PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 0, Fasc. 0 (0000), pp. 000–000
doi:10.19195/0208-4147.0.0.0

RANDOM ITERATION WITH PLACE DEPENDENT PROBABILITIES

BY

RAFAŁ KAPICA (KRAKÓW) AND MACIEJ ŚLĘCZKA (KATOWICE)

Abstract. Markov chains arising from random iteration of functions $S_\theta : X \to X$, $\theta \in \Theta$, where X is a Polish space and Θ is an arbitrary set of indices are considered. At $x \in X$, θ is sampled from a distribution ϑ_x on Θ and ϑ_x are different for different x. Exponential convergence to a unique invariant measure is proved. This result is applied to the case of random affine transformations on $\mathbb{R}^d$ giving the existence of exponentially attractive perpetuities with place dependent probabilities.

2010 AMS Mathematics Subject Classification: Primary: 60J05; Secondary: 37A25.

Key words and phrases: random iteration of functions, exponential convergence, invariant measure, perpetuities

1. INTRODUCTION

We consider the Markov chain of the form $X_0 = x_0$, $X_1 = S_{\theta_0}(x_0)$, $X_2 = S_{\theta_1} \circ S_{\theta_0}(x_0)$ and inductively

$$X_{n+1} = S_{\theta_n}(X_n), \tag{1.1}$$

where S_{θ_0}, S_{θ_1},...,S_{θ_n} are randomly chosen from a family $\{S_\theta : \theta \in \Theta\}$ of functions that map a separable metric space X into itself. If the chain is at $x \in X$ then $\theta \in \Theta$ is sampled from a distribution ϑ_x on Θ, where ϑ_x are different for different x. The chain $(X_n)_{n\in\mathbb{N}_0}$ is defined in a rigorous way in Section 3. We are interested

in the rate of convergence to a stationary distribution μ_* on X, i.e.

$$P\{X_n \in A\} \to \mu_*(A) \quad \text{as} \quad n \to \infty. \tag{1.2}$$

When the above convergence is exponential then $(X_n)_{n\in\mathbb{N}_0}$ is called exponentially ergodic. In the case of constant probabilities, i.e. $\vartheta_x = \vartheta_y$ for $x, y \in X$, the basic tool when studying asymptotics of (1.1) are backward iterations $Z_{n+1} = S_{\theta_0} \circ S_{\theta_1} \circ \cdots \circ S_{\theta_n}(x_0)$. Since X_n and Z_n are identically distributed and, under suitable conditions, Z_n converge almost surely at exponential rate to some random element Z, one obtains exponential ergodicity of $(X_n)_{n\in\mathbb{N}_0}$ (see [6]). For place dependent ϑ_x we need different approach because distributions of X_n and Z_n are not equal.

The simplest case when $\Theta = \{1, \ldots, n\}$ is treated in [2] and [29], where the existence of a unique attractive invariant measure is established. Similar result holds true when $\Theta = [0, T]$ and ϑ_x are absolutely continuous (see [15]). Recently it was shown that the rate of convergence in the case of $\Theta = \{1, \ldots, n\}$ is exponential (see [30]).

In this paper we treat the general case of place dependent ϑ_x for arbitrary Θ and prove the existence of a unique exponentially attractive invariant measure for (1.1). Our approach is based on the coupling method which can be briefly described as follows. For arbitrary starting points $x_0, \bar{x}_0 \in X$ we consider chains $(X_n)_{n\in\mathbb{N}_0}$, $(\bar{X}_n)_{n\in\mathbb{N}_0}$ with $X_0 = x_0$, $\bar{X}_0 = \bar{x}_0$ and try to build correlations between $(X_n)_{n\in\mathbb{N}_0}$ and $(\bar{X}_n)_{n\in\mathbb{N}_0}$ in order to make their trajectories as close as possible. This can be done because the transition probability function $\mathbf{B}_{x,y}(A) = P\{(X_{n+1}, \bar{X}_{n+1}) \in A \,|\, (X_n, \bar{X}_n) = (x, y)\}$ of the coupled chain $(X_n, \bar{X}_n)_{n\in\mathbb{N}_0}$ taking values in X^2 can be decomposed (see [11]) in the following way

$$\mathbf{B}_{x,y} = \mathbf{Q}_{x,y} + \mathbf{R}_{x,y},$$

where subprobability measures $\mathbf{Q}_{x,y}$ are contractive, i.e. $\int_{X^2} d(u, v)\, \mathbf{Q}_{x,y}(du, dv) \leqslant \alpha d(x, y)$ for some constant $\alpha \in (0, 1)$.

Since transition probabilities for (1.1) can be mutually singular for even very close points, one cannot expect that chains $(X_n)_{n\in\mathbb{N}_0}$ and $(\bar{X}_n)_{n\in\mathbb{N}_0}$ couple in finite time (i.e. $X_n = \bar{X}_n$ for some $n \in \mathbb{N}_0$) as in classical coupling constructions ([21]) leading to the convergence in the total variation norm. On the contrary, they only couple at infinity (i.e. $d(X_n, \bar{X}_n) \to 0$ as $n \to \infty$) so this method is sometimes called asymptotic coupling ([13]) and gives convergence in weak topology.

There are two reasons which make random iterations with place dependent probabilities in some sense the critical case. First, when passing from constant to place dependent probabilities one loses the backward iterations argument, which combined with mutual singularity of transition probabilities makes establishing exponential ergodicity a difficult task. Secondly, such iterations are an example of the simplest systems where exponential ergodicity must be understood in the sense of convergence in weak topology. The other example is much more sophisticated: stochastic partial differential equations (SPDEs). In both cases the main difficulty is the same: mutual singularity of transition probabilities. The emerging theory of weak exponential ergodicity consists of many results which can be found in vast SPDEs literature (see [19], [20], [12], [34]), some of them ([13]) indicating analogies to existing theory of strong ergodicity summarized in [23]. However, to our best knowledge, none of these results applies in our case.

One of the most important consequences of exponential ergodicity (understood here in the sense of weak topology) is that it implies the Law of Large Numbers and the Central Limit Theorem. Shirikyan showed in [27] that if the process is exponentially ergodic in norm equivalent to the Fortet-Mourier norm then the strong version of LLN holds. Since convergence we obtain in this paper is in Fortet-Mourier norm, Shirikyan's result can be applied in our case. Establishing the CLT is much more delicate problem. As Komorowski and Walczuk proved in [18], exponential ergodicity implies the CLT, however they use stronger Wassertein norm.

Recent results [14], [16] establish the CLT and exponential ergodicity simultaneously, using methods similar to ours. It is thus reasonable to expect that general theorem in the spirit of Komorowski and Walczuk and concerning the Fortet-Mourier norm can be formulated.

The paper is organized as follows. In Section 2 we formulate and prove a theorem which assures exponential convergence to an invariant measure for a class of Markov chains. This theorem is applied in Section 3 to chains generated by random iteration of functions. In Section 4 we discuss special class of such functions, random affine transformations on $\mathbb{R}^d$, thus generalizing the notion of *perpetuity* to the place dependent case.

2. AN EXPONENTIAL CONVERGENCE RESULT

2.1. Notation and basic definitions.

Let (X, d) be a *Polish* space, i.e. a complete and separable metric space and denote by $\mathcal{B}_X$ the σ-algebra of Borel subsets of X. By $B_b(X)$ we denote the space of bounded Borel-measurable functions equipped with the supremum norm, $C_b(X)$ stands for the subspace of bounded continuous functions. Let $\mathcal{M}_{fin}(X)$ and $\mathcal{M}_1(X)$ be the sets of Borel measures on X such that $\mu(X) < \infty$ for $\mu \in \mathcal{M}_{fin}(X)$ and $\mu(X) = 1$ for $\mu \in \mathcal{M}_1(X)$. The elements of $\mathcal{M}_1(X)$ are called *probability measures*. The elements of $\mathcal{M}_{fin}(X)$ for which $\mu(X) \leqslant 1$ are called *subprobability measures*. By $supp\,\mu$ we denote the support of the measure μ. We also define

$$\mathcal{M}_L(X) = \{\mu \in \mathcal{M}_1(X) : \int_X L(x)\mu(dx) < \infty\}$$

where $L : X \to [0, \infty)$ is an arbitrary Borel measurable function and

$$\mathcal{M}_1^1(X) = \{\mu \in \mathcal{M}_1(X) : \int_X d(\bar{x}, x)\mu(dx) < \infty\},$$

where $\bar{x} \in X$ is fixed. By the triangle inequality the definition of $\mathcal{M}_1^1(X)$ is independent of the choice of $\bar{x}$.

The space $\mathcal{M}_1(X)$ is equipped with the *Fortet-Mourier metric*:

$$\|\mu_1 - \mu_2\|_{FM} = \sup\{|\int_X f(x)(\mu_1 - \mu_2)(dx)| : f \in \mathcal{F}\},$$

where $\mathcal{F} = \{f \in C_b(X) : |f(x) - f(y)| \leqslant 1 \text{ and } |f(x)| \leqslant 1 \quad \text{for} \quad x, y \in X\}$. The space $\mathcal{M}_1^1(X)$ is equipped with the *Wasserstein metric*:

$$\|\mu_1 - \mu_2\|_W = \sup\{|\int_X f(x)(\mu_1 - \mu_2)(dx)| : f \in \mathcal{W}\},$$

where $\mathcal{W} = \{f \in C_b(X) : |f(x) - f(y)| \leqslant 1 \text{ for } x, y \in X\}$. By $\|\cdot\|$ we denote the total variation norm. If a measure μ is nonnegative then $\|\mu\|$ is simply the total mass of μ. In Section 4 we will use Euclidean norm $|\cdot|$ in $\mathbb{R}^d$ and operator norm $||\cdot||_{op}$ given by $||m||_{op} = \sup\{|mx| : x \in \mathbb{R}^d, |x| = 1\}$.

