Riesz transform characterization of Hardy spaces associated with Schrödinger operators with compactly supported potentials

Marcin Preisner (Wrocław University)

joint work with Jacek Dziubański

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Let $P_t f(x) = P_t * f(x)$, $P_t(x) = (4\pi t)^{-d/2} \exp(-|x|^2/4t)$.

**Theorem (Fefferman, Stein, Coifman, Latter, ...)**

A function $f \in L^1(\mathbb{R}^d)$ belongs to the Hardy space $H^1(\mathbb{R}^d)$ when one of the following equivalent conditions holds:
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- \( \mathcal{M} f(x) = \sup_{t > 0} |P_t f(x)| \in L^1(\mathbb{R}^d) \),
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- $Mf(x) = \sup_{t>0} |P_t f(x)| \in L^1(\mathbb{R}^d)$,
- $R_j f(x) = c_d \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{x_j- y_j}{|x-y|^{d+1}} f(y) \, dy \in L^1(\mathbb{R}^d)$ for $j = 1, 2, \ldots, d$,
- $\exists \{\lambda_n, a_n\}_{n=1}^{\infty} f(x) = \sum_{n=1}^{\infty} \lambda_n a_n(x)$, where $\sum_{n=1}^{\infty} |\lambda_n| < \infty$ and $a_n$ are atoms (i.e. $\text{supp} \, a_n \subset B_n$, $\|a_n\|_\infty \leq |B_n|^{-1}$, $\int a = 0$)
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Each of the above conditions define a complete norm of $H^1(\mathbb{R}^d)$ and all of them are comparable, i.e.,

$$\|\mathcal{M}f\|_{L^1(\mathbb{R}^d)} \sim \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \|\mathcal{R}_j f\|_{L^1(\mathbb{R}^d)} \sim \inf_{f=\sum_n \lambda_n a_n} \sum_n |\lambda_n|.$$
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$$M_Lf(x) = \sup_{t>0} |K_tf(x)|.$$

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**Definition**

We say that an $L^1(\mathbb{R}^d)$-function $f$ belongs the Hardy space $H^1_L$ if

$$\|f\|_{H^1_L} = \|M_L f\|_{L^1(\mathbb{R}^d)} < \infty.$$
Recall that in the classical case for $f \in L^1(\mathbb{R}^d)$ we have

$$\mathcal{R}_j f(x) = \sqrt{\pi} \frac{1}{\partial x_j} (-\Delta)^{-1/2} = c_d \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) \, dy$$

$$= \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} P_t(x-y) f(y) \, dy \, \frac{dt}{\sqrt{t}},$$
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\]

**Definition**

Thus, for \( f \in L^1(\mathbb{R}^d) \), we define Riesz transforms \( R_j \) associated with \( L \) for \( j = 1, 2, \ldots, d \) by setting

\[
R_j f = \sqrt{\pi} \frac{\partial}{\partial x_j} L^{-1/2} f = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1/\varepsilon} \frac{\partial}{\partial x_j} K_t f \, dt \sqrt{t},
\]

where the limit is understood in the sense of distributions.
Having all this prepared we present the main theorem:

Theorem

Assume that $f \in L^1(\mathbb{R}^d)$. Then $f$ is in the Hardy space $H^1_L$ if and only if $R_j f \in L^1(\mathbb{R}^d)$ for every $j = 1, \ldots, d$. Moreover, there exists $C > 0$ such that $C^{-1} \|f\|_{H^1_L} \leq \|f\|_{L^1(\mathbb{R}^d)} + d \sum_{j=1}^{d} \|R_j f\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{H^1_L}$. 
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$$C^{-1} \| f \|_{H^1_L} \leq \| f \|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \| R_j f \|_{L^1(\mathbb{R}^d)} \leq C \| f \|_{H^1_L}.$$
Hardy spaces $H^1_L$ (with the same assumptions on potential $V$) were studied in the paper:

