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OCCUPATION TIME PROBLEM FOR MULTIFRACTIONAL BROWNIAN MOTION

BY

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Abstract. In this paper, by using a Fourier analytic approach, we investigate sample path properties of the fractional derivatives of multifractional Brownian motion local times. We also show that those additive functionals satisfy a property of local asymptotic self similarity. As a consequence, we derive some local limit theorems for the occupation time of multifractional Brownian motion in the space of continuous functions.

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1. INTRODUCTION

A fractional Brownian motion (fBm for short) $B^H = \{B^H(t),\ t \geqslant 0\}$ with Hurst index $H \in (0,1)$ is a real-valued, centered Gaussian process with covariance function given by

$$\mathbb{E}\left(B^H(t)B^H(s)\right) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}), \qquad \text{for all } s,t \geqslant 0.$$

Several properties of fBm such as self-similarity and the stationarity of increments make it important in theory and also a useful model in various applications, such as in telecommunication, finance, image analysis among others. Note that in the case $H = \frac{1}{2}$; we retrieve the well known Brownian motion. The fact that the Hurst parameter H is independent of time t, makes the Hölder regularity of fBm constant along the paths. This property restricts its applications when modelling phenomena whose regularity evolves in time, such as Internet traffic and some highly textured images with strong global organization see for example Lévy-Véhel [16] and Pesquet-Popescu and Lévy-Véhel [18].

The multifractional Brownian motion, (mBm for short), was introduced as a naturel extension of fBm to overcome this limitation. The basic idea is to replace the Hurst parameter H by a function $H(t) : [0, \infty) \to [a, b] \subset (0, 1)$, which is a Hölder continuous function of exponent $\beta > 0$, i.e., there exists a finite positive constant C such that

$$|H(t) - H(s)| \leqslant C|t - s|^{\beta}, \quad \text{for all } s, t \geqslant 0.$$

1. **Moving average representation of mBm (Lévy-Véhel and Peltier [17]):**

$$\widetilde{B}^{H(t)}(t) = \frac{1}{\Gamma\left(H(t) + \frac{1}{2}\right)} \Big(\int_{-\infty}^{0} \left[(t-u)^{H(t)-\frac{1}{2}} - (-u)^{H(t)-\frac{1}{2}}\right] W(du) + \int_{0}^{t} (t-u)^{H(t)-\frac{1}{2}} W(du) \Big), \quad t \geqslant 0,$$

where W is a standard Brownian motion defined on $\mathbb{R}$.

2. **Harmonizable representation of mBm (Benassi et *al*. [4]):**

$$\widehat{B}^{H(t)}(t) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+\frac{1}{2}}} \widehat{W}(d\xi), \quad t \geqslant 0,$$

where $\widehat{W}(\xi)$ is the Fourier transform of the series representation of white noise with respect to an orthonormal basis of $L^2(\mathbb{R})$.
In the sequel, the notation $B := (B(t),\ t \geqslant 0)$ means that both previous representations of mBm may be chosen.
When $H(.)$ varies with time, the mBm is no longer self-similar, even though it is seen as an intricacy, Lévy-Véhel and Peltier [17] proved that if $H(.)$ satisfies the assumption:

$$(\mathcal{H}_\beta) : \big\{H(.) \text{ is } \beta\text{-Hölder continuous and } \sup_{t \in \mathbb{R}^+} H(t) < \beta\big\},$$

then the mBm verify a property called local asymptotic self-similarity, (LASS in short), defined as follows:

$$\lim_{\rho \to 0^+} law \left\{ \frac{B(t + \rho u) - B(t)}{\rho^{H(t)}},\ u \in \mathbb{R} \right\} = law \left\{ B^{H(t)}(u),\ u \in \mathbb{R} \right\},$$

where $B^{H(t)}$ is a fBm with the Hurst parameter $H(t)$.
By using the concept of local nondeterminism and the assumption $(\mathcal{H}_\beta)$, Boufoussi et *al*. ([6] proved the existence of a jointly continuous local time of mBm and studied its Hölder regularity in time and space. In Boufoussi et *al*. [7], the authors

proved that the local time of mBm has a kind of local asymptotic self similarity (LASS) property. Through this result, they obtained some local limit theorems corresponding to the mBm (first order limit theorems).

The aim of this work is to establish some local limit theorems of occupation time of mBm. To prove our main results we are led to study the regularities of the fractional derivative of local time of mBm, by using a representation based on Fourier analytic approach.

The rest of this paper is organized as follows. In Section 2, we give some basic facts about local time. In Section 3, we prove some Hölder regularities of the fractional derivative of local time of mBm. Section 4 is devoted to the asymptotic results; More precisely: Subsection 4.1 to the LASS property of those additive functionals and in Subsection 4.2, we study some local limit theorems for the occupation time of mBm.
Most of the estimates in the sequel contain unspecified finite positive constants. We use the same symbol for these constants, even when they vary from one line to the next.

2. LOCAL TIMES

In this section we recall the definition and some properties of local times of Gaussian processes and a result on mBm that will be needed in the sequel.

