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ENERGY OF TAUT STRINGS ACCOMPANYING RANDOM WALK

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Abstract. We consider the kinetic energy of the taut strings accompanying the trajectories of a Wiener process and a random walk. Under certain assumptions on the band width, it is shown that the energy of the taut string accompanying the random walk within a band satisfies the same strong law of large numbers as one proved earlier for a Wiener process and a fixed band width. New results for Wiener process are also obtained.

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1. INTRODUCTION

Consider a time interval $[0,T]$ and some continuous functional boundaries $g_1(t) \leqslant g_2(t)$, $0 \leqslant t \leqslant T$. A *Taut string* is a function h_* that has the remarkable property of universal optimization: for every convex function $\varphi(\cdot)$ it minimizes the functional

$$\int_0^T \varphi(h'(s))\,ds,$$

over all absolutely continuous functions h having the same values at 0 and T and satisfying the inequalities $g_1(t) \leqslant h(t) \leqslant g_2(t), 0 \leqslant t \leqslant T$. In particular, such

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functionals as the *kinetic energy* $\int_0^T h'(t)^2 dt$ and the graph length $\int_0^T \sqrt{1+h'(t)^2}dt$ are minimized by the taut string, see [10, Theorem 5.2, Remark 5.2], [9, Theorems 4.1,5.1,5.2], [19, Theorem 4.35, p.141].

If the values at 0 and/or T are not fixed, the universality is maintained for the smaller set of *even* convex functions.

Recall that for every function φ the solution of this problem exists, while for strictly convex functions φ the solution is unique; however, if φ is not strictly convex, the uniqueness may break down. For example, for $\varphi(x) = |x|$ there exist curious alternative solutions called "lazy functions", see [13]. In this article we deal only with the kinetic energy, therefore uniqueness is not a problem.

Taut strings appeared in the literature for the first time in G. Dantzig's article [1] in connection with some problems of optimal control. For further developments and applications of taut strings in optimal planning, discrete optimization, Statistics, image handling, and information transmission, we refer to [2], [3], [9], [15], [16], [19], [20].

In [12], M. Lifshits and E. Setterqvist studied the taut strings accompanying Wiener process. This work opened the way to further research studying taut strings related to other random processes, other types of energy, and to other distances between the string and the approximated process, see [4], [5], [6], [18].

In this work, we will consider the taut strings accompanying *a random walk* and extend to this case the results of [12], that we briefly recall now. Let us consider the taut strings running through a band of constant width around a sample path of Wiener process W, i.e. for some $r > 0$ we define the functional boundaries as follows: $g_1(t) := W(t) - r, g_2(t) := W(t) + r$, see Fig. 1. The results of [12] show that, as $T \to \infty$, the string expends an asymptotically constant amount of the kinetic energy $r^{-2}\mathcal{C}^2$ per unit of time. More precise statements are cited below in Theorems 2.1 and 2.2.

In this article we establish an analogous result for the energy of the taut strings going through the bands of different widths, i.e., we consider a system of problems with boundaries $g_1(t) := W(t) - r_T, g_2(t) := W(t) + r_T, 0 \leqslant t \leqslant T$. The precise statement is given in Theorem 2.3.

Then the obtained result is transferred to the case of random walk. Notice that the assumptions imposed on the function $T \mapsto r_T$, depend on the moment properties of the walk's step. The precise statement is given in Theorem 2.4.

All proofs are collected in Section 4.

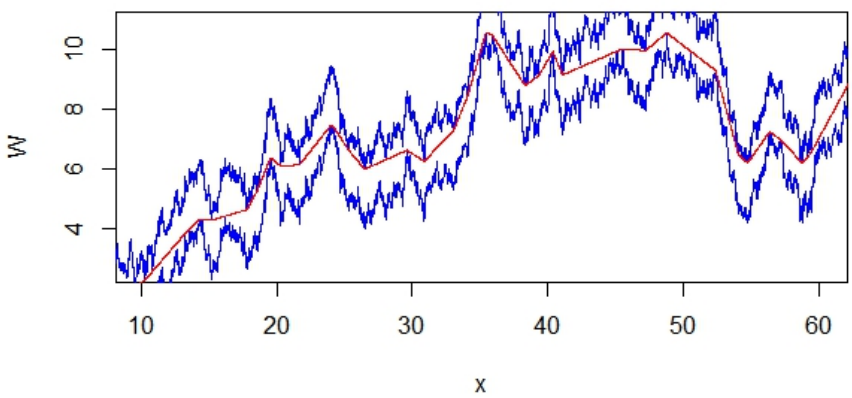

FIGURE 1. A taut string accompanying Wiener process

2. ENERGY OF ACCOMPANYING TAUT STRINGS

2.1. Wiener process. Throughout the article we consider the uniform norms

$$||h||_T := \sup_{0 \leqslant t \leqslant T} \{|h(t)|, \ \ h \in \mathbb{C}[0,T]\}$$

and the Sobolev norms

$$|h|_T^2 := \int_0^T h'(t)^2 dt, \ \ h \in \mathrm{AC}[0,T],$$

where $AC[0,T]$ denotes the space of absolutely continuous functions defined on the time interval [0,T]. We call $|h|_T^2$ the *kinetic energy* or simply the energy of a function h.