Let $P : B_b(X) \to B_b(X)$ be a *Markov operator*, i.e. a linear operator satisfying $P\mathbf{1}_X = \mathbf{1}_X$ and $Pf(x) \geqslant 0$ if $f \geqslant 0$. Denote by P^* the the dual operator, i.e. operator $P^* : \mathcal{M}_{fin}(X) \to \mathcal{M}_{fin}(X)$ defined as follows

$$P^*\mu(A) := \int_X P\mathbf{1}_A(x)\mu(dx) \quad \text{for} \quad A \in \mathcal{B}_X.$$

We say that a measure $\mu_* \in \mathcal{M}_1(X)$ is *invariant* for P if $\int_X Pf(x)\mu_*(dx) = \int_X f(x)\mu_*(dx)$ for every $f \in B_b(X)$ or, alternatively, we have $P^*\mu_* = \mu_*$. By $\{\mathbf{P}_x : x \in X\}$ we denote a *transition probability function* for P, i.e. a family of measures $\mathbf{P}_x \in \mathcal{M}_1(X)$ for $x \in X$ such that the map $x \mapsto \mathbf{P}_x(A)$ is measurable for every $A \in \mathcal{B}_X$ and $Pf(x) = \int_X f(y)\mathbf{P}_x(dy)$ for$x \in X \quad \text{and} \quad f \in B_b(X)$ or equivalently $P^*\mu(A) = \int_X \mathbf{P}_x(A)\mu(dx)$ for $A \in \mathcal{B}_X$ and $\mu \in \mathcal{M}_{fin}(X)$.

2.2. Formulation of the theorem.

DEFINITION 2.1. A coupling for $\{\mathbf{P}_x : x \in X\}$ is a family $\{\mathbf{B}_{x,y} : x, y \in X\}$ of probability measures on $X \times X$ such that for every $B \in \mathcal{B}_{X^2}$ the map $X^2 \ni (x, y) \mapsto \mathbf{B}_{x,y}(B)$ is measurable and

$$\mathbf{B}_{x,y}(A \times X) = \mathbf{P}_x(A), \quad \mathbf{B}_{x,y}(X \times A) = \mathbf{P}_y(A)$$

for every $x, y \in X$ and $A \in \mathcal{B}_X$.

In the following we assume (see [11]) that there exists a family $\{\mathbf{Q}_{x,y} : x, y \in X\}$ of subprobability measures on X^2 such that the map $(x, y) \mapsto \mathbf{Q}_{x,y}(B)$ is measurable for every $B \in \mathcal{B}_{X^2}$ and $\mathbf{Q}_{x,y}(A \times X) \leqslant \mathbf{P}_x(A)$ and $\mathbf{Q}_{x,y}(X \times A) \leqslant \mathbf{P}_y(A)$ for every $x, y \in X$ and $A \in \mathcal{B}_X$.

Measures $\{\mathbf{Q}_{x,y} : x, y \in X\}$ allow us to construct a coupling for $\{\mathbf{P}_x : x \in X\}$. Define on X^2 the family of measures $\{\mathbf{R}_{x,y} : x, y \in X\}$ which on rectangles $A \times B$ are given by

$$\mathbf{R}_{x,y}(A \times B) = \frac{1}{1 - \mathbf{Q}_{x,y}(X^2)}(\mathbf{P}_x(A) - \mathbf{Q}_{x,y}(A \times X))(\mathbf{P}_y(B) - \mathbf{Q}_{x,y}(X \times B)),$$

when $\mathbf{Q}_{x,y}(X^2) < 1$ and $\mathbf{R}_{x,y}(A \times B) = 0$ otherwise. A simple computation shows that the family $\{\mathbf{B}_{x,y} : x, y \in X\}$ of measures on X^2 defined by

$$\mathbf{B}_{x,y} = \mathbf{Q}_{x,y} + \mathbf{R}_{x,y} \quad \text{for} \quad x, y \in X \tag{2.1}$$

is a coupling for $\{\mathbf{P}_x : x \in X\}$.

Now we list our assumptions on Markov operator P and transition subprobabilities $\{\mathbf{Q}_{x,y} : x, y \in X\}$.

A0 *P is a Feller operator, i.e. $P(C_b(X)) \subset C_b(X)$.*

A1 *There exists a Lyapunov function for P, i.e. continuous function $L : X \to [0, \infty)$ such that L is bounded on bounded sets, $\lim_{x\to\infty} L(x) = +\infty$ (for bounded X this condition is omitted) and for some $\lambda \in (0, 1)$, $c > 0$*

$$PL(x) \leqslant \lambda L(x) + c \quad for \quad x \in X.$$

A2 *There exist $F \subset X^2$ and $\alpha \in (0, 1)$ such that supp $\mathbf{Q}_{x,y} \subset F$ and*

$$\int_{X^2} d(u, v)\mathbf{Q}_{x,y}(du, dv) \leqslant \alpha d(x, y) \quad for \quad (x, y) \in F. \tag{2.2}$$

A3 *There exist $\delta > 0$, $l > 0$ and $\nu \in (0, 1]$ such that*

(2.3)

$$1 - \|\mathbf{Q}_{x,y}\| \leqslant l d(x, y)^\nu \quad \text{and} \quad \mathbf{Q}_{x,y}(\{(u, v) \in X^2 : d(u, v) < \alpha d(x, y)\}) \geqslant \delta$$

for $(x, y) \in F$.

A4 *There exist* $\beta \in (0,1)$, $\tilde{C} > 0$ *and* $R > 0$ *such that for*

$$\kappa(\,(x_n, y_n)_{n\in\mathbb{N}_0}\,) = \inf\{n \in \mathbb{N}_0 : (x_n, y_n) \in F \quad \text{and} \quad L(x_n) + L(y_n) < R\},$$

where $(x_n, y_n)_{n\in\mathbb{N}_0}$ *is a sequence of elements of* $X \times X$, *we have*

$$\mathbb{E}_{x,y}\beta^{-\kappa} \leqslant \tilde{C} \quad \textit{whenever} \quad L(x) + L(y) < \frac{4c}{1-\lambda},$$

where $\mathbb{E}_{x,y}$ *denotes here the expectation with respect to the chain starting from* (x, y) *and with transition function* $\{\mathbf{B}_{x,y} : x, y \in X\}$.

Before proceeding to the formulation of the theorem we briefly comment above conditions. Assumption **A0** is standard. Existence of a Lyapunov function in **A1** allows to reduce the dynamics to some bounded region of X. Assumption **A2** is well known *contraction on average* condition, written in the language of coupling. Measures $\mathbf{Q}_{x,y}$ are thus contractive part of coupled transition probabilities $\mathbf{B}_{x,y}$ with $\mathbf{R}_{x,y}$ being the uncontrollable part. Some version of such condition seems to be necessary for exponential ergodicity (see [25]). The first part of **A3** means that the total mass of the contractive part $\mathbf{Q}_{x,y}$ is close to 1 for x close to y. Some higher order regularity here is necessary to ensure the uniqueness of invariant measure (see [28]). The second part of **A3** guarantees that some part of contractive on average measure $\mathbf{Q}_{x,y}$ is contractive in the strict sense. Condition **A4** means that the dynamics quickly enters the domain of contractivity F. In this paper we discuss Markov chains generated by random iteration of functions for which always $F = X^2$ and $L(x) = d(x, \bar{x})$ with some fixed $\bar{x} \in X$, so **A4** is trivially fulfilled when $R = \frac{4c}{1-\lambda}$. There are, however, examples of random dynamical systems for which F is a proper subset of X^2. Indeed, in *contractive Markov systems* introduced by I. Werner in [33] we have $X = \sum_{i=1}^{n} X_i$ but $F = \sum_{i=1}^{n} X_i \times X_i$. They are studied in [31].

Now we formulate the main result of this section. Its proof is given in Section 2.4.

THEOREM 2.1. *Assume* **A0** – **A4**. *Then operator P possesses a unique invariant measure $\mu_* \in \mathcal{M}_L(X)$, which is attractive, i.e.*

$$\lim_{n\to\infty} \int_X P^n f(x)\,\mu(dx) = \int_X f(x)\,\mu(dx) \quad \textit{for} \quad f \in C_b(X),\ \mu \in \mathcal{M}_1(X).$$

Moreover, there exist $q \in (0,1)$ and $C > 0$ such that

$$\|P^{*n}\mu - \mu_*\|_{FM} \leqslant q^n C(1 + \int_X L(x)\mu(dx)) \tag{2.4}$$

for $\mu \in \mathcal{M}_L(X)$ and $n \in \mathbb{N}$.

Remark. In [13], Theorem 4.8, authors formulate sufficient conditions for the existence of a unique exponentially attractive invariant measure for continuous-time Markov semigroup $\{P(t)\}_{t\geqslant 0}$, that do not refer to coupling. One of assumptions is that there exists distance-like (i.e. symmetric, lower semi-continuous and vanishing only on the diagonal) function $d : X \times X \to [0,1]$ which is contractive for some $P(t_*)$, i.e. there exists $\alpha < 1$ such that for every $x, y \in X$ with $d(x,y) < 1$ we have $d(\mathcal{P}(x,\cdot), \mathcal{P}(y,\cdot)) \leqslant \alpha d(x,y)$, where $\mathcal{P}(\cdot,\cdot) : X \times \mathcal{B}_X \to [0,1]$ is transition kernel for $P(t_*)$. This assumption is stronger than **A2**, since measures $\mathbf{R}_{x,y}$ in (2.1) need not be contractive (i.e. $\int_{X^2} d(u,v)\,\mathbf{R}_{x,y}(du,dv) \leqslant \alpha d(u,v)$) for any distance-like function d.

