PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 0, Fasc. 0 (0000), pp. 000–000
doi:10.19195/0208-4147.0.0.0

# WEYL MULTIFRACTIONAL ORNSTEIN-UHLENBECK PROCESSES MIXED WITH A GAMMA DISTRIBUTION

BY

KHALIFA ES-SEBAIY (KUWAIT CITY), FATIMA-EZZAHRA FARAH (MARRAKESH), AND ASTRID HILBERT (VÄXJÖ)

*Abstract.* The aim of this paper is to study the asymptotic behavior of the aggregated Weyl multifractional Ornstein-Uhlenbeck processes mixed with Gamma random variables. This allows us to introduce a new process, so-called Gamma-mixed Weyl multifractional Ornstein-Uhlenbeck process (GWmOU), and study its elementary properties such as Hausdorff dimension, locally self-similarity and short range dependence. We also prove that the GWmOU process is a new candidate to approach the multifractional Brownian motion.

**2010 AMS Mathematics Subject Classification:** Primary: 60G22; Secondary: 60G17.

**Key words and phrases:** Weyl multifractional Ornstein-Uhlenbeck process, Gamma distribution, Aggregated process, Multifractional Brownian motion

## 1. INTRODUCTION

The fractional Ornstein-Uhlenbeck (fOU) process is one of the most well studied and widely applied stochastic process [8]. Recently, in [10], an interesting process considered as a good candidate to be a model for various applications, has been introduced employing a sequence of fOU processes with random coefficients.

Let us first present a brief summary of its construction. Let $B^H = \{ B^H(t),\, t \in \mathbb{R} \}$ be a fractional Brownian motion (fBm) with Hurst index $H > 1/2$, defined on the probability space $(\Omega_{B^H}, \mathcal{F}_{B^H}, \mathcal{P}_{B^H})$. Consider a sequence of stationary fOU processes with random coefficients $X_k,\ k \geqslant 1$, defined by the

following stochastic integral

$$X_t^k = \int_{-\infty}^{t} e^{\gamma_k(t-s)} dB_s^H, \qquad t \in \mathbb{R}, \tag{1.1}$$

with initial condition $X_0^k = \int_{-\infty}^{0} e^{\gamma_k(t-s)} dB_s^H$. The random coefficients $\gamma_k$, $k \geqslant 1$, are independent random variables on the probability space $(\Omega_\gamma, \mathcal{F}_\gamma, \mathcal{P}_\gamma)$ and for any $k \geqslant 1$, $-\gamma_k \sim \Gamma(1-h, \lambda)$ with $0 < h < 1 - H$ and $\lambda > 0$.

Assume that the family $\{\gamma_k,\, k \geqslant 1\}$ is independent of $B^H$. The processes $X_k, k \geqslant 1$, defined above are $P_\gamma$ almost surely fOU processes, see [8]. Let

$$Y_n(t) = \frac{1}{n} \sum_{k=1}^{n} X_k(t), \qquad t \in \mathbb{R}, \tag{1.2}$$

denote the so-called aggregated process. It has been proven that the sequence $(Y_n)_{n \geqslant 1}$ converges weakly and in $L^2(\Omega_{B^H})$ for fixed time, $P_\gamma$ almost surely, when $n$ goes to infinity, to a stochastic process denoted by $Y^\lambda := \{\, Y^\lambda(t), t \in \mathbb{R}\}$, given by the following stochastic integral

$$Y^\lambda(t) = \int_{-\infty}^{t} \left( \frac{\lambda}{\lambda + t - s} \right)^{1-h} dB_s^H, \qquad t \in \mathbb{R}. \tag{1.3}$$

The limiting process $Y^\lambda$ is stationary, almost self-similar and exhibits long range dependence (see [13] or [10]). The asymptotic behavior of the process $Y^\lambda$ with respect to the parameter $\lambda$ is also studied, as $\lambda$ varies between $\infty$ and $0$. The process $Y^\lambda$ ranges from a fBm with index $H$ to a fBm with index $h + H$.

When $B^H$ is a standard Brownian motion (i.e. $H = 1/2$), the Gamma-mixed Ornstein-Uhlenbeck process has been studied in [13].

Our goal is to construct a new process, called Gamma-mixed Weyl multifractional Ornstein-Uhlenbeck (GWmOU), in analogy to the limiting procedure that lead to the process defined in equation (1.3). In our construction we replace the processes $X_k$, $1 \leqslant k \leqslant n$, in the aggregated process (1.2) by Weyl multifractional Ornstein-Uhlenbeck (WmOU)processes mixed with Gamma random variables defined by the Wiener integral

$$X_{\alpha(t)}^k(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} (t-s)^{\alpha(t)-1} e^{\gamma_k(t-s)} dB_s, \qquad t \in \mathbb{R},$$

$B = \{\, B(s),\, s \in \mathbb{R}\,\}$ is a Brownian motion on $(\Omega_B, \mathcal{F}_B, P_B)$. $\gamma_k, k \geqslant 1$, are independent random variables on $(\Omega_\gamma, \mathcal{F}_\gamma, P_\gamma)$, also independent by B, and for any $k \geqslant 1$, $-\gamma_k \sim \Gamma(1-h, \lambda)$ with $0 < h < 1$ and $\lambda > 0$. $\alpha$ is a Hölder continuous function with exponent $0 < \beta \leqslant 1$. The processes $X_k$, $k \geqslant$

1 are $P_\gamma$ almost surely WmOU processes, see Section 2.

We will define the GWmOU, denoted $Y_\alpha^\lambda$, by

$$Y_{\alpha(t)}^\lambda(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} \left(\frac{\lambda}{\lambda + t - s}\right)^{1-h} (t-s)^{\alpha(t)-1} dB_s, \qquad t \in \mathbb{R}.$$

It is non stationary, locally asymptotically self-similar and exhibits short range dependence. we will also study the Hölder exponent, the box and the Hausdorff dimension of the process $Y_\alpha^\lambda$. In addition, we will investigate the asymptotic behavior of the process $Y_\alpha^\lambda$ with respect to the parameter $\lambda$, we will prove that the process $Y_\alpha^\lambda$ is a new candidate to approach the multifractional Brownian motion, see [17], when $\lambda$ goes to infinity while its integrated renormalized process

$$\hat{Y}_\alpha^\lambda(t) = \lambda^{h-1} \int_0^t Y_\alpha^\lambda(s) ds, \qquad t \geqslant 0,$$

here we suppose that the function $\alpha$ is constant, converges when $\lambda$ goes to zero, to modulo constant fractional Brownian motion.

The motivation of this work came from two facts. On one hand, Gamma-mixed processes are good models for various applications; for example, the limiting process $Y^\lambda$ defined by equation 1.3 is a successful model of heart rate variability and could also be a good model of a lot of Gaussian stationary data with long range dependence, see [10], [13] for more details. Moreover, the so-called Gamma-mixed Poisson, also named Polya process has many practical applications, one of them is the study of the reliability of engineering systems [9]. On the other hand, multifractional Ornstein-Uhlenbeck processes are omnipresent in physics, as effective models of many physical systems. For further detailed and references, we refer the reader to [15]. Also, for more details about construction and study of several classes of multifractional processes, see e.g. [3], [2], [5], [4], [17], [19]. In the light of the above reasons the motivation emerged to mix the multifractional Ornstein-Uhlenbeck process (Weyl version) with Gamma random variables, in order to introduce the GWmOU process, as a counterpart of the limiting process $Y^\lambda$, a new candidate to model several short range, variable fractal dimension and non stationary physical phenomena.

The paper is structured as follows. Section 2 is devoted to present a short summary of results on the WmOU process. In section 3 we introduce the GWmOU process as a limit of the aggregated Weyl multifractional Ornstein-Uhlenbeck processes mixed with Gamma distribution random variables. Finally, Section 4 contains some interesting properties of the GWmOU process including its asymptotic behavior.

## 2. PRELIMINARIES

The WmOU process has been introduced as multifractional generalization of the Weyl fractional Ornstein-Uhlenbeck process (WfOU).

Let us begin with a brief reminder of the WfOU process (see [14]). First, we need to recall some elementary definitions of the fractional calculus (see [16], [18]). The Weyl fractional derivative of order $\alpha > 0$, denoted by ${}_aD_t^{\alpha}$, for $a = -\infty$, can be defined by its inverse using the Weyl fractional integral,

$$ {}_aD_t^{-\alpha} f(t) = {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \qquad t \geqslant a. $$

For $n-1 \leqslant \alpha < n$, we have that ${}_aD_t^{\alpha}$ is defined as the ordinary derivative of order $n$ of the Weyl fractional integral of order $n-\alpha$

$$ {}_aD_t^{\alpha} = (d/dt)^n \, {}_aD_t^{\alpha-n}. $$

The WfOU is a stochastic process obtained as solution of the following fractional Langevin equation

$$ ({}_aD_t + w)^{\alpha} X(t) = W(t), \qquad \alpha > 0, \, w > 0, $$

where $W(t)$ is a Gaussian White Noise. It is defined explicitly by the following stochastic integral

$$ X_{\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-s)^{\alpha-1} e^{-w(t-s)} dB_s, \qquad t \in \mathbb{R}, $$

where $B = \{B(s), s \in \mathbb{R}\}$ is the standard Brownian motion and $\alpha > 1/2$ to ensure that $X_{\alpha}(t)$ has finite variance.

Similarly to the generalization of fractional Brownian motion to multifractional Brownian motion ( see [17]), the extension of WfOU process is obtained by replacing the parameter $\alpha$ by a Hölder continuous function with exponent $0 < \beta \leqslant 1$, i.e. there exists a constant $K$ such that

$$ |\, \alpha(t) - \alpha(s) \,| \leqslant K |\, t - s \,|^{\beta} \qquad \forall \, s, t, $$

and $\alpha(t) > 1/2$ for all $t$.

Let us recall the WmOU process and its properties needed in the sequel. For more details we refer the reader to [15].

The WmOU process is the Gaussian process defined by the Wiener integral

$$ X_{\alpha(t)}(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^t (t-s)^{\alpha(t)-1} e^{-w(t-s)} dB_s, \qquad t \in \mathbb{R}. $$

We have

$$E_B\left[\left(X_{\alpha(t)}(t+s)-X_{\alpha(t)}(t)\right)^2\right]=\frac{-|s|^{2\alpha(t)-1}}{\Gamma(2\alpha(t))\cos(\pi\alpha(t))}-2|s|^2w^{3-2\alpha(t)}S_{\alpha(t)}(w|s|), \tag{2.1}$$

where $S_\vartheta(x)$ is a continuous function given explicitly by

$$\begin{aligned} S_\vartheta(x) &= -\frac{\sqrt{\pi}}{8\Gamma(\vartheta)cos(\pi\vartheta)}\Big[\sum_{m=0}^{\infty}\frac{x^{2m}}{2^{2m}(m+1)!\Gamma(m+5/2-\vartheta)} \\ &\qquad -\left(\frac{x}{2}\right)^{2\vartheta-1}\sum_{m=0}^{\infty}\frac{x^{2m}}{2^{2m}(m+1)!\Gamma(m+3/2+\vartheta)}\Big], \end{aligned}$$

for every $x>0$ and $1/2<\vartheta<3/2$.