Let $(X(t), t \in \mathbb{R}^+)$ be a separable random process with Borel sample function. The occupation measure of X is defined as follows:

$$\mu(A, B) = \lambda\left(\{s \in A, X(s) \in B\}\right) \quad \forall A \in \mathcal{B}(\mathbb{R}^+),\ \forall B \in \mathcal{B}(\mathbb{R}),$$

where λ is the Lebesgue measure on $\mathbb{R}^+$. If $\mu(A, .)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, we say that X has local time on A and define the local time $L(A, .)$ as the Radon Nikodym derivative of $\mu(A, .)$. Sometimes, we write L_t^x instead of $L([0, t], x)$.
The following property of local time is called the occupation density formula :
For every $t \in \mathbb{R}^+$ and every measurable function $f : \mathbb{R} \mapsto \mathbb{R}^+$,

$$\int_0^t f(X_s)ds = \int_{\mathbb{R}} f(x)L_t^x dx.$$

The well-known Fourier analytic approach introduced by Berman [5], states that for a fixed sample function at fixed t, the Fourier transform on x of L_t^x is the function:

$$F(u) = \int_{\mathbb{R}} e^{iux} L_t^x dx.$$

Using the occupation density formula and the inverse Fourier transform of this

function, we have:

$$L_t^x = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_0^t e^{iu(X(s)-x)} ds \right) du. \tag{2.1}$$

The following result due to Boufoussi et *al.* ([6], Lemma 3.3) will be extensively used in the sequel.

LEMMA 2.1. *Under the assumption* $(\mathcal{H}_\beta)$*, for any even integer* $m \geqslant 2$*, there exist positive and finite constants* $C_m > 0$ *and* $\delta > 0$ *such that, for any* $t, s \geqslant 0$ *with* $|t-s| < \delta$*, any* $x, y \in \mathbb{R}$ *and any* $0 < \xi < \min(1, \frac{1-\lambda}{2\lambda})$*, where* $\lambda = \sup_{u \in [s,t]} H(u)$*, we have*

$$\mathbb{E}\left[L_t^x - L_s^x\right]^m \leqslant C_m \frac{|t-s|^{m(1-\lambda)}}{\Gamma(1+m(1-\lambda))}. \tag{2.2}$$

$$\mathbb{E}\left[L_t^y - L_s^y - L_t^x + L_s^x\right]^m \leqslant C_m |y-x|^{\xi m} \frac{|t-s|^{m(1-\lambda(1+\xi))}}{\Gamma(1+m(1-\lambda(1+\xi))}. \tag{2.3}$$

REMARK 2.1. *Estimates similar to (2.2) and (2.3) were proved for the multi-fractional Brownian sheet by Meerschaert et al. [15], Lemmas 3.5 and 3.7.*

3. FRACTIONAL DERIVATIVE OF LOCAL TIME OF MBM

The fractional derivative have many use such as fractional integro-differentiation which has now become a significant topic in mathematical analysis. Fractional derivatives of local time have been discussed for physical purposes in the paper by Ezawa et *al.* [9]. For a complete survey on the fractional derivative we refer the reader to Hardy and Littelewood [13] and the book by Samko et *al.* [19].

DEFINITION 3.1. Let $0 < \theta < 1$ and $f : \mathbb{R} \to \mathbb{R}$ be a function that belongs to $\mathcal{C}^\theta \cap L^1(\mathbb{R})$, where $\mathcal{C}^\theta$ is the space of locally $\theta-$Hölder continuous functions on $\mathbb{R}$. For $0 < \gamma < \theta$, we define $D_\pm^\gamma f$ by:

$$D_\pm^\gamma f(x) := \frac{\gamma}{\Gamma(1-\gamma)} \int_0^{+\infty} \frac{f(x) - f(x \mp y)}{y^{1+\gamma}} dy.$$

The operators D_+^γ and D_-^γ are called, respectively, right-handed and left-handed Marchaud fractional derivatives of order γ. We put $D^\gamma := D_+^\gamma - D_-^\gamma$.

REMARK 3.1. 1. *D_+^γ and D_-^γ satisfy the switching identity:*

$$\int_{\mathbb{R}} f(x) D_-^\gamma g(x) dx = \int_{\mathbb{R}} g(x) D_+^\gamma f(x) dx,$$

for any $f, g \in \mathcal{C}^\theta(\mathbb{R}) \cap L^1(\mathbb{R})$ *and* $0 < \gamma < \theta$.

2. *$D^{\gamma}_{\pm}f$ is $(\theta-\gamma)-$Hölder continuous whenever f is $\theta-$Hölder continuous for any $0<\gamma<\theta$.*

3. *For $h:\mathbb{R}\to\mathbb{R}$ and $a>0$, we denote by h_a the function $x\to h(ax)$, then*

$$D^{\gamma}_{\pm}(h_a)=a^{\gamma}(D^{\gamma}_{\pm}h)_a, \quad \textit{for all } \gamma>0.$$

4. *The fractional derivatives of the local time are particular continuous additive functionals of zero energy, in the sense of Fukushima [11].*

Lemma 2.1 allows us to define the fractional derivative of order γ of local time of mBm as follows:

$$D^{\gamma}L(t,.)(x):=\frac{\gamma}{\Gamma(1-\gamma)}\int_0^{\infty}\frac{L_t^{x+a}-L_t^{x-a}}{a^{1+\gamma}}da, \qquad \text{for all } \ 0<\gamma<\xi.$$

Using (2.1), we get

$$D^{\gamma}L(t,.)(x):=\frac{\gamma}{2\pi\Gamma(1-\gamma)}\int_0^{+\infty}\int_{\mathbb{R}}\int_0^{t}\frac{e^{iu(B(s)-(x+a))}-e^{iu(B(s)-(x-a))}}{a^{1+\gamma}}dsduda$$

$$=\frac{\gamma}{2\pi\Gamma(1-\gamma)}\int_0^{+\infty}\int_{\mathbb{R}}\int_0^{t}e^{iuB(s)}[e^{-iu(x+a)}-e^{-iu(x-a)}]\frac{1}{a^{1+\gamma}}dsduda. \tag{3.1}$$

Here are the main results of this section.