2.3. Measures on the pathspace.

For fixed $(x_0, y_0) \in X^2$ the next step of a chain with transition probability function $\mathbf{B}_{x,y} = \mathbf{Q}_{x,y} + \mathbf{R}_{x,y}$ can be drawn according to $\mathbf{Q}_{x_0,y_0}$ or according to $\mathbf{R}_{x_0,y_0}$. To distinguish these two cases we introduce (see [11]) the augmented space $\widehat{X} = X^2 \times \{0,1\}$ and the transition function $\{\widehat{\mathbf{B}}_{x,y,\theta} :\ (x,y,\theta) \in \widehat{X}\}$ on $\widehat{X}$ given by $\widehat{\mathbf{B}}_{x,y,\theta} = \widehat{\mathbf{Q}}_{x,y,\theta} + \widehat{\mathbf{R}}_{x,y,\theta}$, where $\widehat{\mathbf{Q}}_{x,y,\theta} = \mathbf{Q}_{x,y} \times \delta_1$ and $\widehat{\mathbf{R}}_{x,y,\theta} = \mathbf{R}_{x,y} \times \delta_0$. The parameter $\theta \in \{0,1\}$ is responsible for choosing measures $\mathbf{Q}_{x,y}$ and $\mathbf{R}_{x,y}$. If

a Markov chain with transition function $\{\widehat{\mathbf{B}}_{x,y,\theta} : (x,y,\theta) \in \widehat{X}\}$ stays in the set $X^2 \times \{1\}$ at time n it means that the last step was drawn according to $\mathbf{Q}_{u,v}$, for some $(u,v) \in X^2$.

For every $x \in X$ finite-dimensional distributions $\mathbf{P}_x^{0,\dots,n} \in \mathcal{M}_1(X^{n+1})$ are defined by

$$\mathbf{P}_x^{0,\dots,n}(B) = \int_X \delta_x(dx_0) \int_X \mathbf{P}_{x_1}(dx_2) \dots \int_X \mathbf{P}_{x_{n-1}}(dx_n) \mathbf{1}_B(x_0,\dots,x_n)$$

for $n \in \mathbb{N}_0$, $B \in \mathcal{B}_{X^{n+1}}$, where δ_x is the Dirac measure at x. By the Kolmogorov extension theorem we obtain the measure $\mathbf{P}_x^\infty$ on the pathspace X^∞. Similarly we define measures $\mathbf{B}_{x,y}^\infty$, $\widehat{\mathbf{B}}_{x,y,\theta}^\infty$ on $(X \times X)^\infty$ and $\widehat{X}^\infty$. These measures have the following interpretation. Consider the Markov chain $(X_n, Y_n)_{n\in\mathbb{N}_0}$ on $X \times X$, starting from (x_0, y_0), with the transition function $\{\mathbf{B}_{x,y} : x, y \in X\}$, obtained by canonical Kolmogorov construction, i.e. $\Omega = (X \times X)^\infty$ is the sample space equipped with the probability measure $\mathbb{P} = \mathbf{B}_{x_0,y_0}^\infty$, $X_n(\omega) = x_n$, $Y_n(\omega) = y_n$, where $\omega = (x_k, y_k)_{k\in\mathbb{N}_0} \in \Omega$, and $n \in \mathbb{N}_0$. Then $(X_n)_{n\in\mathbb{N}_0}$, $(Y_n)_{n\in\mathbb{N}_0}$ are Markov chains in X, starting from x_0 and y_0, with the transition function $\{\mathbf{P}_x : x \in X\}$, and $\mathbf{P}_x^\infty$, $\mathbf{P}_y^\infty$ are their measures on the pathspace X^∞.

In this paper we often consider marginals of measures on the pathspace. If μ is a measure on a measurable space X and $f : X \to Y$ is a measurable map, then $f^\# \mu$ is the measure on Y defined by $f^\# \mu(A) = \mu(f^{-1}(A))$. So, if we denote by pr the projection map from a product space to its component, then $pr^\# \mu$ is simply the marginal of μ on this component.

In the following we consider Markov chains on $\widehat{X}$ with the transition function $\{\widehat{\mathbf{B}}_{x,y,\theta} : x, y \in X, \theta \in \{0,1\}\}$. We adopt the convention that $\theta_0 = 1$, so Φ always starts from $X^2 \times \{1\}$, and define $\widehat{\mathbf{B}}_{x,y}^\infty := \widehat{\mathbf{B}}_{x,y,1}^\infty$.

For $b \in \mathcal{M}_{fin}(X^2)$ we write: $\widehat{\mathbf{B}}_b^\infty(B) = \int_X \widehat{\mathbf{B}}_{x,y}^\infty(B)\, b(dx,dy)$, $B \in \mathcal{B}_{\widehat{X}^\infty}$, $\mathbf{Q}_b(A) = \int_{X^2} \mathbf{Q}_{x,y}(A)\, b(dx,dy)$, $A \in \mathcal{B}_{X^2}$ and $\mathbf{Q}_{x,y}^n(A) = \mathbf{Q}_{\mathbf{Q}_{x,y}^{n-1}}(A)$, $A \in \mathcal{B}_{X^2}$. When studying the asymptotics of a chain $(X_n)_{n\in\mathbb{N}_0}$ with a transition function

$\{\mathbf{P}_x : x \in X\}$ it is particularly interesting whether a coupled chain $(X_n, Y_n)_{n\in\mathbb{N}_0}$ is moving only according to the contractive part $\mathbf{Q}_{x,y}$ of the transition function $\mathbf{B}_{x,y}$. For every subprobability measure $b \in \mathcal{M}_{fin}(X^2)$ we define finite-dimensional subprobability distributions $\mathbf{Q}_b^{0,\dots,n} \in \mathcal{M}_{fin}((X \times X)^{n+1})$

$$\mathbf{Q}_b^{0,\dots,n}(B) = \int_{X^2} b(dx_0, dy_0) \int_{X^2} \mathbf{Q}_{x_0,y_0}(dx_1, dy_1) \dots$$
$$\dots \int_{X^2} \mathbf{Q}_{x_{n-1},y_{n-1}}(dx_n, dy_n) \mathbf{1}_B((x_0, y_0), \dots, (x_n, y_n)),$$

where $B \in \mathcal{B}_{(X\times X)^{n+1}}$, $n \in \mathbb{N}_0$. For $n \geqslant 1$ define $\iota_n : (X \times X)^n \to \widehat{X}^n$,

$$\iota_n((x_1, y_1), ..., (x_n, y_n)) = ((x_1, y_1, 1), ..., (x_n, y_n, 1))$$

and take $\widehat{\mathbf{Q}}_b^{0,\dots,n} = \iota_n^{\#} \mathbf{Q}_b^{0,\dots,n}$. Since $\mathbf{Q}_b^{0,\dots,n}((X \times X)^{n+1}) > \mathbf{Q}_b^{0,\dots,n+1}((X \times X)^{n+2})$ the family $\{\widehat{\mathbf{Q}}_b^{0,\dots,n} : n \in \mathbb{N}_0\}$ is not consistent and we cannot use the Kolmogorov extension theorem to obtain a measure on the whole pathspace $\widehat{X}^\infty$. However, defining for every $b \in \mathcal{M}_{fin}(X^2)$ the measure $\mathbf{Q}_b^\infty \in \mathcal{M}_{fin}(\widehat{X}^\infty)$ by $\mathbf{Q}_b^\infty(B) = \widehat{\mathbf{B}}_b^\infty(B \cap (X^2 \times \{1\})^\infty)$, where $B \in \mathcal{B}_{\widehat{X}^\infty}$, one can easily check that for every cylindrical set $B = A \times \widehat{X}^\infty$, $A \in \mathcal{B}_{\widehat{X}^{n+1}}$, we have

$$\mathbf{Q}_b^\infty(B) = \lim_{n\to\infty} \widehat{\mathbf{Q}}_b^{0,\dots,n}(A). \tag{2.5}$$

2.4. Proof of Theorem 2.1.

Before proceeding to the proof of Theorem 2.1 we formulate two lemmas. The first one is partially inspired by the reasoning which can be found in [26].