- J. Dziubański and J. Zienkiewicz, "Hardy space $H^1$ for Schrödinger operators with compactly supported potentials", Ann. Mat. Pura Appl., 184 (2005), 315–326.
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It was proved there that the elements of the space $H^1_L$ admit special atomic decompositions. In order to define the atoms the authors introduced the function

$$\omega(x) = \lim_{t \to \infty} \int K_t(x, y) \, dy$$

which turns out to be $L$-harmonic, Lipschitz, and $0 < \varepsilon < \omega \leq 1$. 
Previous results

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The atoms of the Hardy space for the Schrödinger operator satisfy the same conditions as in the classical case, the only difference is that the cancellation condition holds with respect to $\omega$, i.e., $\int a(x)\omega(x)\,dx = 0$. 
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In other words the mapping

$$H^1_L \ni f \mapsto f \cdot \omega \in H^1(\mathbb{R}^d)$$

is an isomorphism of the Hardy space for the Schrödinger operator and the classical one.
An isomorphism with classical Hardy space

In order to prove this result Dziubański and Zienkiewicz first obtained another isomorphism between $H^1_L$ and the classical Hardy space $H^1(\mathbb{R}^d)$.

Remark

Having these results, in order to prove the Riesz transform characterization of $H^1_L$, it suffices to show that $R_j(I - VL^{-1}) - R_j$, $j = 1, 2, \ldots, d$, are bounded operators on $L^1(\mathbb{R}^d)$. 
In order to prove this result Dziubański and Zienkiewicz first obtained another isomorphism between $H^1_L$ and the classical Hardy space $H^1(\mathbb{R}^d)$. Namely, they proved that the operator

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The operator $I - VL^{-1}$ is also an isomorphism on $L^1(\mathbb{R}^d)$. 
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Also, corresponding norms are equivalent, i.e.,

$$\|(I - VL^{-1})f\|_{H^1(\mathbb{R}^d)} \simeq \|f\|_{H^1_L} \quad \text{for} \quad f \in H^1_L.$$
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*Having these results, in order to prove the Riesz transform characterization of $H^1_L$, it suffices to show that $R_j(I - VL^{-1}) - R_j, j = 1, 2..., d$, are bounded operators on $L^1(\mathbb{R}^d)$.**
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\[ P_t = K_t + \int_0^t P_{t-s} V K_s \, ds \]
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from which the following two identities hold

\[ P_t(I - VL^{-1}) - K_t = \int_0^t P_{t-s} VK_s \, ds - P_t VL^{-1} = \widetilde{W}_t - \widetilde{Q}_t. \]

\[ P_t(I - VL^{-1}) - K_t = \int_0^t (P_{t-s} - P_t) VK_s \, ds - \int_t^\infty P_t VK_s \, ds = W_t - Q_t. \]
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Derivating both formulas with respect to \( x_j \), integrating the first one from \( \varepsilon \) to 1 and the second one from 1 to \( \varepsilon^{-1} \) (with respect to the weight \( dt/\sqrt{t} \)) we get:
Crucial difference

\[
\left( \int_{\varepsilon}^{\varepsilon^{-1}} \frac{\partial}{\partial x_j} P_t \frac{dt}{\sqrt{t}} \right) (I - VL^{-1}) - \int_{\varepsilon}^{\varepsilon^{-1}} \frac{\partial}{\partial x_j} K_t \frac{dt}{\sqrt{t}}
\]

\[
= \int_{\varepsilon}^{1} \frac{\partial}{\partial x_j} \tilde{W}_t \frac{dt}{\sqrt{t}} - \int_{\varepsilon}^{1} \frac{\partial}{\partial x_j} \tilde{Q}_t \frac{dt}{\sqrt{t}} + \int_{\varepsilon}^{1} \frac{\partial}{\partial x_j} W_t \frac{dt}{\sqrt{t}} - \int_{1}^{\varepsilon} \frac{\partial}{\partial x_j} Q_t \frac{dt}{\sqrt{t}}
\]

\[
= \tilde{W}^\varepsilon_j - \tilde{Q}^\varepsilon_j + W^\varepsilon_j - Q^\varepsilon_j.
\]
The operator on the LHS when we apply to $f \in L^1(\mathbb{R}^d)$ tends to $\mathcal{R}_j(I - VL^{-1})f - R_j f$ as $\varepsilon \to 0$, so it is enough to study the limits of the four terms on the RHS.
The limits of $\tilde{W}_j^\varepsilon$, $\tilde{Q}_j^\varepsilon$, $W_j^\varepsilon$