THEOREM 3.1. *Let $0<\gamma<\xi$ and $D\in\{D^{\gamma},D^{\gamma}_{\pm}\}$. Under the assumption $(\mathcal{H}_{\beta})$, there exist finite and positive constants $C_m>0$, and $\delta>0$ such that for every $t,s\geqslant 0$ with $|t-s|<\delta$, any $x\in\mathbb{R}$ and any $0<\xi<\min(1,\frac{1-\lambda}{2\lambda})$, where $\lambda=\sup_{u\in[s,t]}H(u)$, we have,*

$$\|DL(t,.)(x)-DL(s,.)(x)\|_{2m}\leqslant C_m|t-s|^{1-\lambda(1+\gamma)}, \tag{3.2}$$

where $\|.\|_{2m}=\left[\mathbb{E}|.|^{2m}\right]^{\frac{1}{2m}}$.

Proof. We will prove (3.2) for D^{γ} since the other cases may be treated in the same way. For all integers $m\geqslant 1$, we have for any $b>0$

$$||D^{\gamma}L_t^{\cdot}(x)-D^{\gamma}L_s^{\cdot}(x)||_{2m}=\left\|\int_0^{+\infty}\frac{\gamma}{\Gamma(1-\gamma)}\frac{L_t^{x+u}-L_t^{x-u}-L_s^{x+u}+L_s^{x-u}}{u^{1+\gamma}}du\right\|_{2m}$$

$$\leqslant C(I_1(b)+I_2(b)),$$

where

$$I_1(b):=\int_0^{b}\frac{||L_t^{x+u}-L_t^{x-u}-L_s^{x+u}+L_s^{x-u}||_{2m}}{u^{1+\gamma}}du,$$

and

$$I_2(b) := \int_b^{+\infty} \frac{||L_t^{x+u} - L_t^{x-u} - L_s^{x+u} + L_s^{x-u}||_{2m}}{u^{1+\gamma}} du.$$

We estimate $I_1(b)$ and $I_2(b)$ separately.
In view of (2.3), we deduce

$$\begin{aligned} I_1(b) &\leqslant C_m |t-s|^{1-\lambda(1+\xi)} \int_0^b u^{\xi-\gamma-1} du \\ &\leqslant C_m |t-s|^{1-\lambda(1+\xi)} b^{\xi-\gamma}. \end{aligned}$$

Now, we deal with $I_2(b)$. We use (2.2) to conclude that

$$\begin{aligned} I_2(b) &\leqslant C_m |t-s|^{1-\lambda} \int_b^{+\infty} \frac{1}{u^{1+\gamma}} du \\ &\leqslant C_m |t-s|^{1-\lambda} b^{-\gamma}. \end{aligned}$$

Finally, by choosing $b = |t-s|^{\lambda}$, we obtain the desired result. ■

THEOREM 3.2. *Let $0 < \gamma < \xi$ and $D \in \{D^\gamma, D^\gamma_\pm\}$. Under the assumption $(\mathcal{H}_\beta)$, there exists a finite and positive constant $C_m > 0$, such that for every $t, s \geqslant 0$ with $|t-s| < \delta$, any $(x,y) \in \mathbb{R}^2$ and any $0 < \xi < \min(1, \frac{1-\lambda}{2\lambda})$, where $\lambda = \sup_{u\in[s,t]} H(u)$, we have,*

$$\begin{aligned} &||DL(t,.)(x) - DL(t,.)(y) - DL(s,.)(x) + DL(s,.)(y)||_{2m} \\ &\leqslant C_m |t-s|^{1-\lambda(1+\xi)} |x-y|^{\xi-\gamma}. \end{aligned} \tag{3.3}$$

REMARK 3.2. *Similar results are obtained respectively, by Ait Ouahra and Eddahbi [2] and Ait Ouahra [1], for the local time and the fractional derivative of local time of a symmetric stable process of index $1 < \alpha \leqslant 2$. The key ingredient in their proofs was the Markov property of the symmetric stable process, which is'nt satisfied by the mBm.*

Proof. of Theorem 3.2. For any integer $m \geqslant 1$, we have

$$\begin{aligned} &\|DL(t,.)(x) - DL(t,.)(y) - DL(s,.)(x) + DL(s,.)(y)\|_{2m} \\ &= \left\| \int_{[0,\infty)} \frac{L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} - L_t^{y+u} + L_s^{y+u} + L_t^{y-u} - L_s^{y-u}}{u^{1+\gamma}} du \right\|_{2m} \\ &\leqslant C \int_{[0,\infty)} \left\| \frac{L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} - L_t^{y+u} + L_s^{y+u} + L_t^{y-u} - L_s^{y-u}}{u^{1+\gamma}} \right\|_{2m} du \\ &\leqslant C \left(J_1(b) + J_2(b) \right). \end{aligned}$$

where

$$J_1(b) := \int_{[0,b]} \left\| \frac{L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} - L_t^{y+u} + L_s^{y+u} + L_t^{y-u} - L_s^{y-u}}{u^{1+\gamma}} \right\|_{2m} du,$$

$$J_2(b) := \int_{[b,+\infty[} \left\| \frac{L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} - L_t^{y+u} + L_s^{y+u} + L_t^{y-u} - L_s^{y-u}}{u^{1+\gamma}} \right\|_{2m} du,$$

for some $b > 0$.