Let W be a Wiener process. We are interested in its following approximation characteristics

$$I_W(T,r) := \inf\{|h|_T; h \in AC[0,T], ||h-W||_T \leqslant r, h(0)=0\}$$

and

$$I_W^0(T,r) := \inf\{|h|_T; h \in AC[0,T], ||h-W||_T \leqslant r, h(0)=0, h(T)=W(T)\}.$$

The unique functions at which these infima are attained are called *the taut string* and *the taut string with fixed end*, respectively.

The main results obtained in [12] are as follows.

THEOREM 2.1. *There exists a constant $\mathcal{C} \in (0,\infty)$ such that, as $\frac{r_T}{T^{1/2}} \to 0$, it is true that*

$$\begin{aligned} \frac{r_T}{T^{1/2}} I_W(T,r_T) &\xrightarrow{L_q} \mathcal{C}, \\ \frac{r_T}{T^{1/2}} I_W^0(T,r_T) &\xrightarrow{L_q} \mathcal{C} \end{aligned}$$

for every $q > 0$.

THEOREM 2.2. *For every fixed $r > 0$, as $T \to \infty$, we have*

$$\begin{aligned} \frac{r}{T^{1/2}} I_W(T,r) &\to \mathcal{C} \qquad a.s., \\ \frac{r}{T^{1/2}} I_W^0(T,r) &\to \mathcal{C} \qquad a.s.. \end{aligned}$$

The constant $\mathcal{C}^2$ shows how much energy per unit of time must expend an absolutely continuous trajectory, if it is bounded to stay within unit distance of the non-differentiable sample path of the Wiener process W. The precise value of $\mathcal{C}$ remains unknown, while computer simulation yields the approximate value

$\mathcal{C} \approx 0.63$; moreover, in [12] the following theoretical bounds for $\mathcal{C}$ are obtained: $0.381 \leqslant \mathcal{C} \leqslant \frac{\pi}{2}$. An alternative theoretical approach to $\mathcal{C}$ is given in [18].

In the present work we need an intermediate result between Theorems 2.1 and 2.2. There, the band width will be varying as in Theorem 2.1 but the convergence with probability one will be obtained, as in Theorem 2.2. The zone of the admissible variation of the band width turns out to be slightly more narrow than in Theorem 2.1.

THEOREM 2.3. *Let W be a Wiener process. Assume that a band width r_T is non-decreasing but it is true that*

$$\frac{r_T\,(\ln\ln T)^{1/2}}{T^{1/2}} \to 0 \ \text{ as } T \to \infty, \tag{2.1}$$

the left hand side being non-increasing. Then it is true that

$$\lim_{T\to\infty} \frac{r_T}{T^{1/2}}\, I_W(T, r_T) = \mathcal{C} \ \text{a.s.}, \tag{2.2}$$

$$\lim_{T\to\infty} \frac{r_T}{T^{1/2}}\, I_W^0(T, r_T) = \mathcal{C} \ \text{a.s.} \tag{2.3}$$

where $\mathcal{C} \in (0, \infty)$ is the constant appearing in Theorem 2.1.

REMARK 2.1. *The claim of the theorem obviously remains true if we replace the band width r_T by $\rho_T = r_T(1 + o(1))$. This observation essentially enables to drop the monotonicity assumptions.*

REMARK 2.2. *It follows from the proof of the theorem that the claim* (2.3) *remains valid, if the condition of the arrival at the end point $W(T)$ is replaced by the arrival at some given point $W(T) + x_T$, whenever $x_T = o(r_T)$.*

REMARK 2.3. *In Theorem* 2.3 *a system of the bands of constant widths is considered. However, one can derive from it an information about the energy of the taut strings running through a band of varying width. Namely, let a function*

$T \mapsto r_T$ satisfy the assumptions of Theorem 2.3. *Let*

$$I_W(T, r.) := \inf\{|h|_T; h \in AC[0,T], h(0) = 0, |h(t) - W(t)| \leqslant r_t, 0 \leqslant t \leqslant T\}.$$

Then, as $T \to \infty$, it is true that

$$I_W(T, r.)^2 \sim \mathcal{C}^2 \int_0^T \frac{dt}{r_t^2} \qquad a.s.$$

The proof of this result requires considerable amount of computations and will be provided in a subsequent publication.

Assumption (2.1) may look strange at first glance but the following proposition shows that the iterated logarithm is essential.

PROPOSITION 2.1. *Let $M > 0$ and $r_T = M(T/\ln\ln T)^{1/2}$. Then*

$$\limsup_{T\to\infty} \frac{r_T}{T^{1/2}} I_W(T, r_T) \geqslant \sqrt{2}\, M \quad a.s.$$

2.2. Random walk.

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. real random variables. Define their partial sums by $S_0 := 0$ and

$$S_k := \sum_{j=1}^{k} X_j, \qquad k \geqslant 1.$$

Define on $[0, \infty)$ a random broken line $S(\cdot)$ as

$$S(t) := \begin{cases} S_k, & t = k,\ k = 0, 1, \ldots, \\ (k+1-t)S_k + (t-k)S_{k+1}, & t \in (k, k+1),\ k = 0, 1, \ldots. \end{cases}$$

We will consider the taut strings running through the band of width r around the broken line S. Introduce the approximation characteristics for S similar to those introduced previously for the Wiener process:

$$\begin{aligned} I_S(T, r) &:= \inf\{|h|_T; h \in AC[0,T], ||h - S||_T \leqslant r, h(0) = 0\}, \\ I_S^0(T, r) &:= \inf\{|h|_T; h \in AC[0,T], ||h - S||_T \leqslant r, h(0) = 0, h(T) = S(T)\}. \end{aligned}$$

In the following, just for writing simplicity, we only consider $T \in \mathbb{N}$. Then the following result on the energy of the taut string accompanying the random broken line S (random walk) is true.