Then, its variance is equal to

$$E([X_{\alpha(t)}(t)]^2)=\frac{(2w)^{1-2\alpha(t)}\Gamma(2\alpha(t)-1)}{\Gamma(\alpha(t))^2}.$$

On the other hand, for $s<t$ the covariance of the WmOU is given by

$$E(X_{\alpha(t)}(t)\,X_{\alpha(s)}(s))=\frac{e^{-w(t-s)}(t-s)^{\alpha(t)+\alpha(s)-1}}{\Gamma(\alpha(t))}\psi(\alpha(s),\alpha(s)+\alpha(t);2w(t-s)),$$

where $\psi(\alpha,\gamma;z)$ is the confluent hypergeometric function. The variance and the covariance functions are divergent when $w$ goes to zero. However, when we set $Z_{\alpha(t)}(t)=X_{\alpha(t)}(t)-X_{\alpha(t)}(0)$, it has been proven in [15] that for $\alpha(t)\in(1/2,3/2)$ and by identifying $\alpha(t)$ with $H(t)+1/2$, the process $Z_{\alpha(t)}(t)$ approaches when $w$ goes to zero (in the sense of finite-dimensional distributions) $B_{H(t)}(t)$ the multifractional Brownian motion (of moving average definition) defined in [17] by

$$B_{H(t)}(t)=\frac{1}{\Gamma(H(t)+1/2)}\left(\int_{-\infty}^{0}(t-s)^{H(t)-1/2}-(-s)^{H(t)-1/2}dB_s+\int_0^t(t-s)^{H(t)-1/2}dB_s\right).$$

For the basic properties of the WmOU process such as short-range dependence, local self-similarity and Hausdorff dimension, we refer the reader to [15].

Let us now recall a sufficient criteria for weak convergence, which will be needed in the sequel. By virtue of Prohorov's theorem, the convergence of the finite-dimensional distributions and tightness yield weak convergence. For processes $X,\ X_n,\ n\geqslant 1$, with paths in $C([a,b],\mathbb{R})$, one has the following sufficient criterion, cf. (Billingsley [6, Theorem 12.3], or Billingsley [7]).

THEOREM 2.1. *Suppose that the convergence of the finite-dimensional distributions of the family $(X_n)_{n\geqslant1}$ to those of $X$ holds true. If, in addition, there exist constants $\zeta>0$, $\theta>1$ and $c_{\zeta,\theta}$ depending only on $\zeta$ and $\theta$, such that for all $s,t\in[a,b]$, $a,b\in\mathbb{R}$, such that $a<b$*

$$E\left[|X_n(t)-X_n(s)|^\zeta\right]\leqslant c_{\zeta,\theta}|t-s|^\theta$$

*for all $n \geqslant 1$, then the family $(X_n)_{n\geqslant 1}$ is tight and consequently*

$$X_n \to X \qquad \textit{weakly in} \quad C[a,b]$$

*as $n \to \infty$.*

## 3. AGGREGATED WEYL MULTIFRACTIONAL ORNSTEIN-UHLENBECK PROCESSES MIXED WITH GAMMA DISTRIBUTION

Let us now consider a sequence of WmOU processes mixed with Gamma distribution random variables $X^k_\alpha := \left\{X^k_{\alpha(t)}(t), t \in \mathbb{R}\right\}$ defined by the following Wiener integral

$$X^k_{\alpha(t)}(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} (t-s)^{\alpha(t)-1} e^{\gamma_k(t-s)} dB_s, \qquad t \in \mathbb{R}, \tag{3.1}$$

where $B = \{B(s), s \in \mathbb{R}\}$ is the Brownian motion defined on the probability space $(\Omega_B, \mathcal{F}_B, P_B)$ and for any $k \geqslant 1$, $-\gamma_k \sim \Gamma(1-h, \lambda)$ with $0 < h < 1$ and $\lambda > 0$ are independent random variables, also independent by B, defined on the probability space $(\Omega_\gamma, \mathcal{F}_\gamma, P_\gamma)$.

The processes $X^k_\alpha$, $k \geqslant 1$, are $P_\gamma$ almost surely WmOU processes defined on the probability space $(\Omega_B, \mathcal{F}_B, P_B)$. We define the empirical mean of these processes by

$$Y^n_{\alpha(t)}(t) = \frac{1}{n} \sum_{k=1}^{n} X^k_{\alpha(t)}(t),$$

for every $t \in \mathbb{R}$ and $n \geqslant 1$.

Throughout the paper we assume that

$$1/2 < \alpha_{inf} \leqslant \alpha_{sup} < 3/2, \tag{3.2}$$

where $\alpha_{inf} := \inf_{t\in\mathbb{R}} \alpha(t)$ and $\alpha_{sup} := \sup_{t\in\mathbb{R}} \alpha(t)$.

We will also need the following notations,

$$m_\alpha[a,b] = min\{\alpha(t),\, t \in [a,b]\}; \qquad M_\alpha[a,b] = max\{\alpha(t),\, t \in [a,b]\}$$

for all real $a, b$ such that $a < b$. $E_B$ and $E_\gamma$ denote the expectations with respect to $P_B$ and $P_\gamma$ respectively.

$C$ denotes a generic constant depending only on the interval $[a,b]$, $\lambda$ and $h$.

$C^{x,y}$ denote a generic constant depending on $[a,b]$, $\lambda$, $h$, $x$ and $y$ such that $0 < x < 2m_\alpha[a,b] - 1$ and $0 < y < 3/2 - h - M_\alpha[a,b]$.

$C^{x,y}_\eta$ denotes a generic constant depending on $[a,b]$, $\lambda$, $h$, $x$ and $y$ such that $0 < x < 2m_\alpha[a,b] - 1$, $0 < y < 3/2 - h - M_\alpha[a,b]$ and $0 \leqslant \eta < m_\alpha[a,b] - 1/2$.

**3.1. The limit of the aggregated process.**

If $0 < h < 3/2 - \alpha_{\sup}$, we define the zero mean Gaussian process $Y_\alpha^\lambda := \{Y_{\alpha(t)}^\lambda(t),\, t \in \mathbb{R}\}$ by

$$Y_{\alpha(t)}^\lambda(t) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t} \left(\frac{\lambda}{\lambda + t - s}\right)^{1-h} (t-s)^{\alpha(t)-1} dB_s, \qquad t \in \mathbb{R}. \tag{3.3}$$

It is easy to see that the Wiener integral in 3.3 is well-defined.

The process $Y_\alpha^\lambda$ will be called Gamma-mixed Weyl multifractional Ornstein-Uhlenbeck process, abbreviated as GWmOU.

Given a compact interval $[a, b] \subset \mathbb{R}$, the following result proves that $P_\gamma$−a.s., $Y_{\alpha(t)}^n(t)$ converges to $Y_{\alpha(t)}^\lambda(t)$ in $L^2(\Omega_B)$, uniformly in $t \in [a, b]$, as $n \to \infty$.

THEOREM 3.1. *Fix real numbers $a, b$ such that $a < b$. If $0 < h < 3/2 - M_\alpha[a, b]$, then $P_\gamma$−a.s.,*

$$Y_{\alpha(t)}^n(t) \underset{n\to\infty}{\longrightarrow} Y_{\alpha(t)}^\lambda(t) \qquad \text{in } L^2(\Omega_B) \tag{3.4}$$

*uniformly in $t \in [a, b]$.*

*In particular, if $0 < h < 3/2 - \alpha_{\sup}$, then $P_\gamma$−a.s., for every $t \in \mathbb{R}$,*

$$Y_{\alpha(t)}^n(t) \longrightarrow Y_{\alpha(t)}^\lambda(t) \qquad \text{in } L^2(\Omega_B) \tag{3.5}$$

*as $n \to \infty$.*

P r o o f. We prove (3.4). Let us first introduce the following notations, for every $x > 0$, $n \geqslant 1$,

$$f_n(x) := \frac{1}{n} \sum_{k=1}^{n} e^{\gamma_k x}; \qquad c(x) := E_\gamma(e^{\gamma_1 x}) = \left(\frac{\lambda}{\lambda + x}\right)^{1-h}.$$

By the law of large numbers, we have $P_\gamma$−a.s., for every $x > 0$,

$$f_n(x) = \frac{1}{n} \sum_{k=1}^{n} e^{\gamma_k x} \underset{n\to\infty}{\longrightarrow} c(x), \tag{3.6}$$

and for every $c > 0$, $d < 3/2 - h$,

$$\frac{1}{n} \sum_{k=1}^{n} \frac{e^{\gamma_k c}}{(-\gamma_k)^{d-1/2}} \underset{n\to\infty}{\longrightarrow} E_\gamma\left(\frac{e^{\gamma_1 c}}{(-\gamma_1)^{d-1/2}}\right) = \frac{\lambda^{1-h}\Gamma(3/2 - d - h)}{\Gamma(1-h)(\lambda + c)^{3/2-d-h}}. \tag{3.7}$$

Using the change of variable $u = t - s$, we can write

$$\begin{aligned} E_B\left[\left(Y_{\alpha(t)}^n(t) - Y_{\alpha(t)}^\lambda(t)\right)^2\right] &= \frac{1}{\Gamma(\alpha(t))^2} E_B\left[\left(\int_{-\infty}^{t} (t-s)^{\alpha(t)-1}(f_n(t-s) - c(t-s))dB_s\right)^2\right] \\ &= \frac{1}{\Gamma(\alpha(t))^2} \int_{-\infty}^{t} (t-s)^{2\alpha(t)-2} \left(f_n(t-s) - c(t-s)\right)^2 ds \\ &= \frac{1}{\Gamma(\alpha(t))^2} \int_{0}^{\infty} u^{2\alpha(t)-2} \left(f_n(u) - c(u)\right)^2 du. \end{aligned}$$

Hence, for every $m \geqslant 2$, $t \in [a,b]$, we have

$$
\begin{aligned}
(3.8) \qquad & E_B\left(Y^n_{\alpha(t)}(t) - Y^\lambda_{\alpha(t)}(t)\right)^2 \\
&= \frac{1}{\Gamma(\alpha(t))^2}\left[\int_0^1 u^{2\alpha(t)-2}\left(f_n(u)-c(u)\right)^2 du + \int_1^m u^{2\alpha(t)-2}\left(f_n(u)-c(u)\right)^2 du\right. \\
&\quad \left. + \int_m^\infty u^{2\alpha(t)-2}\left(f_n(u)-c(u)\right)^2 du\right] \\
&\leqslant K\left[\int_0^1 u^{2m_\alpha[a,b]-2}\left(f_n(u)-c(u)\right)^2 du + \int_1^m u^{2M_\alpha[a,b]-2}\left(f_n(u)-c(u)\right)^2 du\right. \\
&\quad \left. + \int_m^\infty u^{2M_\alpha[a,b]-2}\left(f_n(u)-c(u)\right)^2 du\right] \\
(3.9) \qquad &:= K\left[A(n,m)+B(n,m)+C(n,m)\right],
\end{aligned}
$$

where $K$ is the maximum of the continuous function $z \mapsto 1/\Gamma(z)$ on $[m_\alpha[a,b], M_\alpha[a,b]]$. Combining (3.6), $f_n(u) \leqslant 1, c(u) \leqslant 1$ and (3.2) together with Lebesgue's dominated convergence theorem, we can conclude that $P_\gamma - a.s$, for every $m \geqslant 2$,

$$
(3.10) \qquad A(n,m) \underset{n\to\infty}{\longrightarrow} 0, \qquad B(n,m) \underset{n\to\infty}{\longrightarrow} 0.
$$