First, let us estimate $J_1(b)$. Combining the Minkowski inequality, and (2.3) we obtain

$$\begin{aligned} & \left\| L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} - L_t^{y+u} + L_s^{y+u} + L_t^{y-u} - L_s^{y-u} \right\|_{2m} \\ \leqslant \; & \left\| L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} \right\|_{2m} + \left\| L_t^{y+u} - L_s^{y+u} - L_t^{y-u} + L_s^{y-u} \right\|_{2m} \\ \leqslant \; & C_m \frac{|u|^{\xi} |t-s|^{(1-\lambda(1+\xi))}}{\Gamma(1+2m(1-\lambda(1+\xi)))}. \end{aligned}$$

Therefore,

$$\begin{aligned} J_1(b) \; & \leqslant \; C_m \frac{|t-s|^{(1-\lambda(1+\xi))}}{\Gamma(1+2m(1+\lambda(1+\xi))} \int_{[0,b]} \frac{u^{\xi}}{u^{1+\gamma}} du \\ & \leqslant \; C_m \frac{|t-s|^{(1-\lambda(1+\xi))}}{\Gamma(1+2m(1-\lambda(1+\xi)))} b^{\xi-\gamma}. \end{aligned}$$

Now, we are going to estimate $J_2(b)$. Using the Minkowski inequality, and decomposing otherwise, we obtain

$$\begin{aligned} & \left\| L_t^{x+u} - L_s^{x+u} - L_t^{x-u} + L_s^{x-u} - L_t^{y+u} + L_s^{y+u} + L_t^{y-u} - L_s^{y-u} \right\|_{2m} \\ \leqslant \; & \left\| L_t^{x+u} - L_s^{x+u} - L_t^{y+u} + L_s^{y+u} \right\|_{2m} + \left\| L_t^{x-u} - L_s^{x-u} - L_t^{y-u} + L_s^{y-u} \right\|_{2m} \\ \leqslant \; & C_m \frac{|x-y|^{\xi} |t-s|^{(1-\lambda(1+\xi))}}{\Gamma(1+2m(1-\lambda(1+\xi)))}. \end{aligned}$$

Therefore, $J_2(b)$ is dominated by

$$b^{-\gamma} \times C_m \frac{|x-y|^{\xi} |t-s|^{(1-\lambda(1+\xi))}}{\Gamma(1+2m(1-\lambda(1+\xi)))}.$$

By choosing $b = |x-y|$, we deduce that,

$$\begin{aligned} & \| DL(t,.)(x) - DL(t,.)(y) - DL(s,.)(x) + DL(s,.)(y) \|_{2m} \\ & \leqslant C_m \, |t-s|^{(1-\lambda(1+\xi))} \, |x-y|^{(\xi-\gamma)}, \end{aligned}$$

which completes the proof of the theorem. ■

4. ASYMPTOTIC RESULTS

It is well known that techniques for proving limit theorems related to self similar processes use the self similarity of their local times. It is natural to expect the same when dealing with processes satisfying the LASS property. The answer to the preceding question is affirmative in the case of the mBm and the result is given by the following lemma. (See Theorem 5.1 in Boufoussi et *al*. [7]).
In the sequel, we will use the following notation $B^{H(t)}(t) = B(t)$ for the sake of simplicity.

LEMMA 4.1. *Under the assumption* $(\mathcal{H}_\beta)$, *for any fixed* t_0, *the local time of mBm is locally asymptotically self similar with parameter* $1 - H(t_0)$, *in the sense that for every* $x \in \mathbb{R}$, *the family processes* $\{Y_\rho(t,x), t \in [0,1]\}_{\rho>0}$, *defined by:*

$$Y_\rho(t,x) = \frac{L(t_0 + \rho t, \rho^{H(t_0)}x + B(t_0)) - L(t_0, \rho^{H(t_0)}x + B(t_0))}{\rho^{1-H(t_0)}}, \tag{4.1}$$

converges in law to the local time $\{l(t,x), t \in [0,1]\}$, *of the fBm* $B^{H(t_0)}$ *with the Hurst parameter* $H(t_0)$.

We will introduce the proof of this lemma in the Appendix, to clarify a passage in the estimate of $I_1^{\varepsilon,\rho}$ in page 864 in Boufoussi et al. [6].