LEMMA 2.1. *Let Y be a metric space and let $(Y_n^y)_{n\in\mathbb{N}_0}$ be a family of Markov chains indexed by starting point $y \in Y$, with common transition function $\{\pi_y : y \in Y\}$. Let $V : Y \to [0, \infty)$ be a Lyapunov function for $\{\pi_y : y \in Y\}$. Assume that for some bounded and measurable $A \subset Y$ there exist $\lambda \in (0, 1)$ and $C_\rho > 0$ such that for*

$$\rho((y_n)_{n\in\mathbb{N}_0}) = \inf\{n \geqslant 1 : y_n \in A\}$$

we have

$$\mathbb{E}_y \lambda^{-\rho} \leqslant C_\rho (V(y_0)+1) \quad \textit{for} \quad y \in Y,$$

where $\mathbb{E}_y$ *is the expectation with respect to the measure* $\mathbb{P}_y$ *on* Y^∞ *induced by* $(Y_n^y)_{n\in\mathbb{N}_0}$. *Moreover, assume that for some measurable* $B \subset Y$ *and*

$$\epsilon((y_n)_{n\in\mathbb{N}_0}) = \inf\{n \geqslant 1 : \, y_n \notin B\}$$

there exist constants $p \in (0,1)$, $\beta \in (0,1)$ *and* $C_\epsilon > 0$ *such that*

$$\mathbb{P}_y(\{(y_n)_{n\in\mathbb{N}_0} : \, \forall_{n\geqslant 1} \, y_n \in B\}) > p \quad \textit{and} \quad \mathbb{E}_y \mathbf{1}_{\{\epsilon<\infty\}} \beta^{-\epsilon} \leqslant C_\epsilon,$$

for every $y \in A$. *Then there exist* $\gamma \in (0,1)$ *and* $C > 0$ *such that for*

$$\tau((y_n)_{n\in\mathbb{N}_0}) = \inf\{n \geqslant 1 : \, \forall_{k\geqslant n} \, y_k \in B\}$$

we have

$$\mathbb{E}_y \gamma^{-\tau} \leqslant C(V(y)+1) \quad \textit{for} \quad y \in Y.$$

Proof of Lemma 2.1. Define $\kappa = \epsilon + \rho \circ T_\epsilon$, where $T_n((y_k)_{k\in\mathbb{N}_0}) = (y_{k+n})_{k\in\mathbb{N}_0}$. Fix $y \in Y$, $\alpha \in (0,1)$ and $r > 1$ such that $(\lambda\alpha)^{-\frac{1}{r-1}} < \beta^{-1}$. The strong Markov property and the Hölder inequality for every $y \in Y$ give

$$\begin{aligned}
\mathbb{E}_y \mathbf{1}_{\{\epsilon<\infty\}} \lambda^{-\frac{\kappa}{r}} &\leqslant [\mathbb{E}_y(\mathbf{1}_{\{\epsilon<\infty\}}(\lambda\alpha)^{-\frac{\epsilon}{r}})^{\frac{r}{r-1}}]^{\frac{r-1}{r}} [\mathbb{E}_y(\mathbf{1}_{\{\epsilon<\infty\}}\alpha^\epsilon \lambda^{-\rho\circ T_\epsilon})]^{\frac{1}{r}} \\
&\leqslant (\mathbb{E}_y \mathbf{1}_{\{\epsilon<\infty\}} \beta^{-\epsilon})^{\frac{r-1}{r}} [\mathbb{E}_y(\mathbf{1}_{\{\epsilon<\infty\}}\alpha^\epsilon \mathbb{E}_y(\lambda^{-\rho\circ T_\epsilon}|\mathcal{F}_\epsilon))]^{\frac{1}{r}} \\
&= (\mathbb{E}_y \mathbf{1}_{\{\epsilon<\infty\}} \beta^{-\epsilon})^{\frac{r-1}{r}} [\mathbb{E}_y(\mathbf{1}_{\{\epsilon<\infty\}}\alpha^\epsilon \mathbb{E}_{Y_\epsilon^y}(\lambda^{-\rho}))]^{\frac{1}{r}} \\
&\leqslant (\mathbb{E}_y \mathbf{1}_{\{\epsilon<\infty\}} \beta^{-\epsilon})^{\frac{r-1}{r}} [\mathbb{E}_y(\mathbf{1}_{\{\epsilon<\infty\}}\alpha^\epsilon C_\rho(V(Y_\epsilon^y)+1))]^{\frac{1}{r}},
\end{aligned}$$

where $\mathcal{F}_\epsilon$ is the σ-algebra generated by ϵ. Since $\sup_{y\in A} V(y) < \infty$ and V satisfies

$$\mathbb{E}_y(\mathbf{1}_{\{\epsilon<\infty\}}\alpha^\epsilon V(Y_\epsilon^y)) \leqslant C_1(V(y)+1) \quad \text{for} \quad y \in Y,$$

for some $C_1 > 0$, taking $c = \lambda^{\frac{1}{r}}$ we obtain $\mathbb{E}_y \mathbf{1}_{\{\epsilon<\infty\}} c^{-\kappa} \leqslant C_2$ whenever $y \in A$, for some constant $C_2 > 0$. Define $\epsilon_0 = 0$, $\kappa_0 = \rho$ and

$$\begin{aligned}
\epsilon_n &= \kappa_{n-1} + \epsilon \circ T_{\kappa_{n-1}}, \\
\kappa_n &= \kappa_{n-1} + \kappa \circ T_{\kappa_{n-1}} \quad \text{for} \quad n \geqslant 1.
\end{aligned}$$

Observe that $Y^y_{\kappa_n} \in A$, $Y^y_{\epsilon_n} \notin B$, $\epsilon_n \leqslant \kappa_n \leqslant \epsilon_{n+1}$ and $\kappa_n \nearrow \infty$. We have

$$\begin{aligned}\mathbb{E}_y \mathbf{1}_{\{\kappa_{n+1}<\infty\}} c^{-\kappa_{n+1}} &= \mathbb{E}_y[\mathbf{1}_{\{\kappa_n<\infty\}} c^{-\kappa_n} \mathbb{E}_y((\mathbf{1}_{\{\kappa<\infty\}} c^{-\kappa}) \circ T_{\kappa_n} | \mathcal{F}_{\kappa_n})] \\ &= \mathbb{E}_y[\mathbf{1}_{\{\kappa_n<\infty\}} c^{-\kappa_n} \mathbb{E}_{Y^y_{\kappa_n}} (\mathbf{1}_{\{\kappa<\infty\}} c^{-\kappa})] \\ &\leqslant C_2 \mathbb{E}_y \mathbf{1}_{\{\kappa_n<\infty\}} c^{-\kappa_n}\end{aligned}$$

and thus

$$\mathbb{E}_y \mathbf{1}_{\{\kappa_n<\infty\}} c^{-\kappa_n} \leqslant C_2^n C_\rho (V(y)+1) \quad \text{for} \quad y \in Y.$$

Define $E = Y^\infty \setminus (Y \times B^\infty)$ and $B_n = \{\epsilon_n < \infty\}$. Observe that $B_{n+1} = T^{-1}_{\kappa_n} E$ and $B_n \in \mathcal{F}_{\epsilon_n} \subset \mathcal{F}_{\kappa_n}$. For $y \in Y$ we have

$$\begin{aligned}\mathbb{P}_y(B_{n+1}) = \mathbb{P}_y(B_n \cap B_{n+1}) &= \mathbb{E}_y(\mathbf{1}_{B_n} \mathbb{E}_y(\mathbf{1}_E \circ T_{\kappa_n} | \mathcal{F}_{\kappa_n})) \\ &= \int_{B_n} \mathbb{P}_{x_{\kappa_n}}(E)\, \mathbb{P}_y(dx) \leqslant (1-p)\mathbb{P}_y(B_n),\end{aligned}$$

where $x = (x_0, x_1, \dots) \in Y^\infty$. It follows that

$$\mathbb{P}_y(B_n) \leqslant (1-p)^n \quad \text{for} \quad y \in Y, n \geqslant 1.$$

Define $\widehat{\tau}((y_n)_{n\in\mathbb{N}_0}) = \inf\{n \geqslant 1 :\ y_n \in A,\ \forall_{k>n}\, y_k \in B\}$ and $D_0 = \{\widehat{\tau} = \kappa_0\}$, $D_n = \{\kappa_{n-1} < \widehat{\tau} \leqslant \kappa_n < \infty\}$, for $n \geqslant 1$. Since $B_n = \{\widehat{\tau} > \kappa_{n-1}\}$, we have $D_n \subset B_n$ for $n \geqslant 0$ and $\mathbb{P}_y(\widehat{\tau} = \infty) = \lim_{n\to\infty} \mathbb{P}_y(\widehat{\tau} > \kappa_{n-1}) = \lim_{n\to\infty} \mathbb{P}_y(B_n) \leqslant \lim_{n\to\infty}(1-p)^n = 0$ for $y \in Y$. Finally, by the Hölder inequality, for $s > 1$ and $y \in Y$ we obtain

$$\begin{aligned}\mathbb{E}_y \lambda^{-\frac{\widehat{\tau}}{s}} &\leqslant \sum_{n=0}^\infty \mathbb{E}_y(\mathbf{1}_{\{\kappa_n<\infty\}} \lambda^{-\frac{\kappa_n}{s}} \mathbf{1}_{D_n}) \\ &\leqslant \sum_{n=0}^\infty [\mathbb{E}_y \mathbf{1}_{\{\kappa_n<\infty\}} \lambda^{-\kappa_n}]^{\frac{1}{s}} \mathbb{P}_y(D_n)^{1-\frac{1}{s}} \\ &\leqslant \sum_{n=0}^\infty [C_2^n C_\rho (V(y)+1)]^{\frac{1}{s}} (1-p)^{n(1-\frac{1}{s})} \\ &\leqslant C_\rho^{\frac{1}{s}} (1+V(y)) \sum_{n=0}^\infty [(\frac{C_2}{1-p})^{\frac{1}{s}} (1-p)]^n.\end{aligned}$$

Choosing sufficiently large s and setting $\gamma = \lambda^{\frac{1}{s}}$ we have $\mathbb{E}_y \gamma^{-\widehat{\tau}} \leqslant C(1+V(y))$ for $y \in Y$. Since $\tau < \widehat{\tau}$, the proof is complete. $\square$