Now we will estimate $C(n,m)$ for all $m \geqslant 2$. We have

$$
\begin{aligned}
C(n,m) &= \int_m^\infty \left(f_n(u)-c(u)\right)^2 u^{2M_\alpha[a,b]-2} du \\
&\leqslant 2\int_m^\infty f_n(u)^2\, u^{2M_\alpha[a,b]-2} du + 2\int_m^\infty c^2(u)\; u^{2M_\alpha[a,b]-2} du.
\end{aligned}
$$

Moreover, by the change of variable $v = (-\gamma_j - \gamma_k)u$ and $2\sqrt{(-\gamma_j)(-\gamma_k)} \leqslant -\gamma_j - \gamma_k$,

$$
\begin{aligned}
\int_m^\infty f_n(u)^2 \quad u^{2M_\alpha[a,b]-2}\, du &= \frac{1}{n^2}\sum_{k,j=1}^n \int_m^\infty e^{\gamma_j u}\, e^{\gamma_k u}\, u^{2M_\alpha[a,b]-2} du \\
&= \frac{1}{n^2}\sum_{k,j=1}^n \frac{1}{(-\gamma_j-\gamma_k)^{2M_\alpha[a,b]-1}} \int_{m(-\gamma_j-\gamma_k)}^\infty v^{2M_\alpha[a,b]-2} e^{-v} dv \\
&\leqslant \frac{2^{1-2M_\alpha[a,b]}}{n^2}\sum_{k,j=1}^n \frac{e^{-\frac{m}{2}(-\gamma_j-\gamma_k)}}{\left[(-\gamma_j)(-\gamma_k)\right]^{M_\alpha[a,b]-1/2}} \int_{m(-\gamma_j-\gamma_k)}^\infty v^{2M_\alpha[a,b]-2} e^{-v/2} dv \\
&\leqslant \Gamma\left(2M_\alpha[a,b]-1\right)\left(\frac{1}{n}\sum_{j=1}^n \frac{e^{-\frac{m}{2}(-\gamma_j)}}{(-\gamma_j)^{M_\alpha[a,b]-1/2}}\right)^2.
\end{aligned}
$$

Combining this with ((3.7)), we get $P_\gamma - a.s.$

$$
\begin{aligned}
\limsup_{n\to\infty} \int_m^\infty f_n(u)^2\, u^{2M_\alpha[a,b]-2} du &\leqslant \Gamma(2M_\alpha[a,b]-1)\left(\frac{\lambda^{1-h}\Gamma(3/2-M_\alpha[a,b]-h)}{\Gamma(1-h)(\lambda+\frac{m}{2})^{3/2-M_\alpha[a,b]-h}}\right)^2 \\
&\longrightarrow 0
\end{aligned}
$$

as $m \longrightarrow \infty$. On the other hand, since

$$\int_0^\infty \left(\frac{\lambda}{\lambda+u}\right)^{2-2h} u^{2M_\alpha[a,b]-2}du \;=\; \lambda^{2M_\alpha[a,b]-1}\,\beta(3-2M_\alpha[a,b]-2h, 2M_\alpha[a,b]-1) < \infty,$$

$$\begin{aligned}\int_m^\infty c^2(u)\; u^{2M_\alpha[a,b]-2}du &= \int_m^\infty \left(\frac{\lambda}{\lambda+u}\right)^{2-2h} u^{2M_\alpha[a,b]-2}du \\ &\longrightarrow 0\end{aligned}$$

as $m \longrightarrow \infty$, which implies that $P_\gamma - a.s.$

$$\limsup_{n\to\infty} C(n,m) \underset{m\to\infty}{\longrightarrow} 0. \tag{3.11}$$

Therefore, by inserting the convergences (3.10) and (3.11) in (3.9) we deduce that $P_\gamma - a.s.$

$$\limsup_{n\to\infty} \sup_{t\in[a,b]} E_B\left(Y^n_{\alpha(t)}(t) - Y^\lambda_{\alpha(t)}(t)\right)^2 = 0,$$

which finishes the proof of (3.4).
Finally, the convergence (3.5) is a direct consequence of (3.4), (3.2) and $0 < h < 3/2 - \alpha_{\sup}$. ■

The weak convergence of the sequence $(Y^n_\alpha)_{n\geqslant 1}$ is established by our next theorem.

THEOREM 3.2. *Fix real numbers $a, b$ such that $a < b$.*
*If $0 < h < 3/2 - M_\alpha[a,b]$ and $\min\{2m_\alpha[a,b]-1, 2\beta\} < 1$, then $P_\gamma$–a.s.,*

$$Y^n_\alpha \underset{n\to\infty}{\to} Y^\lambda_\alpha \;\; \textit{in}\;\; C[a,b], \tag{3.12}$$

*where $C[a,b]$ is the space of continuous functions on $[a,b]$.*

P r o o f. First, note since $P_\gamma$ almost surely, $Y^n_\alpha$ and $Y^\lambda_\alpha$ are zero mean Gaussian processes whose finite dimensional distributions are determined by their covariances, (3.4) implies the convergence $P_\gamma$ almost surely of the finite-dimensional distributions of the family $(Y^n_\alpha)_{n\geqslant 1}$ to those of $Y^\lambda_\alpha$. Then, in order to prove (3.12) it remains to prove the $P_\gamma - a.s.$ tightness of the family $(Y^n_\alpha)_{n\geqslant 1}$ by using Theorem (2.1).
Throughout the proof all the results are given $P_\gamma$ almost surely.
Let $t,\, t+\tau \in [a,b]$ such that $|\tau| < \min(\lambda/2, 1)$.

$$\begin{aligned} E_B\left[\left(Y^n_{\alpha(t+\tau)}(t+\tau) - Y^n_{\alpha(t)}(t)\right)^2\right] &= E_B\left[\left(\frac{1}{n}\sum_{k=1}^n \left(X^k_{\alpha(t+\tau)}(t+\tau) - X^k_{\alpha(t)}(t)\right)\right)^2\right] \\ &\leqslant 2E_B\left[\left(\frac{1}{n}\sum_{k=1}^n U^k_t(\tau)\right)^2\right] + 2E_B\left[\left(\frac{1}{n}\sum_{k=1}^n V^k_t(\tau)\right)^2\right], \end{aligned} \tag{3.13}$$

where

$$U^k_t(\tau) := X^k_{\alpha(t)}(t+\tau) - X^k_{\alpha(t)}(t); \qquad V^k_t(\tau) := X^k_{\alpha(t+\tau)}(t+\tau) - X^k_{\alpha(t)}(t+\tau).$$

We will first prove that for every $n \geqslant 1$,

$$E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} U_t^k(\tau)\right)^2\right] \leqslant C|\tau|^{2m_\alpha[a,b]-1}. \tag{3.14}$$

To this end, by using Hölder's inequality and (2.1), we can write

$$\begin{aligned} E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} U_t^k(\tau)\right)^2\right] &\leqslant \frac{1}{n}\sum_{k=1}^{n} E_B\left[\left(U_t^k(\tau)\right)^2\right] \\ &= \frac{-|\tau|^{2\alpha(t)-1}}{\Gamma(2\alpha(t))\cos(\pi\alpha(t))} - 2|\tau|^2\frac{1}{n}\sum_{k=1}^{n}(-\gamma_k)^{3-2\alpha(t)}S_{\alpha(t)}(-\gamma_k|\tau|). \end{aligned}$$

Since $1/2 < \alpha(t) < 3/2$ and $cos(\pi\alpha(t)) < 0$, we get

$$\begin{aligned} &-S_{\alpha(t)}(-\gamma_k|\tau|) \\ &= \frac{\sqrt{\pi}}{8\Gamma(\alpha(t))cos(\pi\alpha(t))}\sum_{m=0}^{\infty}\frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}(m+1)!\Gamma(m+5/2-\alpha(t))} - \frac{\sqrt{\pi}}{8\Gamma(\alpha(t))cos(\pi\alpha(t))} \\ &\quad \left(\frac{-\gamma_k|\tau|}{2}\right)^{2\alpha(t)-1}\sum_{m=0}^{\infty}\frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}(m+1)!\Gamma(m+3/2+\alpha(t))} \\ &\leqslant -\frac{\sqrt{\pi}}{8\Gamma(\alpha(t))cos(\pi\alpha(t))}\left(\frac{-\gamma_k|\tau|}{2}\right)^{2\alpha(t)-1}\sum_{m=0}^{\infty}\frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}(m+1)!\Gamma(m+3/2+\alpha(t))} \\ &\leqslant C\,(-\gamma_k|\tau|)^{2\alpha(t)-1}\sum_{m=0}^{\infty}\frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}((m+1)!)^2}, \end{aligned}$$

where the last inequality comes from $\Gamma(m+3/2+\alpha(t)) \geqslant (m+1)!$ and the fact that the functions $\Gamma(x)$ and $\cos(\pi x)$ are continuous at every $1/2 < m_\alpha[a,b] \leqslant x \leqslant M_\alpha[a,b] < 3/2$.
As consequence,

$$E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} U_t^k(\tau)\right)^2\right] \leqslant C|\tau|^{2\alpha(t)-1}\left(1+\frac{1}{n}\sum_{k=1}^{n}(-\gamma_k)^2\sum_{m=0}^{\infty}\frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}((m+1)!)^2}\right). \tag{3.15}$$

Moreover, by the law of the large numbers, we obtain

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(-\gamma_k)^2\sum_{m=0}^{\infty}\frac{(-\gamma_k|\tau|)^{2m}}{2^{2m}((m+1)!)^2} &= E_\gamma\left[(-\gamma_1)^2\sum_{m=0}^{\infty}\frac{(-\gamma_1|\tau|)^{2m}}{2^{2m}((m+1)!)^2}\right] \\ &= \frac{1}{\Gamma(1-h)\lambda^2}\sum_{m=0}^{\infty}\frac{\Gamma(2m+3-h)}{2^{2m}((m+1)!)^2}\left(\frac{|\tau|}{\lambda}\right)^{2m} \\ &\leqslant C\sum_{m=0}^{\infty}\frac{\Gamma(2m+3-h)}{2^{2m}((m+1)!)^2}\left(\frac{1}{2}\right)^{2m} < \infty, \end{aligned} \tag{3.16}$$

where we used that the radius of convergence of the power series $\sum_{m=0}^{\infty}\frac{\Gamma(2m+3-h)}{2^{2m}((m+1)!)^2}x^m$ is equal to 1. Thus, by combining (3.15) and (3.16), we obtain (3.14).
Let us now turn to approach the second term in (3.13), it remains to prove that for every $n \geqslant 1$,

$$E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} V_t^k(\tau)\right)^2\right] \leqslant C^{\delta,\rho}|\tau|^{2\beta}. \tag{3.17}$$