REMARK 4.1. *The previous result can also be obtained by using propositions (4.1) and (4.2) in Jolis and Viles [14], which allows to identify the limits law as local time.*

4.1. LASS for the fractional derivative of local time of mBm.

PROPOSITION 4.1. *Under the assumption* $(\mathcal{H}_\beta)$, *for any fixed* t_0, *the fractional derivative of local time of mBm is LASS with the parameter* $1-(1+\gamma)H(t_0)$, *in the sense that for every* $x \in \mathbb{R}$, *the family of processes* $\{D_\rho(t,x), t \in [0,1]\}_{\rho\geqslant 0}$ *defined by:*

$$D_\rho(t,x) := \frac{D^\gamma L(t_0 + t\rho, .)(\rho^{H(t_0)}x + B(t_0)) - D^\gamma L(t_0, .)(\rho^{H(t_0)}x + B(t_0))}{\rho^{1-H(t_0)(1+\gamma)}}$$

converges in law to the fractional derivative $\{D^\gamma l_t^{\cdot}(x), t \in [0,1]\}$ *of local time of fBm* $B^{H(t_0)}$.

Proof. To prove the convergence in law, we proceed in two steps. First, we prove the tightness of the family $\{D_\rho(t,x), t \in [0,1]\}_{\rho\geqslant 0}$ in the space of continu-

ous functions. According to Theorem 3.1. we obtain

$$\begin{aligned}
&\mathbb{E}|D_\rho(t,x) - D_\rho(s,x)|^m \\
&= \frac{\mathbb{E}|D^\gamma L(t_0+\rho t, \rho^{H(t_0)}x + B(t_0)) - D^\gamma L(t_0+s\rho, \rho^{H(t_0)}x + B(t_0))|^m}{\rho^{(1-(1+\gamma)H(t_0))m}} \\
&\leqslant C\frac{|t\rho - s\rho|^{(1-(1+\gamma)H(t_0))m}}{\rho^{(1-(1+\gamma)H(t_0))m}} \\
&\leqslant C|t-s|^{m(1-H(t_0)(1+\gamma))}.
\end{aligned}$$

Then to prove the tightness, it suffices to take $m > \frac{1}{1-H(t_0)(1+\gamma)}$.

It now remains to prove the convergence of the finite-dimensional distributions of D_ρ, as ρ tends to 0, to those of the fractional derivative of local time of the fBm $B^{H(t_0)}$ with Hurst parameter $H(t_0)$.

We note $C_\gamma = \dfrac{\gamma}{2\pi\Gamma(1-\gamma)}$ and $Y^\rho(t_0) = \rho^{H(t_0)}x + B(t_0)$. In view of the definition of $D^\gamma L$, we get:

$$D^\gamma L(t_0+t\rho,.)(Y^\rho(t_0)) - D^\gamma L(t_0,.)(Y^\rho(t_0))$$

$$= C_\gamma \int_0^\infty \int_{\mathbb{R}} \int_{t_0}^{t_0+t\rho} e^{iuB(s)}\big[e^{-iu(Y^\rho(t_0)+a)} - e^{-iu(Y^\rho(t_0)-a)}\big]\frac{1}{a^{1+\gamma}}daduds$$

$$= C_\gamma \int_0^\infty \int_{\mathbb{R}} \int_0^t e^{iuB(\rho r+t_0)}\big[e^{-iu(Y^\rho(t_0)+a)} - e^{-iu(Y^\rho(t_0)-a)}\big]\frac{1}{a^{1+\gamma}}\rho dadudr$$

$$= C_\gamma \int_0^\infty \int_{\mathbb{R}} \int_0^t e^{i\frac{v}{\rho^{H(t_0)}}B(\rho r+t_0)}\big[e^{-i\frac{v}{\rho^{H(t_0)}}(Y^\rho(t_0)+a)} - e^{-i\frac{v}{\rho^{H(t_0)}}(Y^\rho(t_0)-a)}\big]\frac{\rho^{1-H(t_0)}}{a^{1+\gamma}}dadvdr$$

$$= C_\gamma \int_0^\infty \int_{\mathbb{R}} \int_0^t e^{iv\frac{B(\rho r+t_0)-B(t_0)}{\rho^{H(t_0)}}}\big[e^{-iv(x+b)} - e^{-iv(x-b)}\big]\frac{1}{b^{1+\gamma}}\rho^{1-(1+\gamma)H(t_0)}dbdvdr.$$

We have used the change of variable $s = \rho r + t_0, u = \frac{v}{\rho^{H(t_0)}}$ and $b = \frac{a}{\rho^{H(t_0)}}$.

Consequently, we have

$$D_\rho(t,x) = C_\gamma \int_0^\infty \int_{\mathbb{R}} \int_0^t e^{iv\frac{B(\rho r+t_0)-B(t_0)}{\rho^{H(t_0)}}}(e^{-iv(x-b)} - e^{-iv(x+b)})\frac{1}{b^{1+\gamma}}drdvdb,$$

which is the fractional derivative of local time of the Gaussian process B^ρ.

