ON THE EXACT ASYMPTOTICS OF EXIT TIME FROM A CONE OF AN ISOTROPIC $\alpha$-SELF-SIMILAR MARKOV PROCESS WITH A SKEW-PRODUCT STRUCTURE

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Abstract. In this paper we identify the asymptotic tail of the distribution of the exit time $\tau_C$ from a cone $C$ of an isotropic $\alpha$-self-similar Markov process $X_t$ with a skew-product structure, that is $X_t$ is a product of its radial process and independent time changed angular component $\Theta_t$. Under some additional regularity assumptions, the angular process $\Theta_t$ killed on exiting from the cone $C$ has the transition density that can be expressed in terms of a complete set of orthogonal eigenfunctions with corresponding eigenvalues of an appropriate generator. Using this fact and some asymptotic properties of the exponential functional of a killed Lévy process related with Lamperti representation of the radial process, we prove that

$$P_x(\tau_C > t) \sim h(x)t^{-\kappa_1}$$

as $t \to \infty$ for $h$ and $\kappa_1$ identified explicitly. The result extends the work of DeBlassie [6] and Bañuelos and Smits [1] concerning the Brownian motion.

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1. INTRODUCTION

For a dimension $d \geq 2$ and an index $\alpha > 0$ on some probability space $(\Omega, \mathcal{F}, P_x)$ we consider an $\mathbb{R}^d$-valued $\alpha$-self-similar isotropic Markov process $\{X_t, t \geq 0\}$, where $P_x(\cdot) = \mathbb{P}(\cdot | X_0 = x)$. We recall that process $X$ is said to be $\alpha$-self-similar if for every $x \in \mathbb{R}^d$ and $\lambda > 0$,

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the law of \((\lambda X_{\lambda t-\alpha t}, t \geq 0)\) under \(P_x\) is the same as \(P_{\lambda x}\).

Moreover, this process is said to be isotropic (or \(O(d)\)-invariant), if for any \(x \in \mathbb{R}^d\) and \(g \in O(d)\),

\[\text{the law of } (g(X_t), t \geq 0) \text{ under } P_x \text{ is the same as } P_{g(x)},\]

where \(O(d)\) is the group of orthogonal transformations on \(\mathbb{R}^d\). In this paper we assume that the radial process \(R_t = |X_t|\) and the angular process \(X_t/R_t\) do not jump at the same time. Then by Liao and Wang [16, Theorem 1] the process \(X_t\) observed up to its first hitting time of 0 has a skew-product structure:

\[(1.1) \quad X_t = R_t \Theta_{A(t)},\]

where \(A(t)\) is a strictly increasing continuous process defined by

\[(1.2) \quad A(t) = \int_0^t R_s^{-\alpha} ds, \quad t < T_0,\]

for

\[(1.3) \quad T_0 = \inf\{t > 0 : X_t = 0\} = \inf\{t > 0 : R_t = 0\},\]

and \(\Theta_t\) is an \(O(d)\)-invariant Markov process on the unit sphere \(S^{d-1}\) and is independent of the radial process \(R_t\). The classical example concerns \(d\)-dimensional Brownian motion that may be expressed as a product of a Bessel process and a time changed spherical Brownian motion. Moreover, the Bessel process is independent of the spherical Brownian motion. More generally, any continuous isotropic Markov process will have above representation (1.1) with possibly different time change; see [9]. In particular, a self-similar diffusion will have it. Note that an isotropic self-similar Markov process might not satisfy above representation (1.1) though. The most famous examples are the symmetric \((1/\alpha)\)-stable Lévy processes for \(\alpha > 1/2\). Their Lévy measures are absolutely continuous on \(\mathbb{R}^d \setminus \{0\}\), so their radial and angular parts may jump together, and thus do not possess a skew product structure as defined above.

We will also consider an open cone \(C\) in \(\mathbb{R}^d\) generated by a domain \(D\) in the unit sphere \(S^{d-1}\), that is \(C = \cup_{r > 0} rD\). We define the first exit time of \(X_t\) from the cone \(C\) by

\[(1.4) \quad \tau_C = \inf\{t > 0 : X_t \notin C\}.\]

The purpose of this paper is to study the asymptotic behavior of the exit probability \(P_x(\tau_C > t)\) as \(t \to \infty\) for \(x \in \mathbb{C}\). In fact we prove that

\[(1.5) \quad P_x(\tau_C > t) \sim h(x)t^{-\kappa_1}\]
as $t \to \infty$ for $h$ and $\kappa_1$ identified explicitly, where we write $f(t) \sim g(t)$ for some positive functions $f$ and $g$ iff $\lim_{t \to \infty} f(t)/g(t) = 1$.

