ON THE EXACT DIMENSION OF MANDELBROT MEASURE

BY

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Abstract. We develop, in the context of the boundary of a supercritical Galton-Watson tree, a uniform version of the argument used by Kahane in [16] on homogeneous trees to estimate almost surely and simultaneously the Hausdorff and packing dimensions of the Mandelbrot measure over a suitable set $J$. As an application, we compute, almost surely and simultaneously, the Hausdorff and packing dimensions of the level sets $E(\alpha)$ of infinite branches of the boundary of the tree along which the averages of the branching random walk have a given limit point.

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1. INTRODUCTION AND MAINS RESULTS

Let $(N,W_1,W_2,\ldots)$ be a random vector taking values in $\mathbb{N}_+ \times \mathbb{R}^{+\mathbb{N}_+}$. Then consider, $\{(N_{u0}, W_{u1}, W_{u2}, \ldots)\}_{u \in \bigcup_{n \geq 0} \mathbb{N}_+^n}$, a family of independent copies of this random vector indexed by the finite sequences $u = u_1 \ldots u_n$, $n \geq 0$, $u_i \in \mathbb{N}_+$ ($n = 0$ corresponds to the empty sequence denoted $\emptyset$). Let $T$ be the Galton-Watson tree with defining element $\{N_u\}$: we have $\emptyset \in T$ and, if $u \in T$ and $i \in \mathbb{N}_+$, then $ui$, the concatenation of $u$ and $i$, belongs to $T$ if and only if $1 \leq i \leq N_u$. Similarly, for each $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, denote by $T(u)$ the Galton-Watson tree rooted at $u$ and defined by $\{N_{uv}\}$, $v \in \bigcup_{n \geq 0} \mathbb{N}_+^n$.

For each $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, we denote by $|u|$ its length, i.e. the number of letters of $u$, and $[u]$ the cylinder $u \cdot N_+^{+\mathbb{N}_+}$, i.e. the set of $t \in \mathbb{N}_+^n$ such that $t_1 \ldots t_{|u|} = u$. If $t \in \mathbb{N}_+^n$, we set $|t| = \infty$, and the set of prefixes of $t$ consists of $\emptyset \cup \{t_1 \ldots t_n : n \geq 1\} \cup \{t\}$. Also we set $t_{|n|} = t_1 \ldots t_n$ if $n \geq 1$ and $t_{|0|} = \emptyset$.

The probability space over which the previous random variables are built is denoted $(\Omega, A, P)$, and the expectation with respect to $P$ is denoted $E$.

We assume that $E(N) > 1$ so that the Galton-Watson tree is supercritical. Without loss of generality, we also assume that the probability of extinction equals
0, so that \( \mathbb{P}(N \geq 1) = 1 \).

The boundary of \( T \) is the subset of \( \mathbb{N}_+^N \) defined as \( \partial T = \bigcap_{n \geq 1} \bigcup_{u \in T_n} [u] \), where \( T_n = T \cap \mathbb{N}_+^n \). \( \mathbb{N}_+^n \) is endowed with the standard ultrametric distance

\[
d_1 : (s, t) \mapsto \exp(-|s \wedge t|),
\]

where \( s \wedge t \) stands for the longest common prefix of \( s \) and \( t \), and with the convention that \( \exp(-\infty) = 0 \). Endowed with the induced distance the set \( \partial T \) is almost surely (a.s.) compact.

For the sake of simplicity we will assume throughout that the logarithmic moment generating function

\[
\tau(q) = \log \mathbb{E} \left( \sum_{i=1}^{N} W_1^q \right)
\]
is finite over \( \mathbb{R} \). Then, we define, for \( u \in \bigcup_{n \geq 0} \mathbb{N}_+^n \), the random variable

\[
W_{q,u} = \frac{W_n^q}{\mathbb{E} \left( \sum_{i=1}^{N} W_1^q \right)} = W_n^q e^{-\tau(q)}.
\]

Consider the set

\[
J = \{ q \in \mathbb{R}, \ \tau(q) - q \tau'(q) > 0 \} = \{ q \in \mathbb{R}, \ \tau^*(\tau(q)) > 0 \},
\]

where \( \tau^* \) is the Legendre transform of the function \( \tau \) defined as, \( \forall \alpha \in \mathbb{R} \)

\[
\tau^*(\alpha) = \inf_{q \in \mathbb{R}} \left( \tau(q) - q\alpha \right).
\]

Let

\[
\Omega^1_\gamma = \text{int} \{ q : \mathbb{E}[\sum_{i=1}^{N} W_1^q] < \infty \}, \quad \Omega^1 = \bigcup_{\gamma \in (1,2]} \Omega^1_\gamma \quad \text{and} \quad \mathcal{J} = J \cap \Omega^1.
\]

Then, we define, for \( n \geq 1 \) and \( u \in \mathbb{N}_+^n \), the sequence \( (Y_p(q, u))_{p \geq 1} \) defined as

\[
Y_p(q, u) = \sum_{v \in T_p(u)} \prod_{k=1}^{n} W_{q,u,v_1\ldots v_k},
\]

when \( u = \emptyset \) this quantity will be denoted by \( Y_1(q) \) and, when \( n = 0 \), its value equal 1.
Since, for all \( q \in J \), we have
\[
\begin{align*}
\mathbb{E} \left( \sum_{i=1}^{N} W_{q,i} \right) &= 1 \\
\mathbb{E} \left( \sum_{i=1}^{N} W_{q,i} \log W_{q,i} \right) &= q \tau'(q) - \tau(q) < 0 \\
\mathbb{E} \left( \sum_{i=1}^{N} W_{q,i} \log^+ \left( \sum_{i=1}^{N} W_{q,i} \right) \right) &< \infty,
\end{align*}
\]
then, with probability 1, \((Y_p(q, u))\) converges to a positive limit \( Y(q, u) \), while the limit exists and vanishes if the condition is violated. This fact was proven by Kahane in [15] when \( N \) is constant and Biggins in [5] in general. Then, we can associate the Mandelbrot measure defined on the \( \sigma \)-field \( \mathcal{C} \) generated by the cylinders of \( \mathbb{N}_+^N \) as
\[
(1.1) \quad \mu_q([u]) = \begin{cases} 
W_{q,u_1} W_{q,u_2} \cdots W_{q,u_n} Y(q, u) & \text{if } u \in T_n \\
0 & \text{otherwise}
\end{cases}
\]
and supported on \( \partial T \). Moreover, under the property \( E(Y(q) \log^+ Y(q)) < \infty \), hence in particular when \( E(Y(q)^h) < \infty \) for some \( h > 1 \), where \( Y(q) = Y(q, \emptyset) \), we have, following [15], [19], [3], for all \( q \in J \), a.s., for \( \mu_q \)-almost every \( t \in \partial T \),
\[
\liminf_{n \to \infty} \frac{\log \mu_q([t_n])}{-n} \geq \tau(q) - q \tau'(q).
\]
Hence, for all \( q \in J \), a.s., the lower Hausdorff dimension of \( \mu_q \):
\[
\dim \mu_q \geq \tau(q) - q \tau'(q),
\]
see section 6 for the definition.