To this end, from ((3.1)) we can write

$$V_t^k(\tau) = V_{t,1}^k(\tau) + V_{t,2}^k(\tau), \tag{3.18}$$

where

$$V_{t,1}^k(\tau) = \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right) \int_{-\infty}^{t+\tau} (t+\tau-u)^{\alpha(t+\tau)-1} e^{\gamma_k(t+\tau-u)} dB_u$$

and

$$V_{t,2}^k(\tau) = \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t+\tau} \left((t+\tau-u)^{\alpha(t+\tau)-1} - (t+\tau-u)^{\alpha(t)-1}\right) e^{\gamma_k(t+\tau-u)} dB_u.$$

Then, we have

$$E_B\left[\left(\frac{1}{n}\sum_{k=1}^n V_t^k(\tau)\right)^2\right] \leqslant 2E_B\left[\left(\frac{1}{n}\sum_{k=1}^n V_{t,1}^k(\tau)\right)^2\right] + 2E_B\left[\left(\frac{1}{n}\sum_{k=1}^n V_{t,2}^k(\tau)\right)^2\right].$$

Combining mean value theorem and the fact that any continuous function has a maximum on any compact interval, we get

$$\left|\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right|^2 \Gamma(2\alpha(t+\tau)-1) \quad \leqslant \quad C\left|\alpha(t+\tau)-\alpha(t)\right|^2.$$

Furthermore, since $\alpha$ is $\beta-$Hölder continuous, $2\sqrt{(-\gamma_j)(-\gamma_k)} \leqslant -\gamma_j - \gamma_k$ and $1 - 2\alpha(t+\tau) < 0$, we have

$$\begin{aligned}
& E_B\left[\left(\frac{1}{n}\sum_{k=1}^n V_{t,1}^k(\tau)\right)^2\right] \\
= \quad & \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right)^2 \frac{1}{n^2}\sum_{j,k=1}^n \int_{-\infty}^{t+\tau} (t+\tau-u)^{2\alpha(t+\tau)-2} e^{(\gamma_j+\gamma_k)(t+\tau-u)} du \\
= \quad & \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right)^2 \Gamma(2\alpha(t+\tau)-1)\frac{1}{n^2}\sum_{j,k=1}^n (-\gamma_j-\gamma_k)^{1-2\alpha(t+\tau)} \\
\leqslant \quad & C|\tau|^{2\beta}\left[\frac{1}{n}\sum_{k=1}^n (-\gamma_k)^{1/2-\alpha(t+\tau)}\right]^2.
\end{aligned}$$

Moreover,

$$\frac{1}{n}\sum_{k=1}^n (-\gamma_k)^{1/2-\alpha(t+\tau)} \underset{n\to\infty}{\longrightarrow} \frac{\lambda^{\alpha(t+\tau)-1/2}}{\Gamma(1-h)}\Gamma\left(3/2-\alpha(t+\tau)-h\right) < \infty.$$

Thus, we can conclude that for every $n \geqslant 1$,

$$E_B\left[\left(\frac{1}{n}\sum_{k=1}^n V_{t,1}^k(\tau)\right)^2\right] \quad \leqslant \quad C|\tau|^{2\beta}. \tag{3.19}$$

On the other hand, by the change of variable $t+\tau-u=x$, we have

$$\begin{aligned}
& E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} V_{t,2}^{k}(\tau)\right)^2\right] \\
= & \frac{1}{\Gamma(\alpha(t))^2}\frac{1}{n^2}\sum_{j,k=1}^{n}\int_{-\infty}^{t+\tau}\left[(t+\tau-u)^{\alpha(t+\tau)-1}-(t+\tau-u)^{\alpha(t)-1}\right]^2 e^{(\gamma_j+\gamma_k)(t+\tau-u)}du \\
= & \frac{1}{\Gamma(\alpha(t))^2}\frac{1}{n^2}\sum_{j,k=1}^{n}\int_{0}^{\infty}\left[x^{\alpha(t+\tau)-1}-x^{\alpha(t)-1}\right]^2 e^{(\gamma_j+\gamma_k)x}dx \\
= & \frac{[\alpha(t+\tau)-\alpha(t)]^2}{\Gamma(\alpha(t))^2}\frac{1}{n^2}\sum_{j,k=1}^{n}\int_{0}^{\infty}\left[\log(x)\,x^{c_{t,\tau}^{x}-1}\right]^2 e^{(\gamma_j+\gamma_k)x}dx
\end{aligned}$$

for some $c_{t,\tau}^{x}\in(m_\alpha[a,b],M_\alpha[a,b])$, where the last equality comes from mean value theorem.
Let $0<\delta<2m_\alpha[a,b]-1$, $0<\rho<3/2-M_\alpha[a,b]-h$ and set $\mu=1/(2m_\alpha[a,b]-1-\delta)$. Since $\alpha$ is $\beta-$Hölder continuous, we can write

$$\begin{aligned}
& E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} V_{t,2}^{k}(\tau)\right)^2\right] \\
= & \frac{[\alpha(t+\tau)-\alpha(t)]^2}{\Gamma(\alpha(t))^2}\frac{1}{n^2}\sum_{j,k=1}^{n}\int_{0}^{\infty}\log(x)^2\,x^{2c_{t,\tau}^{x}-2}\,e^{(\gamma_j+\gamma_k)x}dx \\
= & \frac{[\alpha(t+\tau)-\alpha(t)]^2}{\Gamma(\alpha(t))^2}\frac{1}{n^2}\sum_{j,k=1}^{n}\Big(\int_{0}^{1}\log^2(x)\,x^{2c_{t,\tau}^{x}-2}\,e^{(\gamma_j+\gamma_k)x}dx \\
& +\int_{1}^{\infty}\log^2(x)\,x^{2c_{t,\tau}^{x}-2}\,e^{(\gamma_j+\gamma_k)x}dx\Big) \\
\leqslant & C|\tau|^{2\beta}\left(\int_{0}^{1}x^{2m_\alpha[a,b]-2-\delta}dx+\frac{1}{n^2}\sum_{j,k=1}^{n}\int_{1}^{\infty}x^{2M_\alpha[a,b]-2+2\rho}\,e^{(\gamma_j+\gamma_k)x}dx\right) \\
\leqslant & C|\tau|^{2\beta}\left(\mu+\frac{1}{n^2}\sum_{j,k=1}^{n}\int_{0}^{\infty}x^{2M_\alpha[a,b]-2+2\rho}\,e^{(\gamma_j+\gamma_k)x}dx\right) \\
\leqslant & C|\tau|^{2\beta}\left(\mu+\frac{1}{n^2}\sum_{j,k=1}^{n}\frac{\Gamma(2M_\alpha[a,b]-1+2\rho)}{(-\gamma_j-\gamma_k)^{2M_\alpha[a,b]-1+2\rho}}\right) \\
\leqslant & C|\tau|^{2\beta}\left(\mu+\frac{2^{1-2\rho-2M_\alpha[a,b]}}{n^2}\sum_{j,k=1}^{n}\frac{\Gamma(2M_\alpha[a,b]-1+2\rho)}{\left(\sqrt{(-\gamma_j)(-\gamma_k)}\right)^{2M_\alpha[a,b]-1+2\rho}}\right) \\
= & C|\tau|^{2\beta}\left(\mu+2^{1-2\rho-2M_\alpha[a,b]}\Gamma(2M_\alpha[a,b]-1+2\rho)\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{(-\gamma_k)^{M_\alpha[a,b]-1/2+\rho}}\right)^2\right).
\end{aligned}$$

Combining this with

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{(-\gamma_k)^{M_\alpha[a,b]-1/2+\rho}}\underset{n\to\infty}{\longrightarrow}\frac{\lambda^{M_\alpha[a,b]-1/2+\rho}}{\Gamma(1-h)}\Gamma\left(3/2-M_\alpha[a,b]-h-\rho\right)<\infty,$$

we then deduce that for every $n \geqslant 1$,

$$E_B\left[\left(\frac{1}{n}\sum_{k=1}^{n} V_{t,2}^{k}(\tau)\right)^2\right] \leqslant C^{\delta,\rho}|\tau|^{2\beta}. \tag{3.20}$$

Thus, combining (3.19) and (3.20), we get (3.17).
Therefore, using (3.13), (3.14) and (3.17), we obtain that for every $n \geqslant 1$

$$E_B\left[\left(Y^n_{\alpha(t+\tau)}(t+\tau) - Y^n_{\alpha(t)}(t)\right)^2\right] \leqslant C^{\delta,\rho}|\tau|^{\min\{2m_\alpha[a,b]-1,2\beta\}}. \tag{3.21}$$

Let $[a,b]$ be given interval for arbitrary $a,b \in \mathbb{R}$ such that $a < b$. Now for $s < t \in [a,b]$, we can find $2k+2$ points $u_1,\ldots,u_{2k+2} \in [s,t]$ with $b-a = k\min(\lambda/2,1)+c$, $0 \leqslant c < \min(\lambda/2,1)$ and $0 < u_{i+1}-u_i < \min(\lambda/2,1)$ such that $[t,s] = \bigcup_{i=1}^{2k+2}[u_i,u_{i+1}]$.
Using Minkowski's Inequality, (3.21) and Proposition. (4.1) (because $0 < \min\{2m_\alpha[a,b]-1,2\beta\} < 1$ ) we can conclude that for every $n \geqslant 1$ and $s,t \in [a,b]$

$$E_B\left(Y^n_{\alpha(t)}(t) - Y^n_{\alpha(s)}(s)\right)^2 \leqslant C^{\delta,\rho}|\,t-s\,|^{\min\{2m_\alpha[a,b]-1,2\beta\}}.$$

Consequently, Given $r > 0$ and using again the fact that $Y^n_\alpha$ is $P_\gamma$ almost surely Gaussian process, there exists a constant $C_r$ depending only on r such that

$$\begin{aligned} E_B\left|Y^n_{\alpha(t)}(t) - Y^n_{\alpha(s)}(s)\right|^r &= C_r\left(E_B[Y^n_{\alpha(t)}(t) - Y^n_{\alpha(s)}(s)]^2\right)^{r/2} \\ &\leqslant C_r(C^{\delta,\rho})^{r/2}|t-s|^{r\min\{m_\alpha[a,b]-1/2,\beta\}}, \end{aligned}$$

for all $n \geqslant 1$ and $s,t \in [a,b]$. Choosing r so that $r\min\{m_\alpha[a,b]-1/2,\beta\} > 1$, Theorem (2.1), implies that the family $(Y^n_\alpha)_{n\geqslant 1}$ is tight. We conclude therefore the desired result. ■

### 3.2. Properties of the GWmOU process and asymptotic behavior with respect to the parameter $\lambda$.

In this section we study several interesting properties of the GWmOU process, $Y^\lambda_\alpha$, such as the Hölder exponent and short-range dependence. In addition, we investigate the asymptotic behavior of the process $Y^\lambda_\alpha$ when $\lambda$ goes to infinity, and when $\lambda$ goes to zero.

