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ON THE CARRYING DIMENSION OF OCCUPATION MEASURES FOR SELF-AFFINE RANDOM FIELDS*

BY

PETER KERN (DÜSSELDORF) AND ERCAN SÖNMEZ (DÜSSELDORF)

Abstract. Hausdorff dimension results are a classical topic in the study of path properties of random fields. This article presents an alternative approach to Hausdorff dimension results for the sample functions of a large class of self-affine random fields. The aim is to demonstrate the following interesting relation to a series of articles by U. Zähle [50], [51], [52], [53]. Under natural regularity assumptions, we prove that the Hausdorff dimension of the graph of self-affine fields coincides with the carrying dimension of the corresponding self-affine random occupation measure introduced by U. Zähle. As a remarkable consequence we obtain a general formula for the Hausdorff dimension given by means of the singular value function.

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1. INTRODUCTION

Let $U \in \mathbb{R}^{d\times d}$ and $V \in \mathbb{R}^{m\times m}$ be contracting, non-singular matrices, i.e. $|\rho| \in (0,1)$ for all eigenvalues ρ of U, respectively V. According to [53, Definition 4.1] a random field $X = \{X(t)\}_{t\in\mathbb{R}^d}$ on $\mathbb{R}^m$ defined on a probability space $(\Omega, \mathcal{A}, P)$ is called (U,V)-*self-affine* if the following four conditions hold:

(i) The field obeys the scaling relation

$$\{X(Ut)\}_{t\in\mathbb{R}^d} \stackrel{\mathrm{fd}}{=} \{VX(t)\}_{t\in\mathbb{R}^d},$$

where "$\stackrel{\mathrm{fd}}{=}$" denotes equality of all finite-dimensional marginal distributions.

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(ii) X has *stationary increments*, i.e. $X(t)-X(s)\stackrel{\mathrm{d}}{=}X(t-s)$ for all $s,t\in\mathbb{R}^d$, where "$\stackrel{\mathrm{d}}{=}$" denotes equality in distribution.

(iii) X is *proper*, i.e. $X(t)$ is not supported on any lower dimensional hyperplane of $\mathbb{R}^m$ for all $t\neq 0$.

(iv) The mapping $X:\mathbb{R}^d\times\Omega\to\mathbb{R}^m$ is $(\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{A})-\mathcal{B}(\mathbb{R}^m)$-measurable with respect to the Borel-σ-algebras of $\mathbb{R}^d$ and $\mathbb{R}^m$.

Several examples of such random fields are discussed in Section 4 below. Note that (i) implies $X(0)=0$ almost surely and that (iv) is fulfilled if X has continuous sample functions (or right-continuous sample paths in case $d=1$). Inductively, from (i) we get $\{X(U^nt)\}_{t\in\mathbb{R}^d}\stackrel{\text{f.d.}}{=}\{V^nX(t)\}_{t\in\mathbb{R}^d}$ for every $n\in\mathbb{Z}$ and thus self-affinity weakens the assumption of self-similarity [31], [30], [17], [12] to a discrete scaling property, which is also called semi-selfsimilarity [35] in the context of stochastic processes, where $d=1$ and usually we have the restriction $t\geqslant 0$.

Over the last decades there has been increasing attention in such random fields in theory as well as in applications. Possible applications can be found in such diverse fields as engineering, finance, physics, hydrology, image processing or network analysis; e.g., see [2], [3], [8], [10], [13], [14], [22], [32], [40], [42], [48] and the literature cited therein. Particularly, in the study of sample function behavior, it is of considerable interest to determine fractal dimensions such as Hausdorff dimension of random sets depending on the sample functions of a self-affine random field. E.g., we refer to [19], [37] for a comprehensive introduction to fractal geometry and the notion of Hausdorff dimension.

The main objective in this paper is the occupation measure τ_X of a self-affine random field X, measuring the size in $\mathbb{R}^d$ the graph of X spends in a Borel set of $\mathbb{R}^d\times\mathbb{R}^m$ with respect to Lebesgue measure. In a series of papers [51], [52], [53] U. Zähle investigated this object in detail, which serves as a starting point of our considerations. In particular, Zähle [53] showed that the occupation measure τ_X of a self-affine random field X is Palm distributed and itself a self-affine random measure, see Section 2 for details. This allows to study Hausdorff dimension results through the notion of carrying dimension of τ_X introduced in [51]. Heuristically, the carrying dimension of a Borel measure is the minimal Hausdorff dimension for Borel sets assigning positive measure; see Definition 2.1 below for the precise mathematical description. In this article we are able to show that under a natural condition the carrying dimension of τ_X always exists and indeed almost surely coincides with the Hausdorff dimension of the graph of X. Under the mild additional assumption of boundedly continuous intensity, see Definition 2.2 below, U. Zähle's result [53] shows that the carrying dimension of τ_X can be calculated by means of the singular value function, which has already been used as a tool in fractal geometry to derive the Hausdorff dimension of self-affine sets arising from iterated function systems [18], [20], [21]. This also enables us to show the applicability of the singular value function in order to derive dimension results for the range and graph of sample functions given by self-affine random fields.

Much effort has been made in the last decades in order to calculate the Hausdorff dimension of the range and the graph of the paths arising from several special classes of self-affine random fields. Classically, the Hausdorff dimension is determined by calculating an upper and a lower bound separately. This approach requires an a priori educated guess on the true value of the Hausdorff dimension. A typical method in the calculation of an upper bound is to find an efficient covering of the graph for example by using sample function properties such as Hölder continuity or independent increments, whereas the calculation of a lower bound is usually related to potential theoretic methods. Let us mention that for various special cases of self-affine random fields the Hausdorff dimensions of the graph and the range have been calculated separately case by case, with expression that by first sight seem not to fit consistently into a master formula. As a consequence of our main result, we provide a general formula in closed form for these expressions in case the contracting non-singular operators U and V are given by exponential matrices, that is $U = c^E$ and $V = c^D$, where $0 < c < 1$ and $E \in \mathbb{R}^{d\times d}$, $D \in \mathbb{R}^{m\times m}$ are matrices with positive real parts of their eigenvalues. In many situations the appearance of exponential scaling matrices is quite natural in the context of self-similar or self-affine random fields and processes; see [25], [38], [33], [12]. Thus we also provide candidates for the Hausdorff dimension of the graph and the range of general self-affine random fields, useful to be applied to open cases. Under the condition of boundedly continuous intensity these candidates always serve as lower bounds for the Hausdorff dimension of the graph and the range of self-affine random fields. Known methods to derive corresponding upper bounds heavily depend on further properties of the field such as Hölder continuity or independent increments and should be derived individually elsewhere. In our approach we particularly elucidate the intuition that the Hausdorff dimension of the graph and the range over sample functions of self-affine random fields should only depend on the real parts of the eigenvalues of E and D as well as their multiplicity.

The rest of this article is structured as follows. Section 2 basically serves as an introduction to self-affine random measures as given in [26], [51], [52], [53] which will be applied in order to establish the main result of this paper. Here, we adopt some notation and repeat fundamental notions and results from [51], [52], [53] concerning self-affine random measures, Palm distributions, the carrying dimension and the boundedly continuous intensity condition. Section 3 is the core part of this article, where we formulate and prove the above mentioned main result. Finally, in Section 4 we show that our results can be applied to large classes of self-affine random fields, namely to operator-self-similar stable random fields introduced by Li and Xiao [33], and to operator semistable Lévy processes. For these particular classes of self-affine random fields, our candidates derived by means of the singular value function in Section 3 are in fact the true values for the Hausdorff dimension of the graph and the range as recently shown in [45], [46], [28], [47]. Furthermore, our results may be useful to derive Hausdorff dimension results for classes of random processes and fields, for which this still remains an open question, e.g. for multiparameter operator semistable Lévy processes or certain semi-selfsimilar Markov processes.