The Mandelbrot measure \( \mu_q \) is naturally considered when studying the multifractal analysis of some random sets [7, 10, 13, 3, 7, 1, 2]. By exploiting the simultaneous construction of the Mandelbrot measure \( \mu_q, q \in J \) and using a uniform version of the argument used by Kahane in [16] on homogeneous trees we get the following result.

**Theorem 1.1.** With probability 1, for all \( q \in J \), \( \dim \mu_q \geq \tau(q) - q \tau'(q) \).

As an application we study, for \( q \in J \), the set \( E(\tau'(q)) \) associated with the branching random walk with \( (X_i = \log(W_i))_{1 \leq i \leq N} \) (see section 4). Since, with probability 1, for all \( q \in J \), the set \( E(\tau'(q)) \) is supported by \( \mu_q \) and its packing dimension is smaller than \( \tau^*(\tau'(q)) \) (see proposition 2.7 in [2]), we get
\[
a.s., \forall q \in J, \quad \dim \mu_q \leq \tau(q) - q \tau'(q),
\]
where \( \dim \mu_q \) is the upper packing dimension of \( \mu_q \) (see section 6 for the definition). As a consequence, we get that the measure are exact dimensional.
COROLLARY 1.1. With probability 1, for all \( q \in J \),
\[
\dim \mu_q = \dim \mu_q = \tau(q) - q^{\tau(q)},
\]
where \( \dim \mu_q \) and \( \dim \mu_q \) denote receptively the Hausdorff and packing dimension of \( \mu_q \).

REMARK 1.1. These results are known (see [1], [3]). Using a uniform version of a percolation argument, we will give a new proof of the sharp lower bounds for the lower Hausdorff dimension of these measures.

2. PRELIMINARY

Given an increasing sequence \( \{A_n\}_{n \geq 1} \) of sub-\( \sigma \)-fields of \( \mathcal{A} \) and a sequence of random functions \( \{P_n(t, \omega)\}_{n \geq 1} \) \( (t \in \partial T) \) such that

1. \( P_n(t) = P_n(t, \omega) \) are non-negative and independent process; \( P_n(\cdot, \omega) \) is borelian for almost all \( \omega \); \( P_n(t, \cdot) \) is \( A_n \)-mesurable for each \( t \).

2. \( \mathbb{E}(P_n(t)) = 1 \) for all \( t \in \partial T \).

such a sequence \( \{P_n\} \) is called a sequence of weights adopted to \( \{A_n\} \). Let

\[
Q_n(t) = Q_n(t, \omega) = \prod_{k=1}^{n} P_k(t, \omega).
\]

For any \( n \geq 1 \) and any positive Radon measure \( \sigma \) on \( \partial T \) (we write \( \sigma \in \mathcal{M}^+(\partial T) \)), we consider the random measures \( Q_n \sigma \) defined as

\[
Q_n\sigma(A) = \int_A Q_n(t) d\sigma(t) \quad (A \in \mathcal{B}(\partial T)),
\]

where \( \mathcal{B}(\partial T) \) is the Borel field on \( \partial T \). For all \( A \in \mathcal{B}(\partial T) \), \( Q_n\sigma(A) \) is a positive martingale so it converges almost surely. Also we have, for all \( \sigma \in \mathcal{M}^+(\partial T) \), almost surely, the random measure \( Q_n \sigma \) converge weakly to random measure \( Q \sigma \).

There are two possible extreme cases. The first one is that \( Q_n(\partial T) \) converges almost surely to zero, i.e. \( Q \sigma = 0 \) a.s.. In this case, we say that \( Q \) degenerates on \( \sigma \) or \( \sigma \) is said to be \( Q \)-singular. The second one is that \( Q_n(\partial T) \) converges in \( L^1 \) so that \( \mathbb{E}(Q_n(\partial T)) = \sigma(\partial T) \). In this case we say that \( Q \) fully acts on \( \sigma \) or \( \sigma \) is said to be \( Q \)-regular.

THEOREM 2.1. Let \( \alpha \) be a positive number such that \( \mathcal{H}^\alpha(\partial T) < \infty \) where \( \mathcal{H}^\alpha \) denote the \( \alpha \)-dimensional Hausdorff measure. Let \( 0 < h < 1 \) and \( C > 0 \). Suppose

\[
(2.1) \quad \sup_{t \in B}(Q_n(t)^h) \leq C |B|^{(1-h)\alpha}
\]

for all balls \( B \) and some \( n = n(B) \) depending on \( B \). Then \( Q \) is completely degenerate, that is \( Q \sigma = 0 \) a.s. for all \( \sigma \in \mathcal{M}^+(\partial T) \).
This provides a good tool to verify the $Q$–singularity of $\sigma$. Indeed, if a measure is not killed, it means that it has a lower Hausdorff dimension at least $\alpha$.

3. PROOF OF THEOREM 1.1

For each $\beta \in (0, 1]$, let $W_\beta$ be a random variable taking the value $1/\beta$ with probability $\beta$ and the value $0$ with probability $1 - \beta$. Then let $\{W_\beta,u\}_{u \in \bigcup_{n \geq 0} N^n_+}$ be a family of independent copies of $W_\beta$. Denote by $(\Omega_\beta, \mathcal{A}_\beta, \mathbb{P}_\beta)$ the probability space on which this family is defined.

