conclude by Lemma 3.1 (i) that

$$
\sigma_{ij}^{(x)}(p, q; s, r || N) = O \left( \frac{1}{N} \sum_{k \neq l, T \neq 0} a_k(\lambda_i^{(N)} | 2\pi p/T) a_k(\lambda_j^{(N)} | 2\pi q/T) a_k(\lambda_j^{(N)} | 2\pi r/T) \right)
$$

$$
= O \left( \frac{1}{N} \right)
$$
In the case that $\lambda_i = \lambda_j$, suppose $\lambda_i^{(N)} \rightarrow \lambda_i$. Thus from (7) we can write

$$\sigma^{(X)}_{ij}(p, q, s, r | N) = \frac{3}{4} \sum_{t \in C_1} a_k(\lambda_i^{(N)}) + 2\pi p.T a_k(\lambda_i^{(N)}) + 2\pi q.T a_k(\lambda_i^{(N)}) + 2\pi s.T a_k(\lambda_i^{(N)}) + 2\pi r.T a_k(\lambda_i^{(N)}) + 2\pi T a_k(\lambda_i^{(N)}) + 2\pi T a_k(\lambda_i^{(N)}) + 2\pi T a_k(\lambda_i^{(N)}) + 2\pi T a_k(\lambda_i^{(N)}) + 2\pi T a_k(\lambda_i^{(N)})$$

where

$$C_1 = \{(k, k', l, l') \in \{0, \ldots, T - 1\} : p - k = q - k' = s - l = r - l'\}$$

$$C_2 = \{(k, k', l, l') \in \{0, \ldots, T - 1\} : (p - k = r - l') \neq (q - k' = s - l)\}$$

$$C_3 = \{(k, k', l, l') \notin C_1 \cup C_2\}$$

According to Lemma 3.1(iii) and continuity of $a_k(\cdot)$, we can obtain

$$B_1 = \left[1 + O\left(\frac{1}{N}\right)\right] \sum_{k=0}^{T-1} a_k(\lambda_i^{(N)}) + 2\pi p.T a_k(\lambda_i^{(N)}) + 2\pi q.T a_k(\lambda_i^{(N)}) + 2\pi s.T a_k(\lambda_i^{(N)}) + 2\pi r.T a_k(\lambda_i^{(N)})$$

$$= \left[1 + O\left(\frac{1}{N}\right)\right] \sum_{k=0}^{T-1} a_k(\theta a_k(\lambda_i^{(N)})) + 2\pi p.T a_k(\lambda_i^{(N)}) + 2\pi q.T a_k(\lambda_i^{(N)}) + 2\pi s.T a_k(\lambda_i^{(N)}) + 2\pi r.T a_k(\lambda_i^{(N)})$$

$$= R(p, q, s, r) + O\left(\frac{1}{N}\right)$$

where $\theta = \lambda_i^{(N)} + 2\pi p.T$. 

$$B_2 = \sum_{k=0}^{T-1} a_k(\lambda_i^{(N)}) + 2\pi p.T a_k(\lambda_i^{(N)}) + 2\pi q.T a_k(\lambda_i^{(N)}) + 2\pi s.T a_k(\lambda_i^{(N)}) + 2\pi r.T a_k(\lambda_i^{(N)})$$

$$= \left[1 + O\left(\frac{1}{N}\right)\right] \sum_{k=0}^{T-1} a_k(\theta a_k(\lambda_i^{(N)})) + 2\pi p.T a_k(\lambda_i^{(N)}) + 2\pi q.T a_k(\lambda_i^{(N)}) + 2\pi s.T a_k(\lambda_i^{(N)}) + 2\pi r.T a_k(\lambda_i^{(N)})$$

$$= R(p, q, s, r) + O\left(\frac{1}{N}\right)$$

where $\theta' = \lambda_i^{(N)} + 2\pi q.T$. 

$$B_3 = \left[O\left(\frac{1}{N}\right)\right] \sum_{k=0}^{T-1} a_k(\lambda_i^{(N)}) + 2\pi p.T a_k(\lambda_i^{(N)}) + 2\pi q.T a_k(\lambda_i^{(N)}) + 2\pi s.T a_k(\lambda_i^{(N)}) + 2\pi r.T a_k(\lambda_i^{(N)})$$

$$= O\left(\frac{1}{N}\right)$$.
Therefore, \( \sigma_{ij}(p, q; s, r) \)

\[
\sigma_{ij}(p, q; s, r) = \frac{1}{T} \sum_{k=0}^{r-1} a_k(\theta) w_{k+r-p} \left( \theta + \frac{2\pi(r-p)}{T} \right) - R(p, q, r, s) + O \left( \frac{1}{N} \right) = f_{r-q} \left( \lambda_i + \frac{2\pi p}{T} \right) + O \left( \frac{1}{N} \right).
\]