LEMMA 2.2. *Let $(Y_n^y)_{n\in\mathbb{N}_0}$ with $y \in Y$ be a family of Markov chains on a metric space Y. Suppose that $V : Y \to [0,\infty)$ is a Lyapunov function for their transition function $\{\pi_y : y \in Y\}$, i.e. there exist $a \in (0,1)$ and $b > 0$ such that*

$$\int_Y V(x)\pi_y(dx) \leqslant aV(y) + b \quad \textit{for} \quad y \in Y.$$

Then there exist $\lambda \in (0,1)$ and $\tilde{C} > 0$ such that for

$$\rho((y_k)_{k\in\mathbb{N}_0}) = \inf\{k \geqslant 1 : V(y_k) < \frac{2b}{1-a}\}$$

we have

$$\mathbb{E}_y\lambda^{-\rho} \leqslant \tilde{C}(V(y_0)+1) \quad \textit{for} \quad y \in Y.$$

Proof of Lemma 2.2. Let us suppose that the chains $(Y_n^y)_{n\in\mathbb{N}_0}, y \in Y$ are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Fix $\alpha \in (\frac{1+a}{2}, 1)$ and set $V_0 = \frac{b}{\alpha - a}$. Define $\tilde{\rho}((y_k)_{k\in\mathbb{N}_0}) = \inf\{k \geqslant 1 : V(y_k) \leqslant V_0\}$. Fix $y \in Y$. Let $\mathcal{F}_n \subset \mathcal{F}$, $n \in \mathbb{N}_0$ be the filtration induced by $(Y_n^y)_{n\in\mathbb{N}_0}$. Define

$$A_n = \{\omega \in \Omega : V(Y_i^y(\omega)) > V_0 \quad \text{for} \quad i = 0,1,\ldots,n\}, \quad n \in \mathbb{N}_0.$$

Observe that $A_{n+1} \subset A_n$ and $A_n \in \mathcal{F}_n$. By the definition of V_0 we have $\mathbf{1}_{A_n}\mathbb{E}(V(Y_{n+1}^y)|\mathcal{F}_n) \leqslant \mathbf{1}_{A_n}(aV(Y_n^y)+b) < \alpha\mathbf{1}_{A_n}V(Y_n^y)$ $\mathbb{P}$-a.e. in Ω. This gives

$$\begin{aligned}\int_{A_n} V(Y_n^y)d\mathbb{P} &\leqslant \int_{A_{n-1}} V(Y_n^y)d\mathbb{P} = \int_{A_{n-1}} \mathbb{E}(V(Y_n^y)|\mathcal{F}_{n-1})d\mathbb{P} \\ &\leqslant \int_{A_{n-1}} (aV(Y_{n-1}^y)+b)d\mathbb{P} \leqslant \alpha \int_{A_{n-1}} V(Y_{n-1}^y)d\mathbb{P}.\end{aligned}$$

By the Chebyshev inequality

$$\begin{aligned}\mathbb{P}(V(Y_0^y) > V_0, \ldots, V(Y_n^y) > V_0) &= \int_{A_{n-1}} \mathbb{P}(V(Y_n^y) > V_0|\mathcal{F}_{n-1})d\mathbb{P} \\ &\leqslant V_0^{-1}\int_{A_{n-1}} \mathbb{E}(V(Y_n^y)|\mathcal{F}_{n-1})d\mathbb{P} \leqslant \alpha^{n-1}V_0^{-1}(aV(y)+b),\end{aligned}$$

thus for some $C > 0$ we have

$$\mathbb{P}_y(\tilde{\rho} > n) \leqslant \alpha^n C(V(y)+1), \quad n \in \mathbb{N}_0.$$

Fix $\gamma \in (0,1)$ and observe that for $\lambda = \alpha^\gamma$ we have

$$\begin{aligned}\mathbb{E}_y\lambda^{-\tilde\rho} &\leqslant 2 + \sum_{n=1}^{\infty} \mathbb{P}_y(\lambda^{-\tilde\rho} > n) = 2 + \sum_{n=1}^{\infty} \mathbb{P}_y(\tilde\rho > \log_\alpha(n^{-\frac{1}{\gamma}}))\\ &\leqslant 2 + \sum_{n=1}^{\infty} \mathbb{P}_y(\tilde\rho > \lfloor \log_\alpha(n^{-\frac{1}{\gamma}})\rfloor) \leqslant 2 + \frac{C(V(y)+1)}{\alpha}\sum_{n=1}^{\infty} n^{-\frac{1}{\gamma}}\\ &= \tilde C(V(y)+1)\end{aligned}$$

for properly chosen $\tilde C$. Since $\rho \leqslant \tilde\rho$, the proof is finished. □

Proof of Theorem 2.1.

For every $r > 0$ define $D_r = \{(x,y) \in X^2 :\ d(x,y) < r\ \}$.

Step I: To simplify calculations define a new metric $\bar d(x,y) = d(x,y)^\nu$ and observe that for $\bar D_r = \{(x,y) \in X^2 :\ \bar d(x,y) < r\}$ we have $D_R = \bar D_{\bar R}$ with $\bar R = R^\nu$. By the Jensen inequality (2.2) takes form

$$\int_{X^2} \bar d(u,v)\mathbf{Q}_{x,y}(du,dv) \leqslant \bar\alpha \bar d(x,y) \quad for \quad (x,y) \in F, \tag{2.6}$$

with $\bar\alpha = \alpha^\nu$. Assumption **A3** implies that

$$1 - \|\mathbf{Q}_{x,y}\| \leqslant l\bar d(x,y) \quad \text{and} \quad \mathbf{Q}_{x,y}(D_{\bar\alpha \bar d(x,y)}) \geqslant \delta \tag{2.7}$$

for $(x,y) \in F$.

Step II: Observe, that if $b \in \mathcal{M}_{fin}(X^2)$ satisfies $supp\, b \subset F$ then (2.7) implies

$$\|\mathbf{Q}_b\| \geqslant \|b\| - l\int_{X^2} \bar d(u,v)b(du,dv).$$

Since $supp\, b \subset F$, we have $supp\, \mathbf{Q}_b^{0,\dots,n} \subset F^{n+1}$. Iterating the above inequality and using (2.6) we obtain $\|\mathbf{Q}_b^{0,\dots,n}\| \geqslant \|b\| - \frac{l}{1-\bar\alpha}\int_{X^2} \bar d(u,v)b(du,dv)$. If $supp\, b \subset \{(u,v) \in X^2 :\ \bar d(u,v) < \frac{1-\bar\alpha}{2l}\} \cap F$ then from (2.5) it follows that

$$\|\mathbf{Q}_b^\infty\| \geqslant \frac{1}{2}\|b\|. \tag{2.8}$$

Set $R_0 = \sup\{\bar d(x,y) :\ L(x) + L(y) < R\} < \infty$ and $n_0 = \min\{n \in \mathbb{N}_0 :\ \bar\alpha^n R_0 < \frac{1-\bar\alpha}{2l}\}$. Now (2.7) implies that for $(x,y) \in F$ such that $L(x) + L(y) < R$ we have

$$\mathbf{Q}_{x,y}^{n_0}(\{(u,v) \in X^2 :\ \bar d(u,v) < \frac{1-\bar\alpha}{2l}\} \cap F) > \delta^{n_0}$$

and finally (2.8) gives

$$\|\mathbf{Q}^{\infty}_{x,y}\| \geqslant \frac{1}{2}\delta^{n_0}. \tag{2.9}$$

Step III: Define $\tilde{\rho}((x_n, y_n)_{n\in\mathbb{N}_0}) = \inf\{n \geqslant 1 : L(x_n) + L(y_n) < \frac{4c}{1-\lambda}\}$. Since $L(x) + L(y)$ is a Lyapunov function for a Markov chain in X^2 with transition probabilities $\{\mathbf{B}_{x,y} : x, y \in X\}$, Lemma 2.2 shows that there exist constants $\lambda_0 \in (0, 1)$ and C_0 such that

$$\mathbb{E}_{x,y}\, \lambda_0^{-\tilde{\rho}} \leqslant C_0(L(x) + L(y) + 1) \quad \text{for} \quad (x, y) \in X^2. \tag{2.10}$$

Define $A = \{(x, y, \theta) \in \widehat{X} : (x, y) \in F \quad \text{and} \quad L(x) + L(y) < R\}$ and

$$\rho((x_n, y_n, \theta_n)_{n\in\mathbb{N}_0}) = \inf\{n \in \mathbb{N}_0 : (x_n, y_n, \theta_n) \in A\}.$$

Slightly abusing notation, we identify $\kappa : (X \times X)^{\infty} \to \mathbb{N}_0 \cup \{\infty\}$, defined in **A4**, with its extension to $\widehat{X}^{\infty}$: $\kappa((x_n, y_n, \theta_n)_{n\in\mathbb{N}_0}) = \kappa((x_n, y_n)_{n\in\mathbb{N}_0})$. Since $\rho \leqslant \tilde{\rho} + \kappa \circ T_{\tilde{\rho}}$, where $T_{\tilde{\rho}}((x_n, y_n, \theta_n)_{n\in\mathbb{N}_0}) = (x_{n+\tilde{\rho}}, y_{n+\tilde{\rho}}, \theta_{n+\tilde{\rho}})_{n\in\mathbb{N}_0}$, an argument similar to that in the proof of Lemma 2.1 shows that there exist $\lambda \in (0, 1)$ such that

$$\mathbb{E}_{x,y,\theta}\, \lambda^{-\rho} \leqslant \tilde{C}C_0(L(x) + L(y) + 1) \quad \text{for} \quad x, y \in X, \theta \in \{0, 1\}.$$