Let us first compute the variance and the covariance of $Y^\lambda_\alpha$ . An easy computation shows that, for all $t \in \mathbb{R}$ the variance is given by

$$\begin{aligned} E_B\left[(Y^\lambda_{\alpha(t)}(t))^2\right] &= \frac{\lambda^{2-2h}}{\Gamma(\alpha(t))^2}\int_{-\infty}^{t}(\lambda+t-s)^{2h-2}(t-s)^{2\alpha(t)-2}ds \\ &= \frac{\lambda^{2\alpha(t)-1}}{\Gamma(\alpha(t))^2}\,\beta(3-2h-2\alpha(t),2\alpha(t)-1), \end{aligned} \tag{3.22}$$

where $\beta$ is a beta function defined by $\beta(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}du$ for $x,y > 0$. We then deduce that the GWmOU process is in general not stationary.

In addition, for $s < t$, using the change of variable $z = \lambda/(\lambda + s - u)$, the covariance $Y^{\lambda}_{\alpha}$ is given by

$$\begin{aligned} & E_B(Y^{\lambda}_{\alpha(t)}(t)Y^{\lambda}_{\alpha(s)}(s)) \\ &= \frac{1}{\Gamma(\alpha(t))\Gamma(\alpha(s))} \int_{-\infty}^{s} \left(\frac{\lambda}{\lambda+t-u}\right)^{1-h} \left(\frac{\lambda}{\lambda+s-u}\right)^{1-h} (t-u)^{\alpha(t)-1}(s-u)^{\alpha(s)-1}du \\ &= \frac{\lambda^{\alpha(t)+\alpha(s)-1}}{\Gamma(\alpha(t))\Gamma(\alpha(s))} G(\alpha(t),\alpha(s),h,t-s/\lambda), \end{aligned} \tag{3.23}$$

with

$$G(a,b,c,d) = \int_0^1 \frac{(1+d\,z)^{c-1}}{(1+[d-1]z)^{1-a}}(1-z)^{b-1}z^{2-[a+b]-2c}dz.$$

In order to study the local properties of the GWmOU process we will need the following result.

PROPOSITION 3.1. *Fix a compact interval* $[a,b] \subset \mathbb{R}$.

1. *If* $0 < h < 3/2 - M_\alpha[a,b]$, *then there exists a constant* $C^{\delta,\rho}$ *such that*

$$E_B([Y^{\lambda}_{\alpha(t+\tau)}(t+\tau) - Y^{\lambda}_{\alpha(t)}(t)]^2) \leqslant C^{\delta,\rho}\,|\,\tau\,|^{\min\{2m_\alpha[a,b]-1,2\beta\}} \tag{3.24}$$

*for all* $t, t+\tau \in [a,b]$ *satisfying* $|\,\tau\,| < \min(\lambda/2, 1)$.

*2 If* $0 < h < 3/2 - M_\alpha[a,b]$, $M_\alpha[a,b] < 1$ *and we add the assumption* $\alpha(t) - 1/2 < \beta$ *for all t, then*

*2.1 There exist constants* $C_2$ *and* $\epsilon < 1$ *such that*

$$E_B([Y^{\lambda}_{\alpha(t+\tau)}(t+\tau) - Y^{\lambda}_{\alpha(t)}(t)]^2) \geqslant C_2/2\,|\,\tau\,|^{2M_\alpha[a,b]-1} \tag{3.25}$$

*for all* $t, t+\tau \in [a,b]$ *satisfying* $|\,\tau\,| < \epsilon$.

*2.1.1 as* $\tau \to 0$

$$E_B([Y^{\lambda}_{\alpha(t+\tau)}(t+\tau) - Y^{\lambda}_{\alpha(t)}(t)]^2) = C_2|\tau|^{\alpha(t)-1/2} + O(|\tau|^{2\alpha(t)-1}). \tag{3.26}$$

Proof. The inequality (3.24) is a direct consequence of (3.21) and (3.4).
Let us now prove (3.25). For convenience, let us first introduce the following notation. For all $t, t+\tau \in [a,b]$ satisfying $|\tau| < 1$, we set

$$\begin{aligned} U^{\lambda}_t(\tau) &= Y^{\lambda}_{\alpha(t)}(t+\tau) - Y^{\lambda}_{\alpha(t)}(t); \qquad V^{\lambda}_t(\tau) = Y^{\lambda}_{\alpha(t+\tau)}(t+\tau) - Y^{\lambda}_{\alpha(t)}(t+\tau) \\ &= V^{\lambda}_{t,1}(\tau) + V^{\lambda}_{t,2}(\tau), \end{aligned}$$

where

$$\begin{aligned} V^{\lambda}_{t,1}(\tau) &= \left(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\right) \int_{-\infty}^{t+\tau} (t+\tau-u)^{\alpha(t+\tau)-1} \frac{\lambda^{1-h}}{(\lambda+t+\tau-u)^{1-h}} dB_u, \\ V^{\lambda}_{t,2}(\tau) &= \frac{1}{\Gamma(\alpha(t))} \int_{-\infty}^{t+\tau} \left[(t+\tau-u)^{\alpha(t+\tau)-1} - (t+\tau-u)^{\alpha(t)-1}\right] \frac{\lambda^{1-h}}{(\lambda+t+\tau-u)^{1-h}} dB_u. \end{aligned}$$

Hence we have

$$\begin{aligned} E_B(Y^{\lambda}_{\alpha(t+\tau)}(t+\tau) - Y^{\lambda}_{\alpha(t)}(t))^2 &\geqslant E_B(U^{\lambda}_t(\tau)^2) + 2E_B[U^{\lambda}_t(\tau)V^{\lambda}_t(\tau))] \\ &\geqslant E_B(U^{\lambda}_t(\tau)^2) - 2E(U^{\lambda}_t(\tau)^2)^{1/2}E(V^{\lambda}_t(\tau)^2)^{1/2}. \end{aligned}$$

(3.27)

The last inequality comes by Cauchy-Schwarz Inequality. Using Lemma (4.1) and the inequality (4.10) in Lemma (4.2), there exist constant $C_1$ and $C_2$ depending only on $[a, b]$, $\lambda$ and $h$ such that

$$C_2|\,\tau\,|^{2\alpha(t)-1} \leqslant E([U^{\lambda}_t]^2) \leqslant C_1\,|\,\tau\,|^{2\alpha(t)-1}. \tag{3.28}$$

On the other hand, we have

$$E_B(V^{\lambda}_t(\tau)^2) \leqslant 2(E_B[V^{\lambda}_{t,1}(\tau)^2] + E_B[V^{\lambda}_{t,2}(\tau)^2]).$$

A standard computation combined with mean value theorem and the fact that any continuous function has maximum on any compact interval, we obtain

$$\begin{aligned} E_B(V^{\lambda}_{t,1}(\tau)^2) &= \Big(\frac{1}{\Gamma(\alpha(t+\tau))} - \frac{1}{\Gamma(\alpha(t))}\Big)^2 \lambda^{2\alpha(t+\tau)-1}\beta(3-2\alpha(t+\tau)-2h,\, 2\alpha(t+\tau)-1)) \\ &\leqslant C|\,\alpha(t+\tau)-\alpha(t)\,|^2 \\ &\leqslant C|\tau\,|^{2\beta}. \end{aligned}$$

Besides, by the change of variable $x = t+\tau-u$, we have

$$\begin{aligned} E_B(V^{\lambda}_{t,2}(\tau)^2) &= \frac{\lambda^{2-2h}}{\Gamma(\alpha(t))^2}\int_{-\infty}^{t+\tau}[(t+\tau-u)^{\alpha(t+\tau)-1} - (t+\tau-u)^{\alpha(t)-1}]^2(\lambda+t+\tau-u)^{2h-2}du \\ &= \frac{\lambda^{2-2h}}{\Gamma(\alpha(t))^2}\int_0^{\infty}[x^{\alpha(t+\tau)-1} - x^{\alpha(t)-1}]^2(\lambda+x)^{2h-2}dx \\ &= \frac{\lambda^{2-2h}[\alpha(t+\tau)-\alpha(t)]^2}{\Gamma(\alpha(t))^2}\int_0^{\infty} log(x)^2 x^{2a^x_{t,\tau}-2}\,(\lambda+x)^{2h-2}du \end{aligned}$$

for some $a^x_{t,\tau} \in (m_\alpha[a,b], M_\alpha[a,b])$, the last equality comes from mean value theorem. Let $0 < \sigma < 2m_\alpha[a,b]-1$ and $0 < \varsigma < 3/2 - M_\alpha[a,b] - h$. Then since $\alpha$ is $\beta$-Hölder continuous, for $c = 3 - 2h - 2M_\alpha[a,b] - 2\varsigma$ and $d = 2M_\alpha[a,b] - 1 + 2\varsigma$, we have

$$\begin{aligned} E_B(V^{\lambda}_{t,2}(\tau)^2) &= \frac{\lambda^{2-2h}[\alpha(t+\tau)-\alpha(t)]^2}{\Gamma(\alpha(t))^2}\Big(\int_0^1 \log^2(x)x^{2c^x_{t,\tau}-2}(\lambda+x)^{2h-2}dx \\ &\qquad + \int_1^{\infty}\log^2(x)\,x^{2a^x_{t,\tau}-2}(\lambda+x)^{2h-2}dx\Big) \\ &\leqslant C|\,\tau\,|^{2\beta}\left(\int_0^1 x^{2m_\alpha[a,b]-2-\sigma}dx + \int_1^{\infty} x^{2M_\alpha[a,b]-2+2\varsigma}(\lambda+x)^{2h-2}dx\right) \\ &\leqslant C|\,\tau\,|^{2\beta}\,(1/(2m_\alpha[a,b]-1-\sigma) + \beta(c,\,d)) \\ &\leqslant C^{\sigma,\varsigma}|\,\tau\,|^{2\beta}. \end{aligned}$$

We then deduce that

$$E_B(V^{\lambda}_t(\tau)^2) \leqslant C^{\sigma,\varsigma}|\,\tau\,|^{2\beta}. \tag{3.29}$$

Combining (3.28), (3.29) and by Cauchy-Schwarz Inequality,

$$\begin{aligned} |\, E(U_t^\lambda(\tau)V_t^\lambda(\tau))\,| &\leqslant E(U_t^\lambda(\tau)^2)^{1/2}E(V_t^\lambda(\tau)^2)^{1/2} \\ &\leqslant C^{\sigma,\varsigma}|\,\tau\,|^{\beta+\alpha(t)-1/2} \end{aligned} \tag{3.30}$$

Thus, by plugging (3.28) and (3.30) in (3.27), we get

$$\begin{aligned} E_B(Y^\lambda_{\alpha(t+\tau)}(t+\tau) - Y^\lambda_{\alpha(t)}(t))^2 &\geqslant C_2|\,\tau\,|^{2\alpha(t)-1} - C^{\sigma,\varsigma}\,|\,\tau\,|^{\alpha(t)-1/2+\beta} \\ &\geqslant |\,\tau\,|^{2M_\alpha[a,b]-1}(C_2 - C^{\sigma,\varsigma}|\,\tau\,|^{\beta-M_\alpha[a,b]+1/2}) \end{aligned}$$

By assuming that $\alpha(t) - 1/2 \leqslant M_\alpha[a,b] - 1/2 < \beta$, the function

$$g : \tau \mapsto C_2 - C^{\sigma,\varsigma}|\,\tau\,|^{\beta-M_\alpha[a,b]+1/2}$$

is continuous in $\tau$ and converges to $C_2$ when $\tau$ goes to 0. So there exists $\epsilon > 0$ so that for $|\tau| < \epsilon$, the function g is greater than $C = C_2/2$. Which gives the inequality (3.25).
On the other hand by the assumption $\alpha(t) - 1/2 \leqslant M_\alpha[a,b] - 1/2 < \beta$ and using the equivalence (4.11) in Lemma (4.2), (3.29) and (3.30), we obtain immediately (3.26). ■

In the following, we state interesting properties of the GWmOU process such as continuity, Hölder exponent at point t, Hausdorff dimension and local asymptotic self-similarity. The same properties hold for the WmOU process, the proofs of which are based on [15], Lemma 3.1, of which Proposition (3.1) is the counterpart for the GWmOU process. having Proposition (3.1) at hand, the proofs of the GWmOU process are pursued analogously to the one in [15]. Therefore, we refer the reader to [15] and omit the proofs.