\( \square \)

**Proof of Theorem 4.1.** (a): It is immediate from (12) and the continuity of \( f(.) \) that

\[
E \left( \tilde{r}(\lambda_1) \right) = E \left( \sum_{l \mid n \mid m} W_n(k) \tilde{r}(\lambda_1 + \lambda_k) \right) = \sum_{l \mid n \mid m} W_n(k) \tilde{t}(\lambda_1).
\]

(b): Using (14), we get

\[
\text{cov}(\tilde{p}_q(\lambda_i), \tilde{p}_r(\lambda_j)) = \sum_{|g|, |t| \leq m} W_n(g) W_n(t) \text{cov}(\tilde{p}_q(\lambda_i + \lambda_g), \tilde{p}_r(\lambda_j + \lambda_t)).
\]

If \( \lambda_i \neq \lambda_j, \lambda_i + \lambda_j \neq \frac{2\pi}{T} \) and \( N \) be sufficiently large then \( \lambda_i + \lambda_g \neq \lambda_j + \lambda_t \) for all \( |g|, |t| \leq n \). Therefore, by using corollary 3.1, we can obtain

\[
|\text{cov}(\tilde{p}_q(\lambda_i), \tilde{p}_r(\lambda_j))| \leq O \left( \frac{1}{N} \right) \left( \sum_{|l| \leq m} W_n(t) \right)^2.
\]

It follows from (13) that the covariance approaches to zero.

Also, in the case that \( \lambda_i + \lambda_j = \frac{2\pi}{T} \), we have

\[
\text{cov}(\tilde{p}_q(\lambda_i), \tilde{p}_r(\lambda_j)) = \text{cov} \left( \sum_{l \mid n \mid m} W_n(g) \tilde{p}_q(\lambda_i + \lambda_g), \sum_{l \mid n \mid m} W_n(t) \tilde{p}_r(\lambda_j + \lambda_t) \right).
\]

Let \( D_1 = \{(t, g) : |t| + |g| = ml\} \) and \( D_2 = \{(t, g) : |t| + |g| \neq ml\} \). Using Lemma 3.1 (ii), we can conclude that

\[
\text{cov}(\tilde{p}_q(\lambda_i), \tilde{p}_r(\lambda_j)) = B(\lambda_i) \left( 1 - O(N^{-1}) \right) \sum_{D_1} W_n(t) W_n(g) + O(N^{-1}) \sum_{D_2} W_n(t) W_n(g).
\]

where

\[
B(\lambda_i) = \left[ \sum_{k} a_k(\theta) a_{p+s-(T-1)-k} \left( \theta + \frac{2\pi(p+s-(T-1))}{T} \right) \right] \times \left[ \sum_{k} a_k(\theta') a_{q+r-(T-1)-k'} \left( \theta' + \frac{2\pi(q+r-(T-1))}{T} \right) \right].
\]
and \( \theta = \lambda_i + \frac{2\pi p}{T}, \theta' = \lambda_i + \frac{2\pi q}{T} \).

Since for large \( N \), \( \lambda_k + \lambda_j \to 0 \), we have therefore \( D_1 \to \emptyset \). Thus we conclude that
\[
\text{Cov}(f_{pq}(\lambda_i), f_{sr}(\lambda_j)) \to 0.
\]

(c): We note that
\[
\text{cov}(f_{pq}(\lambda_i), f_{sr}(\lambda_j)) = \left( \sum_{|t| \leq m} W_n^2(t) \right) \left( f_{r-p}(\lambda_i) + \frac{2\pi p}{T} f_{s-q}(\lambda_i + \frac{2\pi p}{T}) + O(N^{-1}) \right)
+ \sum_{g \neq t} W_n(g) W_n(t) O(N^{-1})
\]
\[
= \left( \sum_{|t| \leq m} W_n^2(t) \right) \left( f_{r-p}(\lambda_i) + \frac{2\pi p}{T} f_{s-q}(\lambda_i + \frac{2\pi p}{T}) \right)
+ O \left( \frac{1}{N} \right) \left( \sum_{|t| \leq m} W_n^2(t) \right) \left( \sum_{|t| \leq m} W_n^2(t) \right) (2m + 1).
\]

According to the properties (11)-(13), all term tends to zero, for \( N \) sufficiently large. Moreover in this case
\[
\left( \sum_{|t| \leq m} W_n^2(t) \right)^{-1} \text{Cov}(f_{pq}(\lambda_i), f_{sr}(\lambda_j)) \to f_{r-p}(\lambda_i) + \frac{2\pi p}{T} f_{s-q}(\lambda_i + \frac{2\pi p}{T}),
\]
giving (d). \( \square \)

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Estimation of spectral density matrix of PC processes

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Received on 2.12.2016;
revised version on 16.1.2017