THEOREM 2.4. *Let S be the random broken line based as above on the partial sums of i.i.d. random variables X_j having zero expectation and unit variance.*

Let X_j have the finite moment of order $p > 2$ and let r_T satisfy assumption (2.1) *and $T^{1/p} = O(r_T)$. Then*

$$\lim_{T\to\infty} \frac{r_T}{T^{1/2}}\, I_S(T, r_T) = \mathcal{C} \text{ a.s.},$$
$$\lim_{T\to\infty} \frac{r_T}{T^{1/2}}\, I_S^0(T, r_T) = \mathcal{C} \text{ a.s.},$$

where $\mathcal{C} \in (0, \infty)$ is the constant from Theorem 2.1.

Moreover, if the variables X_j have a finite exponential moment, then the above mentioned equalities are true under assumptions (2.1) *and $\ln T = o(r_T)$.*

3. A TOOL FOR THE TRANSFER TO RANDOM WALK

We recall now the main tool for the transfer of the results known for Wiener process to a random walk.

Let $X = \{X_1, ..., X_j, ...\}$ be a sequence of independent random variables with finite second moments and let $Y = \{Y_1, ..., Y_j, ...\}$ be a sequence of independent Gaussian random variables such that Y_j has the same expectation and variance as X_j. We want to construct (on some common probability space) the sequences $\bar{X} = \{\bar{X}_1, ..., \bar{X}_j, ...\}$ and $\bar{Y} = \{\bar{Y}_1, ..., \bar{Y}_j, ...\}$ equidistributed with X and Y, respectively, so that the discrepancy

$$\max_{1\leqslant k\leqslant n} \left| \sum_{j=1}^{k} \bar{X}_j - \sum_{j=1}^{k} \bar{Y}_j \right|$$

is small in the sense of a.s.-behavior: as $n \to \infty$, with probability one the discrepancy should increase not faster than some known function; the latter is defined by

the moment characteristics of the sequence X. Such closeness of two sequences is called *strong approximation*.

The optimal rate of strong approximation for sums of i.i.d. random variables was obtained by J. Komlos, P. Major and G. Tusnady. Here is a statement of their result, called by the authors' names KMT-theorem.

THEOREM 3.1. *Let $X = \{X_1, ..., X_j, ...\}$ be a sequence of i.i.d. random variables having finite moment of order $p > 2$. Then one may jointly construct on some probability space a sequence $\bar{X} = \{\bar{X}_1, ..., \bar{X}_j, ...\}$ equidistributed with X and a sequence of independent Gaussian random variables $\{\bar{Y}_1, ..., \bar{Y}_j, ...\}$ having the same expectation and variance such that*

$$\sum_{j=1}^{n} \bar{X}_j - \sum_{j=1}^{n} \bar{Y}_j = o(n^{1/p}) \qquad a.s.$$

If the variables X_j have a finite exponential moment, then one can obtain

$$\sum_{j=1}^{n} \bar{X}_j - \sum_{j=1}^{n} \bar{Y}_j = O(\ln n) \qquad a.s.$$

For the first claim of the theorem, see [14, p.214] for $2 < p < 3$ and [8] for $p > 2$. For the second claim see [7, p.112]. We also refer to [17], [21] for various extensions of the KMT-approach.

4. PROOFS

4.1. Proof of Theorem 2.3.

Before starting the proof, recall the following useful technical result. Let $m(T, r)$ denote the median of the random variable $I_W(T, r)$. The concentration of $I_W(T, r)$ and the limiting properties of $m(T, r)$ are described in the following lemma, cf. [12, Corollary 3.2 and p.408].

LEMMA 4.1. *a) For all $T > 0, r > 0, \rho > 0$ it is true that*

$$\mathbb{P}\left(|I_W(T, r) - m(T, r)| \geqslant \rho\right) \leqslant \exp\{-\rho^2/2\}. \tag{4.1}$$

b) If a function $T \mapsto r_T$ is satisfies $\lim_{T\to\infty} \frac{r_T}{T^{1/2}} = 0$, then

$$\lim_{T\to\infty} \frac{r_T}{T^{1/2}}\, m(T,r) = \mathcal{C}.$$

Now we are ready to start proving (2.2) and (2.3).

Step 1: the proof of (2.2).

Consider first an exponentially growing sequence of time instants $T_k := a^k$, where $a > 1$ is an arbitrary fixed number, as well as the corresponding sequence of band widths r_{T_k}.