The main idea of the proof is based on the following steps. In the first one we give the following representation

$$q_D(t, \theta, \eta) = \sum_{j=1}^{\infty} e^{-\lambda_j t} m_j(\theta)m_j(\eta),$$

of the transition density for the angular process $\Theta_t$ killed upon exiting from the cone $C$ in terms of orthogonal eigenfunctions $m_j$ with corresponding eigenvalues $\lambda_j$ of $-S|_D$ for the generator $S$ of $\Theta_t$ restricted to $D$ with Dirichlet boundary condition. Then

$$P_x(\tau_C > t) = \sum_{j=1}^{\infty} m_j(x/|x|) \left( \int_D m_j d\sigma \right) \mathbb{E}[x] \left[ e^{-\lambda_j A(t)}, t < T_0 \right]$$

for $\sigma$ being the normalized surface measure on $S^{d-1}$. Using the Lamperti [12] transformation we can express process $\{R_t, t < T_0\}$ as a time change of the exponential of an $\mathbb{R} \cup \{-\infty\}$-valued Lévy process, that is, there exists an $\mathbb{R} \cup \{-\infty\}$-valued Lévy process $\xi_t$ starting from 0 and with lifetime $\zeta$, whose law does not depend on $|x|$, such that

$$R_t = |x| \exp(\xi A(t)), \quad 0 \leq t < T_0.$$  

This gives the following representation of the tail exit probability:

$$P_x(\tau_C > t) = \sum_{j=1}^{\infty} m_j(x/|x|) \left( \int_D m_j d\sigma \right) \mathbb{P}(I_{\alpha \lambda_j}(\alpha \xi) > |x|^{-\alpha t}),$$

where

$$I_t(\alpha \xi) = \int_0^t \exp(\alpha \xi_s) ds,$$

$\alpha \lambda_j$ is an independent exponential random variable with the intensity $\lambda_j$ and $I(\alpha \xi) = \lim_{t \to -\infty} I_t(\alpha \xi)$ is an exponential functional. The final result (1.5) follows from Rivero [19, Lemma 4] and Maulik and Zwart [18, Theorem 3.1] concerning the tail asymptotics of the exponential functional $I(\alpha \xi)$. In this case $\kappa_1$ solves equation

$$\phi(\alpha \kappa_1) = \lambda_1$$

for the Laplace exponent of the process $\xi$.

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The asymptotic (1.5) determines also the critical exponents of integrability of the exit time \( \tau_C \). In this sense it generalizes series of papers concerning the \( \alpha \)-stable process, see Kulczycki [13] and Bañuelos and Bogdan [2] and references therein.

The paper is organized as follows. In Preliminaries we give and prove main facts used later. In the next Section 3 we give the main result and its proof.

2. PRELIMINARIES

2.1. Skew-product structure. Let \( X_t \) be an \( \alpha \)-self-similar isotropic Markov process with skew-product representation (1.1). The process \( \Theta_t \) is an \( O(d) \)-invariant Markov process on \( S^{d-1} \) with transition semigroup \( Q_t \) and infinitesimal generator \( S \).

Throughout this paper, we assume that

**Assumption 2.1.** \( \Theta_t \) possesses a bounded transition density \( q(t, \theta, \eta) \) with respect to \( \sigma \), the normalized surface measure on \( S^{d-1} \), and there exist positive constants \( C \) and \( \beta \) such that

\[
q(t, \theta, \eta) \leq C t^{-\beta}
\]

for all \( (t, \theta, \eta) \in (0, 1) \times S^{d-1} \times S^{d-1} \).

**Example 2.1 (Brownian motion).** In the case when \( \Theta_t \) is a Brownian motion the Assumption 2.1 is satisfied. Indeed, the generator \( S \) of \( \Theta_t \) is a multiple of the Laplace-Beltrami operator \( \Delta_{S^{d-1}} \) on \( S^{d-1} \). Moreover, it is known that the transition density \( h(t, \theta, \eta) \) of \( \Theta_t \) has the Gaussian upper bound:

\[
h(t, \theta, \eta) \leq c_1 t^{-(d-1)/2} e^{-c_2 d(\theta, \eta)^2}, \quad t > 0, \ \theta, \eta \in S^{d-1}
\]

for some positive constants \( c_1 \) and \( c_2 \).

**Example 2.2 (Subordinate Brownian motion).** Fix \( \gamma \in (0, 1) \). Let \( W_t \) be a Brownian motion on \( S^{d-1} \) with transition density \( h(t, \theta, \eta) \). Let \( S_t \) be a \( \gamma \)-stable subordinator, i.e., a Lévy process in \( \mathbb{R} \), supported by \( [0, \infty) \), with Laplace transform

\[
\mathbb{E} \left[ e^{-\vartheta S_t} \right] = \exp(-t \vartheta^\gamma), \quad \vartheta > 0.
\]

The Assumption 2.1 is also satisfied when \( \Theta_t = W_{S_t} \). Indeed, in this case \( \Theta_t \) is an \( O(d) \)-invariant pure jump Markov process on \( S^{d-1} \) with transition density

\[
q(t, \theta, \eta) = \int_0^\infty h(u, \theta, \eta) p_t(u) du,
\]

where \( p_t(u) \) is the probability density of \( S_t \). By Theorem 37.1 of Doetsch [8],

\[
\lim_{u \to \infty} p_t(u) u^{1+\gamma} = \frac{\gamma}{\Gamma(1 - \gamma)}.
\]
The limit (2.3) together with the scaling property

\[ p_t(u) = t^{-\frac{1}{\gamma}} p_{t^{-\frac{1}{\gamma}} u} \]

give the following upper bound:

\[ p_t(u) \leq c_1 t u^{-1-\gamma}, \quad t, u > 0. \]

Using (2.2) one can observe now that

\[ q(t, \theta, \eta) \leq c_3 t^{-\frac{d-1}{2\gamma}} \wedge \frac{t}{d(\theta, \eta)^{d-1+2\gamma}} \]

for \((t, \theta, \eta) \in (0, \infty) \times S^{d-1} \times S^{d-1}.

