Uniwersytet Wrocławski
Wydział Matematyki i Informatyki

Marcin Preisner

Przestrzenie Hardy'ego związane z półgrupami operatorów liniowych

rozprawa doktorska

Promotor rozprawy
prof. dr hab. Jacek Dziubański

Wrocław, grudzień 2011

University of Wrocław

Faculty of Mathematics and Computer Science

Marcin Preisner

Hardy spaces related to semigroups of operators

PhD dissertation

Supervisor

prof. dr hab. Jacek Dziubański

Wrocław, December 2011

Acknowledgements

First and foremost, I would like to thank my supervisor, prof. Jacek Dziubański, who was extremely helpful and offered invaluable assistance, support and guidance.

Secondly, I have learnt a great deal from numerous people over the years and gratefully acknowledge my debt to them, especially to: Prof. Piotr Biler, Dr hab. Dariusz Buraczewski, Prof. Krzysztof Bogdan, Prof. Marek Bożejko, Prof. Ewa Damek, Dr hab. Krzysztof Dębicki, Dr hab. Jan Dymara, Prof. Paweł Głowacki, Prof. Waldemar Hebisch, Prof. Andrzej Hulanicki, Prof. Grzegorz Karch, Prof. Paweł Krupski, Dr Mateusz Kwaśnicki, Dr Mariusz Mirek, Prof. Władysław Narkiewicz, Dr hab. Adam Nowak, Dr Maciej Paluszyński, Dr hab. Tadeusz Pezda, Prof. Grzegorz Plebanek, Prof. Krzysztof Stempak, Dr hab. Robert Stańczy, Prof. Ryszard Szwarc, Dr hab. Piotr Śniady, Prof. Jacek Świątkowski - all from Wrocław University and Wrocław University of Technology, and also to Mgr Ryszard Misiek - my high school teacher. Specifically, I would like to express my gratitude for Dr hab. Jacek Zienkiewicz. I deeply appreciated his comments.

Last but not least, very special thanks to my parents, all my family and friends. I dedicate this thesis to my fiancée Kasia who has been a great source of inspiration.

The research project was supported by Polish Government funds for science - grants N N201 397137 and N N201 412639, MNiSW

CONTENTS

1. *Introduction* . . . 1

1.1 Hardy spaces - background . . . 1

1.2 Definitions of Hardy spaces associated with semigroups of operators . . . 4

1.3 Results and organization of the thesis . . . 5

1.3.1 Schrödinger operators I . . . 5

1.3.2 Schrödinger operators II . . . 6

1.3.3 Laguerre and Bessel operators . . . 6

1.4 Final remarks . . . 7

2. *Hardy spaces related to Schrödinger operators I* . . . 9

2.1 Background and main result . . . 9

2.2 Remarks and examples . . . 11

2.3 Auxiliary lemmas . . . 13

2.4 Proof of Theorem 2.1.6 . . . 19

2.5 Appendix . . . 20

2.5.1 Proof of Lemma 2.3.1 . . . 20

3. *Hardy spaces related to Schrödinger operators II* . . . 23

3.1 Background and main results . . . 23

3.2 Auxiliary lemmas . . . 25

3.3 Proof of Theorem 3.1.2 . . . 29

3.4 Proof of Theorem 3.1.4 . . . 34

3.5 Proof of Theorem 3.1.8 . . . 38

3.6 Appendix . . . 42

3.6.1 Proof of Corollary 3.2.6 . . . 42

4. *Chapter 4: Hardy spaces related to the Laguerre operator* . . . 49

4.1 Background and main result . . . 49

4.2 Hardy spaces in the Bessel setting 52
4.2.1 Global Hardy space 52
4.2.2 Local Hardy spaces 54
4.3 The Riesz transform in the Laguerre setting 58
4.4 Proof of Theorem 4.1.8 60
4.5 Proofs of Propositions 4.2.4 and 4.3.1 61
4.5.1 Proof of Proposition 4.2.4 62
4.5.2 Proof of Proposition 4.3.1 65

Bibliography 73

1. INTRODUCTION

1.1 Hardy spaces - background

The Hardy spaces did appear in the classical theorem of Burkholder, Gundy and Silverstein [6]. The paper considers real-valued, harmonic functions $u(z) = u(x+iy)$ defined in the upper half-plane $\{z = x+iy : y > 0\}$. It is proved there, that for fixed p, $0 < p < \infty$, a function $u(z) = u(x+iy) = u(x,y)$ is the real part of a holomorphic function $F(z) = u(z) + iv(z)$ that satisfies the H^p-property

$$\sup_{y>0} \int_{\mathbb{R}} |F(x+iy)|^p dx < \infty \tag{1.1.1}$$

if and only if the maximal function

$$u^*(x) = \sup_{|x-x'|<y} |u(x',y)| \tag{1.1.2}$$

belongs to $L^p(\mathbb{R})$.

The work of Feferman and Stein [23] gives other characterizations of the H^p-property by means of the real harmonic analysis on $\mathbb{R}$. These lead to the notion of the real Hardy spaces H^p, which can be defined not only on $\mathbb{R}$, but also on $\mathbb{R}^d$ (see, e.g. [38]), or even more generally on spaces of homogeneous type (see, e.g. [8], [29], [40]).

Let

$$\exp(-t\sqrt{-\Delta})f(x) = c_d \int_{\mathbb{R}^d} \frac{t}{(|x-y|^2+t^2)^{(d+1)/2}} f(y)\, dy$$

be the Poisson semigroup related to the Laplace operator

$$\mathbf{\Delta} = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

on $\mathbb{R}^d$. By definition a tempered distribution f is *bounded* if and only if for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ (the Schwartz class) we have $f * \phi \in L^\infty(\mathbb{R}^d)$. Recall that $\exp(-t\sqrt{-\Delta})f$ is well-defined smooth function whenever f is a bounded distribution (see [38, Chapter III, Section 1.1]).

Definition 1.1.3. *A bounded distribution f is an element of the real Hardy space $H^p_{\sqrt{-\mathbf{\Delta}}}(\mathbb{R}^d)$ if the maximal function $\mathbf{M}_{\sqrt{-\mathbf{\Delta}}} f = \sup_{t>0} |\exp(-t\sqrt{-\Delta})f|$ belongs to $L^p(\mathbb{R}^d)$.*

Set $u(x,y) = \exp(-y\sqrt{-\Delta})f(x)$. The result of [23] states that if $f \in H^p_{\sqrt{-\mathbf{\Delta}}}(\mathbb{R}^d)$, then $u^* \in L^p(\mathbb{R}^d)$. Conversely, every harmonic function $u(x,y)$, $x \in \mathbb{R}^d$, $y > 0$, with $u^* \in L^p(\mathbb{R}^d)$ can be obtained in this way from some $f \in H^p_{\sqrt{-\mathbf{\Delta}}}(\mathbb{R}^d)$.

The notion of the classical Hardy spaces remains the same if instead of using the Poisson semigroup we use the heat semigroup $\{\mathbf{P}_t\}_{t>0}$,

$$\mathbf{P}_t f(x) = \int_{\mathbb{R}^d} P_t(x-y)f(y)\,dy, \qquad P_t(x) = (4\pi t)^{-d/2}\exp(-|x|^2/4t) \tag{1.1.4}$$

or more generally, the family of operators

$$\mathbf{\Phi}_t f(x) = \int_{\mathbb{R}^d} \Phi_t(x-y)f(y)\,dy, \qquad \Phi_t(x) = t^{-d}\Phi(x/t),$$

where Φ is any fixed Schwartz class function, such that $\int_{\mathbb{R}^d}\Phi \neq 0$. To be more precise let

$$\mathbf{M}_{\mathbf{\Delta}} f(x) = \sup_{t>0} |\mathbf{P}_t f(x)|, \tag{1.1.5}$$

be the maximal function related to the heat semigroup.

Definition 1.1.6. *The Hardy space $H^p_{\mathbf{\Delta}}(\mathbb{R}^d)$ consists of all tempered distributions f for which $\|\mathbf{M}_{\mathbf{\Delta}} f\|_{L^p(\mathbb{R}^d)}$ is finite.*

It is also a result of [23] that the spaces $H^p_{\sqrt{-\mathbf{\Delta}}}(\mathbb{R}^d)$ and $H^p_{\mathbf{\Delta}}(\mathbb{R}^d)$ coincide. If we equip $H^p_{\mathbf{\Delta}}(\mathbb{R}^d)$ and $H^p_{\sqrt{-\mathbf{\Delta}}}(\mathbb{R}^d)$ with the norms (quasi-norms)

$$\|f\|_{H^p_{\mathbf{\Delta}}(\mathbb{R}^d)} = \|\mathbf{M}_{\mathbf{\Delta}} f\|_{L^p(\mathbb{R}^d)}, \qquad \|f\|_{H^p_{\sqrt{-\mathbf{\Delta}}}(\mathbb{R}^d)} = \|\mathbf{M}_{\sqrt{-\mathbf{\Delta}}} f\|_{L^p(\mathbb{R}^d)}, \tag{1.1.7}$$

then $\|f\|_{H^p_{\mathbf{\Delta}}(\mathbb{R}^d)} \sim \|f\|_{H^p_{\sqrt{-\mathbf{\Delta}}}(\mathbb{R}^d)}$. The same remains true if one considers the family of operators $\{\mathbf{\Phi}_t\}_{t>0}$ (see [38, Chapter III]).

Since in the dissertation we restrict our attention to $p = 1$, we state further characterizations of the classical Hardy spaces only for this case.

The Hardy space $H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$ can be characterized by means of certain singular integral operators, namely the Riesz transforms $\mathbf{R}_j^{\mathbf{\Delta}}$, $j = 1, ..., d$. Formally,

$$\mathbf{R}_j^{\mathbf{\Delta}} f = \frac{\partial}{\partial x_j}(-\mathbf{\Delta})^{-1/2} f. \tag{1.1.8}$$

More precisely, we define $\mathbf{R}_j^{\mathbf{\Delta}}\phi$ for $\phi \in \mathcal{S}(\mathbb{R}^d)$ by one of the following expressions:

$$\mathbf{R}_j^{\mathbf{\Delta}}\phi(x) = c_d \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} \frac{(x_j-y_j)\phi(y)}{|x-y|^{d+1}}\,dy,$$

$$\mathbf{R}_j^{\mathbf{\Delta}}\phi(x) = c_d' \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\varepsilon^{-1}} \frac{\partial}{\partial x_j}\mathbf{P}_t\phi(x)\frac{dt}{\sqrt{t}}. \tag{1.1.9}$$

Assume $f \in L^1(\mathbb{R}^d)$. Then $\mathbf{R}_j^{\mathbf{\Delta}} f$ is a tempered distribution given by $\langle \mathbf{R}_j^{\mathbf{\Delta}} f, \phi \rangle = \langle f, \mathbf{R}_j^{\mathbf{\Delta}} \phi \rangle$.

These operators were investigated by a great number of authors and appeared to be very useful in many circumstances. In particular, $\mathbf{R}_j^{\mathbf{\Delta}}$ are properly defined and bounded on $L^p(\mathbb{R}^d)$ for $p \in (1, \infty)$. Another classical result (see [38, p. 123]) gives the characterization of $H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$ in terms of $\mathbf{R}_j^{\mathbf{\Delta}}$.