By using the concentration inequality (4.1) together with (2.1), for arbitrary $\varepsilon > 0$, $M > 0$ for sufficiently large k_0 we have

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} \mathbb{P}\left(\frac{r_{T_k}}{T_k^{1/2}} \left| I_W(T_k, r_{T_k} a^{1/2}) - m(T_k, r_{T_k} a^{1/2}) \right| > \varepsilon \right) \\
\leqslant\ & \sum_{k=k_0}^{\infty} \exp\left\{ -\frac{T_k \varepsilon^2}{2 r_{T_k}^2} \right\} \leqslant \sum_{k=k_0}^{\infty} \exp\left\{ -\varepsilon^2 M \ln\ln T_k / 2 \right\} \\
=\ & \sum_{k=k_0}^{\infty} \exp\left\{ -\varepsilon^2 M (\ln\ln a + \ln k)/2 \right\},
\end{aligned}$$

and this sum turns out to be finite if we choose $M = M(\varepsilon) > 2\varepsilon^{-2}$. By the Borel–Cantelli lemma we obtain

$$\lim_{k\to\infty} \frac{r_{T_k}}{T_k^{1/2}} \left(I_W(T_k, r_{T_k} a^{1/2}) - m(T_k, r_{T_k} a^{1/2}) \right) = 0 \quad \text{a.s.} \tag{4.2}$$

Next, by the assumption of our theorem, it is true that $\frac{a^{1/2} r_{T_k}}{T_k^{1/2}} \to 0$, therefore, Lemma 4.1 yields the convergence of medians,

$$\frac{r_{T_k}}{T_k^{1/2}}\, m(T_k, a^{1/2} r_{T_k}) \to a^{-1/2}\mathcal{C}.$$

Taking into account the bound for the oscillations (4.2), we obtain

$$\lim_{k\to\infty} \frac{r_{T_k}}{T_k^{1/2}} I_W(T_k, a^{1/2} r_{T_k}) = a^{-1/2}\mathcal{C} \quad \text{a.s.} \tag{4.3}$$

Similarly, while considering the sequence $a^{-1/2}r_{T_k}$, we get:

$$\lim_{k\to\infty} \frac{r_{T_k}}{T_k^{1/2}} I_W(T_k, a^{-1/2}r_{T_k}) = a^{1/2}\mathcal{C} \quad \text{a.s.} \tag{4.4}$$

Let us now consider arbitrary values of the time parameter T. By the assumption of our theorem, the function

$$T \mapsto \frac{r_T}{T^{1/2}} = \frac{r_T(\ln\ln T)^{1/2}}{T^{1/2}} \cdot \frac{1}{(\ln\ln T)^{1/2}}$$

is non-increasing. Therefore, for every $T \in [T_{k-1}, T_k]$ it is true that

$$\frac{r_{T_k}}{T_k^{1/2}} \leqslant \frac{r_T}{T^{1/2}} \leqslant \frac{r_{T_{k-1}}}{T_{k-1}^{1/2}},$$

hence,

$$a^{-1/2}r_{T_k} \leqslant r_T \leqslant a^{1/2}r_{T_{k-1}}.$$

By using the first of these inequalities as well as the fact that $I_W(\cdot, r)$ is non-decreasing, while $I_W(T, \cdot)$ is non-increasing, we obtain the bound:

$$I_W(T, r_T) \leqslant I_W(T_k, a^{-1/2}r_{T_k}).$$

Since r_T is non-decreasing, the limiting relation (4.4) yields

$$\limsup_{T\to\infty} \frac{r_T I_W(T, r_T)}{T^{1/2}} \leqslant \limsup_{k\to\infty} \frac{r_{T_k} I_W(T_k, a^{-1/2}r_{T_k})}{(T_k/a)^{1/2}} \leqslant a\,\mathcal{C}.$$

Similarly, on the other hand,

$$I_W(T, r_T) \geqslant I_W(T_{k-1}, a^{1/2}r_{T_{k-1}})$$

and it follows from (4.3) that

$$\liminf_{T\to\infty} \frac{r_T I_W(T, r_T)}{T^{1/2}} \geqslant \liminf_{k\to\infty} \frac{r_{T_{k-1}} I_W(T_{k-1}, a^{1/2}r_{T_{k-1}})}{(aT_{k-1})^{1/2}} \geqslant a^{-1}\mathcal{C}.$$

By letting a tend to one in the obtained asymptotic bounds, we get (2.2).

Step 2: the proof of (2.3).

We will check now the convergence of $I^0_W(T, r_T)$. It follows from the definition that

$$I_W(T, r) \leqslant I^0_W(T, r) \quad \text{for all } T, r > 0.$$

This yields *the lower bound* in (2.3):

$$\liminf_{T\to\infty} \frac{r_T}{T^{1/2}} I^0_W(T, r_T) \geqslant \lim_{T\to\infty} \frac{r_T}{T^{1/2}}\, I_W(T, r_T) = \mathcal{C}.$$

The proof of *the upper bound* in (2.3) requires another approach because $I^0_W(\cdot, r)$ is not necessarily monotone.

Let us fix a $\delta \in (0, 1/3)$. For each interval length T we write a decomposition $T = T_* + L_T$, where $L_T \approx r_T^2 \ll T$ is small compared to T. The choice of L_T will be made precise later on.