Finally, by using Lemma 5.2 in Eddahbi and Vives [8], concerning the continuity of the fractional derivative, we end the proof of this result. ■

4.2. Local limit theorems. Limit theorems of fractional derivative of local time was studied by Yamada [21] for Brownian motion, Fitzsimmons and Getoor [10] for symmetric stable process of index $1 < \alpha \leqslant 2$ and Shieh [20] for the fBm with the Hurst parameter $H \in (0,1)$ where this last author proved that if f is in the range of fractional derivative transform, i.e., $f = D^{\gamma}_{\pm} g$ with $g \in \mathcal{C}^{\beta}$ and $\int g(x)dx \neq 0$, then the family of processes,

$$\left\{ \frac{1}{A^{1-(1+\gamma)H}} \int_0^{At} f(X_s)ds, \qquad t \geqslant 0 \right\}$$

converges in law, in the space of continuous functions, as $A \to +\infty$, to the process,

$$\left\{ \int g(x)dx D^{\gamma}_{\pm} L^{\cdot}_t(0) \right\}.$$

Ait Ouahra and Eddahbi [2] generalized the result of Fitzsimmons and Getoor [10] in Hölder space and Ait Ouahra and Ouali [3] extended the result of Shieh [20] to Besov space.

The main tool in the proof of all these results was the self similarity of the process. In this section, by using the LASS property, we investigate local limit theorems of occupation time of the mBm.

THEOREM 4.1. *Suppose $f = D^{\gamma}_{\pm} g$, where $g \in C^{\beta}$ with compact support for some β such that $0 < \gamma < \beta < \xi < \min(1, \frac{1-\lambda}{2\lambda})$. Under the assumption $(\mathcal{H}_{\beta})$, the following convergence in law holds,*

$$\lim_{A \to +\infty} \lim_{\rho \to 0^+} \frac{1}{A^{1-(1+\gamma)H(t_0)}} \int_0^{At} f\left(\frac{B(\rho s + t_0) - B(t_0)}{\rho^{H(t_0)}} \right) ds = \int_{\mathbb{R}} g(x)dx D^{\gamma}_{\mp} l^{\cdot}_t(0).$$

THEOREM 4.2. *Suppose $f = D^{\gamma}_{\pm} g$, where $g \in C^{\beta}$ with compact support for some β such that $0 < \gamma < \beta < \xi < \min(1, \frac{1-\lambda}{2\lambda})$. Under the assumption $(\mathcal{H}_{\beta})$, we have,*

$$\frac{1}{\psi(\rho)} \int_{t_0}^{\rho t + t_0} f\left(\frac{B(s) - B(t_0) - \rho^{H(t_0)} y}{\theta(\rho)} \right) ds \xrightarrow[\rho \to 0^+]{\mathcal{L}} \int_{\mathbb{R}} g(x)dx D^{\gamma}_{\mp} l^{\cdot}_t(y),$$

with $\int |g(x)||x|^{\xi-\gamma}dx < \infty$, $\frac{\psi(\rho)}{\theta(\rho)} = \rho^{1-(1+\gamma)H(t_0)}$ and $\frac{\theta(\rho)}{\rho^{H(t_0)}} = o(1)$.

Proof. of Theorem 4.1. Combining the LASS property of mBm and Theorem VI.4.2 in Gihman and Skorohod [12], we obtain,

$$\int_0^{At} f\left(\frac{B(s\rho + t_0) - B(t_0)}{\rho^{H(t_0)}} \right) ds \xrightarrow[\rho \to 0^+]{\mathcal{L}} \int_0^{At} f(B^{H(t_0)}(s))ds,$$

and by using the result of Shieh [20], the family of processes

$$\left\{ \frac{1}{A^{1-(1+\gamma)H(t_0)}} \int_0^{At} f(B^{H(t_0)}(s))ds \right\}_{t\geqslant 0}$$

converges in law to the process

$$\left\{ \int_{\mathbb{R}} g(x)dx D^{\gamma}_{\mp} l^{\cdot}_t(0) \right\}_{t\geqslant 0}.$$

Then the theorem is proved. ■

P r o o f. of Theorem 4.2. We prove only the case $f = D^{\gamma}_{+} g$ since the proof of the other case is similar. We put $A_\rho = \theta(\rho)x + \rho^{H(t_0)}y + B(t_0)$ and $B_\rho = \rho^{H(t_0)}y + B(t_0)$. By the occupation time formula and the switching identity, we have,

$$\begin{aligned}
&\frac{1}{\psi(\rho)} \int_{t_0}^{\rho t+t_0} D^{\gamma}_{+} g\left(\frac{B(s) - B(t_0) - \rho^{H(t_0)}y}{\theta(\rho)} \right) ds \\
&= \int_{\mathbb{R}} f(x) \frac{L(\rho t + t_0, A_\rho) - L(t_0, A_\rho)}{\rho^{1-(1+\gamma)H(t_0)}} dx \\
&= \int_{\mathbb{R}} g(x) \frac{D^{\gamma}_{-} L(\rho t + t_0, .)(A_\rho) - D^{\gamma}_{-} L(t_0, .)(A_\rho)}{\rho^{1-(1+\gamma)H(t_0)}} dx.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&D^{\gamma}_{-} L(\rho t + t_0, .)(A_\rho) - D^{\gamma}_{-} L(t_0, .)(A_\rho) \\
&= D^{\gamma}_{-} L(\rho t + t_0, .)(A_\rho) - D^{\gamma}_{-} L(\rho t + t_0, .)(B_\rho) - D^{\gamma}_{-} L(t_0, .)(A_\rho) \\
&\qquad - D^{\gamma}_{-} L(t_0, .)(B_\rho) + D^{\gamma}_{-} L(\rho t + t_0, .)(B_\rho) + D^{\gamma}_{-} L(t_0, .)(B_\rho) \\
&= D^{\gamma}_{-} L(\rho t + t_0, .)(B_\rho) - D^{\gamma}_{-} L(t_0, .)(B_\rho) + D^{\gamma}_{-} L(J_0, .)(A_\rho) - D^{\gamma}_{-} L(J_0, .)(B_\rho)
\end{aligned}$$