Theorem 1.1.10. *Assume that $f \in L^1(\mathbb{R}^d)$. Then $f \in H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$ if and only if $\mathbf{R}_j^{\mathbf{\Delta}} f$ belongs to $L^1(\mathbb{R}^d)$ for $j = 1, ..., d$. In addition, there exists $C > 1$ such that*

$$C^{-1} \|f\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|\mathbf{R}_j^{\mathbf{\Delta}} f\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{L^1(\mathbb{R}^d)}.$$

In order to state one more characterization of $H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$ we shall need the notion of atoms.

Definition 1.1.11. *A function a is called an atom (more precisely, $(1, \infty)$-atom) if there exists a ball B in $\mathbb{R}^d$ such that:*

- $\operatorname{supp} a \in B$,
- $\|a\|_\infty \leq |B|^{-1}$,
- $\int_B a(x)\, dx = 0$.

The results obtained in [23] were used by Coifman [7] in the one-dimensional case and by Latter [27] in $\mathbb{R}^d$ to prove the following atomic decompositions of the elements of $H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$.

Theorem 1.1.12. *For a function $f \in H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$ there exist complex numbers $\{\lambda_k\}_{k=1}^\infty$ and atoms $\{a_k\}_{k=1}^\infty$ such that $f(x) = \sum_{k=1}^\infty \lambda_k a_k(x)$ and $\sum_{k=1}^\infty |\lambda_k| < \infty$. Moreover, one can chose $\{\lambda_k\}_{k=1}^\infty$ and $\{a_k\}_{k=1}^\infty$ so that*

$$C^{-1} \sum_{k=1}^{\infty} |\lambda_k| \leq \|f\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)} \leq C \sum_{k=1}^{\infty} |\lambda_k|, \tag{1.1.13}$$

where the constant $C > 0$ does not depend on f.

In our investigations we will concentrate our attention on the three definitions presented above, so we do not state numerous other definitions. However, we would like to mention that the real Hardy space $H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$ can be also described by, e.g. other maximal functions, square functions, area integrals. For details we refer the reader to [38], [10], [23], [39].

1.2 Definitions of Hardy spaces associated with semigroups of operators

Natural questions rise. What can be said about the Hardy space H^1 if we replace the classical heat semigroup by another semigroup of linear operators in Definition 1.1.6? Can then the space be characterized by appropriate singular integral operators? Does it admit relevant atomic decompositions?

In [18] and [19] Dziubański and Zienkiewicz started a project of studying Hardy spaces associated with Schrödinger operators. Some results of this dissertation are continuations of their works. In the thesis we investigate Hardy spaces related to semigroups generated by certain differential operators. In fact, we shall deal with Schrödinger operators with various potentials, Laguerre and Bessel operators. All these operators are self-adjoint and positive on an appropriate $L^2(X)$-space, where $X = (X, \mu)$ denotes the measure space. Throughout the whole thesis $\mathbf{L}$ denotes one of these operators. The related semigroups $\{\mathbf{K}_t\}_{t>0} = \{\exp(-t\mathbf{L})\}_{t>0}$ possess symmetric real-valued integral kernels $K_t(x,y) = K_t(y,x)$, i.e.

$$\mathbf{K}_t f(x) = \int_X K_t(x,y) f(y)\, d\mu(y)$$

holds for a.e. $x \in X$ when $f \in L^p(X)$, $1 \leq p \leq \infty$.

In this general context we define the Hardy space $H^1_{\mathbf{L},max}$ in the following way. Let $\mathbf{M_L}$ be the maximal operator related to $\{\mathbf{K}_t\}_{t>0}$, that is

$$\mathbf{M_L} f(x) = \sup_{t>0} |\mathbf{K}_t f(x)|. \tag{1.2.1}$$

Definition 1.2.2. *We say that an $L^1(X)$-function f belongs to $H^1_{\mathbf{L},max}$ if and only if $\mathbf{M_L} f$ is in $L^1(X)$. The norm of the space $H^1_{\mathbf{L},max}$ is given by*

$$\|f\|_{H^1_{\mathbf{L},max}} = \|\mathbf{M_L} f\|_{L^1(X)}.$$

The second definition makes use of the Riesz transforms related to $\mathbf{L}$, which are formally the operators

$$\mathbf{R}_j^{\mathbf{L}} = \frac{\partial}{\partial x_j} \mathbf{L}^{-1/2}. \tag{1.2.3}$$

Let us mention, that we have either $X = \mathbb{R}^d$ or $X = (0, \infty)$ so that $\frac{\partial}{\partial x_j}$ are simply partial derivatives. In each situation we first clarify the sense of $\mathbf{R}_j^{\mathbf{L}} f$ for $f \in L^1(X)$, and then define the space $H^1_{\mathbf{L},Riesz}$ as follows.

Definition 1.2.4. *Assume that $f \in L^1(X)$ and d is the Euclidean dimension of X. By definition, f is in $H^1_{\mathbf{L},Riesz}$, exactly when $f, \mathbf{R}_j^{\mathbf{L}} f \in L^1(X)$ for $j = 1, .., d$. For $f \in H^1_{\mathbf{L},Riesz}$*

we set

$$\|f\|_{H^1_{\mathbf{L},Riesz}} = \|f\|_{L^1(X)} + \sum_{j=1}^{d} \|\mathbf{R}_j^{\mathbf{L}}\|_{L^1(X)}.$$

The third description we are interested in is associated with atomic decompositions. Let us assume, that a definition of **L**-atoms is given (see Definitions 2.1.2, 3.1.5, 4.1.6, 4.2.1, 4.2.12 for precise assumptions on **L**-atoms in a particular case) and each **L**-atom a satisfy $\|a\|_{L^1(X)} \leq 1$.

Definition 1.2.5. *The atomic Hardy space $H^1_{\mathbf{L},at}$ is a subspace of $L^1(X)$ consisting of functions f for which there exist complex numbers $\{\lambda_k\}_{k=1}^{\infty}$ and **L**-atoms $\{a_k\}_{k=1}^{\infty}$ such that $f = \sum_{k=1}^{\infty} \lambda_k a_k$ and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. The norm of $H^1_{\mathbf{L},at}$ is given by*

$$\|f\|_{H^1_{\mathbf{L},at}} = \inf \sum_{k=1}^{\infty} |\lambda_k|,$$

where the infimum is taken over all decompositions as above.

1.3 Results and organization of the thesis

In the present section we describe shortly the results contained in the dissertation and present the organization of the paper.

The results included in the thesis are divided into three parts that are enclosed in Chapters 2, 3, and 4. The precise assumptions on **L**, X and $\{\mathbf{K}_t\}_{t>0}$ are given at the beginning of each chapter. We shall use the notion already described in Sections 1.1 and 1.2 in the whole paper, although some other symbols may be used locally, in a particular chapter or even only in a proof.

1.3.1 Schrödinger operators I

Hardy spaces associated with semigroups of linear operators and in particular Schrödinger semigroups associated to

$$\mathbf{L} = -\mathbf{\Delta} + V,$$

where V is a function called *potential*, on $\mathbb{R}^d$ attracted attention of many authors see, e.g. [1], [3], [9], [15], [19], [21], [22], [26] and references therein.

In Chapter 2 we investigate the Schrödinger operators with potentials satisfying the assumptions $(A_1) - (A_3)$, (D), (K) (see Section 2.1) that appeared previously in [21]. It was proved there that Hardy spaces defined by Definitions 1.2.2 and 1.2.5, with a suitable chosen family of **L**-atoms, coincide. Our goal is to prove the Riesz transform characterization of these Hardy spaces (see Definition 1.2.4). The main result of the chapter is Theorem 2.1.6.

We would like to note here, that the assumptions (A_1)–(A_3), (D), (K), with suitable chosen family $\mathcal{Q}$, are satisfied for several important classes of potentials, in particular for all non-negative, $L^1_{loc}(\mathbb{R})$ potentials in one dimension, or for non-negative, Reverse Hölder class potentials in dimension $d \geq 3$. More examples are provided in Section 2.2.

1.3.2 Schrödinger operators II

Chapter 3 is devoted to proving the equivalence of Definitions 1.2.2, 1.2.4, 1.2.5 of the Hardy space assuming $(A_4) - (A_6)$ (see Section 3.1). We define families of **L**-atoms that occur to be different to the families of **L**-atoms from Chapter 2. The results generalize the theorems from [22] and [15]. The main idea is to refine the methods given there and utilize the operator $\mathbf{I} - V\mathbf{L}^{-1}$ which gives an isomorphism of $H^1_{\mathbf{L},max}$ with the classical Hardy space $H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$. The most important theorems of Chapter 3 are Theorems 3.1.2, 3.1.4, 3.1.8.

1.3.3 Laguerre and Bessel operators

In Chapter 4 we work in the context of one of Laguerre systems $\{\psi_k^{(\alpha-1)/2}\}_{k=0}^{\infty}$ (see 4.1.2) related to the Laguerre operator

$$\mathbf{L}f(x) = -f''(x) - \frac{\alpha}{x}f'(x) + x^2 f(x), \qquad x > 0.$$

We give the proper definition of **L**-atoms such that the Hardy spaces $H^1_{\mathbf{L},Riesz}$ and $H^1_{\mathbf{L},at}$ coincide (cf. Definitions 1.2.4 and 1.2.5). The main result is stated in Theorem 4.1.8.

Our method takes advantage of [5], where the Hardy space associated to the Bessel operator

$$\widetilde{\mathbf{L}}f(x) = -f''(x) - \frac{\alpha}{x}f'(x), \qquad x > 0,$$

was investigated. In order to use results from [5] we define and characterize the local Hardy space related to $\widetilde{\mathbf{L}}$ (see Definition 4.2.12 and Theorem 4.2.13) .

One of the crucial points is to find precise formulas for the kernels of the Riesz transforms in the both settings (see Propositions 4.2.4 and 4.3.1), where precise constants, not only asymptotics, are important.

1.4 Final remarks

Let us note that there has been a big progress recently in studying function spaces associated with semigroups of linear operators (see, e.g. bibliography in [26]). In the paper [26] published in 2011 the authors provide a general approach to the Hardy spaces related to the semigroups satisfying the Davies-Gaffney estimates. They define the Hardy spaces H^1 by using square functions and prove very abstract atomic decomposition. Those atoms defined in [26] have totally different nature than these which occur in the doctoral dissertation. Moreover, it is worth to remark that the atomic decompositions of the Hardy spaces which are presented here, whose geometrical and cancellation properties of atoms are indicated (see Definitions 2.1.2, 3.1.5, 4.1.6, 4.2.1, 4.2.12) have useful attributes. They allow very often in a simple and direct way to determine relations among the spaces. It turns out that different operators may lead to the same Hardy spaces. For example H^1-spaces for the Schrödinger operators: $-\mathbf{\Delta}+|x|^4+|x|^2+1$ and $-\mathbf{\Delta}+|x|^4$ do coincide. One can prove this by considering their atomic decompositions (cf. Chapter 2, Example 2.2.4). On the other hand, as it was noticed in [22], in the case of Schrödinger operators with compactly supported potentials on $\mathbb{R}^d$, $d \geq 3$, any small perturbations of the potential lead to essentially different Hardy spaces.

Finally, I strongly believe that the combination of different, recently developed methods may lead to new interesting theorems for larger and larger classes of semigroups.