Let h_1 be the taut string at which $I_W(T_*, (1-3\delta)r_T)$ is attained, i.e. $h_1(0) = 0$ and

$$||h_1 - W||_{T_*} \leqslant (1-3\delta)r_T, \tag{4.5}$$

$$|h_1|_{T_*} = I_W(T_*, (1-3\delta)r_T) \leqslant I_W(T, (1-3\delta)r_T). \tag{4.6}$$

Further, let us introduce an auxiliary Wiener process

$$\widetilde{W}(s) := W(T_* + s) - W(T_*), \qquad 0 \leqslant s \leqslant T - T_* = L_T,$$

and approximate it with the taut string h_2, at which $I_{\widetilde{W}}(L_T, \delta r_T)$ is attained, i.e. $h_2(0) = 0$ and

$$||h_2 - \widetilde{W}||_{L_T} \leqslant \delta r_T, \tag{4.7}$$

$$|h_2|_{L_T} = I_{\widetilde{W}}(L_T, \delta r_T). \tag{4.8}$$

Finally, we define an approximation function h with the fixed end $h(T) = W(T)$ by the relations

$$h(t) := \begin{cases} h_1(t), & 0 \leqslant t \leqslant T_*, \\ h_1(T_*) + h_2(t - T_*) + (t - T_*)\nu, & T_* \leqslant t \leqslant T, \end{cases}$$

where the value of the constant ν in the linear part is found from the equation

$$h(T) = h_1(T_*) + h_2(L_T) + L_T\nu = W(T),$$

i.e.

$$\nu = \frac{W(T) - h_1(T_*) - h_2(L_T)}{L_T}\,.$$

Let us remark that h is continuous at the transition point T_*, since $h_2(0) = 0$. Moreover, h is *absolutely* continuous, as a result of the continuous gluing of the two absolutely continuous pieces.

Furthermore, by (4.5) and (4.7) we have

$$\begin{aligned} |\nu| &= \frac{|[W(T_*) - h_1(T_*)] + [W(T) - W(T_*) - h_2(L_T)]|}{L_T} \\ &\leqslant \frac{|W(T_*) - h_1(T_*)| + \left|\widetilde{W}(L_T) - h_2(L_T)\right|}{L_T} \\ &\leqslant \frac{(1-3\delta)r_T + \delta r_T}{L_T} = \frac{(1-2\delta)r_T}{L_T}\,. \end{aligned} \tag{4.9}$$

Let us evaluate the uniform distance between h and W. For $0 \leqslant t \leqslant T_*$, by using (4.5) we simply have

$$|h(t) - W(t)| = |h_1(t) - W(t)| \leqslant ||h_1 - W||_{T_*} \leqslant (1-3\delta)r_T.$$

For $T_* \leqslant t \leqslant T$, we use an identity that is easy to verify,

$$h(t) - W(t) = h_2(t - T_*) - \widetilde{W}(t - T_*) - \big[L_T - (t - T_*)\big]\nu + \widetilde{W}(L_T) - h_2(L_T).$$

By using (4.7) and (4.9), we obtain

$$|h(t) - W(t)| \leqslant 2|h_2 - \widetilde{W}|_{L_T} + L_T|\nu| \leqslant 2\delta r_T + (1-2\delta)r_T \leqslant r_T.$$

By merging the estimates for both intervals, we find

$$||h - W||_T \leqslant r_T.$$

Let us now evaluate the energy of the function h. In view of (4.6), (4.8), and (4.9), we have

$$\begin{aligned}
|h|_T^2 &= \int_0^{T_*} h'(t)^2 dt + \int_{T_*}^{T} h'(t)^2 dt \\
&= \int_0^{T_*} h_1'(t)^2 dt + \int_{T_*}^{T} (h_2'(t-T_*)+\nu)^2 dt \\
&\leqslant \int_0^{T_*} h_1'(t)^2 dt + 2\int_{T_*}^{T} h_2'(t-T_*)^2 dt + 2\nu^2 L_T \\
&\leqslant I_W(T,(1-3\delta)r_T)^2 + 2I_{\widetilde{W}}(L_T,\delta r_T)^2 + \frac{2r_T^2}{L_T}.
\end{aligned}$$

We may conclude that

$$I_W^0(T,r_T)^2 \leqslant |h|_T^2 \leqslant I_W(T,(1-3\delta)r_T)^2 + 2I_{\widetilde{W}}(L_T,\delta r_T)^2 + \frac{2r_T^2}{L_T}\,.$$

Since by (2.2)

$$\lim_{T\to\infty} \frac{r_T^2}{T}\, I_W(T,(1-3\delta)r_T)^2 = (1-3\delta)^{-2}\mathcal{C}^2 \qquad \text{a.s.,}$$

and δ can be chosen arbitrarily small, for getting the upper bound in (2.3) it is enough to show that

$$\lim_{T\to\infty} \frac{r_T^2}{T}\, I_{\widetilde{W}}(L_T,\delta r_T)^2 = 0, \qquad \text{a.s.,} \tag{4.10}$$

$$\lim_{T\to\infty} \frac{r_T^2}{T}\,\frac{r_T^2}{L_T} = 0. \tag{4.11}$$

Our next goal is a reduction to a discrete set of time instants. To this aim, we construct the time sequence inductively by letting $T_0 := 1$, $T_{k+1} := T_k + r_{T_k}^2$. We denote $r_k := r_{T_k}$ and introduce the auxiliary Wiener processes by $\widetilde{W}_k(s) := W(T_k+s) - W(T_k)$.