Define $B = \{(x, y, \theta) \in \widehat{X} : \theta = 1\}$ and

$$\epsilon((x_n, y_n, \theta_n)_{n\in\mathbb{N}_0}) = \inf\{n \geqslant 1 : (x_n, y_n, \theta_n) \notin B\}.$$

From Step II we obtain $\mathbb{P}_{x,y,\theta}(B) \geqslant \frac{1}{2}\delta^{n_0}$ for $(x, y, \theta) \in A$. From (2.6) and (2.7) it follows that

$$\begin{aligned}
\widehat{\mathbf{B}}_{x,y,\theta}(\epsilon = n) &= \int_{\widehat{X}^n} \widehat{\mathbf{R}}_{z_{n-1}}(\widehat{X})\, \widehat{\mathbf{Q}}^{0,\ldots,n-1}_{x,y,\theta}(dz_0, \ldots, dz_{n-1}) \\
&= ||\mathbf{Q}^{n-1}_{\delta_{(x,y)}}|| - ||\widehat{Q}_{\mathbf{Q}^{n-1}_{\delta_{(x,y)}}}|| \leqslant l \int_{X^2} \bar{d}(u, v)\, \mathbf{Q}^{n-1}_{\delta_{(x,y)}}(du, dv) \\
&\leqslant l\bar{\alpha}^{n-1}\bar{d}(x, y) < \bar{\alpha}^{n-1} l R_0,
\end{aligned}$$

whenever $(x, y, \theta) \in A$. Finally Lemma 2.1 guarantees the existence of constants $\gamma \in (0,1)$, $C_1 > 0$ such that for

$$\tau((x_n, y_n, \theta_n)_{n\in\mathbb{N}_0}) = \inf\{n \geqslant 1 : \forall_{k\geqslant n}(x_k, y_k, \theta_k) \in B\}$$

we have

$$\mathbb{E}_{x,y,\theta}\, \gamma^{-\tau} \leqslant C_1(L(x) + L(y) + 1) \quad \text{for} \quad x, y \in X, \theta \in \{0,1\}.$$

STEP IV: Define sets $G_{\frac{n}{2}} = \{t \in (X^2 \times \{0,1\})^\infty : \tau(t) \leqslant \frac{n}{2}\}$ and $H_{\frac{n}{2}} = \{t \in (X^2 \times \{0,1\})^\infty : \tau(t) > \frac{n}{2}\}$. For every $n \in \mathbb{N}$ we have

$$\widehat{\mathbf{B}}^\infty_{x,y,\theta} = \widehat{\mathbf{B}}^\infty_{x,y,\theta}\,|_{G_{\frac{n}{2}}} + \widehat{\mathbf{B}}^\infty_{x,y,\theta}\,|_{H_{\frac{n}{2}}} \quad \text{for} \quad x, y \in X, \theta \in \{0,1\}.$$

Fix $\theta = 1$ and $(x, y) \in X^2$. From the fact that $\|\cdot\|_{FM} \leqslant \|\cdot\|_W$ it follows that

$$\begin{aligned}
&\|P^{*n}\delta_x - P^{*n}\delta_y\|_{FM} = \|\mathbf{P}^n_x - \mathbf{P}^n_y\|_{FM} \\
&= \sup_{f\in\mathcal{F}} |\int_{X^2} (f(z_1) - f(z_2))(pr^\#_n \mathbf{B}^\infty_{x,y})(dz_1, dz_2)| \\
&= \sup_{f\in\mathcal{F}} |\int_{X^2} (f(z_1) - f(z_2))(pr^\#_{X^2} pr^\#_n \widehat{\mathbf{B}}^\infty_{x,y,\theta})(dz_1, dz_2)| \\
&\leqslant \sup_{f\in\mathcal{W}} |\int_{X^2} (f(z_1) - f(z_2))(pr^\#_{X^2} pr^\#_n (\widehat{\mathbf{B}}^\infty_{x,y,\theta}\,|_{G_{\frac{n}{2}}}))(dz_1, dz_2)| + 2\widehat{\mathbf{B}}^\infty_{x,y,\theta}(H_{\frac{n}{2}}).
\end{aligned}$$

From **A2** we obtain

$$\begin{aligned}
&\sup_{\mathcal{W}} |\int_{X^2} (f(z_1) - f(z_2))(pr^\#_{X^2} pr^\#_n (\widehat{\mathbf{B}}^\infty_{x,y,\theta}\,|_{G_{\frac{n}{2}}}))(dz_1, dz_2)| \\
&\leqslant \int_{X^2} d(z_1, z_2)(pr^\#_{X^2} pr^\#_n (\widehat{\mathbf{B}}^\infty_{x,y,\theta}\,|_{G_{\frac{n}{2}}}))(dz_1, dz_2) \\
&\leqslant \alpha^{\frac{n}{2}} \int_{X^2} d(z_1, z_2)(pr^\#_{X^2} pr^\#_{\frac{n}{2}} (\widehat{\mathbf{B}}^\infty_{x,y,\theta}\,|_{G_{\frac{n}{2}}}))(dz_1, dz_2) \leqslant \alpha^{\frac{n}{2}} R.
\end{aligned}$$

Now Step III and the Chebyshev inequality imply that

$$\widehat{\mathbf{B}}^\infty_{x,y,\theta}(H_{\frac{n}{2}}) \leqslant \gamma^{\frac{n}{2}} C_1(L(x) + L(y) + 1) \quad \text{for} \quad n \in \mathbb{N}.$$

Taking $C_2 = 2C_1 + R$ and $q = \max\{\gamma^{\frac{n}{2}}, \alpha^{\frac{n}{2}}\}$ we obtain

$$\|P^{*n}\delta_x - P^{*n}\delta_y\|_{FM} \leqslant \gamma^n C_1(L(x) + L(y) + 1) \quad \text{for} \quad x, y \in X, n \in \mathbb{N},$$

and so

$$\|P^{*n}\mu - P^{*n}\nu\|_{FM} \leqslant \gamma^n C_1(\int_X L(x)\mu(dx) + \int_X L(y)\nu(dy) + 1) \tag{2.11}$$

for $\mu, \nu \in \mathcal{M}_1^L(X)$ and $n \in \mathbb{N}$.

Step V: Observe that Step IV and **A1** give

$$\begin{aligned}\|P^{*n}\delta_x - P^{*(n+k)}\delta_x\|_{FM} &\leqslant \int_X \|P^{*n}\delta_x - P^{*n}\delta_y\|_{FM} P^{*k}\delta_x(dy)\\ &\leqslant q^n C_2 \int_X (L(x) + L(y))P^{*k}\delta_x(dy) \leqslant q^n C_3(1 + L(x)),\end{aligned}$$

so $(P^{*n}\delta_x)_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in X$. Since $\mathcal{M}_1(X)$ equipped with the norm $\|\cdot\|_{FM}$ is complete (see [8]), assumption **A0** implies the existence of an invariant measure μ_*. Assumption **A1** gives $\mu_* \in \mathcal{M}_L(X)$. Applying inequality (2.11) we obtain (2.4). Observation that $\mathcal{M}_L(X)$ is dense in $\mathcal{M}_1(X)$ in the total variation norm finishes the proof. □

Remark. In steps IV and V of the above proof we follow M. Hairer (see [11]).

3. RANDOM ITERATION OF FUNCTIONS

Let (X, d) be a Polish space and (Θ, Ξ) a measurable space with a family $\vartheta_x \in \mathcal{M}_1(\Theta)$ of distributions on Θ indexed by $x \in X$. Space Θ serves as a set of indices for a family $\{S_\theta : \theta \in \Theta\}$ of continuous functions acting on X into itself. We assume that $(\theta, x) \mapsto S_\theta(x)$ is product measurable. In this section we study some stochastically perturbed dynamical system $(X_n)_{n\in\mathbb{N}_0}$. Its intuitive description is the following: if X_0 starts at x_0, then by choosing θ_0 at random from ϑ_{x_0} we define $X_1 = S_{\theta_0}(x_0)$. Having X_1 we select θ_1 according to the distribution ϑ_{X_1} and we put $X_2 = S_{\theta_1}(X_1)$ and so on. More precisely, the process $(X_n)_{n\in\mathbb{N}_0}$ can

be written as

$$X_{n+1} = S_{Y_n}(X_n), \quad n = 0, 1, \ldots,$$

where $(Y_n)_{n\in\mathbb{N}_0}$ is a sequence of random elements defined on a probability space $(\Omega, \Sigma, prob)$ with values in Θ such that

$$prob\,(Y_n \in B | X_n = x) = \vartheta_x(B) \quad \text{for} \quad x \in X, B \in \Xi, n = 0, 1, \ldots, \tag{3.1}$$

and $X_0 : \Omega \to X$ is a given random variable. Denoting by μ_n the probability law of X_n, we will give a recurrence relation between μ_{n+1} and μ_n. To this end fix $f \in B_b(X)$ and note that $\mathbb{E}f(X_{n+1}) = \int_X f d\mu_{n+1}$. By (3.1) we have

$$\int\limits_A \vartheta_x(B)\mu_n(dx) = prob(\{Y_n \in B\} \cap \{X_n \in A\}) \quad \text{for} \quad B \in \Xi, A \in \mathcal{B}_X,$$

hence

$$\mathbb{E}f(X_{n+1}) = \int\limits_\Omega f(S_{Y_n(\omega)}(X_n(\omega))prob(d\omega) = \int\limits_X \int\limits_\Theta f(S_\theta(x))\vartheta_x(d\theta)\mu_n(dx).$$