**Lemma**

The operators $Q_j^\varepsilon$, $\tilde{W}_j^\varepsilon$, and $W_j^\varepsilon$ converge in the norm-operator topology to bounded operators on $L^1(\mathbb{R}^d)$ as $\varepsilon \to 0$. 
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**Lemma**

*The operators $Q^\varepsilon_j$, $\tilde{\mathcal{W}}^\varepsilon_j$, and $\mathcal{W}^\varepsilon_j$ converge in the norm-operator topology to bounded operators on $L^1(\mathbb{R}^d)$ as $\varepsilon \to 0$.***

*Proof.* In this talk we will only concentrate on $\lim_{\varepsilon \to 0} \mathcal{W}^\varepsilon_j$. 

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Hardy spaces associated with Schrödinger operators
Lemma

The operators $Q_j^\varepsilon$, $\tilde{W}_j^\varepsilon$, and $W_j^\varepsilon$ converge in the norm-operator topology to bounded operators on $L^1(\mathbb{R}^d)$ as $\varepsilon \to 0$.

Proof. In this talk we will only concentrate on $\lim_{\varepsilon \to 0} W_j^\varepsilon$.

The integral kernels of these operators are

$$W_j^\varepsilon(x, y) = \int_1^{\varepsilon^{-1}} \frac{\partial}{\partial x_j} W_t(x, y) \frac{dt}{\sqrt{t}},$$

where

$$W_t(x, y) = \int_0^t \int (P_{t-s}(x - z) - P_t(x - z)) V(z) K_s(z, y) \, dz \, ds.$$
Proof

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Proof

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$$\int \left| \mathcal{W}^{\varepsilon_1}_{j}(x, y) - \mathcal{W}^{\varepsilon_2}_{j}(x, y) \right| \, dx \leq \mathcal{W}'(y) + \mathcal{W}''(y),$$

where

$$\mathcal{W}'(y) = \int \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} \int_{0}^{t^{8/9}} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} \left( P_t(x - z) - P_{t-s}(x - z) \right) \right| V(z) K_s(z, y) \, dz \, ds \, \frac{dt}{\sqrt{t}} \, dx$$

and

$$\mathcal{W}''(y) = \int \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} \int_{0}^{t^{8/9}} \int_{\mathbb{R}^d} \left| \frac{\partial^2}{\partial x_j \partial x_k} \left( P_t(x - z) - P_{t-s}(x - z) \right) \right| V(z) K_s(z, y) \, dz \, ds \, \frac{dt}{\sqrt{t}} \, dx.$$
Proof

Let $0 < \varepsilon_2 < \varepsilon_1 < 1/2$. Then

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Observe that for $0 < s < t^{8/9}$, there exists $0 \leq \phi \in \mathcal{S}(\mathbb{R})$ such that

$$\left| \frac{\partial}{\partial x_j} \left( P_t(x - z) - P_{t-s}(x - z) \right) \right| \leq s \, t^{-3/2} \phi_t(x - z) \quad (\phi_t(x) = t^{-d/2} \phi(x/\sqrt{t}))$$
Proof

Let $0 < \varepsilon_2 < \varepsilon_1 < 1/2$. Then

$$\int \left| W^{\varepsilon_1}_j(x, y) - W^{\varepsilon_2}_j(x, y) \right| dx \leq W'(y) + W''(y), \quad \text{where}$$

$$W'(y) = \int \int \int \frac{\partial}{\partial x_j} \left( P_t(x - z) - P_{t-s}(x - z) \right) \left| V(z) K_s(z, y) \right| dz ds dt dx$$

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Moreover, $sK_s(z, y) \leq C s^{(2-d)/2} \exp \left( -c |z - y|^2 / s \right) \leq C |z - y|^{2-d}$. Thus
Proof