We will now give sufficient conditions for the Assumption 2.1 to be satisfied for general \(O(d)\)-invariant Markov process \(\Theta_t\) on \(S^{d-1}\). If we identity \(S^{d-1}\) with \(O(d)\)/\(O(d-1)\) then \(\Theta_t\) may be viewed as a Lévy process on the compact homogeneous space \(O(d)/O(d-1)\). Furthermore, the generator \(S\) of \(\Theta_t\) was given by Hunt [10] (see also Liao [14]) explicitly. We state this result as follows. Let \(C^\infty(S^{d-1})\) be the space of smooth functions on \(S^{d-1}\) and let \(\pi: O(d) \to S^{d-1}\) be the map \(g \to g o\) for \(o = (0, \ldots, 0, 1) \in \mathbb{R}^d\). Restricted to a sufficient small neighborhood \(V\) of \(o\), the map

\[ \varphi: (y_1, \ldots, y_d) \to \pi\left(\sum_{j=1}^d y_j O_j\right) \]

is a diffeomorphism and \((y_1, \ldots, y_d)\) may be used as local coordinates on \(\varphi(V)\), where \((O_1, \ldots, O_d)\) is a basis of Lie algebra of \(O(d)\). Then by [14, Theorem 2.2] the domain of \(S\) contains \(C^\infty(S^{d-1})\) and for \(f \in C^\infty(S^{d-1})\),

\[ S f(o) = T f(o) + \int_{S^{d-1}} \left( f(\theta) - f(o) - \sum_{j=1}^d y_j(\theta) \frac{\partial f(o)}{\partial y_j}\right) \nu(d\theta), \]

where \(o\) is the origin in \(S^{d-1}\), \(T\) is an \(O(d)\)-invariant second order differential operator on \(S^{d-1}\) and \(\nu\) is an \(O(d-1)\)-invariant measure on \(S^{d-1}\), called the Lévy measure of \(\Theta_t\), that satisfies \(\nu(\{o\}) = 0\) and

\[ \int_{S^{d-1}} \left[\text{dist}(\theta, o)\right]^2 \nu(d\theta) < \infty. \]

Since \(O(d)/O(d-1)\) is irreducible, all the \(O(d)\)-invariant second order differential operators are multiples of \(\Delta_{S^{d-1}}\). Therefore we may rewrite (2.4) as

\[ S f(o) = a \Delta_{S^{d-1}} f(o) + \int_{S^{d-1}} \left( f(\theta) - f(o) - \sum_{j=1}^d y_j(\theta) \frac{\partial f(o)}{\partial y_j}\right) \nu(d\theta) \]
Note that when $\Theta_t$ is the subordinate Brownian motion defined in Example 2.2, we have $a = 0$ and the Lévy measure
\[ \nu \asymp d(\theta, o)^{-d+1-2\gamma} \] near $o$.

As observed in this case the Assumption 2.1 holds true.

This phenomenon holds for more general $\Theta_t$. Using [15, Theorems 3 and 6] we can state the following proposition giving sufficient conditions for the Assumption 2.1 to hold true.

**Proposition 2.1.** $\Theta_t$ is a Lévy process on $S^{d-1}$ with the infinitesimal generator $S$ given by (2.6). Assume that either $a > 0$ or the Lévy measure $\nu$ is asymptotically larger than $d(\theta, o)^{-\gamma}$ near $\theta = o$ for some $\gamma \in (d-1, d-1 + 2)$. Then $\Theta_t$ has a bounded transition density $q(t, \theta, \eta)$ and it satisfies Assumption 2.1.

For any open subset $D \subset S^{d-1}$ we define the first exit time of $\Theta_t$ from $D$ by
\[ \tau_D = \inf\{ t > 0 : \Theta_t \notin D \}. \]