2. PRELIMINARIES

In this section we recall some basic facts on random measures and Palm distributions which will be needed for our approach. Further, the occupation measure of a self-affine random field and its carrying dimension are introduced. Main parts of this section can be found in the articles [51], [52], [53] by U. Zähle and are presented here for the readers' convenience.

2.1. Random measures, Palm distributions and occupation measures.

Let $\mathcal{M}_{\mathbb{R}^n}$ be the set of all locally finite measures on $\mathbb{R}^n$ equipped with the corresponding σ-algebra $\mathcal{B}(\mathcal{M}_{\mathbb{R}^n})$ generated by the mappings $\mathcal{M}_{\mathbb{R}^n} \ni \mu \mapsto \mu(B)$ for all bounded Borel sets $B \subset \mathbb{R}^n$; e.g., see [26] for details. A random variable $\xi : \Omega \to \mathcal{M}_{\mathbb{R}^n}$ is called a *random measure* on $\mathbb{R}^n$. For any random measure ξ denote by P_ξ its distribution. Note that P_ξ is a probability measure on $(\mathcal{M}_{\mathbb{R}^n}, \mathcal{B}(\mathcal{M}_{\mathbb{R}^n}))$. Furthermore, it is clear that the mapping

$$\mathbb{E}[\xi] : \mathcal{B}(\mathbb{R}^n) \to [0,\infty], \quad A \mapsto \mathbb{E}[\xi(A)] = \int\limits_{\mathcal{M}_{\mathbb{R}^n}} \mu(A)\, dP_\xi(\mu)$$

is a (deterministic) Borel measure called the *intensity measure* of ξ.

For any measure $\mu \in \mathcal{M}_{\mathbb{R}^n}$ and $z \in \mathbb{R}^n$ let $T_z\mu$ be the translation measure given by $T_z\mu(B) = \mu(B - z)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$. We say that a random measure on $\mathbb{R}^n$ is *stationary* if $P_\xi = P_{T_z\xi}$ for all $z \in \mathbb{R}^n$. Let $x > 0$ and $W \in \mathbb{R}^{n\times n}$ be a non-singular operator. A random measure ξ is called (W, x)-*self-affine* if

$$\xi(A) \stackrel{\mathrm{d}}{=} x \cdot \xi(W^{-1}A) \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^n).$$

We now turn to the definition of Palm distributions. A σ-finite measure on $\big(\mathcal{M}_{\mathbb{R}^n}, \mathcal{B}(\mathcal{M}_{\mathbb{R}^n})\big)$ shall be called *quasi-distribution*. Again, a translation invariant quasi-distribution Q satisfying

$$dQ(\mu) = dQ(T_z\mu) \quad \text{for every } z \in \mathbb{R}^n$$

is called *stationary*. For a stationary quasi-distribution it is easy to see that the Borel measure $A \mapsto \int_{\mathcal{M}_{\mathbb{R}^n}} \mu(A)\, dQ(\mu)$ is translation invariant and thus there exists a constant $c_Q \in [0,\infty]$ such that

$$\int\limits_{\mathcal{M}_{\mathbb{R}^n}} \mu(A)\, dQ(\mu) = c_Q \cdot \lambda_n(A) \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^n),$$

where λ_n denotes the Lebesgue measure on $\mathbb{R}^n$. Note that for a stationary random measure ξ this implies that there is a constant $c_\xi \in [0,\infty]$ such that the intensity measure satisfies $\mathbb{E}[\xi] = c_\xi \cdot \lambda_n$. Throughout this paper, we will refer to c_ξ and c_Q as the *intensity constants* of ξ, respectively Q. For a stationary quasi-distribution Q the measure Q^0 defined by

$$(2.1)\quad Q^0(G) = \frac{1}{\lambda_n(A)} \int_{\mathcal{M}_{\mathbb{R}^n}} \int_A \mathbb{1}_G(T_{-z}\mu)\, d\mu(z)\, dQ(\mu) \qquad \text{for all } G \in \mathcal{B}(\mathcal{M}_{\mathbb{R}^n}),$$

is independent of the choice of $A \in \mathcal{B}(\mathbb{R}^n)$ as long as $0 < \lambda_n(A) < \infty$ and it is called the *Palm measure* of Q; see [51, page 85]. Note that $Q^0 \ll Q$ by (2.1), i.e. any Q-nullset is also a Q^0-nullset. A random measure ξ is called *Palm distributed* if there exists a stationary quasi-distribution Q with Palm measure Q^0 such that

$$(2.2)\qquad P_\xi = c_Q^{-1} \cdot Q^0,$$

where $c_Q = Q^0(\mathcal{M}_{\mathbb{R}^n}) \in (0, \infty)$ is the intensity constant of Q.

Let $f : \mathbb{R}^d \to \mathbb{R}^m$ be a Borel-measurable function. Then the *occupation measure* of f is a Borel measure τ_f on $\mathbb{R}^d \times \mathbb{R}^m$ uniquely defined by

$$\tau_f(A \times B) = \lambda_d\{t \in A : f(t) \in B\}$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{R}^m)$. Note that τ_f is concentrated on the graph of f and, to be more precise, we may also say that τ_f is the occupation measure of the graph. For any (U, V)-self-affine random field $\{X(t)\}_{t\in\mathbb{R}^d}$ on $\mathbb{R}^m$ we get by the transformation rule

$$\begin{aligned}
\tau_X(A \times B) &= \int_A \mathbb{1}_B(X(t))\, d\lambda_d(t) = \det U \cdot \int_{U^{-1}(A)} \mathbb{1}_B(X(Ut))\, d\lambda_d(t) \\
&\overset{\mathrm{d}}{=} \det U \cdot \int_{U^{-1}(A)} \mathbb{1}_B(VX(t))\, d\lambda_d(t) \\
&= \det U \cdot \int_{U^{-1}(A)} \mathbb{1}_{V^{-1}(B)}(X(t))\, d\lambda_d(t) \\
&= \det U \cdot \tau_X(U^{-1}(A) \times V^{-1}(B)) \\
&= \det U \cdot \tau_X((U \oplus V)^{-1}(A \times B)),
\end{aligned}$$

where $U \oplus V \in \mathbb{R}^{(d+m)\times(d+m)}$ denotes the block-diagonal matrix. Hence the occupation measure τ_X defines a $(U \oplus V, \det U)$-self-affine random measure. Moreover, it is Palm distributed by the following Lemma due to U. Zähle [53].

LEMMA 2.1. *[53, Proposition 5.2] Let $\{X(t)\}_{t\in\mathbb{R}^d}$ be a (U, V)-self-affine random field on $\mathbb{R}^m$ as defined in Section 1. Denote by τ_X its occupation measure on $\mathbb{R}^d \times \mathbb{R}^m$. Then τ_X is a Palm distributed and $(U \oplus V, \det U)$-self-affine random measure.*

2.2. Carrying dimension.

We now introduce the notion of carrying dimension defined in [51] and recall a result from [53] on how the carrying dimension of the occupation measure of self-affine random fields, under certain regularity assumptions, can be explicitely calculated.

Definition 2.1. Let $\mu \in \mathcal{M}_{\mathbb{R}^n}$ be a Borel measure on $\mathbb{R}^n$. We say that μ has *carrying dimension* $\mathfrak{d} \in [0, n]$, in symbols $\mathfrak{d} = \operatorname{cardim} \mu$, if the following two conditions are satisfied.

(i) $\mu(A) > 0$ implies $\dim_{\mathcal{H}} A \geqslant \mathfrak{d}$ for all $A \in \mathcal{B}(\mathbb{R}^n)$.