Let $0 < \varepsilon_2 < \varepsilon_1 < 1/2$. Then

$$
\int \left| \mathcal{W}_{\varepsilon}^{\varepsilon_1}(x, y) - \mathcal{W}_j^{\varepsilon_2}(x, y) \right| \, dx \leq \mathcal{W}'(y) + \mathcal{W}''(y),
$$

where

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\mathcal{W}'(y) = \int \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} \int_0^{t^{8/9}} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} \left( P_t(x - z) - P_{t-s}(x - z) \right) \right| V(z) K_s(z, y) \, dz \, ds \, dt \, dx.
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\left| \frac{\partial}{\partial x_j} \left( P_t(x - z) - P_{t-s}(x - z) \right) \right| \leq s \, t^{-3/2} \phi_t(x - z) \quad (\phi_t(x) = t^{-d/2} \phi(x/\sqrt{t}))
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Moreover, $sK_s(z, y) \leq Cs^{(2-d)/2} \exp \left( -c |z - y|^2 / s \right) \leq C |z - y|^{2-d}$. Thus

$$
\mathcal{W}'(y) \leq \int \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} \int_0^{t^{8/9}} s \, t^{-2} \phi_t(x - z) V(z) K_s(z, y) \, dz \, ds \, dt \, dx
$$

$$
\leq \int_{\varepsilon_1^{-1}}^{\varepsilon_2^{-1}} t^{-10/9} dt \cdot \int V(z) |z - y|^{2-d} \, dz \leq C \varepsilon_1^{1/9}
$$

uniformly in $y$. The last inequality is a simple consequence of the Hölder inequality and the assumption $p > d/2$. 

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Proof

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Moreover, $sK_s(z, y) \leq Cs^{(2-d)/2} \exp \left( -c |z - y|^2 / s \right) \leq C |z - y|^{2-d}$. Thus

\[
\mathcal{W}'(y) \leq \int \int \int_{\mathbb{R}^d} s \, t^{-2} \phi_t(x - z) V(z) K_s(z, y) \, dz \, ds \, dt \, dx
\]

\[
\leq \int \int t^{-10/9} dt \cdot \int \left| V(z) \right| |z - y|^{2-d} \, dz \leq C\varepsilon_1^{1/9}
\]

uniformly in $y$. The last inequality is a simple consequence of the Hölder inequality and the assumption $p > d/2$.

A similar analysis leads to $\mathcal{W}''(y) \leq C\varepsilon_1^{4d/9 - 1}$. 
The limit of $Q_j^\varepsilon f$ for $f \in L^1(\mathbb{R}^d)$

Lemma

Assume that $f \in L^1(\mathbb{R}^d)$. Then the limit $F = \lim_{\varepsilon \to 0} Q_j^\varepsilon f$ exists in the $L^1$-norm. Moreover, $\|F\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{L^1(\mathbb{R}^d)}$ with $C$ independent of $f$. 
Lemma

Assume that $f \in L^1(\mathbb{R}^d)$. Then the limit $F = \lim_{\varepsilon \to 0} \mathcal{Q}_j^\varepsilon f$ exists in the $L^1$-norm. Moreover, $\|F\|_{L^1(\mathbb{R}^d)} \leq C\|f\|_{L^1(\mathbb{R}^d)}$ with $C$ independent of $f$.

Proof. Of course, for any fixed $y \in \mathbb{R}^d$, the function $z \mapsto U(z, y) = V(z)\Gamma(z, y)$ is supported in the unit ball and 
$\|U(z, y)\|_{L^r(dz)} \leq C_r$ for fixed $r \in \left[1, \frac{dp}{dp+d-2p}\right]$ with $C_r$ independent of $y$. 
The limit of $Q_j^\varepsilon f$ for $f \in L^1(\mathbb{R}^d)$

**Lemma**

Assume that $f \in L^1(\mathbb{R}^d)$. Then the limit $F = \lim_{\varepsilon \to 0} \widetilde{Q}_j^\varepsilon f$ exists in the $L^1$-norm. Moreover, $\|F\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{L^1(\mathbb{R}^d)}$ with $C$ independent of $f$.