Let $\Theta^D$ be the killed process of $\Theta$ upon exiting from $D$, that is, $\Theta^D_t = \Theta_t$ if $t < \tau^D_D$ and $\Theta^D_t = \partial$ if $t \geq \tau^D_D$, where $\partial$ is a cemetery state. Its infinitesimal generator is $S|_{D^c}$, the restriction of $S$ to $D$ with the Dirichlet boundary condition. Then
\[ q_D(t, \theta, \eta) = q(t, \theta, \eta) - \mathbb{E}_\theta \left[ q(t - \tau^D_D, \Theta_{\tau^D_D}, \eta); \tau^D_D < t \right] \]

is the transition density of $\Theta^D$. Clearly, $q_D(t, \theta, \eta) \leq q(t, \theta, \eta)$ for all $t > 0$ and $\theta, \eta \in S^{d-1}$. As a consequence, the transition semigroup $Q^D_t$ associated to the sub-process $\Theta^D$ is compact on $L^2$. Let $\{\lambda_j\}_{j=1}^\infty$ be the eigenvalues of $-S|_{D^c}$ written in increasing order and repeated according to its multiplicity, and $m_j$ the corresponding eigenfunctions normalized by $\|m_j\|_2 = 1$. Then by [5, Theorem 2.1.4], $m_j \in L^\infty$ for all $j$ and $Q^D_t$ has a transition density $q_D(t, \theta, \eta)$, which can be represented as the series:
\[ q_D(t, \theta, \eta) = \sum_{j=1}^\infty e^{-\lambda_j t} m_j(\theta)m_j(\eta) \]

that converges uniformly on $[\delta, \infty) \times D \times D$ for all $\delta > 0$.

**Remark 2.1.** Note that we do not assume any regularity condition on the boundary $\partial D$ of $D$. Thus $q_D(t, \theta, \eta)$ (or $m_j(\theta)$) need not vanish continuously on the boundary $\partial D$.

**Remark 2.2.** If $S^{d-1} \setminus \overline{D}$ is not empty, then from the monotonicity of Dirichlet eigenvalues we have that $\lambda_1 > 0$; see [3, Section 1.5] for more details (check also Lemma 2.2 below).
Remark 2.3. Assume that \( q_D(t, \theta, \eta) \) is strictly positive for \( t > 0 \) and \( \theta, \eta \in D \). Then we have from Jentzsch’s theorem ([20, Theorem V.6.6]) that \( \lambda_1 \) is a simple eigenvalue for \(-S\vline_D\). Using the standard arguments, like the ones given in the proof of [4, Theorem 2.4], one can show \( q_D(t, \theta, \eta) \) is strictly positive when \( \Theta_t \) is a Brownian motion on \( S^{d-1} \) and \( D \) is connected or \( \Theta_t \) is a subordinate Brownian motion on \( S^{d-1} \) satisfying the conditions of Example 2.2.

In our analysis the crucial fact is the following lower bound for the eigenvalues \( \lambda_j \) \((j \geq 1)\).

Lemma 2.2. Assume (2.1) holds true. Then for every \( j \geq 1 \), we have

\[
\lambda_j \geq (C\sigma(D))^{-\frac{1}{2} j^{\frac{1}{2}}}.
\]

and

\[
\|m_j\|_{\infty} \leq eC [\sigma(D)]^{\frac{1}{2} \lambda_j^{\beta}}.
\]

Proof. We will follow the same idea as the one that the proof of [7, Lemma 2.7] is based on. In particular, since \( \lambda_j \) is ordered increasingly, we have from (2.9) and (2.1) that

\[
je^{-\lambda_j t} \leq \sum_{i=1}^{\infty} e^{-\lambda_i t} = \int_D q_D(t, \theta, \theta)\sigma(d\theta) \leq C\sigma(D)t^{-\beta}.
\]

Taking \( t = \lambda_j^{-1} \) we obtain \( j \leq C\sigma(D)\lambda_j^{\beta} \) and (2.10) follows immediately.

Note that

\[
m_j(\theta) = e^{\lambda_j t}Q_D m_j(\theta) = e^{\lambda_j t} \int_D q_D(t, \theta, \eta)m_j(\eta)\sigma(d\eta).
\]

By the Cauchy-Schwarz inequality,

\[
\|m_j\|_{\infty} \leq e^{\lambda_j t} \sup_{\theta} \left( \int_D q_D(t, \theta, \eta)^2\sigma(d\eta) \right)^{1/2} \left( \int_D m_j(\eta)^2\sigma(d\eta) \right)^{1/2} \leq C [\sigma(D)]^{\frac{1}{2} e^{\lambda_j t}} t^{-\beta}.
\]

The proof is completed by setting \( t = \lambda_j^{-1} \). ■

2.2. Positive self-similar Markov processes. Recall that \( R_t = |X_t| \) is a positive \((\mathbb{R}_+\text{-valued})\) \( \alpha \)-self-similar Markov process starting at \( |x| \). According to Lamperti [12], up to its first hitting time of \( 0 \), \( R_t \) may be expressed as a time change of the exponential of an \( \mathbb{R} \cup \{-\infty\}\text{-valued} \) Lévy process. More formally, there exists
an $\mathbb{R} \cup \{-\infty\}$-valued Lévy process $\xi_t$ starting from 0 and with lifetime $\zeta$, whose law does not depend on $|x|$, such that

$$
R_t = |x| \exp \left( \xi_{A(t)} \right), \quad 0 \leq t < T_0,
$$

where $T_0$ is the first hitting time of 0 by $R$ defined formally in (1.3) and $A(t)$ is the positive continuous functional given by $A(t) = \int_0^t R_s^\alpha \, ds$. The law of $\xi$ is characterized completely by its Lévy-Khintchine exponent