2. HARDY SPACES RELATED TO SCHRÖDINGER OPERATORS I

2.1 Background and main result

Let us assume that we have a family of closed cubes $\mathcal{Q} = \{Q_i\}_{i=1}^{\infty}$ in $\mathbb{R}^d$. We shall always impose that there exist $C, \beta > 0$ for which the following conditions are satisfied:

(A_1) $|Q_i \cap Q_j| = 0$ for $i \neq j$,

(A_2) $\mathbb{R}^d \setminus \bigcup_{j=1}^{\infty} Q_j$ is of Lebesgue measure zero,

(A_3) if $Q_i^{****} \cap Q_j^{****} \neq \emptyset$ then $d(Q_i) \leq C d(Q_j)$,

where $d(Q)$ is the diameter of Q and Q^* denotes the cube with the same center as Q such that $d(Q^*) = (1+\beta)d(Q)$. Clearly, if (A_1)–(A_3) hold then there is a constant $C > 0$ such that

$$\sum_{j=1}^{\infty} \mathbf{1}_{Q_j^{****}}(x) \leq C. \tag{2.1.1}$$

Now, we recall from [21] the notion of the local atomic Hardy space associated with the collection $\mathcal{Q}$.

Definition 2.1.2. *We say that a function a is an $\mathcal{Q}$-atom if there exists $Q \in \mathcal{Q}$ such that either*

- $a = |Q|^{-1}\mathbf{1}_Q$ *or*
- *a is the classical atom with support contained in Q^* (that is, there exists a cube, such that: $Q' \subset Q^*$, $\operatorname{supp} a \subset Q'$, $\int a = 0$, $\|a\|_\infty \leq |Q'|^{-1}$).*

Then the space $H^1_{\mathcal{Q},at}$ is given by Definition 1.2.5 with $\mathcal{Q}$-atoms given above.

Let

$$\mathbf{L} = -\boldsymbol{\Delta} + V(x),$$

be a Schrödinger operator on $\mathbb{R}^d$, where $V(x)$ is a locally integrable non-negative potential, $V \not\equiv 0$. It is well known that $-\mathbf{L}$ generates the semigroup $\{\mathbf{K}_t\}_{t>0}$ of linear contractions on $L^p(\mathbb{R}^d)$, $1 \le p < \infty$, and self-adjoint on $L^2(\mathbb{R}^d)$. The Feynman-Kac formula (see, e.g. [30] and [26] for references and details concerning definitions) asserts that

$$\mathbf{K}_t f(x) = \int_{\mathbb{R}^d} K_t(x,y) f(y)\, dy = E^x\left(\exp\Big(-\int_0^t V(\mathcal{B}_s)\, ds\Big) f(\mathcal{B}_t)\right), \tag{2.1.3}$$

where $\mathcal{B}_t$ is the d-dimensional Brownian motion. It implies that the integral kernels $K_t(x,y) = K_t(y,x)$ of the semigroup $\{\mathbf{K}_t\}_{t>0}$ satisfy

$$0 \le K_t(x,y) \le P_t(x-y), \tag{2.1.4}$$

where $P_t(x)$ is the classical heat kernel defied in (1.1.4). The Hardy space $H^1_{\mathbf{L},max}$ related to $\mathbf{L}$ is defined in Chapter 1 (see (1.2.1) and Definition 1.2.2), i.e.

$$\|f\|_{H^1_{\mathbf{L},max}} = \|\mathbf{M_L} f\|_{L^1(\mathbb{R}^d)}.$$

We shall see, under some additional assumptions, that the space $H^1_{\mathcal{Q},at}$ coincide with the Hardy space related to the Schrödinger operator. To this end, following [21] we impose two additional assumptions on the potential V and the collection $\mathcal{Q}$, namely:

(D) $\sup_{y\in Q^*} \int K_{2^n d(Q)^2}(x,y) dx \le C n^{-1-\varepsilon}$ for $Q \in \mathcal{Q}, n \in \mathbb{N}$,

(K) $\int_0^{2t} (\mathbf{1}_{Q^{***}} V) * P_s(x) ds \le C\left(t/d(Q)^2\right)^\delta$ for $x \in \mathbb{R}^d,\ Q \in \mathcal{Q},\ t \le d(Q)^2$,

with some $C, \varepsilon, \delta > 0$.

The Hardy space for the Schrödinger operator with a family $\mathcal{Q}$ satisfying $(A_1) - (A_3)$, (D), (K) was considered in the work of Dziubański and Zienkiewicz [21]. It was proved there that the spaces $H^1_{\mathbf{L},max}$ and $H^1_{\mathcal{Q},at}$ coincide (see [21, Theorem 2.2]) and there exists $C > 0$ such that

$$C^{-1}\|f\|_{H^1_{\mathcal{Q},at}} \le \|f\|_{H^1_{\mathbf{L},max}} \le C\|f\|_{H^1_{\mathcal{Q},at}}. \tag{2.1.5}$$

In other words, the result (2.1.5) means that $\mathcal{Q}$-atoms coincide with $\mathbf{L}$-atoms.

Denote this space by $H^1_{\mathbf{L}}$ and equip it with whichever of these norms. For $j = 1, \ldots, d$, let

$$\mathbf{R}^{\mathbf{L}}_j f(x) = \lim_{\varepsilon\to 0} \int_\varepsilon^{\varepsilon^{-1}} \frac{\partial}{\partial x_j} \mathbf{K}_t f(x) \frac{dt}{\sqrt{t}}$$

be the Riesz transform $\frac{\partial}{\partial x_j}\mathbf{L}^{-1/2}$ associated with $\mathbf{L}$, where the limit is understood in the sense of distributions (see Section 2.3).

The main result of this chapter (see [16]) is to prove that the operators $\mathbf{R}^{\mathbf{L}}_j$ characterize the space $H^1_{\mathbf{L}}$, that is, the following theorem holds.

Theorem 2.1.6. *Assume that a potential $V \geq 0$ and a collection of cubes $\mathcal{Q}$ are such that (A_1)–(A_3), (D) and (K) hold. Then there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_{\mathbf{L}}} \leq \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|\mathbf{R}_j^{\mathbf{L}} f\|_{L^1(\mathbb{R}^d)} \leq C\|f\|_{H^1_{\mathbf{L}}}. \tag{2.1.7}$$

2.2 Remarks and examples

The conditions (D) and (K) can appear to the reader as quite technical and complicated. Roughly speaking, (D) means that the integral kernel of the semigroup $\{\mathbf{K}_t\}_{t>0}$ is *small* for $t > d(Q)^2$ and $y \in Q^*$ (see, e.g. Lemma 2.3.5 and [21, Lemma 3.8]), whereas (K) says that $\mathbf{K}_t$ is *close* to $\mathbf{P}_t$ for $t < d(Q)^2$ when we act on functions supported in Q^* (see, e.g. Lemma 2.3.8 and [21, Lemma 3.11]). Before going to the proof we provide some important examples of non-negative potentials V and appropriate families $\mathcal{Q}$ for which we can apply Theorem 2.1.6.

Example 2.2.1. The Hardy space $H^1_{\mathbf{L}}$ associated with one-dimensional Schrödinger operator $-\mathbf{L}$ was studied by Czaja and Zienkiewicz [9]. It was proved there that for any non-negative $V \in L^1_{loc}(\mathbb{R})$ the collection $\mathcal{Q}$ of maximal dyadic intervals Q of $\mathbb{R}$ that are defined by the stopping time condition

$$|Q| \int_{16Q} V(y)\,dy \leq 1 \tag{2.2.2}$$

fulfils (D) for certain small $\beta > 0$ (see [9, Lemma 2.2]). The authors also remarked that (K) is satisfied. Indeed,

$$\int_0^{2t} (\mathbf{1}_{Q^{***}}V) * P_s(x)\,ds \leq \int_0^{2t} \|\mathbf{1}_{Q^{***}}V\|_{L^1}\|\mathbf{P}_s\|_{L^\infty}\,ds \leq \int_0^{2t} |Q|^{-1}\frac{ds}{\sqrt{4\pi s}} \leq C\frac{t^{1/2}}{|Q|},$$

where in the second inequality we have used (2.2.2).

Example 2.2.3. $V(x) = \gamma|x|^{-2}$, $d \geq 3$, $\gamma > 0$. Then for $\mathcal{Q}$ being the Whitney decomposition of $\mathbb{R}^d \setminus \{0\}$ that consists of dyadic cubes the conditions (A_1)–(A_3), (D) and (K) hold (see Theorem 2.8 of [21]).

Example 2.2.4. $d \geq 3$, V satisfies the reverse Hölder inequality with exponent $q > d/2$, that is

$$\left(\frac{1}{|B|}\int_B V(y)^q\,dy\right)^{1/q} \leq C\frac{1}{|B|}\int_B V(y)\,dy \quad \text{for every ball } B \subseteq \mathbb{R}^d. \tag{2.2.5}$$

Clearly, any non-negative nonzero polynomial V satisfies (2.2.5). Define the family $\mathcal{Q}$ by: $Q \in \mathcal{Q}$ if and only if Q is the maximal dyadic cube for which $\operatorname{diam}(Q)^2|Q|^{-1}\int_Q V(y)\,dy \leq$

1. Then the conditions (A_1)–(A_3), (D) and (K) are fulfilled (see [21, Section 8]). Let us mention that the Riesz transforms characterization of the Hardy spaces associated with Schrödinger operators with potentials satisfying the reverse Hölder inequality was proved in [19].

We finish this section by giving the following three remarks that give examples how we can construct in some cases other family of cubes from families that we already know.

Remark 2.2.6. For $\ell > 0$ denote by $\mathcal{Q}_\ell(\mathbb{R}^n)$ any partition of $\mathbb{R}^n$ into cubes whose diameters have length ℓ. Assume that for a locally integrable non-negative potential V_1 on $\mathbb{R}^d$ and a collection $\mathcal{Q}$ of cubes the conditions (D) and (K) hold. Consider the potential $V(x_1,x_2) = V_1(x_1)$, $x_1 \in \mathbb{R}^d$, $x_2 \in \mathbb{R}^n$, and the family $\widetilde{\mathcal{Q}} = \{Q_1 \times Q_2 : Q_1 \in \mathcal{Q},\ Q_2 \in \mathcal{Q}_{d(Q_1)}(\mathbb{R}^n)\}$ of cubes in $\mathbb{R}^{d+n}$. Then the pair $(V, \widetilde{\mathcal{Q}})$ fulfils (D) and (K).