We naturally extend to $(\Omega_\beta \times \Omega, \mathcal{A} \otimes \mathcal{A}, \mathbb{P} \otimes \mathbb{P})$ the random variables $W_{\beta,u}$ and the random vectors $(N_0, W_{u1}, \ldots)$ as

$W_{\beta,u}(\omega, \omega) = W_{\beta,u}(\omega)$

and

$(N_0(\omega, \omega), W_{u1}(\omega, \omega), \ldots) = (N_0(\omega), W_{u1}(\omega), \ldots)$,

so that the families $\{W_{\beta,u}\}_{u \in \bigcup_{n \geq 0} N^n_+}$ and $\{(N_0, W_{u1}, \ldots)\}_{u \in \bigcup_{n \geq 0} N^n_+}$ are independent.

The expectation with respect to $\mathbb{P} \otimes \mathbb{P}$ will be also denoted by $\mathbb{E}$. For $n \geq 1$ and $\beta \in (0, 1]$, we set $\mathcal{F}_n = \sigma((N_u, W_{u1}, W_{u2}, \ldots) : u \in \bigcup_{k=0}^n N_{k+1}^n)$ and $\mathcal{F}_{\beta,n} = \sigma((W_{\beta,u1}, W_{\beta,u2}, \ldots) : u \in \bigcup_{k=0}^n N_{k+1}^n)$. We denote by $\mathcal{F}_0$ and $\mathcal{F}_{\beta,0}$ the trivial $\sigma$-field.

If $\beta \mathbb{E}(N) > 1$, the random variables $N_{\beta,u}(\omega, \omega) = \sum_{i=1}^{N_u(\omega)} 1_{\{\beta^{-1}\}}(W_{\beta,u}(\omega))$ define a new supercritical Galton-Watson process to which are associated the trees $T_{\beta,n} \subset T_n$ and $T_{\beta,u}(u) \subset T_n(u)$, $u \in \bigcup_{n \geq 0} N^n_+$, $n \geq 1$, as well as the infinite tree $T_{\beta} \subset T$ and the boundary $\partial T_{\beta} \subset \partial T$ conditional on non extinction.

For $u \in \bigcup_{n \geq 0} N^n_+$, $1 \leq i \leq N(u)$ and $q \in \mathcal{J}$, we define

$W_{\beta,q,ui} = W_{\beta,u1} W_{q,ui}$.

For $q \in \mathcal{J}$, $\beta \mathbb{E}(N) > 1$, and $n \geq 0$ and $u \in \bigcup_{n \geq 0} N^n_+$, we define

$Y_n(\beta, q, u) = \sum_{v_1 \ldots v_n \in T_n(u)} \prod_{k=1}^n W_{\beta,q,uv_1 \ldots v_k}$.

When $u = 0$ this quantity will be denoted by $Y_n(\beta, q)$ and, when $n = 0$, its value equal $1$.

3.1. A family of measures indexed by $\mathcal{J}$. For $\beta \in (\mathbb{E}(N)^{-1}, 1]$ and $\epsilon > 0$ we set

$\mathcal{J}_{\beta, \epsilon} = \{q \in \mathcal{J} : \tau^*(\tau(q)) > -\log \beta + \epsilon\}$. 
Notice that \( \tau^*(\tau'(q)) \) takes values between 0 and \( \tau(0) = \log(E(N)) \) over \( \mathcal{J} \), then
\[
(3.1) \quad \mathcal{J} = \bigcup_{\beta \in (E(N)^{-1}, 1], \epsilon > 0} \mathcal{J}_{\beta, \epsilon}.
\]

The following propositions will be established in section 5.

**Proposition 3.1.**
1. For all \( u \in \bigcup_{n \geq 0} \mathbb{N}_+^n \), the sequence of continuous functions \( Y_n(\cdot, u) \) converges uniformly, almost surely and in \( L^1 \) norm, to a positive limit \( Y(\cdot, u) \) on \( \mathcal{J} \).
2. With probability 1, for all \( q \in \mathcal{J} \), the mapping
\[
(3.2) \quad q \mapsto e^{Y_n(q; u)} = \sum_{u \in T_n} \left( \prod_{k=1}^n W_{\beta, u_1 \ldots u_k} \right) q(u) \mu_q(|u|)
\]
defines a positive measure on \( \partial T \).

**Proposition 3.2.** Let \( \beta \in (0, 1] \) such that \( \beta E(N) > 1 \). For all \( \epsilon \in \mathbb{Q}_+^* \)
1. the sequence of continuous functions \( Y_n(\beta, \cdot) \) converges uniformly, almost surely and in \( L^1 \) norm, to a positive limit \( Y(\beta, \cdot) \) on \( \mathcal{J}_{\beta, \epsilon} \).
2. the sequence of continuous functions
\[
q \mapsto \tilde{Y}_n(\beta, q) = \sum_{u \in T_n} \left( \prod_{k=1}^n W_{\beta, u_1 \ldots u_k} \right) \mu_q(|u|)
\]
converges uniformly, almost surely and in \( L^1 \) norm, towards \( Y(\beta, \cdot) \) on \( \mathcal{J}_{\beta, \epsilon} \).

3.2. Proof of Theorem 1.1. Let \( \epsilon \in \mathbb{Q}_+^* \) and \( \beta \in (0, 1] \) such that \( \beta E(N) > 1 \). For every \( t \in \partial T \) and \( \omega_{\beta} \in \Omega_{\beta} \) set
\[
Q_{\beta, n}(t, \omega_{\beta}) = \prod_{k=1}^n W_{\beta, t_{\epsilon_k}}
\]
so that for \( q \in \mathcal{J}_{\beta, \epsilon} \), \( \tilde{Y}_n(\beta, q) \) is the total mass of the measure \( Q_{\beta, n}(t, \omega_{\beta}) \cdot d\mu_q(|u|) \).