(ii) There exists a set $B \in \mathcal{B}(\mathbb{R}^n)$ with $\mu(\mathbb{R}^n \setminus B) = 0$ and $\dim_{\mathcal{H}} B \leqslant \mathfrak{d}$.

Note that this definition is closely related to the lower and upper Hausdorff dimension of the Borel measure μ given by

$$\begin{aligned}\dim_* \mu &= \inf\{\dim_{\mathcal{H}} A : A \in \mathcal{B}(\mathbb{R}^n),\ \mu(A) > 0\},\\ \dim^* \mu &= \inf\{\dim_{\mathcal{H}} B : B \in \mathcal{B}(\mathbb{R}^n),\ \mu(\mathbb{R}^n \setminus B) = 0\},\end{aligned}$$

and discussed in [9], [24], [15]. Obviously, the carrying dimension of μ exists if and only if $\dim_* \mu = \dim^* \mu$ and in this case these values are indeed the same.

The following Lemma of U. Zähle [52] is useful to derive a lower bound of the carrying dimension. Its proof can be found in [50, Theorem 1.4].

LEMMA 2.2. *[52, Lemma 2.1] Let* $\mu \in \mathcal{M}_{\mathbb{R}^n}$, $\gamma \geqslant 0$ *and* $B \in \mathcal{B}(\mathbb{R}^n)$. *Suppose that*

$$\int_{\{\|z-x\|<1\}} \|z - x\|^{-\gamma}\, \mu(dz) < \infty$$

for μ*-almost all* $x \in B$. *Then* $\mu(B' \cap B) > 0$ *implies* $\dim_{\mathcal{H}} B' \geqslant \gamma$ *for all Borel sets* $B' \in \mathcal{B}(\mathbb{R}^n)$ *and, consequently,* $\operatorname{cardim} \mu \geqslant \gamma$.

For an explicit calculation of the carrying dimension of the occupation measure τ_X of a (U, V)- self-affine random field $\{X(t)\}_{t \in \mathbb{R}^d}$ the following condition, called the *boundedly continuous intensity* (b.c.i.) condition in [53], is crucial and sufficient.

Definition 2.2. Let ξ be a (W, x)-self-affine random measure. Then ξ is said to satisfy the b.c.i. condition (with respect to W) if there exists a constant $C \in (0, \infty)$, not depending on W, such that

$$\mathbb{E}[\xi](A) \leqslant C \cdot \lambda_{d+m}(A) \qquad \text{for all Borel sets } A \subset [-1, 1]^{d+m} \setminus W([-1, 1]^{d+m}).$$

An easy sufficient condition for the occupation measure to fulfill the b.c.i. condition is the following; cf. also [53, Chapter 11].

LEMMA 2.3. *Suppose that for all $t \in \mathbb{R}^d \setminus \{0\}$ the distribution of $X(t)$ has a density $x \mapsto p_t(x)$ with respect to λ_m, then for $z = (t, x) \in \mathbb{R}^d \times \mathbb{R}^m$ we have*

$$d\mathbb{E}[\tau_X](z) = p_t(x)\, d\lambda_d(t)\, d\lambda_m(x).$$

Furthermore, if there exists a constant $0 < C < \infty$ such that $p_t(x) \leqslant C$ for all $(t, x) \in [-1, 1]^{d+m} \setminus W([-1, 1]^{d+m})$ it follows immediately that the b.c.i. condition is fulfilled.

P r o o f. For any $A \in \mathcal{B}(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{R}^m)$ by Tonelli's theorem we get

$$\begin{aligned} \mathbb{E}[\tau_X](A \times B) &= \int_\Omega \int_A \mathbb{1}_B(X(t))\, d\lambda_d(t)\, dP \\ &= \int_A P\{X(t) \in B\}\, d\lambda_d(t) = \int_{A\times B} dP_{X(t)}(x)\, d\lambda_d(t) \end{aligned} \tag{2.3}$$

and in case $dP_{X(t)}(x) = p_t(x)\, d\lambda_m(x)$ the assertion follows. ∎

Zähle [53] showed that under the b.c.i. condition there is a close relation between the carrying dimension of occupation measures and the singular value function, which is frequently used as a tool in the study of the Hausdorff dimension of self-affine fractals; e.g., see [18], [20], [21]. Following [18], let us briefly introduce the singular value function ϕ_W of a contracting, non-singular matrix $W \in \mathbb{R}^{n\times n}$. Let $1 > \alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n > 0$ denote the singular values of W, i.e. the positive square roots of the eigenvalues of $W^\top W$, where $W^\top$ denotes the transpose of W. Then the singular value function $\phi_W : (0, n] \to (0, \infty)$ of W is given by

$$\phi_W(s) = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{m-1} \cdot \alpha_m^{s-m+1}, \tag{2.4}$$

where m is the unique integer such that $m - 1 < s \leqslant m$.

LEMMA 2.4. *[18, Proposition 4.1] Let $W \in \mathbb{R}^{n\times n}$ be a contracting and non-singular matrix, $0 < x < 1$ and ϕ_W the singular value function of W. Then there exists a unique number $s = s(W, x) > 0$ given by $x^{-1}(\phi_{W^k}(s))^{\frac{1}{k}} \to 1$ as $k \to \infty$. Moreover, it holds that*

$$\begin{aligned} s &= \inf \left\{ r \in (0, n] : \lim_{k\to\infty} x^{-k} \phi_{W^k}(r) = 0 \right\} \\ &= \sup \left\{ r \in (0, n] : \lim_{k\to\infty} x^{-k} \phi_{W^k}(r) = \infty \right\} \end{aligned}$$

with the convention that $\inf \emptyset = n$.

The following result of U. Zähle [53] will be important for our approach and states that under the b.c.i. condition the carrying dimension of the occupation measure of any self-affine random field can be calculated in terms of the singular value function.

THEOREM 2.3. *[53, Theorem 5.3] Let $X = \{X(t)\}_{t\in\mathbb{R}^d}$ be a (U, V)-self-affine random field on $\mathbb{R}^m$ and τ_X its occupation measure. If τ_X satisfies the b.c.i. condition with respect to $W = U \oplus V$ then with probability one*

$$\operatorname{cardim} \tau_X = s(W, \det U),$$

where $s(W, \det U)$ is the unique number given by Lemma 2.4.

3. MAIN RESULT

Throughout this section, let $X = \{X(t)\}_{t\in\mathbb{R}^d}$ be a (U, V)-self-affine random field on $\mathbb{R}^m$ as introduced in Section 1 and denote by τ_X its occupation measure for the graph. Moreover, denote by

$$\operatorname{Gr} X([0,1]^d) = \{(t, X(t)) : t \in [0,1]^d\} \subset \mathbb{R}^{d+m}$$

the graph of X on the unit cube. We now show our main result that, under a natural additional assumption, the carrying dimension of τ_X coincides with the Hausdorff dimension of the graph of X.