Proof. Assume that $Q = Q_1 \times Q_2 \in \widetilde{\mathcal{Q}}$, where $Q_1 \in \mathcal{Q}$ and $d(Q_2) = d(Q_1) = C_0^{-1/2} d(Q)$. Let $K_t(x,y)$ denotes the integral kernel of the semigroup generated by $\mathbf{\Delta} - V$ on $\mathbb{R}^{d+n}$. Then $K_t(x,y) = K_t^{(1)}(x_1,y_1)P_t^{(2)}(x_2,y_2)$, where $K_t^{(1)}(x_1,y_1)$ is the integral kernel of the semigroup related to $\mathbf{\Delta} - V_1$ on $\mathbb{R}^d$ and $P_t^{(2)}$ is the classical heat semigroup on $\mathbb{R}^n$. Let us check (D):

$$\begin{aligned}\sup_{y \in Q^*} \int_{\mathbb{R}^{d+n}} K_{2^m d(Q)^2}(x,y)\,dx &\le \sup_{y_1 \in Q_1^*} \int_{\mathbb{R}^d} K^{(1)}_{c_0 2^m d(Q_1)^2}(x_1,y_1)\,dx_1 \\ &\le C(m + \log_2 c_0)^{-1-\varepsilon} \le C m^{-1-\varepsilon}.\end{aligned}$$

By using $P_s(x) = P_s^{(1)}(x_1)P_s^{(2)}(x_2)$, where $x_1 \in \mathbb{R}^d$ and $x_2 \in \mathbb{R}^n$ we obtain

$$(\mathbf{1}_{Q^{***}} V) * P_s(x) \le (\mathbf{1}_{Q_1^{***}} V_1) * P_s^{(1)}(x_1)$$

which leads directly to (K) since $d(Q) \sim d(Q_1)$ (in fact, we get (K) only for $t < c_0^{-1} d(Q)^2$, where $c_0 > 1$, but one could check that this c_0^{-1} is unimportant to the theory). □

Remark 2.2.7. One can check that Theorem 2.2 of [21] (see 2.1.5) and Theorem 2.1.6 together with their proofs remain true if we replace cubes by rectangles in the definition of atoms and in the conditions (D) and (K), provided the rectangles have all side-lengths comparable to their diameters. As a corollary of this observation we obtain that if $V(x_1,x_2) = V_1(x_1) + V_2(x_2)$, $x_1 \in \mathbb{R}^d$, $x_2 \in \mathbb{R}^n$, where V_1 and V_2 satisfy conditions (D) and (K) for certain collections $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of cubes on $\mathbb{R}^d$ and $\mathbb{R}^n$ respectively, then the Hardy space $H^1_{\mathbf{L},max}$ associated with the operator $\mathbf{L} = -\mathbf{\Delta} + V(x_1,x_2)$ on $\mathbb{R}^{d+n}$ admits (thanks to Theorem 2.1.6) the atomic and the Riesz transforms characterizations.

Indeed, for any $Q_j \in \mathcal{Q}_1$ and $Q_k \in \mathcal{Q}_2$ we divide the rectangle $Q_j \times Q_k$ into rectangles $Q^s_{j,k}$, $s = 1, 2, ..., s_{j,k}$, with side-lengths comparable to $\min(d(Q^1_j), d(Q^2_k))$. The above construction leads to $V(x_1, x_2)$ and the collection $Q^s_{j,k}$ for which (D) and (K) hold.

Proof. Take $Q^s_{j,k}$ as above. We can assume that $d(Q^s_{j,k}) = c^{-1/2}_{j,k} d(Q_1)$, where $0 < c \leq c_{j,k} \leq C$ independently of j and k. Let $K_t(x,y), K^{(1)}_t(x_1,y_1), K^{(2)}_t(x_2,y_2)$ be the integral kernels of the semigroups generated by $\mathbf{\Delta} - V$ (on $\mathbb{R}^{d+n}$), $\mathbf{\Delta} - V_1$ (on $\mathbb{R}^d$), and $\mathbf{\Delta} - V_2$ (on $\mathbb{R}^n$), respectively. Then $K_t(x,y) \leq K^{(1)}_t(x_1,y_1) P_t(x_2,y_2)$ and (D) follows identically as in the proof of Remark 2.2.6. We check (K):

$$\int_0^{2t} (\chi_{(Q^s_{j,k})^{***}} V) * P_s(x) ds \leq \int_0^{2t} (\chi_{Q^{***}_j} V_1) * P^{(1)}_s(x_1) ds + \int_0^{2t} (\chi_{Q^{***}_k} V_2) * P^{(2)}_s(x_2) ds$$
$$\leq C(t/d(Q_j)^2)^\delta + C(t/d(Q_k)^2)^\delta \leq C(t/d(Q^s_{j,k})^2)^\delta.$$

Notice, that we have got (K) only for $t < c_1 d(Q^s_{j,k})^2$ (with $c_1 < 1$ independent of j,k,s) but, as we noticed at the end of the proof of Remark 2.2.6, it is enough. □

Remark 2.2.8. Let V_1, V_2 be non-negative potentials on $\mathbb{R}^d$, which together with families of cubes $\mathcal{Q}_1$ i $\mathcal{Q}_2$ satisfy (D) and (K). Assume additionally that $\mathcal{Q}_1$, $\mathcal{Q}_2$ consist of dyadic cubes. For $Q_1 \in \mathcal{Q}_1$, $Q_2 \in \mathcal{Q}_2$ the cubes are either disjoint or one contains another. Let $Q_1 \wedge Q_2$ denote the smaller one. Then the family $\mathcal{Q} = \{Q_1 \wedge Q_2 \ : \ Q_1 \in \mathcal{Q}_1, Q_2 \in \mathcal{Q}_2\}$ covers $\mathbb{R}^d$ and satisfies (D) and (K) for $V = V_1 + V_2$.

Proof. Denote by $K_t(x,y)$, $K^{(1)}_t(x,y)$, $K^{(2)}_t(x,y)$ the integral kernels for Schrödinger semigroups with potentials V, V_1, V_2, respectively. Then $K_t(x,y) \leq \min(K^{(1)}_t(x,y), K^{(2)}_t(x,y))$. Assume that $Q_1 = Q_1 \wedge Q_2$. For $x \in \mathbb{R}^d$ and $t \leq d(Q_1)^2$ we simply observe that

$$\int_0^{2t} (\chi_{Q_1***}(V_1 + V_2)) * P_s(x)\, ds \leq C \left(\frac{t}{d(Q_1)^2}\right)^\delta + \left(\frac{t}{d(Q_2)^2}\right)^\delta \leq C' \left(\frac{t}{d(Q_1)^2}\right)^\delta$$

and (K) is satisfied. Also, we check (D):

$$\sup_{y \in Q_1*} \int_{\mathbb{R}^d} K_{2^n d(Q_1)^2}(x,y)\, dx \leq \sup_{y \in Q_1*} \int_{\mathbb{R}^d} K^{(1)}_{2^n d(Q_1)^2}(x,y)\, dx \leq C n^{-1-\varepsilon}.$$

□

2.3 Auxiliary lemmas

Lemma 2.3.1. *For every $\alpha > 0$ there exists a constant $C > 0$ (independent of V) such that for $j = 1, \ldots, d$ and $y \in \mathbb{R}^d$ we have*

$$\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} K_t(x,y) \right|^2 \exp(\alpha |x-y|/\sqrt{t})\, dx \leq C t^{-d/2-1}, \tag{2.3.2}$$

$$\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} K_t(x,y) \right| \exp(\alpha |x-y|/\sqrt{t})\, dx \leq C t^{-1/2}. \tag{2.3.3}$$

The lemma seems to be known. For reader's convenience we give a sketch of a proof in Section 2.5.

For $\varepsilon > 0$, $j = 1, \ldots, d$, we define the operator

$$\mathbf{R}^{\mathbf{L}}_{j,\varepsilon} f(x) = \int R^{\mathbf{L}}_{j,\varepsilon}(x,y) f(y)\, dy,$$

where $R^{\mathbf{L}}_{j,\varepsilon}(x,y) = \int_\varepsilon^{\varepsilon^{-1}} \frac{\partial}{\partial x_j} K_t(x,y) \frac{dt}{\sqrt{t}}$.

We will check in a moment that for $f \in L^1(\mathbb{R}^d)$ the limits $\lim_{\varepsilon \to 0} \mathbf{R}^{\mathbf{L}}_{j,\varepsilon} f$ exist in the sense of distributions and define tempered distributions which are denoted by $\mathbf{R}^{\mathbf{L}}_j f$. Moreover, for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$|\langle \mathbf{R}^{\mathbf{L}}_j f, \varphi \rangle| \leq C \|f\|_{L^1(\mathbb{R}^d)} \left(\|\varphi\|_{L^2(\mathbb{R}^d)} + \left\| \frac{\partial}{\partial x_j} \varphi \right\|_{L^\infty(\mathbb{R}^d)} \right). \tag{2.3.4}$$

To see this we write

$$\left(\mathbf{R}^{\mathbf{L}}_{j,\varepsilon}\right)^* \varphi(y) = \int_1^{1/\varepsilon} \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} K_t(x,y) \varphi(x)\, dx \frac{dt}{\sqrt{t}} - \int_\varepsilon^1 \int_{\mathbb{R}^d} K_t(x,y) \frac{\partial}{\partial x_j} \varphi(x)\, dx \frac{dt}{\sqrt{t}}.$$

Since

$$\int_1^\infty \left[\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} K_t(x,y) \right|^2 dx \right]^{\frac{1}{2}} \frac{dt}{\sqrt{t}} \leq C \int_1^\infty t^{-1-d/4}\, dt \leq C$$

and

$$\int_{\mathbb{R}^d} \int_0^1 K_t(x,y) \frac{dt}{\sqrt{t}}\, dx \leq 2$$

(see Lemma 2.3.1), we conclude that $\left(\mathbf{R}^{\mathbf{L}}_{j,\varepsilon}\right)^* \varphi(y)$ converges uniformly, as $\varepsilon \to 0$, to a bounded function which will be denoted by $\left(\mathbf{R}^{\mathbf{L}}_j\right)^* \varphi(y)$, and

$$\left| \left(\mathbf{R}^{\mathbf{L}}_j\right)^* \varphi(y) \right| \leq C \left(\|\varphi\|_{L^2(\mathbb{R}^d)} + \left\| \frac{\partial}{\partial x_j} \varphi \right\|_{L^\infty(\mathbb{R}^d)} \right).$$

For fixed $Q \in \mathcal{Q}$ and $0 < \varepsilon < 1$, let

$$R^{\mathbf{L}}_{j,\varepsilon,Q,0}(x,y) = \begin{cases} \int_\varepsilon^{d(Q)^2} \frac{\partial}{\partial x_j} K_t(x,y) \frac{dt}{\sqrt{t}} & \text{if } \varepsilon < d(Q)^2 < 1/\varepsilon; \\ \int_\varepsilon^{1/\varepsilon} \frac{\partial}{\partial x_j} K_t(x,y) \frac{dt}{\sqrt{t}} & \text{if } d(Q)^2 \geq 1/\varepsilon; \\ 0 & \text{if } d(Q)^2 \leq \varepsilon; \end{cases}$$

$$R^{\mathbf{L}}_{j,\varepsilon,Q,\infty}(x,y) = \begin{cases} \int_{d(Q)^2}^{1/\varepsilon} \frac{\partial}{\partial x_j} K_t(x,y) \frac{dt}{\sqrt{t}} & \text{if } \varepsilon < d(Q)^2 < 1/\varepsilon; \\ 0 & \text{if } d(Q)^2 \geq 1/\varepsilon; \\ \int_\varepsilon^{1/\varepsilon} \frac{\partial}{\partial x_j} K_t(x,y) \frac{dt}{\sqrt{t}} & \text{if } d(Q)^2 \leq \varepsilon. \end{cases}$$

Clearly, $R^{\mathbf{L}}_{j,\varepsilon}(x,y) = R^{\mathbf{L}}_{j,\varepsilon,Q,0}(x,y) + R^{\mathbf{L}}_{j,\varepsilon,Q,\infty}(x,y)$ for every $Q \in \mathcal{Q}$ and $0 < \varepsilon < 1$. For $f \in L^1(\mathbb{R}^d)$ denote

$$\mathbf{R}^{\mathbf{L}}_{j,Q,0} f(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} R^{\mathbf{L}}_{j,\varepsilon,Q,0}(x,y) f(y)\, dy,$$
$$\mathbf{R}^{\mathbf{L}}_{j,Q,\infty} f(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} R^{\mathbf{L}}_{j,\varepsilon,Q,\infty}(x,y) f(y)\, dy,$$

which of course exist in the sense of distributions.