Now, Proposition 3.2 claims that there exists a measurable subset \( A \) of \( \Omega \times \Omega_\beta \) of full probability in the set of those \( (\omega, \omega_{\beta}) \) such that \( (T_{\beta, n})_{n \geq 1} \) survives and for all \( (\omega, \omega_{\beta}) \in A \), for all \( q \in \mathcal{J}_{\beta, \epsilon} \), \( \tilde{Y}_n(\beta, q) \) does not converge to 0. Moreover, since the branching number of the tree \( T \) is \( \mathbb{P} \)-almost surely equal to the constant \( E(N) \) and \( \beta E(N) > 1 \), conditional on \( T \), the \( \mathbb{P}_{\beta} \)-probability of non extinction of \( (T_{\beta, n})_{n \geq 1} \) is positive ([12, Th. 6.2]). Thus, the projection of \( A \) to \( \Omega \) has \( \mathbb{P} \)-probability 1, and there exists a measurable subset \( \Omega(\beta, \epsilon) \) of \( \Omega \), such that \( \mathbb{P}(\Omega(\beta, \epsilon)) = 1 \) and for all \( \omega \in \Omega(\beta, \epsilon) \), there exists \( \Omega_{\beta} \subseteq \Omega_{\beta} \) of positive probability such that for all \( \omega \in \Omega(\beta, \epsilon) \), for all \( q \in \mathcal{J}_{\beta, \epsilon} \), for all \( \omega_{\beta} \in \Omega_{\beta} \), \( \tilde{Y}_n(\beta, q) \) does not converge to 0. In terms of the multiplicative chaos theory developed in [17],
this means, that for all \( \omega \in \Omega(\beta, \epsilon) \) and \( q \in \mathcal{J}_{\beta, \epsilon} \), the set of those \( \omega_{\beta} \) such that the multiplicative chaos \( (Q_{\beta, n}(\cdot, \omega))_{n \geq 1} \) has not killed \( \mu_q \) on the compact set \( \partial T \) has a positive \( \mathbb{P}_\beta \)-probability. Now, the good property of \( (Q_{\beta, n}(\cdot, \omega))_{n \geq 1} \) is that
\[
\mathbb{E}_\beta \left( \sup_{t \in B} (Q_{\beta, n}(t))^{h} \right) = e^{n(1-h)\log(\beta)} = (|B|)^{-(1-h)\log(\beta)}
\]
for any \( h \in (0, 1) \) and any ball \( B \) of generation \( n \) in \( \partial T \), where \( |B| \) stands for the diameter of \( B \) and \( \mathbb{E}_\beta \) stands for the expectation with respect to \( \mathbb{P}_\beta \). Thus, we can apply Theorem 3 of [17] and claim that for all \( \omega \in \Omega(\beta, \epsilon) \) and all \( q \in \mathcal{J}_{\beta, \epsilon} \), no piece of \( \mu_q \) is carried by a Borel set of Hausdorff dimension less than \( -\log(\beta) \).

Let \( \Omega' = \bigcap_{\beta \in (\mathbb{E}(N))^{1-1}} \bigcap_{Q^* \in \mathcal{Q}^*_+} \Omega(\beta, \epsilon) \). This set is of \( \mathbb{P} \)-probability 1. Let \( q \in \mathcal{J} \), by (5.1), there exists a sequence of points \((\beta_n, \epsilon_n) \in (\mathbb{E}(N))^{1-1} \times \mathcal{Q}^*_+ \) such that \( \tau(q) - q\tau(q) > -\log(\beta_n) + \epsilon_n / 2 \) for all \( n \geq 1 \), and \( \lim_{n \to \infty} -\log(\beta_n) = \tau(q) - q\tau(q) \), \( \lim_{n \to \infty} \epsilon_n = 0 \) and \( q \in \bigcap_{n \geq 1} \mathcal{J}_{\beta_n, \epsilon_n} \). Consequently, the previous paragraph implies that for all \( \omega \in \Omega' \),
\[
\dim(\mu^\omega_q) \geq \limsup_{n \to \infty} -\log(\beta_n) = \tau(q) - q\tau(q).
\]

4. APPLICATION

Let \((N, X_1, X_2, \ldots)\) be a random vector taking values in \( \mathbb{N}^+ \times (\mathbb{R})^\mathbb{N}^+ \). Then consider \( \{ (N_u, X_{u1}, X_{u2}, \ldots) \}_{u \in \mathbb{N}^+_1} \) a family of independent copies of the vector \((N, X_1, X_2, \cdots)\) indexed by the set of finite words over the alphabet \( \mathbb{N}^+ \). We assume that \( \mathbb{E}(N) > 1 \) and \( \mathbb{P}(N > 1) = 1 \). Suppose that, for all \( u \in T \), \( X_u \) is integrable and the sequence \( (X_u)_{u \in \mathbb{N}^+_1} \) are i.i.d. Given \( t \in \partial T \), by the strong law of large numbers, we have \( \lim \frac{1}{n} S_n(t) = \mathbb{E}(X_1) \) almost surely, where
\[
S_n(t) = \sum_{k=1}^{n} X_{t_{k1} \cdots t_k}.
\]
Since \( \partial T \) is not countable, the following question naturally arises: are there some \( t \in \partial T \) so that \( \lim_{n \to \infty} \frac{1}{n} S_n(t) = \alpha \neq \mathbb{E}(X_1) \) ? Multifractal analysis is a framework adapted to answer this question. Consider the set \( I \) of those \( \alpha \in \mathbb{R} \) such that
\[
E(\alpha) = \{ t \in \partial T : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_{u_{1} \cdots u_k} = \alpha \} \neq \emptyset.
\]
These level sets can be described geometrically through their Hausdorff dimensions. They have been studied by many authors, see [9, 10, 13, 14, 17, 18, 19]; all these papers also deal with the multifractal analysis of associated Mandelbrot measures (see also [15, 20, 21, 22] for the study of Mandelbrot measures dimension).
Take, for \( u \in \bigcup_{n \geq 0} \mathbb{N}_1^n \), the random variable \( W_u = e^{X_u} \) and we set

\[ I = \{ \tau'(q) ; \ q \in \mathcal{J} \}. \]

**Theorem 4.1.** With probability 1, for all \( \alpha \in I \), the multifractal formalism holds at \( \alpha \), i.e.,

\[ \dim E(\alpha) = \dim E(\alpha) = \tau^*(\alpha), \]

in particular, \( E(\alpha) \neq \emptyset \).