THEOREM 3.1. *Assume that*

$$\int_{[-1,1]^d} \mathbb{E}\big[\big(\|t\| + \|X(t)\|\big)^{-\gamma}\big]\, d\lambda_d(t) < \infty \tag{3.1}$$

for all $\gamma < \dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d)$. Then with probability one the carrying dimension of τ_X exists and we have

$$\operatorname{cardim} \tau_X = \dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d).$$

REMARK 3.1. *Note that by the definition of the carrying dimension, the upper bound* $\operatorname{cardim} \tau_X \leqslant \dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d)$ *almost surely is immediate. Thus we only need to proof the lower bound. Further note that by stationarity of the increments,* (3.1) *is equivalent to*

$$\int_{[0,1]^d\times[0,1]^d} \mathbb{E}\big[\big(\|t-s\| + \|X(t) - X(s)\|\big)^{-\gamma}\big]\, d\lambda_d(t)\, d\lambda_d(s) < \infty.$$

The above integral is the expected value of the γ-energy of τ_X, usually denoted $I_\gamma(\tau_X)$. In case $\gamma < \dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d)$, an application of Frostman's Lemma (e.g., see Theorem 8.8 in [37]), known as the potential theoretic method laid out in Section 4.2 of [19] to derive lower bounds of the Hausdorff dimension, shows the almost sure existence of a (random) probability distribution μ on $\operatorname{Gr} X([0,1]^d)$ *such that $I_\gamma(\mu) < \infty$. However, in general one does not have the information that $\mu = \tau_X$, although $\mu = \tau_X$ is the canonical candidate for the derivation of a lower bound.*

P r o o f. [Proof of Theorem 3.1] As remarked above, we only need to prove the lower bound

$$\operatorname{cardim}\tau_X \geqslant \dim_{\mathcal H}\operatorname{Gr}X([0,1]^d) \quad \text{almost surely.} \tag{3.2}$$

By Lemma 2.1, τ_X is Palm distributed and thus there exists a stationary quasi-distribution Q with intensity constant c_Q and Palm measure Q^0 given by (2.1) such that $P_{\tau_X} = c_Q^{-1}Q^0$ as in (2.2). Moreover, to prove (3.2), by Lemma 2.2 (applied to $B = \operatorname{Gr}X([0,1]^d)$) it suffices to show that for all $\gamma < \dim_{\mathcal H}\operatorname{Gr}X([0,1]^d)$ we have

$$\int_{\{\|z-x\|<1\}} \|z-x\|^{-\gamma}\, d\mu(z) < \infty \tag{3.3}$$

for P_{τ_X}-almost all μ and μ-almost all $x \in \operatorname{Gr}X([0,1]^d)$. If we can show that

$$\mathbb{E}\Big[\int_{\{\|z\|<1\}} \|z\|^{-\gamma}\, d\tau_X(z)\Big] < \infty \tag{3.4}$$

then for Q^0-almost all μ we have

$$\int_{\{\|z\|<1\}} \|z\|^{-\gamma}\, d\mu(z) < \infty,$$

which by [51, Lemma 3.3] is equivalent to

$$\int_{\{\|z-x\|<1\}} \|z-x\|^{-\gamma}\, d\mu(z) < \infty$$

for Q-almost all μ and μ-almost all $x \in \operatorname{Gr}X([0,1]^d)$. Since $Q^0 \ll Q$ by (2.1) and thus $P_{\tau_X} \ll Q$ by (2.2), it follows that (3.3) holds for P_{τ_X}-almost all μ and μ-almost all $x \in \operatorname{Gr}X([0,1]^d)$. Thus it suffices to show (3.4). By (2.3) and our assumption (3.1) we get for some constant $0 < K < \infty$

$$\begin{aligned}
\mathbb{E}\Big[\int_{\{\|z\|<1\}} \|z\|^{-\gamma}\, d\tau_X(z)\Big] &= \int_{\{\|z\|<1\}} \|z\|^{-\gamma}\, d\mathbb{E}[\tau_X](z)\\
&= \int_{\{\|(t,x)\|<1\}} \|(t,x)\|^{-\gamma}\, dP_{X(t)}(x)\, d\lambda_d(t)\\
&\leqslant K \int_{[-1,1]^d}\int_{\mathbb{R}^m} (\|t\|+\|x\|)^{-\gamma}\, dP_{X(t)}(x)\, d\lambda_d(t)\\
&= K \int_{[-1,1]^d} \mathbb{E}\big[\big(\|t\|+\|X(t)\|\big)^{-\gamma}\big]\, d\lambda_d(t) < \infty,
\end{aligned}$$

which shows (3.4) and concludes the proof. ∎

Combining Theorem 3.1 with Theorem 2.3 we immediately get the following result.

COROLLARY 3.2. Assume (3.1) is fulfilled and τ_X satisfies the b.c.i. condition. Then with probability one

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) = s(W, \det U),$$

where $W = U \oplus V$ and $s(W, \det U)$ is the unique number given by Lemma 2.4.

REMARK 3.2. *Note that to be able to check condition* (3.1) *one needs information on the precise value of* $\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d)$. *If one strives to calculate this value by using general energy type arguments, the only result one can expect in case of exponential matrices is precisely the value given in formula* (3.7) *below.*

In case the contracting, non-singular operators U and V are given by exponential matrices $U = c^E$ and $V = c^D$ for some $c \in (0,1)$ and some matrices $E \in \mathbb{R}^{d\times d}$ and $D \in \mathbb{R}^{m\times m}$ with positive real parts of their eigenvalues, we are able to calculate $s(U \oplus V, \det U)$ of Lemma 2.4 explicitly in terms of the real parts of the eigenvalues of E and D as follows.

EXAMPLE 3.1. Let $0 < a_1 \leqslant \cdots \leqslant a_d$ and $0 < b_1 \leqslant \cdots \leqslant b_m$ denote the real parts of the eigenvalues of E, respectively D. Write $0 < \gamma_1 \leqslant \cdots \leqslant \gamma_{d+m}$ for the union of all these quantities in a common order. Then we have

$$\det U = \det c^E = c^Q,$$

where $Q = \operatorname{trace}(E) = a_1 + \cdots + a_d$. For the block-diagonal $W = c^E \oplus c^D = c^{E\oplus D}$ we obtain that the positive square roots of the eigenvalues of the symmetric matrices $(W^n)^\top W^n$ asymptotically behave as $c^{n\gamma_j}$ for $j = 1, \dots, d+m$. More precisely, for all $\varepsilon > 0$ the j-th smallest square root $\eta_n(j)$ of the eigenvalues of $(W^n)^\top W^n$ fulfills

$$c^{n(\gamma_j+\varepsilon)} \leqslant \eta_n(j) \leqslant c^{n(\gamma_j-\varepsilon)} \tag{3.5}$$

for all $j = 1, \dots, d+m$ and $n \in \mathbb{N}$ large enough; e.g. see section 2.2 in [38] for details. Now let $r \in \{1, \dots, d+m\}$ be the unique integer such that

$$\sum_{j=1}^{r-1} \gamma_j < Q \leqslant \sum_{j=1}^{r} \gamma_j \tag{3.6}$$

then by (3.5) for all $r-1 < s \leqslant r$ the singular value function $\phi_{W^n}(s)$ in the above sense asymptotically behaves as

$$c^{n\gamma_1} \cdots c^{n\gamma_{r-1}} c^{n\gamma_r(s-r+1)}$$

and a comparison with $(\det U)^n = c^{nQ}$ together with (3.6) readily shows that

$$s(c^{E\oplus D}, c^Q) = r - 1 + \frac{1}{\gamma_r}\big(Q - \sum_{j=1}^{r-1}\gamma_j\big). \tag{3.7}$$

Note that if $Q = \sum_{j=1}^r \gamma_j$ then $s(W, \det U) = r$ which shows $s(W, \det U) \geqslant d$. Note further that the right-hand side of (3.7) is independent of $c \in (0, 1)$ and only depends on the real parts of the eigenvalues of the scaling exponents E and D.