Proof. Of course, for any fixed $y \in \mathbb{R}^d$, the function $z \mapsto U(z, y) = V(z)\Gamma(z, y)$ is supported in the unit ball and $\|U(z, y)\|_{L^r(dz)} \leq C_r$ for fixed $r \in \left[1, \frac{dp}{dp+d-2p}\right]$ with $C_r$ independent of $y$. Let

$$H_j^\varepsilon(x, z) = \int_0^1 \frac{\partial}{\partial x_j} P_t(x - z) \frac{dt}{\sqrt{t}}, \quad H_j^* g(x) = \sup_{0 < \varepsilon < 1} |H_j^\varepsilon g(x)|.$$
The limit of $Q_j^\varepsilon f$ for $f \in L^1(\mathbb{R}^d)$

**Lemma**

Assume that $f \in L^1(\mathbb{R}^d)$. Then the limit $F = \lim_{\varepsilon \to 0} Q_j^\varepsilon f$ exists in the $L^1$-norm. Moreover, $\|F\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{L^1(\mathbb{R}^d)}$ with $C$ independent of $f$.

Proof. Of course, for any fixed $y \in \mathbb{R}^d$, the function $z \mapsto U(z, y) = V(z)\Gamma(z, y)$ is supported in the unit ball and $\|U(z, y)\|_{L^r(dz)} \leq C_r$ for fixed $r \in \left[1, \frac{dp}{dp+d-2p}\right]$ with $C_r$ independent of $y$. Let

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It follows from the theory of singular integral convolution operators that for $1 < r < \infty$ there exists $C_r$ such that

$$\|H_j^* g\|_{L^r(\mathbb{R}^d)} \leq C_r \|g\|_{L^r(\mathbb{R}^d)} \quad \text{for } g \in L^r(\mathbb{R}^d)$$

and $\lim_{\varepsilon \to 0} H_j^\varepsilon g(x) = H_j g(x)$ a.e. and in $L^r(\mathbb{R}^d)$-norm.
Proof

Note that $\widetilde{Q}^\varepsilon_j(x, y) = H^\varepsilon_j U(\cdot, y)(x)$. Hence, there exists a function $f^\varepsilon_j(x, y)$ such that $\lim_{\varepsilon \to 0} f^\varepsilon_j(x, y) = f^\varepsilon_j(x, y)$ a.e. and

$$\sup_y \int_\mathbb{R}^d \sup_{0 < \varepsilon < 1} \int f^\varepsilon_j(x, y) \, dx \leq C' \varepsilon^{\frac{1}{p'}} \quad \text{for} \ 1 < \frac{1}{p'} \leq \frac{d}{dp} + \frac{d}{d-2}p.$$
Proof

Note that $\widetilde{Q}_j^\varepsilon(x, y) = H_j^\varepsilon U(\cdot, y)(x)$. Hence, there exists a function $\widetilde{Q}_j(x, y)$ such that $\lim_{\varepsilon \to 0} \widetilde{Q}_j^\varepsilon(x, y) = \widetilde{Q}_j(x, y)$ a.e. and

$$\sup_y \int_{\mathbb{R}^d} \sup_{0 < \varepsilon < 1} \left| \widetilde{Q}_j^\varepsilon(x, y) \right|^r \, dx \leq C' \text{ for } 1 < r < \frac{dp}{dp + d - 2p}.$$
Proof

Note that $\widetilde{Q}_{j}^{\varepsilon}(x, y) = H_{j}^{\varepsilon} U(\cdot, y)(x)$. Hence, there exists a function $\widetilde{Q}_{j}(x, y)$ such that $\lim_{\varepsilon \to 0} \widetilde{Q}_{j}^{\varepsilon}(x, y) = \widetilde{Q}_{j}(x, y)$ a.e. and

$$\sup_{y} \int_{\mathbb{R}^{d}} \sup_{0 < \varepsilon < 1} \left| \widetilde{Q}_{j}^{\varepsilon}(x, y) \right|^{r} \, dx \leq C_{r}^{'} \text{ for } 1 < r < \frac{dp}{dp + d - 2p}.$$  