The first property to be mentioned is stated in the following way:

**3.2.1.** *Continuity*

PROPOSITION 3.2. *The process* $\{Y^\lambda_{\alpha(t)}(t), t \in \mathbb{R}\}$ *admits a continuous modification.*

In the following properties, Hölder exponent, Hausdorff dimension and Local asymptotic self-similarity, we make the additional assumptions that $\alpha(t) - 1/2 < \beta$ for all $t$ in the domain of $\alpha$ and $M_\alpha[a,b] < 1$.

**3.2.2.** *Hölder exponent*

PROPOSITION 3.3. *Let* $[a,b] \subset \mathbb{R}$ *be an interval and for any* $0 \leqslant \eta < m_\alpha[a,b] - 1/2$. *With probability one, there exists a constant* $C^{\delta,\rho}_\eta$ *such that*

$$|\,Y^\lambda_{\alpha(t)}(t) - Y^\lambda_{\alpha(s)}(s)\,| \leqslant C^{\delta,\rho}_\eta\,|\,t-s\,|^\eta \qquad \forall\, t,s \in [a,b].$$

We now turn to the Hölder continuity of the GWmOU process. Let us first recall the following definition.

DEFINITION 3.1. A real-valued function is said to have the Hölder exponent $\beta$ at point $t_0$ if and only if

(1) for any $\gamma$ such that $\gamma < \beta$:

$$\lim_{h\to 0}\frac{|f(t_0+h)-f(t_0)|}{|h|^{\gamma}}=0,$$

(2) for any $\gamma > \beta$:

$$\limsup_{h\to 0}\frac{|f(t_0+h)-f(t_0)|}{|h|^{\gamma}}=\infty.$$

PROPOSITION 3.4. *With probability one, the Hölder exponent of $Y^{\lambda}_{\alpha(t)}(t)$ at the point $t_0$ in the domain is $\alpha(t_0)-1/2$.*

**3.2.3.** *Hausdorff dimension* Let $dim_H A$, $\underline{dim}_B A$, and $\overline{dim}_B A$ denote the Hausdorff dimension, the lower box dimension, and the upper box dimension of a set $A$ in $\mathbb{R}^n$, respectively. Given a compact interval $[a,b]\subset\mathbb{R}$, $\mathbf{G}_\alpha[a,b]=\{(t,Y^{\lambda}_{\alpha(t)}(t)) : t\in[a,b]\}$ stands for the graph of the process $Y^{\lambda}_{\alpha(t)}(t)$ restricted to $[a,b]$. For more information of these notions see [11]. We now formulate our result.

PROPOSITION 3.5. *Let $[a,b]$ be an interval in the domain of definition of $\alpha$, with probability one, $dim_H \boldsymbol{G}_\alpha[a,b]=\underline{dim}_B \boldsymbol{G}_\alpha[a,b]=\overline{dim}_B \boldsymbol{G}_\alpha[a,b]=5/2-m_\alpha[a,b]$.*

**3.2.4.** *Local asymptotic self-similarity* The WmOU process is locally asymptotically self-similar, in the following sense as defined in [4].

DEFINITION 3.2. Let $X(t)$ be a Gaussian process. We say that X(t) is locally asymptotically self-similar with parameter H at a point $t_0$ if the limit process

$$\left\{\lim_{h\to 0^+}\frac{X(t_0+hu)-X(t_0)}{h^H},\ u\in\mathbb{R}\right\},$$

exists and is nontrivial for every $t_0$.

This property remains true for the GWmOU process. Before stating this result, let us first recall that the fractional Brownian motion with Hurst index H is the centered Gaussian process with covariance

$$E(B^H(t)B^H(s))=\frac{1}{2}\left[|\,t\,|^{2H}+|\,s\,|^{2H}-|\,t-s\,|^{2H}\right].$$

PROPOSITION 3.6. *For any $t_0$ the stochastic process*

$$\left\{\lim_{h\to 0^+}\frac{Y^{\lambda}_{\alpha(t_0+hu)}(t_0+hu)-Y^{\lambda}_{\alpha(t_0)}(t_0)}{h^{\alpha(t_0)/2-1/4}},\ u\in\mathbb{R}\right\}$$

*is, modulo a constant, fractional Brownian motion with Hurst index $\alpha(t_0)/2-1/4$.*

**3.2.5.** *Short-range dependence* We are now interested in studying the strength of the dependence of the GWmOU process.

DEFINITION 3.3 ([12]). Let $X(t)$ be a Gaussian process with covariance denoted by $c(s,t) = \mathrm{cov}(X(s), X(t))$ and correlation $\rho(s,t)$ defined by

$$\rho(s,t) = \frac{c(s,t)}{\sqrt{c(t,t)\, c(s,s)}}.$$

we say that $X(t)$ is long-range dependence if

$$\int_0^\infty |\, \rho(t, t+\tau)| d\tau = \infty,$$

and it is short-range dependence if the integral is finite.

The following Lemma provides an upper bound for the inverse of the variance of the process $Y_\alpha^\lambda$ with $0 < h < 3/2 - \alpha_{\sup}$ and $1/2 < \alpha(t)$ for all $t$.

LEMMA 3.1. *For all $t$ the function $t \mapsto \dfrac{1}{E_B(Y^\lambda_{\alpha(t)}(t))^2}$ is an upper bounded function.*

P r o o f. From (3.22), we find that

$$\frac{1}{E_B(Y^\lambda_{\alpha(t)}(t)^2)} \quad = \quad \frac{\lambda^{1-2\alpha(t)}[2\alpha(t)-1]\Gamma(\alpha(t))^2\Gamma(2-2h)}{\Gamma(2\alpha(t))\Gamma(3-2h-2\alpha(t))}.$$

The functions $z \mapsto \lambda^{1-2z}$, $z \mapsto 2z-1$, $z \mapsto \Gamma(z)^2$, $z \mapsto \Gamma(2z)$ and $z \mapsto \Gamma(3-2h-2z)$ are continuous for $z \in [1/2, \alpha_{\sup}]$. As a consequence

$$\frac{1}{E_B(Y^\lambda_{\alpha(t)}(t))^2} \leqslant C. \tag{3.31}$$

■

We are thus led to the following short range dependence property of the GWmOU process.

PROPOSITION 3.7. *For $0 < h < 1 - \alpha_{\sup}$, the GWmOU process has short memory.*

P r o o f. Set $y = \tau/\lambda$, using (3.23) and (3.31), we have

$$0 \leqslant \rho_\alpha(t, t+\tau) \quad \leqslant \quad CG(\alpha(t+\tau), \alpha(t), h, y).$$

Since $0 \leqslant u \leqslant 1$ and $1/2 < \alpha(t) < 1$ for all $t$ we obtain

$$\begin{aligned} & G(\alpha(t+\tau), \alpha(t), h, y) \\ &= \int_0^1 u^{2-[\alpha(t)+\alpha(t+\tau)]-2h}(1-u)^{\alpha(t)-1}\,(1+yu)^{h-1}\,(yu+1-u)^{\alpha(t+\tau)-1}\,du \\ &\leqslant y^{\alpha(t+\tau)-1}\,(y+1)^{h-1}\int_0^1 u^{-\alpha(t)-h}(1-u)^{\alpha(t)-1}du. \end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^\infty |\rho_\alpha(t,t+\tau)|d\tau &\leqslant C\int_0^\infty y^{\alpha(t+\lambda y)-1}(y+1)^{h-1}dy\int_0^1 u^{-\alpha(t)-h}(1-u)^{\alpha(t)-1}du.\\ &\leqslant C\beta(1-h-\alpha_{\sup},1/2)\beta(1-h-\alpha(t),\alpha(t))\\ &< \infty,\end{aligned}$$

the last inequality comes from the assumption $0<h<1-\alpha_{\sup}$, which completes the proof. ■

We are now interested in analyzing the asymptotic behavior of the process $Y_\alpha^\lambda$ when $\lambda\to\infty$.

PROPOSITION 3.8. *Let $\{Y^\lambda_{\alpha(t)}(t), t\geqslant 0\}$ be the GWmOU process restricted to $t\geqslant 0$ and set $\alpha(t)=H(t)+1/2$ with $0<h<3/2-\alpha_{\sup}$.*
*Then for fixed t in $\mathbb{R}^+$,*

$$Y^\lambda_{\alpha(t)}(t)-Y^\lambda_{\alpha(t)}(0)\underset{\lambda\to\infty}{\to} B_{H(t)}(t)\qquad \text{in } L^2(\Omega_B).$$

P r o o f. For convenience, let us first introduce the following notation. Set for each $s\leqslant t$, $c_\lambda(t-s)=(\lambda/(\lambda+t-s))^{1-h}$, for each $t\geqslant 0$, let $X^\lambda_{\alpha(t)}(t)=Y^\lambda_{\alpha(t)}(t)-Y^\lambda_{\alpha(t)}(0)$ and denote by

$$\begin{aligned}A^\lambda_{H(t)}(t) &= \int_{-\infty}^0 \left[c_\lambda(t-s)(t-s)^{H(t)-1/2}-c_\lambda(-s)(-s)^{H(t)-1/2}\right]-\\ &\quad \left[(t-s)^{H(t)-1/2}-(-s)^{H(t)-1/2}\right]dB_s.\\ &=: \int_{-\infty}^0 A^\lambda_{1,H(t)}(t,s)-A_{1,H(t)}(t,s)dB_s,\\ D^\lambda_{H(t)}(t) &= \int_0^t (t-s)^{H(t)-1/2}(c_\lambda(t-s)-1)dB_s.\end{aligned}$$

By substituting $\alpha(t)$ with $H(t)+1/2$ we get that

$$\begin{aligned}X^\lambda_{\alpha(t)}(t)-B_{H(t)}(t) &= X^\lambda_{H(t)+1/2}(t)-B_{H(t)}(t)\\ &= \frac{1}{\Gamma(H(t)+1/2)}\left[A^\lambda_{H(t)}(t)+D^\lambda_{H(t)}(t)\right].\end{aligned}$$