From the regularity conditions of our theorem, it easily follows that

$$\lim_{k\to\infty} T_k = \infty; \quad \lim_{k\to\infty} \frac{T_{k+1}}{T_k} = 1; \quad \lim_{k\to\infty} \frac{r_{k+1}}{r_k} = 1.$$

Let us now determine the parameters of the previous construction, by letting for $T \in [T_{k+1}, T_{k+2})$,

$$T_* = T_*(T) := T_k, \qquad L_T := T - T_k.$$

Then for all sufficiently large k we have

$$\frac{r_T}{T^{1/2}} \leqslant \frac{2r_k}{T_k^{1/2}}$$

and

$$L_T \geqslant T_{k+1} - T_k = r_k^2 > r_{k+2}^2/2 \geqslant r_T^2/2,$$

and (4.11) follows from (2.1). On the other hand, for all large k it is true that

$$L_T \leqslant T_{k+2} - T_k = r_k^2 + r_{k+1}^2 < 3r_k^2,$$

which, in combination with (4.10), yields

$$I_{\widetilde{W}_k}(L_T, \delta r_T) \leqslant I_{\widetilde{W}_k}(L_T, \delta r_k) \leqslant I_{\widetilde{W}_k}(3r_k^2, \delta r_k).$$

Therefore, for proving (4.10) it is sufficient to the verify the relation

$$\lim_{k\to\infty} \frac{r_k}{T_k^{1/2}} I_{\widetilde{W}_k}(3r_k^2, \delta r_k)^2 = 0, \qquad \text{a.s.}$$

that only concerns the discrete sequence of time instants.

Notice that the variables $I_{\widetilde{W}_k}(3r_k^2, \delta r_k)$ are equidistributed with $I_W(3, \delta)$ by the self-similarity of the Wiener process. According to the Borel–Cantelli lemma, it is enough to check that for every $\varepsilon > 0$ it is true that

$$\sum_k \mathbb{P}\left(I_W(3, \delta) \geqslant \varepsilon \frac{T_k^{1/2}}{r_k} \right) < \infty. \tag{4.12}$$

By using the Gaussian concentration estimate (4.1) we only need to check that for every $h > 0$

$$\sum_k \exp\left\{ -\frac{hT_k}{r_k^2} \right\} < \infty.$$

By using that the functions $T \mapsto r_T$ and $T \mapsto \frac{T}{r_T^2}$ are non-decreasing, we have the integral bound

$$\begin{aligned}\int\limits_{T_{k-1}}^{T_k} \exp\left\{-\frac{hT}{r_T^2}\right\} r_T^{-2}\, dT &\geqslant (T_k - T_{k-1}) \exp\left\{-\frac{hT_k}{r_k^2}\right\} r_k^{-2} \\ &= \frac{r_{k-1}^2}{r_k^2} \exp\left\{-\frac{hT_k}{r_k^2}\right\} = \exp\left\{-\frac{hT_k}{r_k^2}\right\}(1+o(1)).\end{aligned}$$

It remains to prove that

$$\int^{\infty} \exp\left\{-\frac{hT}{r_T^2}\right\} r_T^{-2} dT < \infty.$$

Let us write the integrand as

$$\exp\left\{-\frac{hT}{r_T^2}\right\} r_T^{-2} = \exp\left\{-\frac{hT}{r_T^2}\right\} h\,\frac{T}{r_T^2} \cdot h^{-1}T^{-1} = \exp\{-u\}\, u \cdot h^{-1}T^{-1},$$

where $u := \frac{hT}{r_T^2}$. Recall that, according to (2.1), the inequality $u > 2\ln\ln T$ is true for sufficiently large T. Therefore,

$$\exp\{-u\}u \leqslant 2\exp\{-2\ln\ln T\}\ln\ln T = 2(\ln T)^{-2}\ln\ln T,$$

and we arrive at an estimate having the form of a convergent integral,

$$2h^{-1}\int^{\infty} \frac{\ln\ln T}{(\ln T)^2}\, T^{-1} dT < \infty.$$

Thus relation (4.12) is proved, hence (4.10) also follows. □

4.2. Proof of Proposition 2.1.

Our reasoning is based on the following elementary bound: for all $T, r > 0$ and every function $w \in \mathbb{C}[0,T]$ it is true that

$$I_w(T,r)^2 \geqslant \frac{[|w(T)-w(0)|-r]_+^2}{T}. \tag{4.13}$$

Indeed, let $||h-w||_T \leqslant r$ and $h(0) = w(0)$. Then the Hölder inequality yields

$$\begin{aligned}|h|_T^2 &= \int\limits_0^T h'(s)^2 ds \geqslant \frac{\left(\int\limits_0^T h'(s)ds\right)^2}{T} \\ &= \frac{(h(T)-h(0))^2}{T} \geqslant \frac{[|w(T)-w(0)|-r]_+^2}{T},\end{aligned}$$

and (4.13) follows.