Since $|H_{j}^{\varepsilon}(x, z)| \leq C_{N}|x - z|^{-N}$ for $|x - z| > 1$,

$$|\widetilde{Q}_{j}^{\varepsilon}(x, y)| = \left| \int_{|z| \leq 1} H_{j}^{\varepsilon}(x, z) U(z, y) \, dz \right| \leq C_{N}|x|^{-N} \text{ for } |x| > 2, \ y \in \mathbb{R}^{d}.$$
Proof

Note that $\tilde{Q}_j^\varepsilon(x, y) = H_j^\varepsilon U(\cdot, y)(x)$. Hence, there exists a function $\tilde{Q}_j(x, y)$ such that $\lim_{\varepsilon \to 0} \tilde{Q}_j^\varepsilon(x, y) = \tilde{Q}_j(x, y)$ a.e. and

$$\sup_y \int_{\mathbb{R}^d} \sup_{0 < \varepsilon < 1} \left| \tilde{Q}_j^\varepsilon(x, y) \right|^r dx \leq C_r' \text{ for } 1 < r < \frac{dp}{dp + d - 2p}.$$

Since $|H_j^\varepsilon(x, z)| \leq C_N |x - z|^{-N}$ for $|x - z| > 1$,

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Applying the Hölder inequality we get

$$\sup_y \int_{\mathbb{R}^d} \sup_{0 < \varepsilon < 1} \left| \tilde{Q}_j^\varepsilon(x, y) \right| dx \leq C,$$

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \left| \tilde{Q}_j^\varepsilon(x, y) - \tilde{Q}_j(x, y) \right| dx = 0 \text{ for every } y.$$

Now the lemma can be easily concluded from the Lebesgue dominated convergence theorem.
Proof of the main theorem

Consider $f \in L^1(\mathbb{R}^d)$. Using above lemmas we get that $R_j f \in L^1(\mathbb{R}^d)$ if and only if $R_j (I - VL - 1)V f \in L^1(\mathbb{R}^d)$ and

$$\|f\|_{L^1(\mathbb{R}^d)} + dX_j = 1 \|R_j f\|_{L^1(\mathbb{R}^d)} \sim \|R_j (I - VL - 1)V f\|_{L^1(\mathbb{R}^d)}.$$ 

Classical characterization implies that $(I - VL - 1)V f \in H^1(\mathbb{R}^d)$ if and only if $(I - VL - 1)V f, R_j (I - VL - 1)V f \in L^1(\mathbb{R}^d)$ for $j = 1, \ldots, d$. Moreover,

$$\|R_j (I - VL - 1)V f\|_{L^1(\mathbb{R}^d)} \sim \|I - VL - 1 f\|_{H^1(\mathbb{R}^d)}.$$ 

Finally, recall that $(I - VL - 1)V$ is the isomorphism of Hardy spaces $H^1L$ and $H^1(\mathbb{R}^d)$. Thus $(I - VL - 1)V f \in H^1(\mathbb{R}^d)$ if and only if $f \in H^1L$ and

$$\|I - VL - 1 f\|_{H^1(\mathbb{R}^d)} \sim \|f\|_{H^1L}.$$ 

Putting these facts together we obtain the proof of the main theorem.
Proof of the main theorem

Consider $f \in L^1(\mathbb{R}^d)$.

- Using above lemmas we get that $R_j f \in L^1(\mathbb{R}^d)$ if and only if $R_j (I - VL^{-1}) f \in L^1(\mathbb{R}^d)$ and

\[
\|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|R_j f\|_{L^1(\mathbb{R}^d)} \sim \|(I - VL^{-1}) f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|R_j (I - VL^{-1}) f\|_{L^1(\mathbb{R}^d)}
\]
Proof of the main theorem

Consider $f \in L^1(\mathbb{R}^d)$.