$$
\Psi(z) = \log \mathbb{E} \left[ e^{iz \xi_t} \right] = -q + ibz - \frac{\sigma^2}{2} z^2 + \int_{-\infty}^{+\infty} \left( e^{izy} - 1 - izy 1_{\{|y|<1\}} \right) \Pi(dy),
$$

where $q \geq 0$, $\sigma \geq 0$, $b \in \mathbb{R}$ and $\Pi$ is a Lévy measure satisfying the condition $\int_{\mathbb{R}} (1 + |y|^2) \Pi(dy) < \infty$. The lifetime $\zeta$ of $\xi$ is an exponential random variable with parameter $q$, with the convention that $\zeta = \infty$ when $q = 0$. Observe that the process $\xi$ does not depend on the starting point of $X$. Hence we will denote the law of $\xi$ by $\mathbb{P}$.

For fixed $\alpha > 0$, we define the exponential functional $I_t(\alpha \xi)$ by (1.7). Then by a change of variable $s = A(u)$,

$$
I_t(\alpha \xi) = \int_0^{A^{-1}(t)} \exp(\alpha \xi_{A(u)}) R_u^{-\alpha} \, du = \int_0^{A^{-1}(t)} |x|^{-\alpha} A^{-1}(t).
$$

Hence $A(|x|^t)$ is the right inverse of the strictly increasing continuous process $I_t(\alpha \xi)$ and we can recover the law of $(R_t, t < T_0)$ from the law of $\xi_t$ for fixed $|x|$ and $\alpha > 0$. In particular, we have

$$
T_0, \mathbb{P}_x \overset{d}{=} (|x|^\alpha I(\alpha \xi), \mathbb{P}).
$$

As mentioned in [12], the probabilities $\mathbb{P}_x(T_0 = +\infty), \mathbb{P}_x(T_0 < +\infty, R_{T_0} = 0)$ and $\mathbb{P}_x(T_0 < +\infty, R_{T_0} > 0)$ are 0 or 1 independently of $x$. Moreover, we have

1. if $\mathbb{P}_x(T_0 = +\infty) = 1$, then $\zeta = +\infty$, $\limsup_{t \to \infty} \xi_t = +\infty$, and $\lim_{t \to \infty} A(t) = +\infty$;
2. if $\mathbb{P}_x(T_0 < +\infty, R_{T_0} = 0) = 1$, then $\zeta = +\infty$, $\lim_{t \to \infty} \xi_t = -\infty$, and $\lim_{t \to T_0^-} A(t) = +\infty$;
3. if $\mathbb{P}_x(T_0 < +\infty, R_{T_0} > 0) = 1$, then $\zeta$ is an exponentially distributed random time with parameter $q > 0$. Moreover, $A(T_0^-)$ has the same distribution as that of $\zeta$, thus the functional $A(t)$ always jumps from a finite value to $+\infty$, that is, $\mathbb{P}_x(A(T_0^-) < +\infty, A(T_0) = +\infty) = 1$. 
Let $e_\lambda$ be an independent exponential random variable with parameter $\lambda$. Then

$$E_{|x|}\left[e^{-\lambda A(t)}, t < T_0\right] = P_{|x|}(A(t) < e_\lambda, t < T_0).$$

Note that by the construction above we have that $t < T_0$ is equivalent to $A(t) < \zeta$. Thus

$$E_{|x|}\left[e^{-\lambda A(t)}, t < T_0\right] = P_{|x|}(A(t) < e_\lambda \land \zeta) = P\left(\int_0^{e_\lambda} \exp(\alpha \xi_s)ds > |x|^{-\alpha} t\right),$$

where in the last equality we used the fact that $|x|^\alpha I_{x}(\alpha \xi)$ is the right inverse of $A(t)$. The equation (2.15) will give (along with the representation (2.9)) another main ingredient of the proof of the main result. In the last step we will need the tail asymptotic behaviour of the exponential function $I_{x}(\alpha \xi)$ described below.

2.3. Exponential functional of a killed Lévy process. Let $\xi^\lambda$ be a Lévy process with Lévy-Khintchine exponent $\Psi$ given by (2.13) killed by the independent exponential time $e_\lambda$ with parameter $\lambda > 0$. Thus the resulted process has a lifetime $\zeta' = e_\lambda \land \zeta$, an exponential random variable with parameter $\lambda + \eta$.

We define the Laplace exponent of $\xi$ via

$$E\left(\exp(\vartheta \xi_t)\right) = \exp(t\phi(\vartheta)), \quad t \geq 0, \quad \vartheta \in \Xi,$$

where $\Xi = \{\vartheta : \phi(\vartheta) < \infty\}$. By (2.13), for $\vartheta \in \Xi$ we have $\phi(\vartheta) = \Psi(-i\vartheta)$. It is easy to see from Hölder inequality that $\phi(\vartheta)$ is a convex function. From now we assume that $\xi$ satisfies the following conditions.

Assumption 2.2. $\xi$ is not arithmetic; that is, there is no $d$ such that support of $\xi_1$ is $d\mathbb{N}$.