For $Q \in \mathcal{Q}$ we define

$$\mathcal{Q}'(Q) = \{Q' \in \mathcal{Q} : Q^{***} \cap (Q')^{***} \neq \emptyset\},$$
$$\mathcal{Q}''(Q) = \{Q'' \in \mathcal{Q} : Q^{***} \cap (Q'')^{***} = \emptyset\}.$$

Lemma 2.3.5. *Assume that (D) holds. Then there exists a constant $C > 0$ such that for every $Q \in \mathcal{Q}$ we have*

$$\int_{\mathbb{R}^d} \sup_{0<\varepsilon<1} |R^{\mathbf{L}}_{j,\varepsilon,Q,\infty}(x,y)|\, dx \leq C \quad \text{for } y \in \bigcup_{Q' \in \mathcal{Q}'(Q)} Q'^{*}. \tag{2.3.6}$$

Proof. Fix $y \in \bigcup_{Q'\in\mathcal{Q}'(Q)} Q'^{*}$. Let $Q' \in \mathcal{Q}'(Q)$ be such that $y \in Q'^{*}$. Denote by S the left-hand side of (2.3.6). Then

$$S \leq \int_{\mathbb{R}^d} \int_{\min(d(Q),\, d(Q'))^2}^{d(Q')^2} \left| \frac{\partial}{\partial x_j} K_t(x,y) \right| \frac{dt}{\sqrt{t}} dx + \int_{\mathbb{R}^d} \int_{d(Q')^2}^{\infty} \left| \frac{\partial}{\partial x_j} K_t(x,y) \right| \frac{dt}{\sqrt{t}} dx$$
$$= S_1 + S_2.$$

Recall that $d(Q) \sim d(Q')$. Using (2.3.3), we get

$$S_1 \leq C \int_{\min(d(Q),\, d(Q'))^2}^{d(Q')^2} t^{-1}\, dt \leq C.$$

Applying (2.3.3) and (D), we obtain

$$S_2 = \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} \int_{2^n d(Q')^2}^{2^{n+1} d(Q')^2} \left| \frac{\partial}{\partial x_j} K_t(x,y) \right| \frac{dt}{\sqrt{t}} dx$$
$$\leq C \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} \int_{2^n d(Q')^2}^{2^{n+1} d(Q')^2} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} T_{t-2^{n-1} d(Q')^2}(x,z) \right| T_{2^{n-1} d(Q')^2}(z,y)\, dz \frac{dt}{\sqrt{t}} dx$$
$$\leq C \sum_{n=0}^{\infty} \int_{2^n d(Q')^2}^{2^{n+1} d(Q')^2} \int_{\mathbb{R}^d} (2^n d(Q')^2)^{-1/2} T_{2^{n-1} d(Q')^2}(z,y)\, dz \frac{dt}{\sqrt{t}}$$
$$\leq C \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} T_{2^{n-1} d(Q')^2}(z,y)\, dz \leq C + C \sum_{n=1}^{\infty} n^{-1-\varepsilon} \leq C,$$

and the lemma is proved. ∎

For $0 \le \varepsilon < d(Q)^2$ let

$$W_{j,\varepsilon,Q}(x,y) = \int_{\varepsilon}^{d(Q)^2} \frac{\partial}{\partial x_j}\big(K_t(x,y) - P_t(x-y)\big)\frac{dt}{\sqrt{t}}. \tag{2.3.7}$$

Set $\mathbf{W}_{j,\varepsilon,Q}f(x) = \int W_{j,\varepsilon,Q}(x,y)f(y)\,dy$, $\mathbf{W}_{j,Q}f = \mathbf{W}_{j,0,Q}f$.

Lemma 2.3.8. *Assuming (K) there exists a constant $C > 0$ such that for every $Q \in \mathcal{Q}$ one has*

$$\sup_{y\in Q^*}\int_{\mathbb{R}^d}\int_0^{d(Q)^2}\left|\frac{\partial}{\partial x_j}\big(K_t(x,y) - P_t(x,y)\big)\right|\frac{dt}{\sqrt{t}}\,dx \le C.$$

Proof. The proof borrows some ideas from the estimates of maximal functions associated with $(\mathbf{K}_t - \mathbf{P}_t)$ in dimension one which are given in [9, Lemma 2.3]. Fix $j \in \{1,\dots,d\}$ and denote

$$J_Q(x,y) = \int_0^{d(Q)^2}\left|\frac{\partial}{\partial x_j}\left(K_t(x,y) - P_t(x-y)\right)\right|\frac{dt}{\sqrt{t}}.$$

The perturbation formula asserts that

$$\mathbf{K}_t - \mathbf{P}_t = -\int_0^t \mathbf{P}_{t-s}V\mathbf{K}_s\,ds. \tag{2.3.9}$$

Therefore

$$\begin{aligned} J_Q(x,y) \le &\int_0^{d(Q)^2}\int_0^{t/2}\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}P_{t-s}(x-z)\right|V_1(z)K_s(z,y)\,dz\,ds\frac{dt}{\sqrt{t}}\\ &+\int_0^{d(Q)^2}\int_{t/2}^{t}\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}P_{t-s}(x-z)\right|V_1(z)K_s(z,y)\,dz\,ds\frac{dt}{\sqrt{t}}\\ &+\int_0^{d(Q)^2}\int_0^{t}\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}P_{t-s}(x-z)\right|V_2(z)K_s(z,y)\,dz\,ds\frac{dt}{\sqrt{t}}\\ =\ &J_1'(x,y) + J_1''(x,y) + J_2(x,y), \end{aligned}$$

where $V_1(x) = V(x)\mathbf{1}_{Q^{***}}$, $V_2(x) = V(x) - V_1(x)$.

To evaluate J_1' observe that

$$\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}P_{t-s}(x-y)\right|dx \le Ct^{-1/2} \quad \text{for } 0 < s < t/2.$$

Thus, using (K), we get

$$\begin{aligned} \int_{Q^{**}} J_1'(x,y)\,dx &\le C\int_0^{d(Q)^2}\int_0^{t/2}\int_{\mathbb{R}^d} t^{-1/2}V_1(z)P_s(z-y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\\ &\le C\int_0^{d(Q)^2} t^{-1/2}\left(\frac{t}{d(Q)^2}\right)^{\delta}\frac{dt}{\sqrt{t}} \le C. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{Q^{**}} J_1''(x,y)\,dx &\le C\int_0^{d(Q)^2}\int_{t/2}^{t}\int_{\mathbb{R}^d}(t-s)^{-1/2}V_1(z)P_t(z-y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\\ &= C'\int_0^{d(Q)^2}\int_{\mathbb{R}^d} V_1(z)P_t(z-y)\,dz\,dt \le C. \end{aligned}$$

In order to estimate J_2 we notice that

$$\left|\frac{\partial}{\partial x_j}P_{t-s}(x-z)\right| \leq Cd(Q)^{-d-1}e^{-c(|x-z|/d(Q))^2} \tag{2.3.10}$$

for $0 < s < t < d(Q)^2$, $z \notin Q^{***}$, $x \in Q^{**}$. Lemma 3.10 of [21] asserts that

$$\sup_{y\in\mathbb{R}^d}\int_0^\infty \int_{\mathbb{R}^d} V(z)K_s(z,y)\,dz\,ds \leq C.$$

Hence, by (2.3.10), we obtain

$$\begin{aligned}\int_{Q^{**}} J_2(x,y)\,dx &\leq C\,d(Q)^{-1}\int_0^{d(Q)^2}\int_0^t\int_{\mathbb{R}^d} V_2(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\\ &\leq C\,d(Q)^{-1}\int_0^{d(Q)^2}\int_0^t\int_{\mathbb{R}^d} V(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\\ &\leq Cd(Q)^{-1}\int_0^{d(Q)^2}\frac{dt}{\sqrt{t}} \leq C.\end{aligned}$$

We now turn to estimate $J_Q(x,y)$ for $x \notin Q^{**}$ and $y \in Q^*$. Clearly,

$$\begin{aligned}\int_{Q^{**c}} J_Q(x,y)\,dx &\leq \int_{Q^{**c}}\int_0^{d(Q)^2}\left|\frac{\partial}{\partial x_j}K_t(x,y)\right|\frac{dt}{\sqrt{t}}\,dx\\ &+\int_{Q^{**c}}\int_0^{d(Q)^2}\left|\frac{\partial}{\partial x_j}P_t(x-y)\right|\frac{dt}{\sqrt{t}}\,dx = \mathcal{J}_Q' + \mathcal{J}_Q''.\end{aligned}$$

By using (2.3.2) combined with the Cauchy-Schwarz inequality we get

$$\begin{aligned}\mathcal{J}_Q' &\leq \int_0^{d(Q)^2}\left(\int_{Q^{**c}}\left|\frac{\partial}{\partial x_j}K_t(x,y)\right|^2 e^{2\frac{|x-y|}{\sqrt{t}}}\,dx\right)^{1/2}\left(\int_{Q^{**c}} e^{-2\frac{|x-y|}{\sqrt{t}}}\,dx\right)^{1/2}\frac{dt}{\sqrt{t}}\\ &\leq C\int_0^{d(Q)^2} t^{-d/4-1/2}\left(\int_{Q^{**c}}\left(\frac{\sqrt{t}}{|x-y|}\right)^N dx\right)^{1/2}\frac{dt}{\sqrt{t}} \leq C.\end{aligned} \tag{2.3.11}$$

The estimates for $\mathcal{J}_Q''$ go in the same way. Hence,

$$\sup_{y\in Q^*}\int_{Q^{**c}} J_Q(x,y)\,dx \leq C,$$

which completes the proof of Lemma 2.3.8. □

Let $\{\phi_Q\}_{Q\in\mathcal{Q}}$ be a family of smooth functions that form a resolution of identity associated with $\{Q^*\}_{Q\in\mathcal{Q}}$, that is: $\phi_Q \in C_c^\infty(Q^*)$, $0 \leq \phi_Q \leq 1$, $|\nabla\phi_Q(x)| \leq Cd(Q)^{-1}$, $\sum_{Q\in\mathcal{Q}}\phi_Q(x) = 1$ a.e.

The following corollary follows easily from Lemma 2.3.8.

Corollary 2.3.12. *For $f \in L^1(\mathbb{R}^d)$ we have:*

$$\lim_{\varepsilon\to 0}\|\mathbf{W}_{j,\varepsilon,Q}(\phi_Q f) - \mathbf{W}_{j,Q}(\phi_Q f)\|_{L^1(\mathbb{R}^d)} = 0,$$

$$\|\mathbf{W}_{j,Q}(\phi_Q f)\|_{L^1(\mathbb{R}^d)} \leq C\|\phi_Q f\|_{L^1(\mathbb{R}^d)}$$

with C independent of Q and f.