with $J_0 = [t_0, t_0 + \rho t]$. Therefore,

$$\begin{aligned}
&\frac{1}{\psi(\rho)} \int_{t_0}^{\rho t+t_0} D^{\gamma}_{+} g\left(\frac{B(s) - B(t_0) - \rho^{H(t_0)}y}{\theta(\rho)} \right) ds \\
&= \frac{D^{\gamma}_{-} L(\rho t + t_0, .)(B_\rho) - D^{\gamma}_{-} L(t_0, .)(B_\rho)}{\rho^{1-(1+\gamma)H(t_0)}} \int_{\mathbb{R}} g(x)dx \\
&\qquad - \int_{\mathbb{R}} \frac{D^{\gamma}_{-} L(J_0, .)(B_\rho) - D^{\gamma}_{-} L(J_0, .)(A_\rho)}{\rho^{1-(1+\gamma)H(t_0)}} g(x)dx \\
&=: (*) - (**).
\end{aligned}$$

By Proposition 4.1 we have $(*)\xrightarrow{\mathcal{L}} D^{\gamma} l_t^{\cdot}(y)\int_{\mathbb{R}} g(x)dx$ as $\rho \to 0$. It only remains to prove that $(**)$ converges to 0. By the Hölder inequality we have:

$$\begin{aligned}
&\mathbb{E}\left|\int_{\mathbb{R}} \frac{D_-^{\gamma} L(J_0,.)(B_\rho)) - D_-^{\gamma} L(J_0,.)(A_\rho)}{\rho^{1-(1+\gamma)H(t_0)}} g(x)dx\right| \\
&\leqslant \int_{\mathbb{R}} |g(x)| \left\| \frac{D_-^{\gamma} L(J_0,.)(B_\rho)) - D_-^{\gamma} L(J_0,.)(A_\rho))}{\rho^{1-(1+\gamma)H(t_0)}} \right\|_2 dx. \qquad (4.2)
\end{aligned}$$

Applying Lemma 2.1, for the process $B(t) - B(t_0)$ instead of mBm, we have:

$$\begin{aligned}
&\left\| \frac{D_-^{\gamma} L(J_0,.)(B_\rho)) - D_-^{\gamma} L(J_0,.)(A_\rho))}{\rho^{1-(1+\gamma)H(t_0)}} \right\|_2 \\
&\leqslant C(t\rho)^{1-H(t_0)(1+\xi)} |x|^{\xi-\gamma} \frac{\theta(\rho)^{\xi-\gamma}}{\rho^{1-(1+\gamma)H(t_0)}} \\
&\leqslant C t^{1-H(t_0)(1+\xi)} |x|^{\xi-\gamma} \left(\frac{\theta(\rho)}{\rho^{H(t_0)}}\right)^{\xi-\gamma}.
\end{aligned}$$

for sufficiently small ρ and $0 < \gamma < \xi < \min(1, \frac{1-\lambda}{2\lambda})$. Hence, (4.2) is dominated by

$$C t^{1-H(t_0)(1+\xi)} \int_{\mathbb{R}} |g(x)||x|^{\xi-\gamma} dx \left(\frac{\theta(\rho)}{\rho^{H(t_0)}}\right)^{\xi-\gamma}.$$

This last integral is finite by the assumption of the theorem and then $(**)$ tends to zero as ρ tends to zero. ■

5. APPENDIX

Proof of Lemma 4.1 : To prove the convergence in law, we proceed in two steps. First we prove the tightness of the family $\{Y_\rho(t,x), t \in [0,1]\}_{\rho>0}$ in the space of continuous functions. By using (2.2) and (2.3), for ρ small enough, we obtain

$$\begin{aligned}
&\mathbb{E}|Y_\rho(t,x) - Y_\rho(s,x)|^m \\
&= \frac{\mathbb{E}[L(t_0+\rho t, \rho^{H(t_0)}x + B(t_0)) - L(t_0+\rho s, \rho^{H(t_0)}x + B(t_0))]}{\rho^{(1-H(t_0))m}} \\
&\leqslant C_m |t-s|^{(1-H(t_0))m}.
\end{aligned}$$

We can take $m > \frac{1}{1-H(t_0)}$, to get the tightness.
Now, we prove the convergence of the finite dimensional distributions of Y_ρ, as ρ tends to 0, to those of the local time ℓ of the fBm, $B^{H(t_0)}$, with Hurst parameter

$H(t_0)$. We need to show that for any $d \geqslant 1$, $a_1, ..., a_d \in \mathbb{R}$ and $t_1, ..., t_d \in [0,1]$, the following convergence holds,

$$\sum_{j=1}^{d} a_j Y_\rho(t_j, x) \xrightarrow{\mathcal{L}} \sum_{j=1}^{d} a_j \ell(t_j, x) \text{ as } \rho \to 0.$$

We will show the convergence of the corresponding characteristic function. More precisely we prove that

$$\big|\mathbb{E}\exp\big[i\lambda \sum_{j=1}^{d} a_j Y_\rho(t_j, x)\big] - \mathbb{E}\exp\big[i\lambda \sum_{j=1}^{d} a_j \ell(t_j, x)\big]\big| \longrightarrow 0 \text{ as } \rho \to 0.$$