Assumption 2.3. There exists a constant $\vartheta^* > \alpha (1 \lor (2\beta))$ such that $\phi(\vartheta) < \infty$ for $0 < \vartheta < \vartheta^*$ and $\lim_{\vartheta \to \vartheta^*} \phi(\vartheta) = \infty$, where $\beta$ is the constant in Assumption 2.1.

Note that under Assumption 2.3, we have that for every $\lambda > 0$, there exists a unique $0 < \kappa < \vartheta^*$ such that

$$\phi(\alpha \kappa) = \lambda.$$

Moreover,

$$E[\xi_1^\lambda e^{\alpha \kappa \xi_1}] = \phi'(\alpha \kappa) < +\infty.$$

In the proof we will use the following crucial result giving the tail asymptotics of the distribution of the exponential functional.
THEOREM 2.3 ([19, Lemma 4], [18, Theorem 3.1]). Suppose that the Assumptions 2.2 and 2.3 are satisfied. Then, as $t \to \infty$,

\begin{equation}
(2.18) \quad t^\kappa \mathbb{P}(I_{e_\lambda}(\alpha \xi) > t) \sim \frac{1}{\alpha \varphi'(\alpha \kappa)} \mathbb{E} \left[ I_{e_\lambda}(\alpha \xi)^{\kappa-1} \right].
\end{equation}

EXAMPLE 2.3 (Linear Brownian motion with drift). Let $\sigma > 0$, $b \in \mathbb{R}$ and $\xi_t = \sigma B_t + bt$, where $B_t$ is a standard linear Brownian motion. Then $\varphi(\vartheta) = \frac{\sigma^2 \vartheta^2 + b \vartheta}{2}$ and for

$$\kappa = \frac{1}{\alpha \sigma^2} \left( \sqrt{2\sigma^2 \lambda + b^2} - b \right),$$

we have $\mathbb{E} \left[ e^{\alpha \sigma (B_t + b)} ; 1 < e_\lambda \right] = 1$. By Theorem 2.3,

$$\lim_{t \to \infty} t^\kappa \mathbb{P}(I_{e_\lambda}(\alpha \xi) > t) = C_\kappa,$$

where

$$C_\kappa = \frac{1}{\alpha \sqrt{2\sigma^2 \lambda + b^2}} \mathbb{E} \left\{ \left[ \int_0^{e_\lambda} \exp (\alpha \sigma B_s + \alpha b s) \, ds \right]^{\kappa-1} \right\}.$$

By the scaling property of Brownian motion, the random variable

$$\int_0^{e_\lambda} \exp (\alpha \sigma B_s + \alpha b s) \, ds$$

has the same distribution as the integral:

$$\frac{\sigma^2 \epsilon_\lambda}{4} \int_0^{e_\lambda} \exp \left( 2B_s + \frac{4b}{\alpha \sigma^2} s \right) \, ds.$$

Note that $\frac{\sigma^2 \epsilon_\lambda}{4}$ is an exponential distributed random variable with parameter $\frac{4}{\epsilon_\lambda^2 \lambda}$ independent of $B_t$. Yor [21] (see also [17, Theorem 4.12]) proved the following identity in law:

\begin{equation}
(2.19) \quad I_{e_\lambda}(\alpha \xi) \overset{d}{=} \frac{Z_{1,a}}{2 \gamma_\kappa},
\end{equation}

where $a = \kappa + \frac{2b}{\alpha \sigma^2}$. $Z_{1,a}$ is a beta variable with parameters $(1, a)$, and $\gamma_\kappa$ is a gamma variable with parameter $\kappa$, which is independent of $Z_{1,a}$. Since

$$\mathbb{E} \left[ Z_{1,a}^{\kappa-1} \right] = \int_0^1 t^{\kappa-1} (1 - t)^{a-1} \, dt = \frac{\Gamma(\kappa) \Gamma(a + 1)}{\Gamma(a + \kappa)}$$
and
\[
\mathbb{E} \left[ \gamma_\kappa^{1-\kappa} \right] = \frac{1}{\Gamma(\kappa)} \int_0^\infty t^{1-\kappa} t^{\kappa-1} e^{-t} dt = \frac{1}{\Gamma(\kappa)},
\]
we have
\[
\mathbb{E} \left[ \left( I_{e_\lambda}(\alpha \xi) \right)^{\kappa-1} \right] = \frac{2^{1-\kappa} \Gamma(a + 1)}{\Gamma(a + \kappa)} = \frac{2^{1-\kappa} \Gamma \left( \kappa + \frac{2b}{\alpha \sigma^2} + 1 \right)}{\Gamma \left( 2\kappa + \frac{2b}{\alpha \sigma^2} + 1 \right)}.
\]
Therefore,
\[
C_\kappa = \frac{4}{\alpha^2 \sigma^2 2^\kappa} \frac{\Gamma \left( \kappa + \frac{2b}{\alpha \sigma^2} + 1 \right)}{\Gamma \left( 2\kappa + \frac{2b}{\alpha \sigma^2} + 1 \right)}.
\]

3. MAIN RESULT

Let
\[
M(x) = \sum_{j : \lambda_j = \lambda_1} \left( \int_D m_j d\sigma \right) m_j(x/|x|)
\]
be a particular eigenfunction corresponding to the eigenvalue \( \lambda_1 \) of the operator \( S \) in \( D \) with Dirichlet boundary condition. Moreover, let \( \kappa_1 \) solve \( \phi(\alpha \kappa_1) = \lambda_1 \); that is, (1.8) is satisfied.