Lemma 2.3.13. *There exists a constant $C > 0$ such that for every $Q \in \mathcal{Q}$ and every $f \in L^1(\mathbb{R}^d)$ such that $\operatorname{supp} f \subset \widetilde{Q} = \bigcup_{Q' \in \mathcal{Q}'(Q)} Q'^*$ we have*

$$\|\mathbf{R}_j^{\mathbf{L}}(\phi_Q f) - \phi_Q \mathbf{R}_j^{\mathbf{L}} f\|_{L^1(\mathbb{R}^d)} \le C\|f\|_{L^1(\widetilde{Q})}. \tag{2.3.14}$$

Proof. Note that

$$\mathbf{R}_j^{\mathbf{L}}(\phi_Q f)(x) - \phi_Q(x)\mathbf{R}_j^{\mathbf{L}} f(x) = \lim_{\varepsilon \to 0} \int_\varepsilon^{1/\varepsilon} \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial x_j} K_t(x,y)\right) (\phi_Q(y) - \phi_Q(x))\, f(y)\, dy \frac{dt}{\sqrt{t}}.$$

From (2.3.3) we conclude

$$\begin{aligned}
\int_{\mathbb{R}^d} \int_0^{d(Q)^2} & \left| \left(\frac{\partial}{\partial x_j} K_t(x,y)\right) (\phi_Q(y) - \phi_Q(x)) \right| \frac{dt}{\sqrt{t}}\, dx \\
&\le \frac{C}{d(Q)} \int_{\mathbb{R}^d} \int_0^{d(Q)^2} \left| \frac{\partial}{\partial x_j} K_t(x,y) \right| \frac{|x-y|}{\sqrt{t}}\, dt\, dx \\
&\le \frac{C}{d(Q)} \int_{\mathbb{R}^d} \int_0^{d(Q)^2} \left| \frac{\partial}{\partial x_j} K_t(x,y) \right| e^{|x-y|/\sqrt{t}}\, dt\, dx \le C.
\end{aligned} \tag{2.3.15}$$

Now (2.3.14) follows from (2.3.6) and (2.3.15). □

Lemma 2.3.16. *There exists a constant $C > 0$ such that*

$$\sum_{Q \in \mathcal{Q}} \left\| \mathbf{1}_{Q^{***}} \mathbf{R}_j^{\mathbf{L}} \Big(\sum_{Q'' \in \mathcal{Q}''(Q)} \phi_{Q''} f \Big) \right\|_{L^1(\mathbb{R}^d)} \le C\|f\|_{L^1(\mathbb{R}^d)}. \tag{2.3.17}$$

Proof. Let S denote the left-hand side of (2.3.17). Applying (2.1.1), we have

$$\begin{aligned}
S &\le \sum_{Q \in \mathcal{Q}} \sum_{Q'' \in \mathcal{Q}''(Q)} \left\| \mathbf{1}_{Q^{***}} \mathbf{R}_j^{\mathbf{L}} (\phi_{Q''} f) \right\|_{L^1(\mathbb{R}^d)} = \sum_{Q'' \in \mathcal{Q}} \sum_{Q \in \mathcal{Q}''(Q'')} \cdots \\
&\le C \sum_{Q'' \in \mathcal{Q}} \left\| \mathbf{R}_j^{\mathbf{L}} (\phi_{Q''} f) \right\|_{L^1((Q'')^{**c})} \\
&\le C \sum_{Q'' \in \mathcal{Q}} \left(\left\| \mathbf{R}_{j,Q'',0}^{\mathbf{L}} (\phi_{Q''} f) \right\|_{L^1((Q'')^{**c})} + \left\| \mathbf{R}_{j,Q'',\infty}^{\mathbf{L}} (\phi_{Q''} f) \right\|_{L^1((Q'')^{**c})} \right).
\end{aligned} \tag{2.3.18}$$

By using (2.3.6) and (2.1.1), we get

$$\sum_{Q'' \in \mathcal{Q}} \left\| \mathbf{R}_{j,Q'',\infty}^{\mathbf{L}} (\phi_{Q''} f) \right\|_{L^1((Q'')^{**c})} \le C \sum_{Q'' \in \mathcal{Q}} \|\phi_{Q''} f\|_{L^1(\mathbb{R}^d)} \le C' \|f\|_{L^1(\mathbb{R}^d)}. \tag{2.3.19}$$

Similarly to (2.3.11), for $y \in (Q'')^*$, we have

$$\int_{(Q'')^{**c}} \int_0^{d(Q'')^2} \left| \frac{\partial}{\partial x_j} K_t(x,y) \right| \frac{dt}{\sqrt{t}}\, dx \le C,$$

which implies

$$\sum_{Q'' \in \mathcal{Q}} \left\| \mathbf{R}_{j,Q'',0}^{\mathbf{L}} (\phi_{Q''} f) \right\|_{L^1((Q'')^{**c})} \le C \sum_{Q'' \in \mathcal{Q}} \|\phi_{Q''} f\|_{L^1(\mathbb{R}^d)} \le C \|f\|_{L^1(\mathbb{R}^d)}. \tag{2.3.20}$$

The lemma is a consequence of (2.3.18)–(2.3.20). □

2.4 Proof of Theorem 2.1.6

In order to prove the second inequality of (2.1.7) it suffices by (2.3.4) and (2.1.5) to verify that there exists a constat $C > 0$ such that

$$\|\mathbf{R}_j^{\mathbf{L}} a\|_{L^1(\mathbb{R}^d)} \leq C \tag{2.4.1}$$

for every $\mathcal{Q}$-atom a and $j = 1, \ldots, d$. Assume that a is an $\mathcal{Q}$-atom supported by a cube Q^*, $Q \in \mathcal{Q}$. Then

$$\begin{aligned}\mathbf{R}_j^{\mathbf{L}} a(x) &= \lim_{\varepsilon \to 0} \left(\mathbf{R}_{j,\varepsilon,Q,0}^{\mathbf{L}} a(x) + \mathbf{R}_{j,\varepsilon,Q,\infty}^{\mathbf{L}} a(x)\right) \\ &= \lim_{\varepsilon \to 0} \left(\mathbf{W}_{j,\varepsilon,Q} a(x) + \mathbf{H}_{j,\varepsilon,Q} a(x) + \mathbf{R}_{j,\varepsilon,Q,\infty}^{\mathbf{L}} a(x)\right),\end{aligned}$$

where $\mathbf{H}_{j,\varepsilon,Q} a(x) = \int_\varepsilon^{d(Q)^2} \frac{\partial}{\partial x_j}(a * \mathbf{P}_t)(x) \frac{dt}{\sqrt{t}}$. Similarly to (2.3.4), the limit

$$\mathbf{H}_{j,Q} a(x) = \lim_{\varepsilon \to 0} \mathbf{H}_{j,\varepsilon,Q} a(x)$$

exists in the sense of distributions. Moreover, by the boundedness of the local Riesz transforms on the local Hardy spaces (see [24]), we have $\|\mathbf{H}_{j,Q} a\|_{L^1(\mathbb{R}^d)} \leq C$ with C independent of a. Using Lemmas 2.3.8 and 2.3.5, we obtain (2.4.1).

We now turn to prove the first inequality of (2.1.7). To this end, by the local Riesz transform characterization of the local Hardy spaces (see [24, Section 2]), it suffices to show that

$$\sum_{Q \in \mathcal{Q}} \|\mathbf{H}_{j,Q}(\phi_Q f)\|_{L^1(Q^{**})} \leq C \left(\|f\|_{L^1(\mathbb{R}^d)} + \|\mathbf{R}_j^{\mathbf{L}} f\|_{L^1(\mathbb{R}^d)}\right), \quad j = 1, ..., d. \tag{2.4.2}$$

Clearly,

$$\mathbf{H}_{j,Q}(\phi_Q f) = -\mathbf{W}_{j,Q}(\phi_Q f) + \mathbf{R}_{j,Q,0}^{\mathbf{L}}(\phi_Q f).$$

Lemma 2.3.8 together with (2.1.1) implies

$$\sum_{Q \in \mathcal{Q}} \|\mathbf{W}_{j,Q}(\phi_Q f)\|_{L^1(\mathbb{R}^d)} \leq C \sum_{Q \in \mathcal{Q}} \|\phi_Q f\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{L^1(\mathbb{R}^d)}. \tag{2.4.3}$$

Note that

$$\begin{aligned}\mathbf{R}_{j,Q,0}^{\mathbf{L}}(\phi_Q f) = &\left[\mathbf{R}_j^{\mathbf{L}}\Big(\phi_Q \sum_{Q' \in \mathcal{Q}'(Q)} (\phi_{Q'} f)\Big) - \phi_Q \mathbf{R}_j^{\mathbf{L}}\Big(\sum_{Q' \in \mathcal{Q}'(Q)} (\phi_{Q'} f)\Big)\right] \\ &- \mathbf{R}_{j,Q,\infty}^{\mathbf{L}}(\phi_Q f) + \phi_Q \mathbf{R}_j^{\mathbf{L}} f - \phi_Q \mathbf{R}_j^{\mathbf{L}}\Big(\sum_{Q'' \in \mathcal{Q}''(Q)} (\phi_{Q''} f)\Big).\end{aligned} \tag{2.4.4}$$

Lemmas 2.3.13, 2.3.5, and 2.3.16 combined with (2.4.4) imply

$$\begin{aligned}\sum_{Q\in\mathcal{Q}} \|\mathbf{R}^{\mathbf{L}}_{j,Q,0}(\phi_Q f)\|_{L^1(Q^{**})} &\le C\sum_{Q\in\mathcal{Q}}\Big(\sum_{Q'\in\mathcal{Q}'(Q)} \|\phi_{Q'}f\|_{L^1(\mathbb{R}^d)} + \|\phi_Q f\|_{L^1(\mathbb{R}^d)} \\ &\quad + \|\phi_Q \mathbf{R}^{\mathbf{L}}_j f\|_{L^1(\mathbb{R}^d)}\Big) + C\|f\|_{L^1(\mathbb{R}^d)} \\ &\le C\left(\|f\|_{L^1(\mathbb{R}^d)} + \|\mathbf{R}^{\mathbf{L}}_j f\|_{L^1(\mathbb{R}^d)}\right).\end{aligned} \tag{2.4.5}$$

Now (2.4.2) follows from (2.4.3) and (2.4.5).

2.5 Appendix

2.5.1 Proof of Lemma 2.3.1

Proof. The argument is based on estimates of the semigroup $\mathbf{K}_t$ acting on weighted L^2 spaces. This technique was utilized, e.g. in [14], [25], [20].