Putting $f = \mathbf{1}_A$, $A \in \mathcal{B}_X$, we obtain $\mu_{n+1}(A) = P^*\mu_n(A)$, where

$$P^*\mu(A) = \int\limits_X \int\limits_\Theta \mathbf{1}_A(S_\theta(x))\vartheta_x(d\theta)\mu(dx) \quad \text{for} \quad \mu \in \mathcal{M}_{fin}(X), A \in \mathcal{B}_X.$$

In other words this formula defines the transition operator for μ_n. Operator P^* is adjoint of the Markov operator $P : B_b(X) \to B_b(X)$ of the form

$$Pf(x) = \int\limits_\Theta f(S_\theta(x))\vartheta_x(d\theta). \tag{3.2}$$

We take this formula as the precise formal definition of considered process. We will show that operator (3.2) has a unique invariant measure, provided the following conditions hold:

B1 *There exists* $\alpha \in (0, 1)$ *such that*

$$\int\limits_\Theta d(S_\theta(x), S_\theta(y))\vartheta_x(d\theta) \leqslant \alpha d(x, y) \quad \text{for} \quad x, y \in X.$$

B2 *There exists* $\bar{x} \in X$ *such that*

$$c := \sup_{x \in X} \int_{\Theta} d(S_\theta(\bar{x}), \bar{x}) \vartheta_x(d\theta) < \infty.$$

B3 *The map* $x \mapsto \vartheta_x$, $x \in X$, *is Hölder continuous in the total variation norm, i.e. there exists* $l > 0$ and $\nu \in (0, 1]$ *such that*

$$\|\vartheta_x - \vartheta_y\| \leqslant l\, d(x, y)^\nu \quad \text{for} \quad x, y \in X.$$

B4 *There exists* $\delta > 0$ *such that*

$$\vartheta_x \wedge \vartheta_y(\{\theta \in \Theta : d(S_\theta(x), S_\theta(y)) \leqslant \alpha d(x, y)\}) > \delta \quad \text{for} \quad x, y \in X,$$

where $\wedge$ *denotes the greatest lower bound in the lattice of finite measures.*

Remark. In the simplest case $\Theta = \{1, ..., n\}$ above conditions are standard and nearly optimal (see [30] for discussion).

PROPOSITION 3.1. *Assume* **B1** – **B4**. *Then operator* (3.2) *possesses a unique invariant measure* $\mu_* \in \mathcal{M}_1^1(X)$, *which is attractive in* $\mathcal{M}_1(X)$. *Moreover there exist* $q \in (0, 1)$ *and* $C > 0$ *such that*

$$\|P^{*n}\mu - \mu_*\|_{FM} \leqslant q^n C(1 + \int_X d(\bar{x}, x)\mu(dx))$$

for $\mu \in \mathcal{M}_1^1(X)$ *and* $n \in \mathbb{N}$.

Proof. Define the operator Q on $B_b(X^2)$ by

$$Q(f)(x, y) = \int_{\Theta} f(S_\theta(x), S_\theta(y)) \vartheta_x \wedge \vartheta_y(d\theta).$$

Since

$$||\vartheta_{x'} \wedge \vartheta_{y'} - \vartheta_x \wedge \vartheta_y|| \leqslant 2(||\vartheta_{x'} - \vartheta_x|| + ||\vartheta_{y'} - \vartheta_y||)$$

it follows that

$$\begin{aligned}|Q(f)(x',y')-Q(f)(x,y)| &\leqslant \int_\Theta |f(S_\theta(x'),S_\theta(y'))|\,||\vartheta_{x'}\wedge\vartheta_{y'}-\vartheta_x\wedge\vartheta_y||(d\theta)\\ &+\int_\Theta |f(S_\theta(x'),S_\theta(y'))-f(S_\theta(x),S_\theta(y))|\vartheta_x\wedge\vartheta_y(d\theta)\\ &\leqslant 2l\sup_{z\in X^2}|f(z)|\,(d(x,x')^\nu+d(y,y')^\nu)\\ &+\int_\Theta |f(S_\theta(x'),S_\theta(y'))-f(S_\theta(x),S_\theta(y))|\vartheta_x\wedge\vartheta_y(d\theta),\end{aligned}$$

for $f\in B_b(X^2)$, $x,y\in X$. Consequently, we see that $Q(C_b(X^2))\subset C_b(X^2)$, by Lebesgue's dominated convergence theorem. Put

$$\mathcal{F}=\{f\in B_b(X^2):\sup_{z\in X^2}|f(z)|\leqslant M, Q(f)\in B_b(X^2)\},$$

where $M>0$ is fixed, and observe that the family $\mathcal{F}$ contains all continuous functions bounded by M and the limit of any convergent sequence of functions in the class, i.e. $\mathcal{F}$ consists, by the definition, of all Baire functions bounded by M. Since the class of Baire functions is identical with the class of Borel functions (see [22, Theorem 4.5.2]) it follows that $Q(B_b(X^2))\subset B_b(X^2)$. In particular, for the family $\{Q_{x,y}:x,y\in X\}$ of (subprobability) measures given by

$$Q_{x,y}(C)=\int_\Theta \mathbf{1}_C(S_\theta(x),S_\theta(y))\vartheta_x\wedge\vartheta_y(d\theta),$$

we have that maps $(x,y)\mapsto Q_{x,y}(C)$ are measurable for every $C\in\mathcal{B}_{X^2}$.

Arguing similarly as above we show that (3.2) is well defined Feller operator. It has Lyapunov function $L(x)=d(x,\bar{x})$, since $\int_\Theta d(S_\theta(x),\bar{x})\vartheta_x(d\theta)\leqslant\alpha d(x,\bar{x})+c$. Now, observe that

$$\begin{aligned}\|Q_{x,y}\|=\vartheta_x\wedge\vartheta_y(\Theta)&=\inf_{A\in\Xi}\{\vartheta_x(A)+\vartheta_y(\Theta\setminus A)\}\\ &=1-\sup_{A\in\Xi}\{\vartheta_y(A)-\vartheta_x(A)\}=1-\|\vartheta_x-\vartheta_y\|\geqslant 1-l\,d(x,y)^\nu\end{aligned}$$

for $x,y\in X$. Moreover, we have

$$\int_{X^2} d(u,v)Q_{x,y}(du,dv)=\int_\Theta d(S_\theta(x),S_\theta(y))\vartheta_x\wedge\vartheta_y(d\theta)\leqslant\alpha d(x,y),$$

and $Q_{x,y}(D_{\alpha d(x,y)}) = \vartheta_x \wedge \vartheta_y(\{\theta \in \Theta : d(S_\theta(x), S_\theta(y)) \leqslant \alpha d(x,y)\}) > \delta$ for $x, y \in X$. In consequence **A0** – **A3** are fulfilled. The use of Theorem 2.1 (see also comments in subsection 2.2 concerning assumption **A4**) ends the proof. □

4. PERPETUITIES WITH PLACE DEPENDENT PROBABILITIES

Let $X = \mathbb{R}^d$ and $G = \mathbb{R}^{d\times d} \times \mathbb{R}^d$, and consider the function $S_\theta : X \to X$ defined by $S_\theta(x) = M(\theta)x + Q(\theta)$, where (M, Q) is a random variable on (Θ, Ξ) with values in G. Then (3.2) may be written as

$$Pf(x) = \int_G f(mx + q)d\vartheta_x \circ (M, Q)^{-1}(m, q) \tag{4.1}$$

This operator is connected with the random difference equation of the form

$$\Phi_n = M_n\Phi_{n-1} + Q_n, \quad n = 1, 2, \ldots, \tag{4.2}$$

where $(M_n, Q_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables distributed as (M, Q). Namely, the process $(\Phi_n)_{n\in\mathbb{N}_0}$ is a homogeneous Markov chain with the transition kernel P given by

$$Pf(x) = \int_G f(mx + q)d\mu(m, q), \tag{4.3}$$

where μ stands for the distribution of (M, Q). Equation (4.2) arises in various disciplines as economics, physics, nuclear technology, biology, sociology (see e.g. [32]). It is closely related to a sequence of backward iterations $(\Psi_n)_{n\in\mathbb{N}}$, given by $\sum_{k=1}^n M_1 \ldots M_{k-1}Q_k, n \in \mathbb{N}$ (see e.g. [9]). Under conditions ensuring the almost sure convergence of the sequence $(\Psi_n)_{n\in\mathbb{N}}$ the limiting random variable

$$\sum_{n=1}^{\infty} M_1 \ldots M_{n-1}Q_n \tag{4.4}$$

is often called *perpetuity*. It turns out that the probability law of (4.4) is a unique invariant measure for (4.3). The name perpetuity comes from perpetual payment

streams and recently gained some popularity in the literature on stochastic recurrence equations (see [7]). In the insurance context a perpetuity represents the present value of a permanent commitment to make a payment at regular intervals, say annually, into the future forever. The Q_n represent annual payments, the M_n cumulative discount factors. Many interesting examples of perpetuities can be found in [1]. Due to significant papers [17], [10], [32] and [9] we have complete (in the dimension one) characterization of convergence of perpetuities. The rate of this convergence has recently been extensively studied by many authors (see for instance [3]-[5], [24]). The main result of this section concerns the rate of convergence of the process $(X_n)_{n\in\mathbb{N}_0}$ associated with the operator $P : B_b(\mathbb{R}^d) \to B_b(\mathbb{R}^d)$ given by

$$Pf(x) = \int_G f(mx+q)d\mu_x(m,q), \tag{4.5}$$

where $\{\mu_x : x \in \mathbb{R}^d\}$ is a family of Borel probability measures on G. In contrast to $(\Phi_n)_{n\in\mathbb{N}_0}$, the process $(X_n)_{n\in\mathbb{N}_0}$ moves by choosing at random θ from a measure depending on x. Taking into considerations the concept of perpetuities we may say that $(X_n)_{n\in\mathbb{N}_0}$ forms *a perpetuity with place dependent probabilities*.