Then, we have

$$\begin{aligned}E_B\left(X^\lambda_{\alpha(t)}(t)-B_{H(t)}(t)\right)^2 &= \frac{1}{\Gamma(H(t)+1/2)^2}E_B(A^\lambda_{H(t)}(t)+D^\lambda_{H(t)}(t))^2\\ &\leqslant \frac{2}{\Gamma(H(t)+1/2)^2}\left(E_B(A^\lambda_{H(t)}(t)^2)+E_B(D^\lambda_{H(t)}(t)^2)\right).\end{aligned}$$

Let us first evaluate the asymptotic behavior of $E_B(A^\lambda_{H(t)}(t)^2)$ when $\lambda$ goes to infinity.
For fixed $t\geqslant 0$, it is easily seen that

$$A^\lambda_{1,H(t)}(t,s)\underset{\lambda\to\infty}{\to} A_{1,H(t)}(t,s). \tag{3.32}$$

Using the elementary inequality, for any $p \geqslant 0$ and $x, y \in \mathbb{R}$

$$|| \, x \, |^p - | \, y \, |^p| \leqslant (p \vee 1) 2^{(p-2)^+} [|x-y|^p + |y|^{(p-1)^+} |x-y|^{(p \wedge 1)}]$$

and the fact that $c_\lambda(x) \leqslant 1$ for all $x > -\lambda$, we have for $s < 0$,

$$\begin{aligned} &\left| c_\lambda(t-s)(t-s)^{H(t)-1/2} - c_\lambda(-s)(-s)^{H(t)-1/2} \right| \\ &\leqslant \; c_\lambda(t-s) \left| (t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2} \right| + (-s)^{H(t)-1/2} |c_\lambda(t-s) - c_\lambda(-s)| \\ &\leqslant \; \left| (t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2} \right| + 2t^{1-h} (-s)^{H(t)-1/2} (t-s)^{h-1}. \end{aligned}$$

Moreover, for fixed $t > 0$ such that $H(t) \neq 1/2$, when $s$ goes to $-\infty$, we get

$$\left( (t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2} \right)^2 \sim (H(t) - 1/2)^2 \, t^2 (-s)^{2H(t)-3}.$$

As a result, $s \mapsto \left( (t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2} \right)^2$ is integrable at $-\infty$, because $2H(t) - 3 < -1$, and as $s$ goes to $0^-$ as well, since $2H(t) - 1 > -1$. Consequently,

$$\int_{-\infty}^{0} \left( (t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2} \right)^2 ds < \infty.$$

Also, by the hypothesis $2 - 2H(t) - 2h > 0$

$$\begin{aligned} \int_{-\infty}^{0} (-s)^{2H(t)-1} (t-s)^{2h-2} ds \; &= \; t^{2H(t)+2h-2} \beta(2 - 2H(t) - 2h, 2H(t)) \\ &< \; \infty. \end{aligned}$$

The dominated convergence theorem gives that for fixed $t \geqslant 0$, $\lim_{\lambda \to \infty} E_B(A^\lambda_{H(t)}(t)^2) = 0$. Similarly, one shows that for fixed $t \geqslant 0$, $\lim_{\lambda \to \infty} E_B(D^\lambda_{H(t)}(t)^2) = 0$, which proves the desired result. ∎

On the other hand, we consider now the asymptotic behavior of $Y^\lambda_\alpha$ when $\lambda$ goes to zero. In the following result, it is assumed that $\alpha(t) = \alpha$ for all $t$, $1 - \alpha < h < 3/2 - \alpha$ and $1/2 < \alpha < 1$.

PROPOSITION 3.9. *Let $\{\hat{Y}^\lambda_\alpha, \, t \geqslant 0\}$ be the process defined by*

$$\hat{Y}^\lambda_\alpha(t) = \lambda^{h-1} \int_0^t Y^\lambda_\alpha(s) \, ds, \qquad t \geqslant 0.$$

*Then*

$$\hat{Y}^\lambda_\alpha(t) \underset{\lambda \to 0}{\longrightarrow} Y_\alpha(t) \qquad \text{in } L^2(\Omega_B),$$

*where*

$$Y_\alpha(t) := \frac{1}{\Gamma(\alpha)(h + \alpha - 1)} \left[ \int_{-\infty}^{0} (t-u)^{h+\alpha-1} - (-u)^{h+\alpha-1} dB_u + \int_0^t (t-u)^{h+\alpha-1} dB_u \right].$$

*Moreover, the process $(Y_\alpha(t))_{t \geqslant 0}$ is (modulo a constant) a fractional Brownian motion with Hurst index $h + \alpha - 1/2$.*

P r o o f. For each $t \geqslant 0$, we have

$$\begin{aligned}\lambda^{h-1}\int_0^t Y_\alpha^\lambda(s)ds &= \frac{1}{\Gamma(\alpha)}\int_{-\infty}^t dB_u \int_{u\vee 0}^t (\lambda+s-u)^{h-1}(s-u)^{\alpha-1}\,ds \\ &= \frac{1}{\Gamma(\alpha)}\int_{-\infty}^0 dB_u \int_0^t (\lambda+s-u)^{h-1}(s-u)^{\alpha-1}ds + \\ &\quad \frac{1}{\Gamma(\alpha)}\int_0^t dB_u \int_u^t (\lambda+s-u)^{h-1}(s-u)^{\alpha-1}ds.\end{aligned}$$

Using the same computations as in the proof of Proposition (3.8), it is easily checked that for every $t \geqslant 0$,

$$\hat{Y}_\alpha^\lambda(t) \underset{\lambda\to 0}{\to} Y_\alpha(t) := \frac{1}{\Gamma(\alpha)(h+\alpha-1)}\left[\int_{-\infty}^0 (t-u)^{h+\alpha-1}-(-u)^{h+\alpha-1}dB_u + \int_0^t (t-u)^{h+\alpha-1}dB_u\right]$$

in $L^2(\Omega_B)$. Moreover, it is obvious that the process $(Y_\alpha(t))_{t\geqslant 0}$ is (modulo a constant) a fractional Brownian motion (of moving average definition) with Hurst index $h+\alpha-1/2$. ■

## 4. APPENDIX

PROPOSITION 4.1. *For all* $0<p<1$ *and* $k \geqslant 2$

$$\sum_{i=1}^k x_i^p \leqslant 2^{(k-1)(1-p)}\left(\sum_{i=1}^k x_i\right)^p, \quad x_i \geqslant 0, \textit{ for } i=1,\dots,k. \tag{4.1}$$

P r o o f. For $k \geqslant 2$, $0<p<1$ and $x_i \geqslant 0$ for all $i=1,\dots,k$, we will denote by $A(k)$ the following inequality

$$A(k):\ \sum_{i=1}^k x_i^p \leqslant 2^{(k-1)(1-p)}\left(\sum_{i=1}^k x_i\right)^p.$$

Let $k=2$, since the function $x \to x^p$, $x \geqslant 0$ is concave for every $0<p<1$, we get

$$x^p + y^p \leqslant 2^{1-p}(x+y)^p,$$

then $A(2)$ holds true.
Let us assume that $A(n-1)$ holds. Using $A(2)$ and $A(n-1)$ and by easy computations, we get $A(n)$. Then by induction the statement $A(k)$ holds for each $k \geqslant 2$, and the proof is complete. ■

Throughout the appendix, it is supposed that $0<h<3/2-M_\alpha[a,b]$, $M_\alpha[a,b]<1$ and $m_\alpha[a,b]>1/2$ for any compact interval $[a,b]\subset\mathbb{R}$.

LEMMA 4.1. *Fix a compact interval* $[a,b]\subset\mathbb{R}$. *There exists a constant* $C$ *depending only on* $[a,b]$, $\lambda$ *and* $h$ *such that*

$$E_B([Y_{\alpha(t)}^\lambda(t+\tau)-Y_{\alpha(t)}^\lambda(t)]^2) \leqslant C\,|\,\tau\,|^{2\alpha(t)-1},$$

*for all* $t, t+\tau \in [a,b]$ *and* $|\tau|<1$.

P r o o f. Set $\eta = 3 - 2h - 2\alpha(t), \nu = 2\alpha(t) - 1$ and $y = |\tau|/\lambda$. Using (3.22), we get

$$\begin{aligned} & E_B([Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t)]^2) \\ &= E_B([Y^\lambda_{\alpha(t)}(t+\tau)]^2) + E_B([Y^\lambda_{\alpha(t)}(t)]^2) - 2E_B(Y^\lambda_{\alpha(t)}(t+\tau)Y^\lambda_{\alpha(t)}(t)) \\ &= \frac{2\lambda^\nu}{\Gamma(\alpha(t))^2}\beta(\eta,\nu) - 2E_B(Y^\lambda_{\alpha(t)}(t+\tau)Y^\lambda_{\alpha(t)}(t)). \end{aligned} \tag{4.2}$$

Let us first evaluate the second term of (4.2). By (3.23) we have,

$$\begin{aligned} & -2E_B(Y^\lambda_{\alpha(t)}(t+\tau)Y^\lambda_{\alpha(t)}(t)) \\ &= \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2}\int_0^1 u^{\eta-1}(1-u)^{\alpha(t)-1}(1+yu)^{h-1}(yu+1-u)^{\alpha(t)-1}du. \end{aligned} \tag{4.3}$$

By applying the mean value theorem to the function $t \mapsto (1+yt)^{h-1}$, for $t \in [0,u]$, we obtain

$$\begin{aligned} & -2E_B(Y^\lambda_{\alpha(t)}(t+\tau)Y^\lambda_{\alpha(t)}(t)) \\ &= \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2}\int_0^1 u^{\eta-1}(1-u)^{\nu-1}\left(1+y\frac{u}{1-u}\right)^{\alpha(t)-1}du + \frac{2\lambda^\nu(1-h)}{\Gamma(\alpha(t))^2}y \\ & \quad \int_0^1 (1+yC_u)^{h-2}u^\eta(1-u)^{\alpha(t)-1}(yu+1-u)^{\alpha(t)-1}du \\ &= A_{\lambda,h}(\alpha(t),y) + B_{\lambda,h}(\alpha(t),y), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} A_{\lambda,h}(\alpha(t),y) &= \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2}\int_0^1 u^{\eta-1}(1-u)^{\nu-1}\left(1+y\frac{u}{1-u}\right)^{\alpha(t)-1}du, \\ B_{\lambda,h}(\alpha(t),y) &= \frac{2\lambda^\nu(1-h)}{\Gamma(\alpha(t))^2}y\int_0^1 (1+yC_u)^{h-2}u^\eta(1-u)^{\alpha(t)-1}(yu+1-u)^{\alpha(t)-1}du. \end{aligned}$$