**Proof.** A simple covering argument yields, with probability 1, for all \( \alpha \in I \),

\[ \dim E(\alpha) \leq \tau^*(\alpha) \] (see for example Proposition 2.7 in [2]). In addition consider the Mandelbrot measure \( \mu_q, q \in \mathcal{J} \), defined by (1.1). It is known (see for example Corollary 2.5 in [1]) that with probability 1,

\[ \mu_q(E(\tau'(q))) = 1. \] In addition, according to Theorem 1.1, we have, with probability 1, for all \( q \in \mathcal{J} \),

\[ \dim \mu_q \geq \tau(q) - q \tau'(q). \] We deduce the result from the mass distribution principle (Theorem 6.2).

**Remark 4.1.** This result has been proved when \( N \) is not random in [3], and in the weaker form, for each fixed \( \alpha \in I \), almost surely \( \dim E(\alpha) = \tau^*(\alpha) \), when \( N \) is random in [7, 10, 13].

**Remark 4.2.** Using Cauchy formula we can prove Theorem 1.1 (see [1]). Then our result gives a new approach to estimate, almost surely and simultaneously the lower Hausdorff dimension of the Mandelbrot measure over \( \mathcal{J} \).

### 5. PROOF OF PROPOSITIONS 3.1 AND 3.2

Define, for \( (q, p, \beta) \in \mathcal{J} \times [1, \infty) \times (0, 1] \), the function

\[ \varphi_\beta(p; q) = \exp (\tau(pq) - p \tau(q) + (1 - p) \log \beta). \]

**Lemma 5.1.** For all nontrivial compact \( K \subset \mathcal{J}_{\beta, \epsilon} \) there exists a real number

\[ 1 < p_K < 2 \] such that for all \( 1 < p \leq p_K \) we have

\[ \sup_{q \in K} \varphi_\beta(p_K, q) < 1. \]

**Proof.** Let \( q \in \mathcal{J}_{\beta, \epsilon} \), one has \( \frac{\partial \varphi_\beta}{\partial p}(1^+, q) < 0 \) and there exists \( p_q > 1 \) such that \( \varphi_\beta(p_q, q) < 1 \). Therefore, in a neighborhood \( V_q \) of \( q \), one has \( \varphi_\beta(p_q, q') < 1 \) for all \( q' \in V_q \). If \( K \) is a nontrivial compact of \( \mathcal{J}_{\beta, \epsilon} \), it is covered by a finite number of such \( V_q \). Let \( p_K = \inf_{q \in K} p_q \). If \( 1 < p \leq p_K \) and \( \sup_{q \in K} \varphi_\beta(p, q) \geq 1 \), there exists \( q \in K \) such that \( \varphi_\beta(p, q) \geq 1 \), and \( q \in V_{q_i} \) for some \( i \). By log-convexity of the mapping \( p \mapsto \varphi_\beta(p, q) \) and the fact that \( \varphi_\beta(1, q) = 1 \), since \( 1 < p \leq p_q \), we have \( \varphi_\beta(p, q) < 1 \), which is a contradiction. \( \blacksquare \)
**Lemma 5.2.** For all compact \( K \subset J \), there exists \( \tilde{p}_K > 1 \) such that

\[
\sup_{q \in K} \mathbb{E}((\sum_{i=1}^N W_i^q)^{	ilde{p}_K}) < \infty.
\]

**Proof.** Since \( K \) is compact and the family of open sets \( J \cap \Omega_1^\gamma \) increases to \( J \) as \( \gamma \) decreases to 1, there exists \( \gamma \in (1, 2] \) such that \( K \subset \Omega_1^\gamma \). Take \( \tilde{p}_K = \gamma \). The conclusion comes from the fact that the function \( q \mapsto \mathbb{E}((\sum_{i=1}^N W_i^q)^{	ilde{p}_K}) \) is continuous over \( \Omega_1^\gamma \).

**Lemma 5.3.** ([6]) If \( \{X_i\} \) is a family of integrable and independent complex random variables with \( \mathbb{E}(X_i) = 0 \), then \( \mathbb{E}|\sum X_i|^p \leq 2^p \sum \mathbb{E}|X_i|^p \) for \( 1 \leq p \leq 2 \).

The same lines as in Lemma 2.11 in [1], we get the following lemma.

**Lemma 5.4.** Let \( (N, V_1, V_2, \ldots) \) be a random vector taking values in \( \mathbb{N}_+ \times \mathbb{C}^{\mathbb{N}_+} \) and such that \( \sum_{i=1}^N V_i \) is integrable and \( \mathbb{E}(\sum_{i=1}^N V_i) = 1 \). Consider a sequence \( \{(N_u, V_{u1}, V_{u2}, \ldots)\}_{u \in \bigcup_{n \geq 0} \mathbb{N}_+} \) of independent copies of \( (N, V_1, \ldots, V_N) \). We define the sequence \( (Z_n)_{n \geq 0} \) by \( Z_0 = 1 \) and for \( n \geq 1 \)

\[
Z_n = \sum_{u \in T_n} (\prod_{k=1}^n V_{uk}).
\]

Let \( p \in (1, 2] \). There exists a constant \( C_p \) depending on \( p \) only such that for all \( n \geq 1 \),

\[
\mathbb{E}(|Z_n - Z_{n-1}|^p) \leq C_p \left(\mathbb{E}\left(\sum_{i=1}^N |V_i|^p\right)^{n-1}\left(\mathbb{E}\left(\sum_{i=1}^N |V_i|^p\right) + 1\right)\right).
\]

**Proof of Proposition 5.2.** (1) Recall that the uniform convergence result uses an argument developed in [6]. Fix a compact \( K \subset J_{\beta, \epsilon} \). By Lemma 5.2, we can fix a compact neighborhood \( K' \) of \( K \) and \( \tilde{p}_{K'} > 1 \) such that

\[
\sup_{q \in K'} \mathbb{E}((\sum_{i=1}^N W_i^q)^{	ilde{p}_{K'}}) < \infty.
\]