COROLLARY 4.1. *Assume that $\{\mu_x : x \in \mathbb{R}^d\}$ is a family of Borel probability measures on G such that*

$$\alpha := \sup_{x\in\mathbb{R}^d} \int_G ||m||_{op}\, d\mu_x(m,q) < 1, \quad c := \sup_{x\in\mathbb{R}^d} \int_G |q| d\mu_x(m,q) < \infty. \tag{4.6}$$

Assume moreover that the map $x \mapsto \mu_x$, $x \in X$, is Hölder continuous in the total variation norm and there exists $\delta > 0$ such that

$$\mu_x \wedge \mu_y(\{(m,q) \in G : ||m||_{op} \leqslant \alpha\}) > \delta \quad \textit{for} \quad x, y \in \mathbb{R}^d.$$

Then operator (4.5) *possesses a unique invariant measure $\mu_* \in \mathcal{M}_1^1(\mathbb{R}^d)$, which is attractive in $\mathcal{M}_1(\mathbb{R}^d)$. Moreover there exist $q \in (0,1)$ and $C > 0$ such that*

$$\|P^{*n}\mu - \mu_*\|_{FM} \leqslant q^n C(1 + \int_{\mathbb{R}^d} |x|\mu(dx))$$

for $\mu \in \mathcal{M}_1^1(\mathbb{R}^d)$ *and* $n \in \mathbb{N}$.

The proof of corollary is straightforward application of Proposition 3.1. We leave the details to the reader. We finish the paper by giving an example to illustrate Corollary 4.1.

Example. Let ν_0, ν_1 be distributions on $\mathbb{R}^2$. Assume that $p, q : \mathbb{R} \to [0,1]$ are Lipschitz functions (with Lipschitz constant L) summing up to 1, and $p(x) = 1$, for $x \leqslant 0$, $p(x) = 0$, for $x \geqslant 1$. Define μ_x by $\mu_x = p(x)\nu_0 + q(x)\nu_1, x \in \mathbb{R}$. Then:
1. $\|\mu_x - \mu_y\| \leqslant 2L|x-y| \quad \text{for} \quad x, y \in \mathbb{R}$.
2. If $\int_{\mathbb{R}^2} |m| d\nu_i(m,q) < 1$ and $\int_{\mathbb{R}^2} |q| d\nu_i(m,q) < \infty$ for $i = 0, 1$, then (4.6) holds.
3. For every $A \in \mathcal{B}_{\mathbb{R}^2}$, $x, y \in \mathbb{R}$ we have: $\mu_x \wedge \mu_y(A) \geqslant \nu_0 \wedge \nu_1(A) = (\nu_0 - \lambda^+)(A) = (\nu_1 - \lambda^-)(A) \geqslant \max\{\nu_0(A), \nu_1(A)\} - \|\nu_0 - \nu_1\|(A)$, where (λ^+, λ^-) is the Jordan decomposition of $\nu_1 - \nu_0$.

Acknowledgment. This work was partially supported by the Faculty of Applied Mathematics AGH UST statutory tasks within subsidy of Ministry of Science and Higher Education.

REFERENCES

[1] G. Alsmeyer, A, Iksanov, U. Rösler, On distributional properties of perpetuities, J. Theoret. Probab. 22 (2009) 666–682.

[2] M.F. Barnsley, S.G. Demko, J.H. Elton, J.S. Geronimo, Invariant measures for Markov processes arising from iterated function systems with place dependent probabilities, Ann. Inst. H. Poincaré 24 (1988) 367–394.

[3] K. Bartkiewicz, A. Jakubowski, T. Mikosch, O. Wintenberger, Stable limits for sums of dependent infinite variance random variables, Probab. Theory Rel. Fields 150 (2011) 337–372.

[4] S. Brofferio, D. Buraczewski, E. Damek, On the invariant measure of the random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case, Ann. I. H. Poincare (B) Probab. Stat. 48 (2012) 377–395.

[5] D. Buraczewski, E. Damek, Y. Guivarc'h, Convergence to stable laws for a class of multidimensional stochastic recursions, Probab. Theory Rel. Fields 148 (2010) 333–402.

[6] P. Diaconis, D. Freedman, Iterated random functions, *SIAM Rev.* 41 (1999) 45–76.

[7] P. Embrechts, C. Klüppelberg, T. Mikosch, Modeling extremal events for insurance and finance, Applications of Mathematics 33, Springer-Verlag, New-York, 1997.

[8] S. Ethier, T. Kurtz, Markov Processes, Wiley, New York, 1986.

[9] C.M. Goldie, R.A. Maller, Stability of Perpetuities, Ann. Probab. 28 (2000) 1195–1218.

[10] A.K. Grincevičjus, On the continuity of the distribution of a sum of dependent variables connected with independent walks on lines, Theory Probab. Appl. 19 (1974) 163–168.

[11] M. Hairer, Exponential mixing properties of stochastic PDEs through asymptotic coupling, Probab. Theory Rel. Fields 124 (2002) 345–380.

[12] M. Hairer, J. Mattingly, Spectral gaps in Wasserstein distances and the 2D stochastic Navier-Stokes equations, Ann. Probab. 36 (2008) pp. 2050-2091.

[13] M. Hairer, J. Mattingly, M. Scheutzow, Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations, Prob. Theory Rel. Fields 149 (2011) 223–259.

[14] S. Hille, K. Horbacz, T. Szarek, H. Wojewódka, Limit theorems for some Markov chains, J Math. Anal. Appl. 443 (2016),385–408.

[15] K. Horbacz, T. Szarek, Continuous iterated function systems on Polish spaces, Bull. Polish Acad. Sci. Math. 49 (2001) 191–202.

[16] K. Horbacz, The Central Limit Theorem for Random Dynamical Systems, Journal of Statistical Physics 164 (2016), 1261-1291.

[17] H. Kesten, Random difference equations and renewal theory for products of random matrices, Acta Math. 131 (1973) 207–248.

[18] T. Komorowski, A. Walczuk, Central limit theorem for Markov processes with spectral gap in the Wasserstein metric, Stoch. Proc. Appl. 122 (2012), 2155-2184.

[19] S. Kuksin, A. Shirikyan, A coupling approach to randomly forced nonlinear PDE's. I, Comm. Math. Phys. 221 (2001) 351-366.

[20] S. Kuksin, A. Piatnitsky, A. Shirikyan A coupling approach to randomly forced nonlinear PDE's. II, Comm. Math. Phys. 230 (2002) 81-85.

[21] T. Lindvall, Lectures on the Coupling Method, John Wiley&Sons, New York, 1992.

[22] St. Łojasiewicz, An introduction to the Theory of Real Function, John Wiley&Sons Chichester New York Brisbane, Toronto, Singapore, 1998.

[23] S. Meyn, R. L. Tweedie, Markov Chains and Stochastic Stability, 2nd ed., Cambridge University Press, New York, 2009.

[24] M. Mirek, Heavy tail phenomenon and convergence to stable laws for iterated Lipschitz maps,

Probab. Theory Rel. Fields 151 (2011) 705–734.

[25] A. Öberg, Approximation of invariant measures for random iterations, Rocky Mountain J. Math. 36 (2006), 273–301.

[26] C. Odasso, Exponential mixing for stochastic PDEs: the non-additive case, Probab. Theory Rel. Fields 140 (2008) 41–82.

[27] A. Shirikyan, A version of the law of large numbers and applications, in Probabilistic Methods in Fluids, Proceedings of the Swansea Workshop held on 14 - 19 April 2002, World Scientific, New Jersey, 2003, pp. 263–271.

[28] Ö. Stenflo, A note on a theorem of Karlin, Stat. Probab. Lett. 54(2) (2001), 183–187.

[29] T. Szarek, Invariant measures for nonexpansive Markov operators on Polish spaces, Diss. Math. 415 (2003) 1–62.

[30] M. Ślęczka, The rate of convergence for iterated functions systems, Studia Math. 205 (2011) 201–214.

[31] M. Ślęczka, Exponential convergence for Markov Systems, Annales Mathematicae Silesianae 29 (2015) 139–149.

[32] W. Vervaat, On a stochastic difference equation and a representation of non-negative infinitely divisible random variables, Adv. Appl. Prob. 11 (1979) 750–783.

[33] I. Werner, Contractive Markov sysems, J. London Math. Soc. 71 (2005) 236–258.

[34] L. Xu, Exponential mixing of 2D SDEs forced by degenerate Levy noises, J. Evolutionary Equations 14 (2014) 249–272.

AGH University of Science and Technology, Faculty of Applied Mathematics
al. A. Mickiewicza 30, 30-059 Krakow, Poland
E-mail: rafal.kapica@agh.edu.pl

Institute of Mathematics, University of Silesia in Katowice
Bankowa 14, 40-007 Katowice, Poland
E-mail: sleczka@math.us.edu.pl

Received on 6.11.2017;
revised version on xx.xx.xxxx