Fix $y_0 \in \mathbb{R}^d$ and $\alpha > 0$. The semigroup $\{\mathbf{K}_t\}_{t>0}$ acting on $L^2(e^{\alpha|x-y_0|}dx)$ has the unique extension to a holomorphic semigroup $\mathbf{K}_\zeta$, $\zeta \in \{\zeta\in\mathbb{C} : |\mathrm{Arg}\,\zeta| < \pi/4\}$ such that

$$\|K_\zeta\|_{L^2(e^{\alpha|x-y_0|}dx)\to L^2(e^{\alpha|x-y_0|}dx)} \le Ce^{c'\alpha^2\Re\zeta} \tag{2.5.1}$$

with C and c' independent of V and y_0 (see, e.g. [20, Section 6]). Let $-\mathbf{A}_{\alpha,y_0}$ denote the infinitesimal generator of $\{\mathbf{K}_t\}_{t>0}$ considered on $L^2(e^{\alpha|x-y_0|}dx)$. The quadratic form $\mathbf{Q} = \mathbf{Q}_{\alpha,y_0}$ associated with $\mathbf{A}_{\alpha,y_0}$ is given by

$$\begin{aligned}\mathbf{Q}(f,g) &= \sum_{j=1}^d \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} f(x)\frac{\partial}{\partial x_j}\overline{g(x)}e^{\alpha|x-y_0|}\,dx + \int_{\mathbb{R}^d} V(x)f(x)\overline{g(x)}e^{\alpha|x-y_0|}\,dx \\ &\quad + \sum_{j=1}^d \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} f(x)\overline{g(x)}\frac{\partial}{\partial x_j} e^{\alpha|x-y_0|}\,dx,\end{aligned}$$

$$D(\mathbf{Q}) = \{f :\ f(x),\ V(x)^{1/2}f(x),\ \frac{\partial}{\partial x_j}f(x) \in L^2(e^{\alpha|x-y_0|}dx),\ j=1,...,d\}.$$

Note that

$$\left|\frac{\partial}{\partial x_j}e^{\alpha|x-y_0|}\right| \le C\alpha e^{\alpha|x-y_0|} \quad \text{for } x\neq y_0.$$

Clearly,

$$|\mathbf{Q}(f,g)| \le C_\alpha \|f\|_{\mathbf{Q}}\|g\|_{\mathbf{Q}}$$

with C_α independent of y_0 and V, where

$$\|f\|^2_{\mathbf{Q}} = \int_{\mathbb{R}^d}\left(\sum_{j=1}^d\left|\frac{\partial}{\partial x_j}f(x)\right|^2 + V(x)|f(x)|^2 + |f(x)|^2\right)e^{\alpha|x-y_0|}dx.$$

Moreover, there exists a constant $C > 0$ independent of V and y_0 such that

$$\|f\|_{\mathbf{Q}}^2 \leq C\mathbf{Q}(f,f). \tag{2.5.2}$$

The holomorphy of the semigroup $\mathbf{K}_t$ combined with (2.5.1) imply

$$\|\mathbf{A}_{\alpha,y_0}\mathbf{K}_t g\|_{L^2(e^{\alpha|x-y_0|}dx)} \leq C' t^{-1} e^{c'' t\alpha^2} \|g\|_{L^2(e^{\alpha|x-y_0|}dx)} \tag{2.5.3}$$

with constants C' and c'' independent of V and y_0. Setting $g(x) = K_{1/2}(x, y_0)$, $f(x) = \mathbf{K}_{1/2}\, g(x) = K_1(x, y_0)$ and using (2.5.2), (2.5.3), (2.5.1), and (2.1.4), we get

$$\begin{aligned} \left\| \frac{\partial}{\partial x_j} K_1(x,y_0) \right\|^2_{L^2(e^{\alpha|x-y_0|}dx)} &\leq \|f\|_{\mathbf{Q}}^2 \\ &\leq C\mathbf{Q}(f,f) \\ &\leq C\|\mathbf{A}_{\alpha,y_0} f\|_{L^2(e^{\alpha|x-y_0|}dx)} \|f\|_{L^2(e^{\alpha|x-y_0|}dx)} \\ &\leq C'' \|g\|^2_{L^2(e^{\alpha|x-y_0|}dx)} \leq C''' \end{aligned} \tag{2.5.4}$$

with C''' independent of y_0 and V. Since $K_t(x,y) = t^{-d/2}\widetilde{K}_1(x/\sqrt{t}, y/\sqrt{t})$, where $\widetilde{K_t}(x,y)$ is the integral kernel of the semigroup $\{\widetilde{\mathbf{K}_t}\}_{t>0}$ generated by $\boldsymbol{\Delta} - tV(\sqrt{t}x)$, we get (2.3.2) from (2.5.4), because C''' is independent of V and y_0. Now (2.3.3) follows from (2.3.2) and the Cauchy-Schwarz inequality. □

3. HARDY SPACES RELATED TO SCHRÖDINGER OPERATORS II

3.1 Background and main results

This chapter is devoted to proving three characterizations (see Theorems 3.1.2, 3.1.4, 3.1.8) of the Hardy space $H^1_{\mathbf{L}}$ related to the Schrödnger operator $\mathbf{L}f(x) = -\mathbf{\Delta} f(x) + V(x)f(x)$, on $\mathbb{R}^d$ for another class of potentials V. The results of this chapter are mostly contained in [17] and [15]. During the whole chapter we assume that V satisfies:

(A_4) there exist $V_i \geq 0$, $V_i \not\equiv 0$, such that

$$V(x) = \sum_{i=1}^{m} V_i(x),$$

(A_5) for every $i \in \{1, ..., m\}$ there exists a linear subspace $\mathbb{V}_i$ of $\mathbb{R}^d$ of dimension $d_i \geq 3$ such that if $\Pi_{\mathbb{V}_i}$ denotes the orthogonal projection on $\mathbb{V}_i$ then

$$V_i(x) = V_i(\Pi_{\mathbb{V}_i} x),$$

(A_6) there exists $\kappa > 0$ such that for $i = 1, ..., m$ we have

$$V_i \in L^r(\mathbb{V}_i)$$

for all r satisfying $|r - d_i/2| \leq \kappa$.

In this chapter we are using the notion already provided in Section 1.2. In particular, $\mathbf{K}_t = \exp(-t\mathbf{L})$ and $\mathbf{P}_t = \exp(t\mathbf{\Delta})$ denote the semigroups of linear operators associated with $-\mathbf{L}$ and $\mathbf{\Delta}$ respectively. Note that the estimate $0 \leq K_t(x,y) \leq P_t(x-y)$ and the perturbation formula $\mathbf{P}_t = \mathbf{K}_t + \int_0^t \mathbf{P}_{t-s} V \mathbf{K}_s \, ds$ still hold (see (2.1.4) and (2.3.9)).

Let $\mathbf{M}_{\mathbf{L}}$ and $\mathbf{M}_{\mathbf{\Delta}}$ be the associated maximal operators (see (1.1.5) and (1.2.1)). Recall, that the Hardy space $H^1_{\mathbf{L},max}$ and the corresponding norm (cf. Definition 1.2.2) are given by

$$f \in H^1_{\mathbf{L},max} \iff \mathbf{M}_{\mathbf{L}} f \in L^1(\mathbb{R}^d), \qquad \|f\|_{H^1_{\mathbf{L},max}} = \|\mathbf{M}_{\mathbf{L}} f\|_{L^1(\mathbb{R}^d)}.$$

The goal of the chapter is to prove three characterizations of the space $H^1_{\mathbf{L},max}$ (see Theorems 3.1.2, 3.1.4, 3.1.8).

Denote by $\mathbf{L}^{-1}$ and $\mathbf{\Delta}^{-1}$ the operators with the integral kernels

$$\Gamma_{\mathbf{L}}(x,y) = \int_0^\infty K_t(x,y)\,dt \quad \text{and} \quad \Gamma_{\mathbf{\Delta}}(x-y) = -\int_0^\infty P_t(x-y)\,dt,$$

respectively. Clearly,

$$0 \leq \int_0^t K_s(z,y)\,ds \leq \Gamma_{\mathbf{L}}(z,y) \leq -\Gamma_{\mathbf{\Delta}}(z-y) = C_d|z-y|^{2-d}. \tag{3.1.1}$$

We shall see that operators $\mathbf{I} - V\mathbf{L}^{-1}$ and $\mathbf{I} - V\mathbf{\Delta}^{-1}$ are bounded on $L^1(\mathbb{R}^d)$. The first main result of Chapter 3 is the following characterization of the Hardy space $H^1_{\mathbf{L},max}$.

Theorem 3.1.2. *Assume $f \in L^1(\mathbb{R}^d)$. Then f belongs to $H^1_{\mathbf{L},max}$ if and only if $(\mathbf{I} - V\mathbf{L}^{-1})f$ belongs to the classical Hardy space $H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$. Moreover,*

$$\|f\|_{H^1_{\mathbf{L},max}} \sim \|(\mathbf{I} - V\mathbf{L}^{-1})f\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)}.$$

We define the auxiliary function ω by

$$\omega(x) = \lim_{t\to\infty} \int_{\mathbb{R}^d} K_t(x,y)\,dy.$$

The above limit exists because, by (2.1.4) and the semigroup property, the function $t \mapsto \mathbf{K}_t\mathbf{1}(x)$ is decreasing and takes values in $[0,1]$. Clearly, for every $t > 0$,

$$\omega(x) = \mathbf{K}_t\omega(x) = \int_{\mathbb{R}^d} K_t(x,y)\omega(y)\,dy. \tag{3.1.3}$$

We shall prove that there exists $\delta > 0$ such that $\delta \leq \omega(x) \leq 1$ (see Proposition 3.2.10).

We are now in a position to state the second main result of this chapter.

Theorem 3.1.4. *Let $f \in L^1(\mathbb{R}^d)$. Then f belongs to $H^1_{\mathbf{L}}$ if and only if ωf belongs to $H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$. Additionally,*

$$\|f\|_{H^1_{\mathbf{L},max}} \sim \|\omega f\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)}.$$

From Theorem 3.1.4 we immediately obtain atomic characterizations of the elements of $H^1_{\mathbf{L},max}$.

Definition 3.1.5. *We call a function a an $\mathbf{L}$-atom if it satisfies:*

- *there exists a ball $B = B(y,r)$ such that* $\operatorname{supp} a \subseteq B$,
- $\|a\|_\infty \leq |B|^{-1}$,
- $\int_{\mathbb{R}^d} a(x)\omega(x)\,dx = 0$.

Corollary 3.1.6. *The spaces $H^1_{\mathbf{L},max}$ and $H^1_{\mathbf{L},at}$ (see Definitions 1.2.5 and 1.2.5) coincide. Also, the corresponding norms are comparable.*

We denote by $\frac{\partial}{\partial x_j}$ the partial derivative in the direction of the j-th canonical coordinate of $\mathbb{R}^d$, where $j = 1, ..., d$. For $f \in L^1(\mathbb{R}^d)$ the Riesz transforms $\mathbf{R}^{\mathbf{L}}_j$ associated to $\mathbf{L}$ (cf. (1.1.9) and (1.2.3)) are given by

$$\mathbf{R}^{\mathbf{L}}_j f = \lim_{\varepsilon \to 0} \int_\varepsilon^{\varepsilon^{-1}} \frac{\partial}{\partial x_j} \mathbf{K}_t f \frac{dt}{\sqrt{t}}. \tag{3.1.7}$$

We shall see that the last limits are well-defined in the sense of distributions.

The third main result of this chapter is to obtain the following characterization of $H^1_{\mathbf{L},max}$.

Theorem 3.1.8. *An $L^1(\mathbb{R}^d)$-function f belongs to $H^1_{\mathbf{L},max}$ if and only if $\mathbf{R}^{\mathbf{L}}_j f$ belong to $L^1(\mathbb{R}^d)$ for $j = 1, ..., d$. Additionally,*

$$\|f\|_{H^1_{\mathbf{L},max}} \sim \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \|\mathbf{R}^{\mathbf{L}}_j f\|_{L^1(\mathbb{R}^d)}.$$

The results included in this chapter generalize the theorems of [22] and [15], where the spaces $H^1_{\mathbf{L}}$ were studied under assumptions: $V \geq 0$, $\operatorname{supp} V$ is compact, $V \in L^r(\mathbb{R}^d)$ for some $r > d/2$. Obviously such potentials V satisfy the conditions $(A_4) - (A_6)$. To prove Theorems 3.1.2, 3.1.4, and 3.1.8 we develop methods of [22] and [15].