Apply now (4.13) to $w = W, r = r_T$ in the framework of our proposition, and obtain by using the law of the iterated logarithm for Wiener process [11, Ch.17],

$$\begin{aligned}\limsup_{T\to\infty} \frac{r_T}{T^{1/2}} I_W(T, r_T) &\geqslant M \limsup_{T\to\infty} \frac{1}{(\ln\ln T)^{1/2}} \frac{[|W(T)| - r_T]_+}{T^{1/2}} \\ &= M \limsup_{T\to\infty} \frac{|W(T)|}{(T\ln\ln T)^{1/2}} = M\sqrt{2}.\end{aligned}$$

□

4.3. Proof of Theorem 2.4.

Recall that for every integer $k \in [0, T]$ we consider the partial sum S_k of i.i.d. random variables X_j, with zero expectations, unit variances and finite moment of order $p > 2$. We approximate these sums by the partial sums W_k of independent standard normal random variables $\bar{Y}_j$ satisfying Theorem 3.1. The random broken line S is defined by the nodes (k, S_k) as follows:

$$S(t) = \begin{cases} S_k, & t = k,\ k = 0, 1, ..., T, \\ (k+1-t)S_k + (t-k)S_{k+1}, & t \in (k, k+1),\ k = 0, 1, ..., T-1. \end{cases}$$

The random broken line $W(t)$ is defined analogously via the nodes (k, W_k). Then Theorem 3.1 yields with probability one $S_k - W_k = o(k^{1/p})$. This means that, as $T \to \infty$, we have

(4.14)

$$||S - W||_T = \sup_{0\leqslant t\leqslant T} \{|S(t) - W(t)|\} = \max_{0\leqslant k\leqslant T} |S_k - W_k| = o(T^{1/p}) \quad \text{a.s.}$$

Now we build a Wiener process $\widetilde{W}(t)$ upon the broken line $W(t)$, by adding independently a Brownian bridge $W_k^0(t-k)$ to every segment, connecting the nodes (k, W_k) and $(k+1, W_{k+1})$. Let us denote by $I_{\widetilde{W}}(T, r_T)$ the energy of the taut string running in the band of width r_T around the process $\widetilde{W}$ on the time interval $[0, T]$.

An upper bound for the energy.

First of all, let us notice that (4.14) and the assumptions of our theorem provide

$$||S - W||_T = o(T^{1/p}) = o(r_T).$$

Let $\rho_T := r_T - ||S - W||_T = r_T(1 + o(1))$.

Let us choose a string h accompanying the Wiener process $\widetilde{W}$ so that $||h - \widetilde{W}||_T \leqslant \rho_T$ and $|h|_T = I_{\widetilde{W}}(T, \rho_T)$. Let $\widehat{h}$ be the random broken line corresponding to the nodes $(k, h(k))$, $0 \leqslant k \leqslant T$. Then it is true that

$$\begin{aligned} ||\widehat{h} - S||_T &\leqslant ||\widehat{h} - W||_T + ||W - S||_T \leqslant ||h - \widetilde{W}||_T + ||W - S||_T \\ &\leqslant \rho_T + ||W - S||_T = r_T \end{aligned}$$

and

$$\begin{aligned} |\widehat{h}|_T^2 &= \sum_{k=0}^{T-1} \int_k^{k+1} \widehat{h}'(s)^2 ds = \sum_{k=0}^{T-1} (\widehat{h}(k+1) - \widehat{h}(k))^2 = \sum_{k=0}^{T-1} (h(k+1) - h(k))^2 \\ &= \sum_{k=0}^{T-1} \left(\int_k^{k+1} h'(s) ds \right)^2 \leqslant \sum_{k=0}^{T-1} \int_k^{k+1} h'(s)^2 ds = |h|_T^2. \end{aligned}$$

By applying Theorem 2.3 and Remark 2.1, we obtain

$$\begin{aligned} I_S(T, r_T) &\leqslant |\widehat{h}|_T \leqslant |h|_T = I_{\widetilde{W}}(T, \rho_T) \\ &= \mathcal{C} \, \frac{T^{1/2}}{\rho_T} \, (1 + o(1)) = \mathcal{C} \, \frac{T^{1/2}}{r_T} \, (1 + o(1)), \end{aligned}$$

which provides the required upper bound.

A lower bound for the energy.

Here, we must additionally take into account that

$$||\widetilde{W} - W||_T = \max_{0 \leqslant k < T} \max_{0 \leqslant s \leqslant 1} |W_k^0(s)| = O((\ln T)^{1/2}) \qquad \text{a.s.}$$

The latter follows from the well known bounds for the maxima of Gaussian processes, see [11, Ch.12]

$$\mathbb{P}\left(\max_{0 \leqslant s \leqslant 1} |W_k^0(s)| \geqslant x \right) = \exp\{-2x^2(1 + o(1))\}, \qquad x \to \infty,$$

combined with the Borel–Cantelli lemma. In particular, under the assumptions of our theorem we have

$$||\widetilde{W} - W||_T = O((\ln T)^{1/2}) = o(T^{1/p}) = o(r_T).$$

Let

$$\rho_T := r_T + ||S - W||_T + ||\widetilde{W} - W||_T = r_T + o(T^{1/p}) + O((\ln T)^{1/2}) = r_T(1 + o(1)).$$