We now turn to the range $X([0, 1]^d) = \{X(t) : t \in [0, 1]^d\}$ of the self-affine random field X. There is an approach analogous to Theorem 3.1 for the graph, combining the Hausdorff dimension of the range with the carrying dimension of random measures supported on $X([0, 1]^d)$ by using the results in [53, Chapter 10], which hint that it seems quite natural that the Hausdorff dimension of the range of the self-affine random field is connected to $s(V, \det U)$. In case $U = c^E$ and $V = c^D$ are given by exponential matrices for some $c \in (0, 1)$ as above, we will now show that $s(V, \det U) = s(c^D, c^Q)$ with $Q = \operatorname{trace}(E)$ always serves as a lower bound for $\dim_{\mathcal{H}} X([0, 1]^d)$ with probability one, provided that the b.c.i. condition for the occupation measure τ_X of the graph is fulfilled. We will first calculate $s(c^D, c^Q)$ explicitly in terms of the real parts of the eigenvalues of E and D.

EXAMPLE 3.2. Let $0 < a_1 \leqslant \cdots \leqslant a_d$ denote the real parts of the eigenvalues of E and let $0 < b_1 < \cdots < b_p$ be the distinct real parts of the eigenvalues of D with multiplicities $m_1, \ldots, m_p$. Then $Q = \operatorname{trace}(E) = \sum_{k=1}^d a_k$ and we distinguish between the following two cases.

Case 1: If for some $\ell \in \{1, \ldots, p\}$ we have

$$\sum_{i=1}^{\ell-1} b_i m_i < Q \leqslant \sum_{i=1}^{\ell} b_i m_i \tag{3.8}$$

then there exists $R \in \{1, \ldots, m_\ell\}$ such that

$$\sum_{i=1}^{\ell-1} b_i m_i + b_\ell(R-1) < Q \leqslant \sum_{i=1}^{\ell-1} b_i m_i + b_\ell R \tag{3.9}$$

so that for $W = c^D$ we have $r = \sum_{i=1}^{\ell-1} m_i + R$ in (3.6). From (3.7) it follows that

$$\begin{aligned} s(c^D, c^Q) &= \sum_{i=1}^{\ell-1} m_i + R - 1 + \frac{1}{b_\ell}\big(Q - \sum_{i=1}^{\ell-1} b_i m_i - b_\ell(R-1)\big) \\ &= \frac{1}{b_\ell}\big(Q + \sum_{i=1}^{\ell}(b_\ell - b_i) m_i\big). \end{aligned} \tag{3.10}$$

Combining (3.10) with the second inequality in (3.8) we see that

$$s(c^D, c^Q) \leqslant \sum_{i=1}^{\ell} m_i \leqslant m. \tag{3.11}$$

Case 2: If $Q > \sum_{i=1}^p b_i m_i$ we choose $\varepsilon > 0$ such that $Q > \sum_{i=1}^p (b_i + \varepsilon) m_i$ then by (3.5) we get as $n \to \infty$

$$c^{-nQ} \phi_{c^{nD}}(m) \geqslant c^{-n(Q - \sum_{i=1}^{p} (b_i + \varepsilon) m_i)} \to \infty,$$

showing that $s(c^D, c^Q) = m$.

Altogether, we have shown that irrespectively of $c \in (0,1)$ we have

$$s(c^D, c^Q) = \begin{cases} \frac{1}{b_\ell} \big(Q + \sum_{i=1}^{\ell} (b_\ell - b_i) m_i \big) & \text{, if } \sum_{i=1}^{\ell-1} b_i m_i < Q \leqslant \sum_{i=1}^{\ell} b_i m_i \\ m & \text{, else.} \end{cases} \tag{3.12}$$

THEOREM 3.3. *Let $X = \{X(t)\}_{t \in \mathbb{R}^d}$ be a (c^E, c^D)-self-affine random field on $\mathbb{R}^m$ for some $c \in (0,1)$ such that its occupation measure τ_X of the graph satisfies the b.c.i. condition. Then with probability one we have*

$$\dim_{\mathcal{H}} X([0,1]^d) \geqslant s(c^D, c^Q),$$

where $Q = \operatorname{trace}(E)$ and $s(c^D, c^Q)$ is given by (3.12).

Proof. As described in Remark 3.1, by Frostman's Lemma (e.g. see [19], [37]) it suffices to show that for all $\gamma < s(c^D, c^Q)$ we have

$$\int_{[0,1]^d \times [0,1]^d} \mathbb{E}\big[\|X(t) - X(s)\|^{-\gamma} \big] \, d\lambda_d(t) \, d\lambda_d(s) < \infty. \tag{3.13}$$

Let $0 < b_1 < \cdots < b_p$ be the distinct real parts of the eigenvalues of D with multiplicities $m_1, \ldots, m_p$. We will use the spectral decomposition with respect to D as laid out in [38]. According to this, in an appropriate basis of $\mathbb{R}^m$, we can decompose $\mathbb{R}^m = V_1 \oplus \cdots \oplus V_p$ into mutually orthogonal subspaces V_i of dimension m_i such that each V_i is D_i-invariant, where the real part of any eigenvalue of D_i is equal to b_i and $D = D_1 \oplus \cdots \oplus D_p$ is block-diagonal. With respect to this spectral decomposition we may write $x \in \mathbb{R}^m$ as $x = x_1 + \cdots + x_p$ with $x_i \in V_i \simeq \mathbb{R}^{m_i}$ and we have $\|x\|^2 = \|x_1\|^2 + \cdots + \|x_p\|^2$ in the associated euclidean norms.

Now let $A = [-1,1]^{d+m} \setminus c^{E \oplus D}([-1,1]^{d+m})$ then

$$\bigcup_{j=0}^{\infty} c^{j(E \oplus D)}(A) = [-1,1]^{d+m} \setminus \{0\}$$

is a disjoint covering. Using a change of variables $(x,t)=(c^{jD}y,c^{jE}s)$ together with the self-affinity of the random field and the b.c.i. condition, we get for some unspecified constant $K>0$ and every $\ell\in\{1,\dots,p\}$

$$\begin{aligned}
&\int\limits_{[0,1]^d\times[0,1]^d} \mathbb{E}\big[\|X(t)-X(s)\|^{-\gamma}\big]\,d\lambda_d(t)\,d\lambda_d(s)\\
&\leqslant \int\limits_{[-1,1]^d} \mathbb{E}\big[\|X(t)\|^{-\gamma}\big]\,d\lambda_d(t)\\
&\leqslant \int\limits_{[-1,1]^d}\int\limits_{[-1,1]^m} \|x\|^{-\gamma}\,dP_{X(t)}(x)\,d\lambda_d(t)+2^d\\
&=\sum_{j=0}^{\infty}\int\limits_{c^{j(E\oplus D)}(A)} \|x\|^{-\gamma}\,dP_{X(t)}(x)\,d\lambda_d(t)+2^d\\
&=\sum_{j=0}^{\infty} c^{jQ}\int\limits_{A} \|c^{jD}y\|^{-\gamma}\,dP_{X(s)}(y)\,d\lambda_d(s)+2^d\\
&=\sum_{j=0}^{\infty} c^{jQ}\int\limits_{A} \|c^{jD}y\|^{-\gamma}\,d\mathbb{E}[\tau_X](s,y)+2^d\\
&\leqslant K\sum_{j=0}^{\infty} c^{jQ}\int\limits_{A} \|c^{jD}y\|^{-\gamma}\,d\lambda_{d+m}(s,y)+2^d\\
&\leqslant K\sum_{j=0}^{\infty} c^{jQ}\int\limits_{[-1,1]^{m_1}}\cdots\int\limits_{[-1,1]^{m_\ell}} \frac{d\lambda_{m_1}(x_1)\cdots d\lambda_{m_\ell}(x_\ell)}{\big(\sum_{i=1}^{\ell}\|c^{jD_i}x_i\|\big)^{\gamma}}+2^d.
\end{aligned}$$