By Lemma 5.1, we can fix \( 1 < p_K \leq \min(2, \tilde{p}_{K'}) \) such that \( \sup_{q \in K} \varphi_\beta(p_K, q) < 1 \). Then for each \( q \in K \), there exists a neighborhood \( V_q \subset \mathbb{C} \) of \( q \), whose projection to \( \mathbb{R} \) is contained in \( K' \), and such that for all \( u \in T \) and \( z \in V_q \), the random variable

\[
W_{\beta, z, u} = W_{\beta, u} \frac{e^{z \log W_u}}{\mathbb{E}(\sum_{i=1}^N e^{z \log W_i})}
\]
is well defined, and we have
\[
\sup_{z \in V_q} \varphi_\beta(p_K, z) < 1,
\]
where for all \( z \in \mathbb{C} \)
\[
\varphi_\beta(p_K, z) = \beta^{1-p_K} \mathbb{E}\left( \sum_{i=1}^N e^{z \log W_i |p_K|} \right)^{-p_K} \mathbb{E}\left( \sum_{i=1}^N e^{z \log W_i |p_K|} \right).
\]
By extracting a finite covering of \( K \) from \( \bigcup_{q \in K} V_q \), we find a neighborhood \( V \subset \mathbb{C} \) of \( K \) such that \( \sup_{z \in V} \varphi_\beta(p_K, z) < 1 \).
Since the projection of \( V \) to \( \mathbb{R} \) is included in \( K' \) and the mapping \( z \mapsto \mathbb{E}\left( \sum_{i=1}^N e^{z \log W_i} \right) \) is continuous and does not vanish on \( V \), by considering a smaller neighborhood of \( K \) included in \( V \) if necessary, we can assume that
\[
A_V = \sup_{z \in V} \mathbb{E}\left( \sum_{i=1}^N e^{z \log W_i |p_K|} \right)^{-p_K} < \infty.
\]
Now, for \( u \in T \), we define the analytic extension to \( V \) of \( Y_n(\beta, q, u) \) given by
\[
Y_n(\beta, z, u) = \sum_{v \in T(u)} \prod_{k=1}^n W_{\beta, z, v_1, \ldots, v_k}.
\]
We denote also \( Y_n(\beta, z, 0) \) by \( Y_n(\beta, z) \). Now, applying Lemma 5.4 with \( V_i = W_{\beta, z, i} \), we obtain
\[
\mathbb{E}\left( |Y_n(\beta, z) - Y_{n-1}(\beta, z)|^{p_K} \right) \leq C_{p_K} \left( \mathbb{E}\left( \sum_{i=1}^N |V_i|^{p_K} \right) \right)^{n-1} \left( \mathbb{E}\left( |\sum_{i=1}^N V_i|^{p_K} \right) + 1 \right).
\]
Notice that \( \mathbb{E}\left( \sum_{i=1}^N |V_i|^{p_K} \right) = \varphi_\beta(p_K, z) \). Then,
\[
\mathbb{E}\left( |Y_n(\beta, z) - Y_{n-1}(\beta, z)|^{p_K} \right) \leq C_{p_K} A_V \sup_{z \in V} \varphi(p_K, z)^{n-1}.
\]
With probability 1, the functions \( z \in V \mapsto Y_n(\beta, z), n \geq 0, \) are analytic. Fix a closed disc \( D(z_0, 2\rho) \subset V \). Theorem (6.1) gives
\[
\sup_{z \in D(z_0, \rho)} |Y_n(\beta, z) - Y_{n-1}(\beta, z)| \leq 2 \int_{[0,1]} |Y_n(\beta, \zeta(\theta)) - Y_{n-1}(\beta, \zeta(\theta))| \, d\theta,
\]
where, for \( \theta \in [0, 1], \ z(\theta) = z_0 + 2\rho e^{2\pi \theta}. \) Furthermore Jensen’s inequality and Fubini’s theorem give

\[
\mathbb{E}\left( \sup_{z \in D(z_0, \rho)} |Y_n(\beta, z) - Y_{\beta, n-1}(z)|^{p_K} \right) \\
\leq \mathbb{E}\left( (2 \int_{[0,1]} |Y_n(\beta, \zeta(\theta)) - Y_{n-1}(\beta, \zeta(\theta))| \, d\theta)^{p_K} \right) \\
\leq 2^{p_K} \mathbb{E}\left( \int_{[0,1]} |Y_n(\beta, \zeta(\theta)) - Y_{n-1}(\beta, \zeta(\theta))|^{p_K} \, d\theta \right) \\
\leq 2^{p_K} \int_{[0,1]} \mathbb{E}|Y_n(\beta, \zeta(\theta)) - Y_{n-1}(\beta, \zeta(\theta))|^{p_K} \, d\theta \\
\leq 2^{p_K} C_{p_K} A_V \sup_{z \in V} \varphi(\beta, p_K, z)^{n-1}.
\]

Since \( \sup_{z \in V} \varphi(p_K, z) < 1, \) it follows that

\[
\sum_{n \geq 1} \left\| \sup_{z \in D(z_0, \rho)} |Y_n(\beta, z) - Y_{\beta, n-1}(z)| \right\|_{p_K} < \infty.
\]

This implies, \( z \mapsto Y_n(\beta, z) \) converge uniformly, almost surely and in \( L^{p_K} \) norm over the compact \( D(z_0, \rho) \) to a limit \( z \mapsto Y(\beta, z). \) This also implies that

\[
\left\| \sup_{z \in D(z_0, \rho)} Y(\beta, z) \right\|_{p_K} < \infty.
\]

Since \( K \) can be covered by finitely many such discs \( D(z_0, \rho) \) we get the uniform convergence, almost surely and in \( L^{p_K} \) norm, of the sequence \( (q \in K \mapsto Y_n(\beta, q))_{n \geq 1} \) to \( q \in K \mapsto Y(\beta, q). \) Moreover, since \( J_{\beta, \epsilon} \) can be covered by a countable union of such compact \( K \) we get the simultaneous convergence for all \( q \in J_{\beta}. \) The same holds simultaneously for all the function \( q \in J_{\beta} \mapsto Y_n(\beta, q, u), \ u \in \bigcup_{n \geq 0} \mathbb{N}_+^*, \) because \( \bigcup_{n \geq 0} \mathbb{N}_+^* \) is countable.