Let's introduce the following notations:

$$B^\rho(t) = \frac{B(t_0 + \rho t) - B(t_0)}{\rho^{H(t_0)}},$$

$$\phi_{\varepsilon,x}(X)(t) = \frac{1}{\varepsilon}\int_0^t \mathbf{1}_{[x,x+\varepsilon]}(X(s))ds,$$

$$I_1^{\varepsilon,\rho} = \big|\mathbb{E}\exp\big[i\lambda \sum_{j=1}^{d} a_j Y_\rho(t_j, x)\big] - \mathbb{E}\exp\big[i\lambda \sum_{j=1}^{d} a_j \phi_{\varepsilon,x}(B^\rho)(t_j)\big]\big|,$$

$$I_2^{\varepsilon,\rho} = \big|\mathbb{E}\exp\big[i\lambda \sum_{j=1}^{d} a_j \phi_{\varepsilon,x}(B^{H(t_0)})(t_j)\big] - \mathbb{E}\exp\big[i\lambda \sum_{j=1}^{d} a_j \phi_{\varepsilon,x}(B^\rho)(t_j)\big]\big|,$$

and

$$I_3^{\varepsilon,\rho} = \big|\mathbb{E}\exp\big[i\lambda \sum_{j=1}^{d} a_j \phi_{\varepsilon,x}(B^{H(t_0)})(t_j)\big] - \mathbb{E}\exp\big[i\lambda \sum_{j=1}^{d} a_j \ell(t_j, x)\big]\big|.$$

Therefore,

(5.1)
$$\big|\mathbb{E}\exp\big[i\lambda \sum_{j=1}^{d} a_j Y_\rho(t_j, x)\big] - \mathbb{E}\exp\big[i\lambda \sum_{j=1}^{d} a_j \ell(t_j, x)\big]\big| \leqslant I_1^{\varepsilon,\rho} + I_2^{\varepsilon,\rho} + I_3^{\varepsilon,\rho}.$$

On the other hand, Y_ρ is the local time of B^ρ and by using the mean value theorem and the occupation density formula we obtain

(5.2)
$$\begin{aligned} I_1^{\varepsilon,\rho} &\leqslant C \max_{1\leqslant j\leqslant d} \mathbb{E}|Y_\rho(t_j, x) - \phi_{\varepsilon,x}(B^\rho)(t_j)| \\ &= C \max_{1\leqslant j\leqslant d} \mathbb{E}\Big|\frac{1}{\varepsilon}\int_x^{x+\varepsilon} Y_\rho(t_j, y)dy - Y_\rho(t_j, x)\Big| \\ &\leqslant C \max_{1\leqslant j\leqslant d} \frac{1}{\varepsilon}\int_x^{x+\varepsilon} \|Y_\rho(t_j, y) - Y_\rho(t_j, x)\|_{L^m(\Omega)}dy. \end{aligned}$$

$$
\begin{aligned}
&||Y_\rho(t_j,y)-Y_\rho(t_j,x)||_{L^m(\Omega)}\\
&=||\frac{L(t_0+\rho t_j,\rho^{H(t_0)}y+B(t_0))-L(t_0,\rho^{H(t_0)}y+B(t_0))}{\rho^{1-H(t_0)}}\\
&-\frac{L(t_0+\rho t_j,\rho^{H(t_0)}x+B(t_0))-L(t_0,\rho^{H(t_0)}x+B(t_0))}{\rho^{1-H(t_0)}}||_{L^m(\Omega)}.
\end{aligned}
$$

On the other hand, by denoting $I_j=[t_0,t_0+\rho t_j]$, we get

$$
\begin{aligned}
&||\frac{L(I_j,\rho^{H(t_0)}y+B(t_0))-L(I_j,\rho^{H(t_0)}x+B(t_0))}{\rho^{1-H(t_0)}}||_{L^m(\Omega)}\\
&\leqslant \frac{C_m(\rho t_j)^{1-H(t_0)(1+\xi)}\rho^{H(t_0)\xi}|x-y|^\xi}{\rho^{1-H(t_0)}}\\
&\leqslant \frac{C_m t_j^{1-H(t_0)(1+\xi)}\rho^{1-H(t_0)}|x-y|^\xi}{\rho^{1-H(t_0)}}\\
&\leqslant C_m|x-y|^\xi.
\end{aligned}
$$

Therefore (5.2) is dominated by

$$
C_m \max_{1\leqslant j\leqslant d}\frac{1}{\epsilon}\int_x^{x+\varepsilon}|x-y|^\xi dy=C_m\varepsilon^\xi.
$$

Then (5.2) converges to zero as ε tends to zero uniformly in ρ.
We deal now with $I_2^{\varepsilon,\rho}$. Since the family of processes $B^\rho(t),t\in[0,1]_{\rho>0}$ converges in distribution to the fBm $B^{H(t_0)}(t),t\in[0,1]$ with Hurst parameter $H(t_0)$, then the second term converges to zero as ρ tends to 0 by Lemma 5.1 in Boufoussi et *al.* [7]. The last term in (5.1) is treated in a similar way as the first and the proof of the finite dimensional convergence is complete.

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