Let begin by providing an upper bound for $A_{\lambda,h}$. Using the following inequality

$$1 - \frac{yu}{1-(1-y)u} \leqslant \left(1+y\frac{u}{1-u}\right)^{\alpha(t)-1},$$

and for $y \neq 0$, we have

$$\begin{aligned} A_{\lambda,h}(\alpha(t),y) &\leqslant \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2}\int_0^1 u^{\eta-1}(1-u)^{\nu-1}du + \frac{2\lambda^\nu y}{\Gamma(\alpha(t))^2}\int_0^1 \frac{u^\eta(1-u)^{\nu-1}}{1-(1-y)u}du \\ &= \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2}\beta(\eta,\nu) + \frac{2\lambda^\nu y}{\Gamma(\alpha(t))^2}\beta(\nu,\eta+1)\,{}_2F_1(1,\eta+1,3-2h,1-y), \end{aligned} \tag{4.5}$$

where ${}_2F_1$ is called the hypergeometric function, and the last equality is due to Euler's representation integral of ${}_2F_1$ see (Theorem 2.2.1, [1]).
Using Euler transformation formula see (Theorem 2.2.5, [1]), we get

$${}_2F_1(1,\eta+1,3-2h,1-y) = y^{2\alpha(t)-2}\,{}_2F_1(2-2h,\nu,3-2h,1-y). \tag{4.6}$$

Set $a = 2 - 2h$, $b = 2m_\alpha[a,b] - 1$ and $c = 3 - 2h - 2M_\alpha[a,b] + 2m_\alpha[a,b]$. For $y \neq 0$, we have

$$\begin{aligned} & {}_2F_1(a,\nu,a+1,1-y) \\ &= \frac{\Gamma(a+1)}{\Gamma(\nu)\Gamma(\eta+1)} \int_0^1 x^{\nu-1}(1-x)^{\eta}(1-(1-y)x)^{-a}dx \\ &\leqslant C \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-(1-y)x)^{-a}dx \\ &= CF(y) \\ &\leqslant C, \end{aligned} \tag{4.7}$$

the last inequality comes from the fact that the function $F$ is continuous function on $[1, \frac{1}{\lambda}]$. By plugging (4.6) in (4.5) and using (4.7), we infer that

$$A_{\lambda,h}(\alpha(t),|\tau|) \leqslant \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2}\beta(\eta,\nu) + C|\tau|^\nu. \tag{4.8}$$

On the other hand, since $\eta, \lambda, C_u > 0$, $0 < h < 1$, and $\alpha(t) < 1$,

$$\begin{aligned} B_{\lambda,h}(\alpha(t),|\tau|) &\leqslant \frac{2\lambda^{\alpha(t)-1}(1-h)}{\Gamma(\alpha(t))^2}\beta(\alpha(t),\alpha(t))|\tau|^{\alpha(t)} \\ &\leqslant M|\tau|^\nu, \end{aligned} \tag{4.9}$$

where $M$ is the maximum of the continuous function

$$z \mapsto (2\lambda^{z-1}(1-h))/(\Gamma(z)^2)\beta(z,z)$$

on $[m_\alpha[a,b], M_\alpha[a,b]]$. Thus, by plugging (4.8) and (4.9) in (4.4), we get

$$-2E_B(Y^\lambda_{\alpha(t)}(t+\tau)Y^\lambda_{\alpha(t)}(t)) \leqslant \frac{-2\lambda^\nu}{\Gamma(\alpha(t))^2}\beta(\eta,\nu) + (M+C)|\tau|^\nu.$$

Then

$$E_B([Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t)]^2]) \leqslant C_1|\tau|^\nu,$$

where $C_1 = M + C$, which establishes the desired result. ■

LEMMA 4.2. *Fix a compact interval $[a,b] \subset \mathbb{R}$. Then,*

*1) There exists a constant $C_2$ depending only on $[a,b]$, $\lambda$ and $h$ such that*

$$E_B([Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t)]^2) \geqslant C_2 \mid \tau \mid^{2\alpha(t)-1}, \tag{4.10}$$

*for all $t, t+\tau \in [a,b]$ satisfying $|\tau| < 1$.*

*2) as $\tau \to 0$*

$$E_B([Y^\lambda_{\alpha(t)}(t+\tau) - Y^\lambda_{\alpha(t)}(t)]^2) = C_2|\tau|^{\alpha(t)-1/2} + O(|\tau|^{2\alpha(t)-1}). \tag{4.11}$$

P r o o f. By following the same notation as in the Lemma (4.1), and since $\alpha(t) < 1$ for all $t$,

$$B_{\lambda,h}(\alpha(t), y) \quad \leqslant \quad E_B([Y^{\lambda}_{\alpha(t)}(t+\tau) - Y^{\lambda}_{\alpha(t)}(t)]^2).$$

For $\tau \neq 0$ and using $C_u \in ]0,1[$, we get

$$\left(\frac{1}{|\tau|} + \frac{1}{\lambda}\right)^{\alpha(t)+h-3} |\tau|^{2\alpha(t)-2} \quad \leqslant \quad (1+y)^{h+\alpha(t)-3} \leqslant (1-u+yu)^{\alpha(t)-1}(1+y\,C_u)^{h-2}. \tag{4.12}$$

Set $h(z,x) = (\lambda\, x)^{3-z-h}(x+\lambda)^{z+h-3}$ a continuous function on $(z,x) \in [m_\alpha[a,b], M_\alpha[a,b]] \times [0,1]$ and let $C_2$ to be the minimum of the function

$$(z,x) \mapsto (2\lambda^{2z-2}(1-h))/(\Gamma(z)^2)\, \beta(4-2h-2z, z)\, h(z,x)$$

when $(z,x) \in [m_\alpha[a,b], M_\alpha[a,b]] \times [0,1]$, then

$$E_B([Y^{\lambda}_{\alpha(t)}(t+\tau) - Y^{\lambda}_{\alpha(t)}(t)]^2) \geqslant C_2\, |\tau|^{\nu},$$

which gives (4.10).
Let us now prove (4.11). If instead of (4.12) we used the following inequality for $y \neq 0$,

$$(1/|\tau| + 1/\lambda)^{\alpha(t)+h-3}\, |\tau|^{\alpha(t)-3/2} \leqslant (1+y)^{\alpha(t)+h-3},$$

the inequality (4.10) becomes

$$E_B([Y^{\lambda}_{\alpha(t)}(t+\tau) - Y^{\lambda}_{\alpha(t)}(t)]^2) \geqslant C_2\, |\tau|^{\alpha(t)-1/2}.$$

Combining the Lemma (4.1) and the last inequality, we get

$$C_2\, |\tau|^{\alpha(t)-1/2} \leqslant E_B([Y^{\lambda}_{\alpha(t)}(t+\tau) - Y^{\lambda}_{\alpha(t)}(t)]^2) \leqslant C_1\, |\tau|^{\alpha(t)-1/2}.$$

To prove (4.11), it remains to prove that $C_2 \leqslant C_1 = M + C$. Since $0 < h < 3/2 - z$ and $z < 1$, we obtain

$$\beta(4-2h-2z, z) \leqslant \beta(z,z) \qquad \text{and} \qquad h(z,x) \leqslant \lambda^{1-z}.$$

Therefore,

$$\begin{aligned} C_2 &\leqslant \frac{2\lambda^{2z-2}(1-h)}{\Gamma(z)^2}\, \beta(4-2h-2z, z) h(z,x) \\ &\leqslant M \\ &\leqslant C_1, \end{aligned}$$

which completes the proof. ■

**REFERENCES**


[1] G. Andrews, R. Askey and R. Roy, *Special functions,* Cambridge University press, 1999.

[2] A. Ayache, S. Jaffard and M.S. Taqqu, *Wavelet construction of generalized multifractional processes,* Rev. Mat. Iberoam. 23 (2007), pp. 327–370.

[3] A. Ayache and M.S. Taqqu, *Multifractional processes with random exponent,* Publ. Mat. 49 (2005), pp. 459–486.

[4] A. Benassi, S. Jaffard and D. Roux, *Elliptic Gaussian random processes,* Rev. Mat. Iberoamericana. 13 (1997), pp. 19–90.

[5] S. Bianchi, *Pathwise identification of the memory function of multifractional Brownian motion with application to finance,* Int. J. Theor. Appl. Finance. 8 (2005), pp. 255–281.

[6] P. Billingsley, *Convergence of Probability Measures,* John Wiley & Sons Inc, New York, 1968.

[7] P. Billingsley, *Convergence of Probability Measures. Wiley Series in Probability and Statistics: Probability and Statistics 2nd edn,* John Wiley & Sons Inc, A Wiley-Interscience Publication, New York, 1999.

[8] P. Cheridito, H. Kawaguchi and M. Maejima, *Fractional Ornstein-Uhlenbeck processes,* Electron. J. Probab. 8 (2003), pp. 1–14.

[9] S. Eryilmaz, *$\delta$-shock model based on Polya process and its optimal replacement policy,* EJOR. 263 (2017), pp. 690–697.

[10] K. Es-Sebaiy and C.A. Tudor, *Fractional Ornstein–Uhlenbeck processes mixed with a Gamma distribution,* Fractals. 23 (2015), pp. 1–10.

[11] K. Falconer, *Mathematical foundations and applications Fractal Geometry 2nd edn,* John Wiley & Sons Ltd, England, 2003.

[12] P. Flandrin , P. Borgnat and P. O. Amblard, *From Stationarity to Self-Semilarity, and Back: Variations on the Lamperti Transformation (Lecture Notes in Physics),* eds G Rangarajan and M Ding. Springer, New York. 621 (2003), pp. 88–117.

[13] E. Iglói and G. Terdik, *Long-range dependence through Gamma-mixed Ornstein-Uhlenbeck process,* Electron. J. Probab. 4 (1999), pp. 1–33.

[14] S. C. Lim and C . H. Eab, *Riemann-Liouville and Weyl fractional oscillator processes,* Phys. Lett. A. 355 (2006), pp. 87–93.

[15] S. C. Lim and L. P. Teo, *Weyl and Riemann-Liouville multifractional Ornstein-Uhlenbeck processes,* J. Phys. A: Math. Theor. 40 (2007), pp. 6035–6060.

[16] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations,* John Wiley and Sons, New York, 1993.

[17] R. F. Peltier and J. Lévy Véhel, *Multifractional Brownian motion: definition and preliminary results,* Research Report RR-2645, INRIA, 1995.

[18] S. Samko, A. A. Kilbas and D. I. Maritchev, *Integrals and Derivatives of the fractional Order and Some of Their Applications,* Gordon and Breach, Amsterdam, 1993.

[19] S. Stoev and M. S. Taqqu, *How rich is the class of multifractional Brownian motions?,* Stochastic Process. Appl. 116 (2006), pp. 200–221.

Department of Mathematics
Faculty of Science
Kuwait University
P.O. Box 5969
Safat -13060, Kuwait
*E-mail*: khalifa.essebaiy@ku.edu.kw

Department of Mathematics
National School of Applied Sciences-Marrakesh
Cadi Ayyad University
BP Guéliz
Boulevard Abdelkrim Al Khattabi
Marrakesh, Morocco
*E-mail*: farah.fatima.ezzahra@gmail.com

Department of Mathematics
Faculty of Technology
Linnaeus University
P G Vejdes väg
351 95 Växjö, Sweden
*E-mail*: astrid.hilbert@lnu.se

*Received on ;*
*revised version on xx.xx.xxxx*