By Theorem 2.2.4 in [38], for all $\varepsilon>0$ there exists $K_1>0$ such that for $j\in\mathbb{N}_0$ and every $i=1,\dots,\ell$ we have $\|c^{jD_i}x_i\|\geqslant\|c^{-jD_i}\|^{-1}\|x_i\|\geqslant K_1c^{j(b_i+\varepsilon)}\|x_i\|$ and hence by change of variables $y_i=c^{j(b_i-b_\ell)}x_i$ we get

$$\begin{aligned}
&\int\limits_{[-1,1]^{m_1}}\cdots\int\limits_{[-1,1]^{m_\ell}} \frac{d\lambda_{m_1}(x_1)\cdots d\lambda_{m_\ell}(x_\ell)}{\big(\sum_{i=1}^{\ell}\|c^{jD_i}x_i\|\big)^{\gamma}}\\
&\leqslant K\int\limits_{[-1,1]^{m_1}}\cdots\int\limits_{[-1,1]^{m_\ell}} \frac{d\lambda_{m_1}(x_1)\cdots d\lambda_{m_\ell}(x_\ell)}{\big(\sum_{i=1}^{\ell}c^{j(b_i+\varepsilon)}\|x_i\|\big)^{\gamma}}\\
&\leqslant K\,c^{-j\gamma(b_\ell+\varepsilon)}\int\limits_{[-1,1]^{m_1}}\cdots\int\limits_{[-1,1]^{m_\ell}} \frac{d\lambda_{m_1}(x_1)\cdots d\lambda_{m_\ell}(x_\ell)}{\big(\sum_{i=1}^{\ell}c^{j(b_i-b_\ell)}\|x_i\|\big)^{\gamma}}\\
&\leqslant K\,c^{-j\left(\gamma(b_\ell+\varepsilon)+\sum_{i=1}^{\ell}(b_i-b_\ell)m_i\right)}\int\limits_{[-1,1]^{\widetilde{m}}}\|y\|^{-\gamma}\,d\lambda_{\widetilde{m}}(y),
\end{aligned}$$

where $\widetilde{m}=m_1+\ldots+m_\ell$ and $y=y_1+\cdots+y_\ell$ with respect to the spectral decomposition of $\widetilde{D} = D_1 \oplus \cdots \oplus D_\ell$. We now distinguish between the two cases considered in Example 3.2.

Case 1: Assume that $\sum_{i=1}^{\ell-1} b_i m_i < Q \leqslant \sum_{i=1}^{\ell} b_i m_i$. For sufficiently small $\varepsilon > 0$ we have $\gamma(b_\ell + \varepsilon) < s(c^D, c^Q) b_\ell$. It suffices to consider large values of $\gamma < s(c^D, c^Q)$ so that combining (3.9) and (3.10) we may assume

$$\sum_{i=1}^{\ell-1} m_i + R - 1 < \gamma \leqslant \sum_{i=1}^{\ell-1} m_i + R$$

for some $R \in \{1, \ldots, m_\ell\}$. Hence for the singular value function by (2.4) and (3.5) we have for sufficiently large $j \in \mathbb{N}$

$$\begin{aligned}\phi_{c^D}(\gamma) &\geqslant c^{j \sum_{i=1}^{\ell-1} (b_i+\varepsilon) m_i} c^{j(b_\ell+\varepsilon)(R-1)} c^{j(b_\ell+\varepsilon)(\gamma - \sum_{i=1}^{\ell-1} m_i - r + 1)} \\ &= c^{j\left((b_\ell+\varepsilon)\gamma + \sum_{i=1}^{\ell-1} (b_i - b_\ell) m_i\right)}.\end{aligned}$$

Note that by (3.11) we have $\gamma < s(c^D, c^Q) \leqslant \widetilde{m}$ and thus

$$\int_{[-1,1]^{\widetilde{m}}} \|y\|^{-\gamma} \, d\lambda_{\widetilde{m}}(y) < \infty.$$

By Lemma 2.4 we further get as $j \to \infty$

$$c^{jQ} c^{-j\left(\gamma(b_\ell+\varepsilon) + \sum_{i=1}^{\ell} (b_i - b_\ell) m_i\right)} \leqslant c^{jQ} \phi_{c^D}^{-1}(\gamma) \to 0,$$

which shows that

$$\sum_{j=0}^{\infty} c^{jQ} c^{-j\left(\gamma(b_\ell+\varepsilon) + \sum_{i=1}^{\ell} (b_i - b_\ell) m_i\right)} \int_{[-1,1]^{\widetilde{m}}} \|y\|^{-\gamma} \, d\lambda_{\widetilde{m}}(y) < \infty.$$

Case 2: Assume that $\sum_{i=1}^{p} b_i m_i < q$ then $s(c^D, c^Q) = m$ and we choose $\ell = p$. For sufficiently small $\varepsilon > 0$ we have $\gamma(b_p + \varepsilon) < s(c^D, c^Q) b_p$. It suffices to consider large values of $\gamma < s(c^D, c^Q)$ so that we may assume

$$\sum_{i=1}^{p-1} m_i + m_p - 1 < \gamma \leqslant \sum_{i=1}^{p-1} m_i + m_p = m = s(c^D, c^Q).$$

Hence for the singular value function by (2.4) and (3.5) we have

$$\begin{aligned}\phi_{c^D}(\gamma) &\geqslant c^{j \sum_{i=1}^{p-1} (b_i+\varepsilon) m_i} c^{j(b_p+\varepsilon)(m_p-1)} c^{j(b_p+\varepsilon)(\gamma - \sum_{i=1}^{p-1} m_i - m_p + 1)} \\ &= c^{j\left((b_p+\varepsilon)\gamma + \sum_{i=1}^{p-1} (b_i - b_p) m_i\right)}\end{aligned}$$

for sufficiently large $j \in \mathbb{N}$. Note that for $\ell = p$ we have $\gamma < s(c^D, c^Q) = m = \widetilde{m}$ and thus $\int_{[-1,1]^m} \|y\|^{-\gamma}\, d\lambda_m(y) < \infty$. By Lemma 2.4 we further get as $j \to \infty$

$$c^{jQ} c^{-j\left(\gamma(b_p+\varepsilon)+\sum\limits_{i=1}^{p}(b_i-b_p)m_i\right)} \leqslant c^{jQ}\phi_{c^D}^{-1}(\gamma) \to 0,$$

which shows that

$$\sum_{j=0}^{\infty} c^{jQ} c^{-j\left(\gamma(b_p+\varepsilon)+\sum\limits_{i=1}^{p}(b_i-b_p)m_i\right)} \int\limits_{[-1,1]^m} \|y\|^{-\gamma}\, d\lambda_m(y) < \infty.$$

Putting things together, we get (3.13) in both cases, concluding the proof. ∎

In a special situation we are able to get an analogue of Theorem 3.1 for the range.

COROLLARY 3.4. If in addition to the assumptions of Theorems 3.1 and 3.3 we have $b_p \leqslant a_1$, where b_p and a_1 are as in Example 3.2, then with probability one

$$\dim_{\mathcal{H}} X([0,1]^d) = s(c^D, c^Q).$$

Proof. By Theorem 3.3 $s(c^D, c^Q)$ is a lower bound for $\dim_{\mathcal{H}} X([0,1]^d)$ almost surely. If $s(c^D, c^Q) = m$ in (3.12) there is nothing to prove. Otherwise, if $\sum_{i=1}^{\ell-1} b_i m_i < Q \leqslant \sum_{i=1}^{\ell} b_i m_i$, a comparison of (3.12) with (3.7) together with the assumption $b_p \leqslant a_1$ directly shows that $s(c^{E\oplus D}, c^Q) = s(c^D, c^Q)$. Thus by Theorems 3.1 and 2.3 we get the upper bound

$$\dim_{\mathcal{H}} X([0,1]^d) \leqslant \dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d) = s(c^{E\oplus D}, c^Q) = s(c^D, c^Q)$$

almost surely, since we assumed (3.1) and the b.c.i. condition. ∎

4. EXAMPLES

To demonstrate the applicability of our main result, we give examples of large classes of self-affine random fields for which (3.1) holds and the precise values of the Hausdorff dimension of the graph and the range are already known.