Recall that \( \tau_C \) is the exit time for the cone \( C \) of the \( \alpha \)-self-similar Markov process \( X_t \) with a skew-product structure (1.1). The main result of this paper is the following asymptotic.

**Theorem 3.1.** Under the Assumptions 2.1, 2.2 and 2.3, we have,
\[
P_x(\tau_C > t) \sim \frac{1}{\alpha \phi'(\alpha \kappa_1)} \mathbb{E} \left[ I_{e_\lambda_1}(\alpha \xi)^{\kappa_1-1} \right] M(x) \left( |x|^{-\alpha} t \right)^{-\kappa_1},
\]
as \( t \to \infty \).

**Remark 3.1.** \( M(x) \) does not depend on the choice of eigenfunction \( m_j \) with \( \lambda_j = \lambda_1 \). Indeed, if we have another choice \( m'_j \), then there exists an orthogonal matrix \( (a_{ik}) \) such that \( m'_i = \sum_k a_{ik} m_k \), which is equivalent to \( m_j = \sum_i a_{ij} m'_i \). Thus,
\[
\sum_i \left( \int_D m'_i d\sigma \right) m'_i = \sum_i \left( \sum_j \int_D a_{ij} m_j d\sigma \right) m'_i
= \sum_j \left( \int_D m_j d\sigma \right) \sum_i a_{ij} m'_i
= \sum_j \left( \int_D m_j d\sigma \right) m_j.
\]
EXAMPLE 3.1. Assume $X_t$ is an isotropic $\alpha$-self-similar diffusion process on $\mathbb{R}^d$. Then the radial process $R_t = |X_t|$ is a positive $\alpha$-self-similar diffusion process and $\Theta_t$ is a (possibly nonstandard) Brownian motion on $\mathbb{S}^{d-1}$ with $a \Delta_{\mathbb{S}^{d-1}}$ as its infinitesimal generator for some $a > 0$. Using the Lamperti’s relation, we have $\xi_t = \sigma B_t + bt$ for some $\sigma > 0$ and $b \in \mathbb{R}$, where $B_t$ is a standard Brownian motion. Clearly, all the Assumptions 2.1, 2.2 and 2.3 are satisfied. It follows from Example 2.3 that

$$\lim_{t \to \infty} t^{-\kappa_1} \mathbb{P}_x(\tau_C > t) = \frac{4}{\alpha^2 \sigma^2} \frac{\Gamma \left( \frac{\kappa_1 + \frac{2b}{\alpha \sigma^2}}{2} + 1 \right)}{\Gamma \left( 2 \kappa_1 + \frac{2b}{\alpha \sigma^2} + 1 \right)} \left( \frac{|x|^2}{2} \right)^{\kappa_1} M(x),$$

where $M(x)$ is defined by (3.1). In particular, when $X_t$ is a $d$-dimensional Brownian motion, we have $\alpha = 2$, $\sigma = 1$, $b = \frac{d}{2} - 1$, and $a = \frac{1}{2}$. Thus

$$\lim_{t \to \infty} t^{-\kappa_1} \mathbb{P}_x(\tau_C > t) = \frac{\Gamma \left( \frac{\kappa_1 + \frac{d}{2}}{2} \right)}{\Gamma \left( 2 \kappa_1 + \frac{d}{2} \right)} \left( \frac{|x|^2}{2} \right)^{\kappa_1} M(x),$$

where

$$\kappa_1 = \frac{1}{2} \left( \sqrt{2 \lambda_1 + \left( \frac{d}{2} - 1 \right)^2} - \left( \frac{d}{2} - 1 \right) \right).$$

This recovers the seminal result of De Blassie [6] (see also [1, Corollary 1]).

Proof of Theorem 3.1. We start the proof from the observation that $\tau_C$ is just the first time $t$ that $R_t = 0$ or the angular process $\Theta_{A(t)} \notin D$, that is:

$$\mathbb{P}_x(\tau_C > t) = \mathbb{P}_x(T_0 > t, \tau_D^\Theta > A(t)).$$

By the assumed independence of $R_t$ and $\Theta_t$ in (1.1) we have,

$$\mathbb{P}_x(\tau_C > t) = \int_0^\infty \mathbb{P}_{x/|x|}(\tau_D^\Theta > u) d_u \mathbb{P}_{|x|}(A(t) \leq u, t < T_0).$$

By (2.9) the exit probability $\mathbb{P}_{\Theta}(\tau_D^\Theta > t)$ can be represented as

$$\mathbb{P}_{\Theta}(\tau_D^\Theta > t) = \int_D q_D(t, \theta, \eta) d\sigma(\eta) = \sum_{j=1}^{\infty} e^{-\lambda_j t} m_j(\theta) \int_D m_j d\sigma.$$