- Using above lemmas we get that $R_j f \in L^1(\mathbb{R}^d)$ if and only if $\mathcal{R}_j(I - VL^{-1}) f \in L^1(\mathbb{R}^d)$ and
  \[
  \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|R_j f\|_{L^1(\mathbb{R}^d)} \sim \|(I - VL^{-1}) f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|\mathcal{R}_j(I - VL^{-1}) f\|_{L^1(\mathbb{R}^d)}
  \]

- Classical characterization implies that $(I - VL^{-1}) f \in H^1(\mathbb{R}^d)$ if and only if $(I - VL^{-1}) f, \mathcal{R}_j(I - VL^{-1}) f \in L^1(\mathbb{R}^d)$ for $j = 1, \ldots, d$. Moreover,
  \[
  \|(I - VL^{-1}) f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|\mathcal{R}_j(I - VL^{-1}) f\|_{L^1(\mathbb{R}^d)} \sim \|(I - VL^{-1}) f\|_{H^1(\mathbb{R}^d)}.
  \]
Proof of the main theorem

Consider \( f \in L^1(\mathbb{R}^d) \).

- Using above lemmas we get that \( R_j f \in L^1(\mathbb{R}^d) \) if and only if \( R_j (I - VL^{-1}) f \in L^1(\mathbb{R}^d) \) and
  \[
  \| f \|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \| R_j f \|_{L^1(\mathbb{R}^d)} \sim \| (I - VL^{-1}) f \|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \| R_j (I - VL^{-1}) f \|_{L^1(\mathbb{R}^d)}
  \]

- Classical characterization implies that \( (I - VL^{-1}) f \in H^1(\mathbb{R}^d) \) if and only if \( (I - VL^{-1}) f, R_j (I - VL^{-1}) f \in L^1(\mathbb{R}^d) \) for \( j = 1, \ldots, d \). Moreover,
  \[
  \| (I - VL^{-1}) f \|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \| R_j (I - VL^{-1}) f \|_{L^1(\mathbb{R}^d)} \sim \| (I - VL^{-1}) f \|_{H^1(\mathbb{R}^d)}.
  \]

- Finally, recall that \( (I - VL^{-1}) \) is the isomorphism of Hardy spaces \( H^1_L \) and \( H^1(\mathbb{R}^d) \). Thus \( (I - VL^{-1}) f \in H^1(\mathbb{R}^d) \) if and only if \( f \in H^1_L \) and
  \[
  \| (I - VL^{-1}) f \|_{H^1(\mathbb{R}^d)} \sim \| f \|_{H^1_L}.
  \]
Proof of the main theorem

Consider $f \in L^1(\mathbb{R}^d)$.

- Using above lemmas we get that $R_j f \in L^1(\mathbb{R}^d)$ if and only if $\mathcal{R}_j (I - VL^{-1}) f \in L^1(\mathbb{R}^d)$ and

  $$\|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \|R_j f\|_{L^1(\mathbb{R}^d)} \sim \|(I - VL^{-1}) f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \|\mathcal{R}_j (I - VL^{-1}) f\|_{L^1(\mathbb{R}^d)}$$

- Classical characterization implies that $(I - VL^{-1}) f \in H^1(\mathbb{R}^d)$ if and only if $(I - VL^{-1}) f, \mathcal{R}_j (I - VL^{-1}) f \in L^1(\mathbb{R}^d)$ for $j = 1, \ldots, d$. Moreover,

  $$\|(I - VL^{-1}) f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \|\mathcal{R}_j (I - VL^{-1}) f\|_{L^1(\mathbb{R}^d)} \sim \|(I - VL^{-1}) f\|_{H^1(\mathbb{R}^d)}.$$  

- Finally, recall that $(I - VL^{-1})$ is the isomorphism of Hardy spaces $H^1_L$ and $H^1(\mathbb{R}^d)$. Thus $(I - VL^{-1}) f \in H^1(\mathbb{R}^d)$ if and only if $f \in H^1_L$ and

  $$\|(I - VL^{-1}) f\|_{H^1(\mathbb{R}^d)} \sim \|f\|_{H^1_L}.$$

Putting these facts together we obtain the proof of the main theorem.