To finish the proof of (1), we must show that a.s., \( q \in K \mapsto Y(\beta, q) \) does not vanish. Without loss of generality we can suppose that \( K = [0, 1]. \) If \( I \) is a dyadic closed subinterval of \( [0, 1], \) we denote by \( E_I \) the event \( \{ \exists q \in I : Y(\beta, q) = 0 \}. \) Let \( I_0, I_1 \) stand for the 2 dyadic subintervals of \( I \) in the next generation. The event \( E_I \) being a tail event of probability 0 or 1, if we suppose that \( P(E_I) = 1, \) there exists \( j \in \{0, 1\} \) such that \( P(E_{I_j}) = 1. \) Suppose now that \( P(E_K) = 1. \) The previous remark allows to construct a decreasing sequence \( (I(n))_{n \geq 0} \) of dyadic subintervals of \( K \) such that \( P(E_{I(n)}) = 1. \) Let \( q_0 \) be the unique element of \( \cap_{n \geq 0} I(n). \) Since \( q \mapsto Y(\beta, q) \) is continuous we have \( P(Y(\beta, q_0) = 0) = 1, \) which contradicts the fact that \( (Y_n(\beta, q_0))_{n \geq 1} \) converge to \( Y(\beta, q_0) \) in \( L^1. \)
(2) Here we develop, in the context of the boundary of a supercritical Galton-Watson tree, a uniform version of the argument used by Kahane in [16] on homogeneous trees, and written in complete rigor in [24]. Fix $\epsilon > 0$ and a compact set $K$ in $\mathcal{J}_{\beta,\epsilon}$. Denote by $E$ the separable Banach space of the real valued continuous functions over $K$ endowed with the supremum norm.

For $n \geq m \geq 1$ and $q \in K$ let

$$Z_{m,n}(\beta, q) = \sum_{u \in \Omega_m} Y_{n-m}(q,u) \prod_{k=1}^{m} W_{\beta;u_1,...,u_k}.$$  

Notice that $Z_{m,n}(\beta, q) = Y_n(\beta, q)$. Moreover, since $Y_n(\beta, \cdot)$ converges a.s. and in $L^1$ norm to $Y(\beta, \cdot)$ as $n \to \infty$, $Y_n(\beta, \cdot)$ belongs to $L^1_E = L^1_E(\Omega_\beta \times \Omega, \mathcal{A}_\beta \times \mathcal{F}_\beta \times \mathcal{P})$ (where we use the notations of [13, Section V-2]), so that the continuous random function $E(Z_{n,n}(\beta, q) | \mathcal{F}_{\beta,m} \otimes \mathcal{F}_n)$ is well defined by [13, Proposition V-2-5]; also, for any fixed $q \in K$, we can deduce from the definitions and the independence assumptions that

$$Z_{m,n}(\beta, q) = E(Z_{n,n}(\beta, q) | \mathcal{F}_{\beta,m} \otimes \mathcal{F}_n)$$

almost surely. By [13, Proposition V-2-5] again, since $g \in E \mapsto g(q)$ is a continuous linear form over $E$, we thus have

$$Z_{m,n}(\beta, q) = E(Z_{n,n}(\beta, \cdot) | \mathcal{F}_{\beta,m} \otimes \mathcal{F}_n)(q)$$

almost surely. By considering a dense countable set of $q$ in $K$, we can conclude that the random continuous functions $Z_{m,n}(\beta, \cdot)$ and $E(Z_{n,n}(\beta, \cdot) | \mathcal{F}_{\beta,m} \otimes \mathcal{F}_n)$ are equal almost surely.

Similarly, since for each $q \in K$ the martingale $(Y_n(\beta, q), \mathcal{F}_{\beta,n} \otimes \mathcal{F}_n)$ converges to $Y(\beta, q)$ almost surely and in $L^1$, and $Y(\beta, \cdot) \in L^1_E$, by using [13, Proposition V-2-5] again we can get almost surely

$$Z_{n,n}(\beta, \cdot) = E(Y(\beta, \cdot) | \mathcal{F}_{\beta,n} \otimes \mathcal{F}_n), \text{ hence } Z_{m,n}(\beta, \cdot) = E(Y(\beta, \cdot) | \mathcal{F}_{\beta,m} \otimes \mathcal{F}_n).$$

Moreover, it follows from Proposition (1) and the definition of $\mu_q([u])$ that $Z_{m,n}(\beta, \cdot)$ converges almost surely uniformly and in $L^1$ norm, as $n \to \infty$, to $Y_m(\beta, \cdot)$. This and (5.1) yield, using [13, Proposition V-2-6],

$$\overline{Y}_m(\beta, \cdot) = \lim_{n \to \infty} Z_{m,n}(\beta, \cdot) = E(Y(\beta, \cdot) | \mathcal{F}_{\beta,m} \otimes \sigma(\bigcup_{n \geq 1} \mathcal{F}_n))$$

and finally

$$\lim_{m \to \infty} \overline{Y}_m(\beta, \cdot) = E(Y(\beta, \cdot) | \sigma(\bigcup_{m \geq 1} \mathcal{F}_{\beta,m}) \otimes \sigma(\bigcup_{n \geq 1} \mathcal{F}_n)) = Y(\beta, \cdot),$$
almost surely (since by construction $Y(\beta, \cdot)$ is $\sigma(\bigcup_{m \geq 1} F_{\beta,m}) \otimes \sigma(\bigcup_{n \geq 1} F_n)$ measurable), where the convergences hold in the uniform norm.

Moreover, since $J_{\beta,\epsilon}$ can be covered by a countable union of such compact $K$, we get the simultaneous convergence for all $q \in J_{\beta,\epsilon}$. ■

**Proof of Proposition 3.1.** The proof of the first point is similar to the proof of Proposition 3.2 (1) ($\beta = 1$). The second point is a consequence of the branching property:

$$Y_{n+1}(q, u) = \sum_{i=1}^{N} W_{q, u_i} Y_n(q, u_i).$$

### 6. APPENDICES

**APPENDIX 1 — CAUCHY FORMULA**

**Definition 6.1.** Let $D(\zeta, r)$ be a disc in $\mathbb{C}$ with centre $\zeta$ and radius $r$. The set $\partial D$ is the boundary of $D$. Let $g \in C(\partial D)$ a continuous function on $\partial D$. We define the integral of $g$ on $\partial D$ as

$$\int_{\partial D} g(\zeta) d\zeta = 2i\pi r \int_{[0,1]} g(\zeta(t)) e^{i2\pi t} dt,$$

where $\zeta(t) = \zeta + re^{i2\pi t}$.