Let us consider the string h accompanying the random broken line S such that $||h - S||_T \leqslant r_T$ and $|h|_T = I_S(T, r_T)$. Then

$$||h - \widetilde{W}||_T \leqslant ||h - S||_T + ||S - W||_T + ||W - \widetilde{W}|| \leqslant \rho_T.$$

Therefore, Theorem 2.3 and Remark 2.1 provide

$$I_S(T, r_T) = |h|_T \geqslant I_{\widetilde{W}}(T, \rho_T) = \mathcal{C}\,\frac{T^{1/2}}{\rho_T}\,(1 + o(1)) = \mathcal{C}\,\frac{T^{1/2}}{r_T}\,(1 + o(1)),$$

which yields the required lower bound.

The asymptotic behavior of $I_S^0(T, r_T)$ is investigated in the same way. One should additionally use Remark 2.2 to Theorem 2.3 with $x_T := S(T) - W(T)$.

The second part of the theorem is proved exactly as the first one but refers to the second part of Theorem 3.1, i.e. instead of (4.14) we use

$$||S - W||_T = \sup_{0 \leqslant t \leqslant T} \{|S(t) - W(t)|\} = \max_{0 \leqslant k \leqslant T} |S_k - W_k| = O(\ln T) \quad \text{a.s.}$$

□

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REFERENCES

[1] G.B. Dantzig, *A control problem of Bellman*, Management Sci. Theory Ser. 17 (1971) no. 9, pp. 542–546.

[2] P.L. Davies and A. Kovac, *Local extremes, runs, strings and multiresolution*, Ann. Statist. 29 (2001), no. 1, pp. 1–65.

[3] L. Dümbgen and A. Kovac, *Extensions of smoothing via taut strings*, Electron. J. Statist. 3 (2009), pp. 41–75.

[4] I.A. Ibragimov, M.A. Lifshits, and Z. Kabluchko, *Some extensions of linear approximation and prediction problems for stationary processes*, Stoch. Proc. Appl. 129 (2019), pp. 2758–2782.

[5] Z. Kabluchko and M. Lifshits, *Least energy approximation for processes with stationary increments*, J. Theoret. Probab. 30 (2017), no. 1, pp. 268—296.

[6] Z. Kabluchko and M. Lifshits, *Adaptive energy saving approximation for stationary processes*, Izvestiya: Mathematics 83 (2019), no.5, pp. 932–956.

[7] J. Komlos, P. Major, and G. Tusnady, *An approximation of partial sums of independent RV'-s and the sample DF. I*, Z. Wahrscheinlichkeitstheor. verw. Geb. 32, (1975), pp. 111–131.

[8] J. Komlos, P. Major, and G. Tusnady, *An approximation of partial sums of independent RV'-s and the sample DF. II*, Z. Wahrscheinlichkeitstheor. verw. Geb. 34 (1976), pp. 34–58.

[9] N. Kruglyak and E. Setterqvist, *Discrete taut strings and real interpolation*, J. Func. Anal. 270 (2016), pp. 671–704.

[10] N. Kruglyak and E. Setterqvist, *Invariant K-minimal sets in the discrete and continuous setting*, J. Fourier Anal. Appl. 23 (2017), pp. 572–611.

[11] M. Lifshits, Gaussian Random Functions, Kluwer, Dordrecht, 1995.

[12] M. Lifshits and E. Setterqvist, *Energy of taut string accompanying Wiener process*, Stoch. Processes Appl. 125 (2015), pp. 401–427.

[13] R.M. Lochowski and P.Miłos, *On truncated variation, upward truncated variation and downward truncated variation for diffusion*, Stoch. Proc. Appl. 123 (2013), pp. 446–474.

[14] P. Major, *The approximation of partial sums of independent r.v.'s*, Z. Wahrscheinlichkeitstheor. verw. Geb. 35 (1976), pp. 213–220.

[15] E. Mammen and S. van de Geer, *Locally adaptive regression splines*, Ann. Statist. 25 (1997), no. 1, pp. 387–413.

[16] F. Modigliani and F.E. Hohn, *Production planning over time and the nature of the expectation and planning horizon*, Econometrica 23 (1955), pp. 46–66.

[17] A.I. Sakhanenko, *Convergence rate in invariance principle for non-identically distributed vari-*

ables with exponential moments. A.A. Borovkov (Ed.), Limit Theorems for Sums of Random Variables, Advances in Probability Theory, Springer, New York (1985), pp. 2–73.

[18] E. Schertzer, *Renewal structure of the Brownian taut string*, Stoch. Proc. Appl. 128 (2018), no. 2, pp. 487–504.

[19] O. Scherzer et al., Variational Methods in Imaging, Ser.: Applied Mathematical Sciences, vol. 167, Springer, New York, 2009.

[20] E. Setterqvist and R. Forchheimer, *Real-time communication systems based on taut strings*, J. Communications and Networks 20 (2018), no. 2, pp. 207–218.

[21] A.Yu. Zaitsev, *The accuracy of strong Gaussian approximation for sums of independent random vectors*, Russian Math. Surveys 68 (2013), no. 4, pp. 721—761.

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