Thus by (2.15):

$$\mathbb{P}_x(\tau_C > t) = \sum_{j=1}^{\infty} m_j(x/|x|) \int_D m_j d\sigma \int_0^\infty e^{-\lambda_j u} d_u \mathbb{P}_{|x|}(A(t) \leq u, t < T_0)$$

$$= \sum_{j=1}^{\infty} m_j(x/|x|) \left( \int_D m_j d\sigma \right) \mathbb{E}_{|x|} \left[ e^{-\lambda_j A(t), t < T_0} \right]$$

$$= \sum_{j=1}^{\infty} m_j(x/|x|) \left( \int_D m_j d\sigma \right) \mathbb{P} \left( I_{\lambda_j} (\alpha \xi) > |x|^{-\alpha} t \right).$$
For $j \geq 1$ let $\kappa_j (j \geq 1)$ be the solutions of

(3.8) \qquad \phi(\alpha \kappa_j) = \lambda_j.

Since $\lambda_j \to \infty$, we have that $\lim \inf_{j \to \infty} \kappa_j \geq \vartheta^*/\alpha$. Fix a $\kappa_0$ with the property

(3.9) \quad 1 \vee \kappa_1 \vee (2\beta) < \kappa_0 < \vartheta^*/\alpha.

Then there are only a finite number of $j$'s (for $j \geq 1$), being in the set, say, $A$, such that $\kappa_1 < \kappa_j \leq \kappa_0$. Applying Theorem 2.3 for $j \in A$ we obtain:

$$\lim_{t \to \infty} t^{\kappa_1} \mathbb{P} \left( I_{e_{\lambda_j}} (\alpha \xi) > |x|^{-\alpha} t \right) = 0.$$ 

Hence

(3.10) \quad \lim_{t \to \infty} t^{\kappa_1} \sum_{j: \kappa_j \leq \kappa_0} m_j(x/|x|) \left( \int_D m_j d\sigma \right) \mathbb{P} \left( I_{e_{\lambda_j}} (\alpha \xi) > |x|^{-\alpha} t \right) 

$$= \frac{1}{\alpha \phi'(\alpha \kappa_1)} \mathbb{E} \left[ I_{e_{\lambda_1}} (\alpha \xi)^{\kappa_1-1} \right] M(x) |x|^{\alpha \kappa_1}.$$ 

Now we consider the summation over the $j \in A^c$, that is, for $\kappa_j > \kappa_0$. By the Markov and Hölder inequalities,

$$t^{\kappa_0} \mathbb{P} \left( I_{e_{\lambda_j}} (\alpha \xi) > |x|^{-\alpha} t \right) \leq |x|^{\alpha \kappa_0} \mathbb{E} \left[ \left( \int_0^{e_{\lambda_j}} \exp(\alpha \xi_s) ds \right)^{\kappa_0} \right] 

\leq |x|^{\alpha \kappa_0} \mathbb{E} \left[ (e_{\lambda_j})^{\kappa_0-1} \int_0^{e_{\lambda_j}} \exp (\alpha \kappa_0 \xi_s) ds \right].$$ 

Using the independence of $e_{\lambda_j}$ and $\xi_s$, we have

$$t^{\kappa_0} \mathbb{P} \left( I_{e_{\lambda_j}} (\alpha \xi) > |x|^{-\alpha} t \right) \leq |x|^{\alpha \kappa_0} \mathbb{E} \left[ (e_{\lambda_j})^{\kappa_0} \int_0^{e_{\lambda_j}} e^{\phi(\alpha \kappa_0) s} ds \right] 

\leq |x|^{\alpha \kappa_0} \mathbb{E} \left[ (e_{\lambda_j})^{\kappa_0} e^{\phi(\alpha \kappa_0) e_{\lambda_j}} \right] 

= |x|^{\alpha \kappa_0} \int_0^\infty e^{\kappa_0 u} e^{\phi(\alpha \kappa_0) u} \lambda_j e^{-\lambda_j u} du 

= |x|^{\alpha \kappa_0} \frac{\lambda_j \Gamma(\kappa_0 + 1)}{\left( \lambda_j - \phi(\alpha \kappa_0) \right)^{\kappa_0 + 1}} 

\leq C |x|^{\alpha \kappa_0} \lambda_j^{-\kappa_0}.$$
for some constant $c > 0$. By (2.10), (2.11) and the fact $\kappa_0 > 2\beta$, we have

$$
\sum_{j: \kappa_j > \kappa_0} |m_j(x/|x|)| \cdot \left| \int_D m_j d\sigma \right| \mathbb{P} \left( \mathcal{I}_{e^{\lambda_j} \alpha \xi} \left( |x|^{-\alpha t} \right) \right) \\
\leq c |x|^\alpha \kappa_0 \left( \sum_j j^{-\frac{\kappa_0 - \beta}{\beta}} \right) t^{-\kappa_0}.
$$

(3.11)

Combining (3.10) and (3.11) completes the proof. □

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