**Theorem 6.1.** Let $D = D(a, r)$ be a disc in $\mathbb{C}$ with a radius $r > 0$, and $f$ be a holomorphic function in a neighborhood of $D$. Then, for all $z \in D$

$$f(z) = \frac{1}{(2i\pi)} \int_{\partial D} \frac{f(\zeta)d\zeta}{\zeta - z}.$$

It follows that

$$\sup_{z \in D(a, r/2)} |f(z)| \leq 2 \int_{[0,1]} |f(\zeta(t))| dt.$$

**APPENDIX 2 — MASS DISTRIBUTION PRINCIPLE**

**Theorem 6.2.** [B] Let $\nu$ be a positive and finite Borel probability measure on a compact metric space $(X, d)$. Assume that $M \subseteq X$ is a Borel set such that $\nu(M) > 0$ and

$$M \subseteq \{ t \in X, \lim_{r \to 0^+} \frac{\log \nu(B(t, r))}{\log r} \geq \delta \}.$$

Then the Hausdorff dimension of $M$ is bounded from below by $\delta$. 
APPENDIX 3 — HAUSDORFF AND PACKING MEASURES AND DIMENSIONS

Given a subset $K$ of $\mathbb{N}_+^+$ endowed with a metric $d$ making it $\sigma$-compact, $g : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous non-decreasing function near $0$ and such that $g(0) = 0$, and $E$ a subset of $K$, the Hausdorff measure of $E$ with respect to the gauge function $g$ is defined as

$$H^g(E) = \lim_{\delta \to 0^+} \inf \left\{ \sum_{i \in \mathbb{N}} g(\text{diam}(U_i)) \right\},$$

the infimum being taken over all the countable coverings $(U_i)_{i \in \mathbb{N}}$ of $E$ by subsets of $K$ of diameters less than or equal to $\delta$.

If $s \in \mathbb{R}_+^*$ and $g(u) = u^s$, then $H^g(E)$ is also denoted $H^s(E)$ and called the $s$-dimensional Hausdorff measure of $E$. Then, the Hausdorff dimension of $E$ is defined as

$$\dim E = \sup \{ s > 0 : H^s(E) = \infty \} = \inf \{ s > 0 : H^s(E) = 0 \},$$

with the convention $\sup \emptyset = 0$ and $\inf \emptyset = \infty$.

Packing measures and dimensions are defined as follows. Given $g$ and $E \subset K$ as above, one first defines

$$P^g(E) = \lim_{\delta \to 0^+} \sup \left\{ \sum_{i \in \mathbb{N}} g(\text{diam}(B_i)) \right\},$$

the supremum being taken over all the packings $(B_i)_{i \in \mathbb{N}}$ of $E$ by balls centered on $E$ and with diameter smaller than or equal to $\delta$. Then, the packing measure of $E$ with respect to the gauge $g$ is defined as

$$P^g(E) = \lim_{\delta \to 0^+} \inf \left\{ \sum_{i \in \mathbb{N}} P^g(E_i) \right\},$$

the infimum being taken over all the countable coverings $(E_i)_{i \in \mathbb{N}}$ of $E$ by subsets of $K$ of diameters less than or equal to $\delta$. If $s \in \mathbb{R}_+^*$ and $g(u) = u^s$, then $P^g(E)$ is also denoted $P^s(E)$ and called the $s$-dimensional packing measure of $E$. Then, the packing dimension of $E$ is defined as

$$\text{Dim} E = \sup \{ s > 0 : P^s(E) = \infty \} = \inf \{ s > 0 : P^s(E) = 0 \},$$

with the convention $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. For more details the reader is referred to [9].

If $\mu$ is a positive and finite Borel measure supported on $K$, then its lower Hausdorff and packing dimensions is defined as

$$\dim(\mu) = \inf \{ \dim F : F \text{ Borel}, \mu(F) > 0 \},$$

$$\text{Dim}(\mu) = \inf \{ \text{Dim} F : F \text{ Borel}, \mu(F) > 0 \},$$
and its upper Hausdorff and packing dimensions are defined as
\[
\overline{\dim}(\mu) = \inf \{ \dim F : F \text{ Borel, } \mu(F) = \|\mu\| \}
\]
\[
\overline{\text{Dim}}(\mu) = \inf \{ \text{Dim} F : F \text{ Borel, } \mu(F) = \|\mu\| \},
\]
We have (see [8], [11])
\[
\dim(\mu) = \operatorname{ess \inf}_\mu \lim_{r \to 0^+} \frac{\log \mu(B(t, r))}{\log(r)},
\]
\[
\overline{\dim}(\mu) = \operatorname{ess \inf}_\mu \lim_{r \to 0^+} \frac{\log \mu(B(t, r))}{\log(r)},
\]
and
\[
\overline{\text{Dim}}(\mu) = \operatorname{ess \sup}_\mu \lim_{r \to 0^+} \frac{\log \mu(B(t, r))}{\log(r)},
\]
\[
\overline{\text{Dim}}(\mu) = \operatorname{ess \sup}_\mu \lim_{r \to 0^+} \frac{\log \mu(B(t, r))}{\log(r)},
\]
where \(B(t, r)\) stands for the closed ball of radius \(r\) centered at \(t\). If \(\dim(\mu) = \overline{\dim}(\mu)\) (resp. \(\overline{\text{Dim}}(\mu) = \overline{\text{Dim}}(\mu)\)), this common value is denoted \(\dim \mu\) (resp. \(\overline{\text{Dim}}(\mu)\)), and if \(\dim \mu = \overline{\text{Dim}} \mu\), one says that \(\mu\) is exact dimensional.

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