4.1. Operator-self-similar stable random fields.

Let $E \in \mathbb{R}^{d\times d}$ and $D \in \mathbb{R}^{m\times m}$ be matrices and assume that the eigenvalues of E and D have positive real part. A random field $\{X(t)\}_{t\in\mathbb{R}^d}$ with values in $\mathbb{R}^m$ is said to be (E, D)-operator-self-similar if

$$\{X(c^E t)\}_{t\in\mathbb{R}^d} \overset{\text{f.d.}}{=} \{c^D X(t)\}_{t\in\mathbb{R}^d} \quad \text{for all } c > 0.$$

These fields have been introduced in [33] as a generalization of both operator scaling random fields [4] and operator-self-similar processes [25], [30]. Moreover, for

$d = m = 1$ one obtains the well-known class of self-similar processes. We say that a random field $\{X(t) : t \in \mathbb{R}^d\}$ is symmetric α-stable (SαS) for some $\alpha \in (0, 2]$ if any linear combination $\sum_{k=1}^{m} \xi_k X(t_k)$ is a symmteric α-stable random vector. In [33, Theorem 2.6] it is shown that a proper, stochastically continuous (E, D)-operator-self-similar SαS random field X with stationary increments can be given by a harmonizable representation, provided that $0 < b_1 \leqslant \ldots \leqslant b_m < 1$ for the real parts of the eigenvalues of D and $1 < a_1 < \ldots < a_q$ for the distinct real parts of the eigenvalues of E. This includes the case of operator fractional Brownian motions studied in [36], [13], [14] and operator scaling stable random fields [4], where corresponding Hausdorff dimension results already were already established in [36], [4], [5]. Note that, since the real parts of the eigenvalues of E and D are assumed to be positive, the matrices c^E and c^D are contracting for all $0 < c < 1$. In particular X is a (c^E, c^D)-self-affine random field for all $0 < c < 1$, since its sample functions are continuous.

We now argue that the occupation measure τ_X satisfies the b.c.i. condition with respect to $W = c^E \oplus c^D = c^{E \oplus D}$. Recall that any symmetric α-stable random variable has a smooth and bounded probability density (see [43]) so that the density $y \mapsto p_t(y)$ of $X(t)$ exists for all $t \neq 0$ and the mapping $(t, y) \mapsto p_t(y)$ is continuous due to stochastic continuity of the field X. By Lemma 2.3 we only have to prove that there is a constant $K > 0$, not depending on t and y, such that $p_t(y) \leqslant K$ for all $(t, y) \in [-1, 1]^{d+m} \setminus W \cdot [-1, 1]^{d+m}$. In order to show this, we will use generalized polar coordinates with respect to E, initially introduced in [4]. For any $t \in \mathbb{R}^d \setminus \{0\}$ one can uniquely write $t = \rho_E(t)^E l_E(t)$ with E-homogeneous radius $\rho_E(t) > 0$ and direction vector $l_E(t) \in S_E = \{t \in \mathbb{R}^d : \rho_E(t) = 1\}$. Note that S_E is compact and does not contain 0. The operator self-similarity implies

$$p_t(y) = (\det c^D)^{-1} p_{c^{-E}t}(c^{-D}y) \quad \text{for all } c > 0,\ t \in \mathbb{R}^d \setminus \{0\},\ y \in \mathbb{R}^m$$

and for $(t, y) \in [0, 1]^{d+m} \setminus W \cdot [0, 1]^{d+m}$ we get

$$\begin{aligned} p_t(y) &= p_{\rho_E(t)^E l_E(t)}(y) = \big(\det \rho_E(t)\big)^{-D} p_{\rho_E(t)^{-E}\rho_E(t)^E l_E(t)}\big(\rho_E(t)^{-D}y\big) \\ &= \rho_E(t)^{-\operatorname{trace}(D)} p_{l_E(t)}\big(\rho_E(t)^{-D}y\big) \\ &\leqslant K_1 \cdot \max_{\theta \in S_E} \sup_{y \in \mathbb{R}^m} p_\theta(y) \leqslant K, \end{aligned}$$

where $K_1, K > 0$ are constants independent of t and y. Hence, the b.c.i. condition holds and Theorem 2.3 allows to compute the carrying dimension of τ_X as

$$\operatorname{cardim} \tau_X = s(W, c^Q) \quad \text{almost surely for all } 0 < c < 1,$$

where $W = c^{E \oplus D}$ and $Q = \operatorname{trace}(E)$.

The Hausdorff dimension of the graph of X has been computed in [45, Theorem 4.1] for $\alpha = 2$ and [46, Theorem 5.1] for $\alpha \in (0, 2)$, where the lower bound in the computation is proven through (3.1). Indeed it is shown that with probability one $\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]^d)$ coincides with

(4.1)
$$\begin{cases} b_\ell^{-1}\big(\sum\limits_{k=1}^{q} a_k d_k + \sum\limits_{i=1}^{\ell}(b_\ell - b_i)\big) & \text{if } \sum\limits_{i=1}^{\ell-1} b_i < \sum\limits_{k=1}^{q} a_k d_k \leqslant \sum\limits_{i=1}^{\ell} b_i, \\ \sum\limits_{j=1}^{\ell} \frac{\tilde a_j}{\tilde a_\ell}\tilde d_j + \sum\limits_{j=\ell+1}^{q} \tilde d_j + \sum\limits_{i=1}^{m}\big(1 - \frac{b_i}{\tilde a_\ell}\big) & \text{if } \sum\limits_{k=1}^{\ell-1} \tilde a_k \tilde d_k \leqslant \sum\limits_{i=1}^{m} b_i < \sum\limits_{k=1}^{\ell} \tilde a_k \tilde d_k, \end{cases}$$

where $d_1, \ldots, d_q$ denote the multiplicities of $a_1, \ldots, a_q$ respectively, $\tilde a_j = a_{q+1-j}$ and $\tilde d_j = d_{q+1-j}$ for $1 \leqslant j \leqslant q$. Since the assumptions of Theorem 3.1 are fulfilled, Corollary 3.2 allows us to state that (4.1) coincides with $s(c^{E\oplus D}, c^Q)$ for all $c \in (0,1)$, which can also be verified by elementary calculations using Example 3.1 as follows.
If $\sum_{i=1}^{\ell-1} b_i < \sum_{k=1}^{q} a_k d_k \leqslant \sum_{i=1}^{\ell} b_i$ then $r = \ell$ in (3.6) and by (3.7) we get

$$\begin{aligned} s(c^{E\oplus D}, c^Q) &= \ell - 1 + \frac{1}{b_\ell}\big(Q - \sum_{i=1}^{\ell-1} b_i\big) \\ &= b_\ell^{-1}\big(\sum_{k=1}^{q} a_k d_k + \sum_{i=1}^{\ell}(b_\ell - b_i)\big). \end{aligned}$$

On the other hand, if $\sum_{k=1}^{\ell-1} \tilde a_k \tilde d_k \leqslant \sum_{i=1}^{m} b_i < \sum_{k=1}^{\ell} \tilde a_k \tilde d_k$, or equivalently

$$\sum_{i=1}^{m} b_i + \sum_{k=1}^{q-\ell} a_k d_k < Q \leqslant \sum_{i=1}^{m} b_i + \sum_{k=1}^{q-\ell+1} a_k d_k$$

then we know that

$$r = m + \sum_{k=1}^{q-\ell} d_k + R \quad \text{for some } R \in \{1, \ldots, d_{q-\ell+1}\}$$

in (3.6) and by (3.7) we get

$$\begin{aligned} s(c^{E\oplus D}, c^Q) &= m + \sum_{k=1}^{q-\ell} d_k + R - 1 \\ &\qquad + \frac{1}{a_{q-\ell+1}}\big(Q - \sum_{i=1}^{m} b_i - \sum_{k=1}^{q-\ell} a_k d_k - a_{q-\ell+1}(R-1)\big) \\ &= \sum_{j=\ell+1}^{q} \tilde d_j + \frac{1}{a_{q-\ell+1}} \sum_{k=q-\ell+1}^{q} a_k d_k + \sum_{i=1}^{m}\big(1 - \frac{b_i}{a_{q-\ell+1}}\big) \\ &= \sum_{j=1}^{\ell} \frac{\tilde a_j}{\tilde a_\ell}\tilde d_j + \sum_{j=\ell+1}^{q} \tilde d_j + \sum_{i=1}^{m}\big(1 - \frac{b_i}{\tilde a_\ell}\big). \end{aligned}$$

Further, by [45, Theorem 4.1] for $\alpha = 2$ and [46, Theorem 5.1] for $\alpha \in (0,2)$ we have almost surely

$$\dim_{\mathcal{H}} X([0,1]^d) = \begin{cases} m & \text{if } \sum_{i=1}^{m} b_i < Q, \\ b_\ell^{-1}\big(Q + \sum_{i=1}^{\ell}(b_\ell - b_i)\big) & \text{if } \sum_{i=1}^{\ell-1} b_i < Q \leqslant \sum_{i=1}^{\ell} b_i. \end{cases}$$

In accordance with Corollary 3.4, a comparison with (3.12) directly shows that this value coincides with $s(c^D, c^Q)$ for all $c \in (0,1)$, indicating that the lower bound in Theorem 3.3 is in fact equal to the Hausdorff dimension of the range for the harmonizable representation of any (E,D)-operator-self-similar stable random field. All the above results also hold for a moving average representation of the random field in the Gaussian case $\alpha = 2$ as shown in [45]. However, for a corresponding moving average representation in the stable case $\alpha \in (0,2)$, constructed in [33], it is questionable if our results are applicable, since these fields do not share the same Hölder continuity properties and thus the joint measurability of sample functions (assumption (iv) in the Introduction) may be violated; cf. [5], [6] for details.

4.2. Operator semistable Lévy processes.

To give an example of random fields that are not operator self-similar but possess the weaker discrete scaling property of self-affinity, we will now consider operator semistable Lévy processes for $d = 1$ with the restriction to $t \geqslant 0$. Let $X = \{X(t)\}_{t \geqslant 0}$ be a strictly operator-semi-selfsimilar process in $\mathbb{R}^m$, i.e.

$$\{X(ct)\}_{t\geqslant 0} \overset{\text{f.d.}}{=} \{c^D X(t)\}_{t \geqslant 0} \quad \text{for some } c \in (0,1), \tag{4.2}$$

where $D \in \mathbb{R}^{m\times m}$ is a scaling matrix. If X is a proper Lévy process, it is called an operator semistable process and it is known that the real part of any eigenvalue of D belongs to $[\frac{1}{2}, \infty)$, where $\frac{1}{2}$ refers to a Brownian motion component; see [38] for details. Hence this process can be considered as a self-affine random field with $d = 1$ and non-singular contractions $U = c$, and $V = c^D$. This includes operator stable Lévy processes, where (4.2) holds for all $c > 0$, and multivariate stable Lévy processes, where additionally D is diagonal. For these particular cases, Hausdorff dimension results for the range and the graph have been established in [1], [39], [7], [41], [49], [23]. Let $\frac{1}{2} \leqslant b_1 < \ldots < b_p$ denote the distinct real parts of the eigenvalues of E with multiplicity $m_1, \ldots, m_p$, then recently Wedrich [47] (cf. also [27]) has shown that for any operator semistable Lévy process X almost surely

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]) = \begin{cases} \max\{b_1^{-1}, 1\} & \text{if } b_1^{-1} \leqslant m_1, \\ 1 + \max\{b_2^{-1}, 1\}(1 - b_1) & \text{else,} \end{cases} \tag{4.3}$$

where the lower bound in the computation is proven through (3.1). Moreover, in view of Lemma 2.3 it follows directly from [28, Lemma 2.2] that τ_X satisfies the b.c.i. condition. Since the assumptions of Theorem 3.1 are fulfilled, Corollary 3.2 allows us to state that (4.3) coincides with $s(c \oplus c^D, c) = \operatorname{cardim} \tau_X$ irrespectively of $c \in (0,1)$, which can also easily be verified by elementary calculations. Further, by Corollary 3.2 and Theorem 3.3 in [28] we have almost surely

$$\dim_{\mathcal{H}} X([0,1]) = \begin{cases} b_1^{-1} & \text{if } b_1^{-1} \leqslant m_1, \\ 1 + b_2^{-1}(1 - b_1) & \text{if } b_1^{-1} > m_1 = 1,\ m \geqslant 2, \\ 1 & \text{if } b_1^{-1} > m_1 = 1,\ m = 1. \end{cases} \tag{4.4}$$

This value coincides with $s(c^D, c)$ as will be shown below, indicating that the lower bound in Theorem 3.3 is in fact equal to the Hausdorff dimension of the range for any operator semistable Lévy process. Since (4.4) shows $\dim_{\mathcal{H}} X([0,1]) \in (0,2]$ almost surely, it suffices to consider the singular value function $\phi_{W^n}(s)$ of $W = c^D$ for $s \in (0,2]$. We distinguish between the following cases.

Case 1: $b_1 \geqslant 1$, then for $s \in (0,1]$ the singular value function $\phi_{W^n}(s)$ asymptotically behaves as $c^{nb_1 s}$ in the sense of (3.5) showing that $s(c^D, c) = b_1^{-1}$.

Case 2: $b_1 < 1$ and $m = 1$, then as in the first case for $s \in (0,1]$ the singular value function $\phi_{W^n}(s)$ asymptotically behaves as $c^{nb_1 s}$ in the sense of (3.5). Due to the restriction $s \leqslant m = 1$ we have $s(c^D, c) = 1$.

Case 3: $b_1 < 1$, $m_1 = 1$ and $m \geqslant 2$, then for $s \in (1,2]$ the singular value function $\phi_{W^n}(s)$ asymptotically behaves as $c^{n(b_1 + b_2(s-1))}$ in the sense of (3.5) showing that $s(c^D, c) = 1 + b_2^{-1}(1 - b_1)$.

The operator semistable Lévy processes may be generalized to multiparameter operator semistable processes with $d \geqslant 2$ as in [16], [49] or to certain operator semi-selfsimilar strong Markov processes as in [34], [11], for which corresponding Hausdorff dimension results of the sample functions are not yet available in full generality from the literature. Our approach will give promising candidates for the Hausdorff dimension of the range and the graph of such fields in terms of the real parts of the eigenvalues of the scaling exponent. These serve at least as lower bounds by Theorems 3.1 and 3.3, while corresponding upper bounds should be pursued elsewhere. We expect that these candidates are the precise Hausdorff dimension values as mentioned in Remark 3.2.

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Mathematical Institute
Heinrich-Heine-University Düsseldorf
Universitätsstr. 1
D-40225 Düsseldorf, Germany
E-mail: kern@hhu.de

Mathematical Institute
Heinrich-Heine-University Düsseldorf
Universitätsstr. 1
D-40225 Düsseldorf, Germany
E-mail: ercan.soenmez@hhu.de

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