3.2 Auxiliary lemmas

We shall use the following notation. For $z \in \mathbb{R}^d$ and a subspace $\mathbb{V}_i$ of $\mathbb{R}^d$ we write

$$z = z_i + \widetilde{z}_i, \quad z_i = \Pi_{\mathbb{V}_i}(z), \quad \widetilde{z}_i = \Pi_{\mathbb{V}_i^\perp}(z), \quad \widetilde{d}_i = \dim \mathbb{V}_i^\perp = d - d_i.$$

Notice that if $\mathbb{V}_i = \mathbb{R}^d$, then, in fact, there is no $\mathbb{V}_i^\perp$.

The following two lemmas state crucial estimates that will be used in many proofs.

Lemma 3.2.1. *There exists $\lambda > 0$ such that*

$$\sup_{y \in \mathbb{R}^d} \|V(\cdot)| \cdot - y|^{2-d+\mu}\|_{L^r(\mathbb{R}^d)} \leq C \quad \textit{for } r \in [1, 1+\lambda] \textit{ and } \mu \in [-\lambda, \lambda]. \tag{3.2.2}$$

Proof. It suffices to prove (3.2.2) for $V = V_1$. For fixed $y \in \mathbb{R}^d$ we have

$$\|V_1(\cdot)| \cdot - y|^{2-d+\mu}\|^r_{L^r(\mathbb{R}^d)} \leq C \int_{\mathbb{V}_1} \int_{\mathbb{V}_1^\perp} \frac{V_1(z_1)^r}{|z_1 - y_1|^{-r(2-d+\mu)} + |\widetilde{z}_1 - \widetilde{y}_1|^{-r(2-d+\mu)}} \, d\widetilde{z}_1 \, dz_1. \tag{3.2.3}$$

Observe that if $\lambda > 0$ is sufficiently small, $r \in [1, 1+\lambda]$, and $\mu \in [-\lambda, \lambda]$ then

$$\begin{aligned}\int_{\mathbb{V}_1^\perp} & \left(|z_1 - y_1|^{-r(2-d+\mu)} + |\widetilde{z}_1 - \widetilde{y}_1|^{-r(2-d+\mu)}\right)^{-1} d\widetilde{z}_1 \\ & \leq C \int_{|z_1-y_1|>|\widetilde{z}_1-\widetilde{y}_1|} |z_1 - y_1|^{r(2-d+\mu)} \, d\widetilde{z}_1 + C \int_{|z_1-y_1|\leq|\widetilde{z}_1-\widetilde{y}_1|} |\widetilde{z}_1 - \widetilde{y}_1|^{r(2-d+\mu)} \, d\widetilde{z}_1 \\ & \leq C|z_1 - y_1|^{r(2-d+\mu)+\widetilde{d}_1}.\end{aligned} \tag{3.2.4}$$

Thus, by (3.2.4),

$$\begin{aligned}\|V_1(\cdot)| \cdot -y|^{2-d+\mu}\|^r_{L^r(\mathbb{R}^d)} \leq & C \int_{|z_1-y_1|\leq 1} V_1(z_1)^r |z_1 - y_1|^{r(2-d+\mu)+\widetilde{d}_1} \, dz_1 \\ & + C \int_{|z_1-y_1|>1} V_1(z_1)^r |z_1 - y_1|^{r(2-d+\mu)+\widetilde{d}_1} \, dz_1.\end{aligned} \tag{3.2.5}$$

Note that by (A_6) there exist $t, s > 1$ such that $V_1^r \in L^t(\mathbb{V}_1) \cap L^s(\mathbb{V}_1)$ and

$$\chi_{\{|z_1|\leq 1\}}(z_1)|z_1|^{r(2-d+\mu)+\widetilde{d}_1} \in L^{t'}(\mathbb{V}_1), \qquad \chi_{\{|z_1|>1\}}(z_1)|z_1|^{r(2-d+\mu)+\widetilde{d}_1} \in L^{s'}(\mathbb{V}_1)$$

for $r \in [1, 1+\lambda]$ and $\mu \in [-\lambda, \lambda]$ provided $\lambda > 0$ is small enough. Thus (3.2.2) follows from the Hölder inequality. □

Corollary 3.2.6. *The operators* $\mathbf{I} - V\mathbf{\Delta}^{-1}$ *and* $\mathbf{I} - V\mathbf{L}^{-1}$ *are bounded on* $L^1(\mathbb{R}^d)$ *and*

$$(\mathbf{I} - V\mathbf{L}^{-1})(\mathbf{I} - V\mathbf{\Delta}^{-1})f = (\mathbf{I} - V\mathbf{\Delta}^{-1})(\mathbf{I} - V\mathbf{L}^{-1})f = f \quad \text{for} \quad f \in L^1(\mathbb{R}^d). \tag{3.2.7}$$

Formally, (3.2.7) can be easily seen, by inserting $V = \mathbf{L} + \mathbf{\Delta}$. However, since we deal with unbounded operators, it is not so straightforward. We provide the detailed proof in the appendix (see Section 3.6).

Lemma 3.2.8. *There exists* $\sigma, \varepsilon > 0$ *such that for* $s \in [1, 1+\varepsilon]$ *and* $R \geq 1$ *we have*

$$\sup_{y\in\mathbb{R}^d} \int_{|z-y|>R} V(z)^s |z - y|^{s(2-d)} \, dz \leq CR^{-\sigma}. \tag{3.2.9}$$

Proof. It is enough to prove (3.2.9) for $V = V_1$. Fix $q > 1$ and $\varepsilon > 0$ such that $d_1/q(1+\varepsilon) - 2 > 0$ and $V_1 \in L^{q(1+\varepsilon)}(\mathbb{V}_1) \cap L^q(\mathbb{V}_1)$ (see (A_6)). Set $\sigma = d_1/q - 2$. For $s \in [1, 1+\varepsilon]$ we have

$$\begin{aligned}\int_{|z-y|>R} V_1(z)^s |z - y|^{s(2-d)} \, dz \leq & \int_{|z_1-y_1|\geq|\widetilde{z}_1-\widetilde{y}_1|} \chi_{\{|z-y|>R\}}(z) V_1(z)^s |z_1 - y_1|^{s(2-d)} \, dz \\ & + \int_{|z_1-y_1|<|\widetilde{z}_1-\widetilde{y}_1|} \chi_{|z-y|>R}(z) V_1(z)^s |\widetilde{z}_1 - \widetilde{y}_1|^{s(2-d)} \, dz \\ = & T(R) + S(R).\end{aligned}$$

If $|z_1 - y_1| \geq |\widetilde{z}_1 - \widetilde{y}_1|$ and $|z-y| > R \geq 1$, then $|z_1 - y_1| > R/2 \geq 1/2$. Thus,

$$\begin{aligned} T(R) &\leq C \int_{|z_1-y_1|>R/2} |z_1-y_1|^{d-d_1} V_1(z_1)^s |z_1-y_1|^{s(2-d)}\, dz_1 \\ &\leq C\|V_1\|^s_{L^{qs}(\mathbb{V}_1)} \left(\int_{|z_1-y_1|>R/2} |z_1-y_1|^{(s(2-d)+d-d_1)q'}\, dz_1 \right)^{1/q'} = CR^{-\sigma}. \end{aligned}$$

Similarly, if $|z_1 - y_1| < |\widetilde{z}_1 - \widetilde{y}_1|$ and $|z-y| > R \geq 1$, then $|\widetilde{z}_1 - \widetilde{y}_1| > R/2 \geq 1/2$ and

$$\begin{aligned} S(R) &\leq C \int_{|\widetilde{z}_1-\widetilde{y}_1|>R/2} \|V_1\|^s_{L^{sq}(\mathbb{V}_1)} \left(\int_{|z_1-y_1|<|\widetilde{z}_1-\widetilde{y}_1|} dz_1 \right)^{1/q'} |\widetilde{z}_1-\widetilde{y}_1|^{s(2-d)}\, d\widetilde{z}_1 \\ &\leq C \int_{|\widetilde{z}_1-\widetilde{y}_1|>R/2} |\widetilde{z}_1-\widetilde{y}_1|^{s(2-d)+d_1/q'}\, d\widetilde{z}_1 = CR^{-\sigma}. \end{aligned}$$

□

We shall need the following properties of the function ω similar to those that hold in the case of compactly supported potentials (cf. [22, Lemma 2.4]).

Proposition 3.2.10. *There exist $\gamma, \delta > 0$ such that for $x, y \in \mathbb{R}^d$ we have*

(a) $|\omega(x) - \omega(y)| \leq C_\gamma |x-y|^\gamma$,

(b) $\delta \leq \omega(x) \leq 1$.

Proof. The property (a) can be proved by a slight modification of the proof of (2.6) in [22]. Indeed, thanks to (3.1.3) and $0 \leq \omega(x) \leq 1$, it suffices to show that there is $C, \gamma > 0$ such that for $|h| < 1$ we have

$$\int_{\mathbb{R}^d} |K_1(x+h,y) - K_1(x,y)|\, dy \leq C|h|^\gamma. \tag{3.2.11}$$

To this purpose, by using (2.3.9), it is enough to establish that

$$\sum_{i=1}^m \int_{\mathbb{R}^d} \left| \int_0^1 \int_{\mathbb{R}^d} (P_s(x+h-z) - P_s(x-z)) V_i(z) K_{1-s}(z,y)\, dz\, ds \right| dy \leq C|h|^\gamma.$$

Consider one summand that contains V_1. Utilizing the fact that $P_s(x) = P_s(x_1)P_s(\widetilde{x}_1)$, where $P_s(x_1)$ and $P_s(\widetilde{x}_1)$ are the heat kernels on $\mathbb{V}_1$ and $\mathbb{V}_1^\perp$ respectively. We have

$$\begin{aligned} I &= \int_{\mathbb{R}^d} \left| \int_0^1 \int_{\mathbb{R}^d} (P_s(x+h-z) - P_s(x-z)) V_1(z) K_{1-s}(z,y)\, dz\, ds \right| dy \\ &\leq \int_0^1 \int_{\mathbb{R}^d} |P_s(x+h-z) - P_s(x-z)| V_1(z)\, dz\, ds \\ &\leq \int_0^1 \int_{\mathbb{R}^d} P_s(x_1+h_1-z_1) \left| P_s(\widetilde{x}_1+\widetilde{h}_1-\widetilde{z}_1) - P_s(\widetilde{x}_1-\widetilde{z}_1) \right| V_1(z_1)\, dz\, ds \\ &\quad + \int_0^1 \int_{\mathbb{R}^d} P_s(\widetilde{x}_1-\widetilde{z}_1) \left| P_s(x_1+h_1-z_1) - P_s(x_1-z_1) \right| V_1(z_1)\, dz\, ds \end{aligned} \tag{3.2.12}$$

By taking $q > d_1/2$ such that $V_1 \in L^q(\mathbb{V}_1)$ and using the Hölder inequality we obtain

$$\begin{aligned} I &\leq \int_0^1 \|P_s(x_1)\|_{L^{q'}(\mathbb{V}_1)} \|V_1(z_1)\|_{L^q(\mathbb{V}_1)} \int_{\mathbb{V}_1^\perp} |P_s(\widetilde{x}_1 + \widetilde{h}_1 - \widetilde{z}_1) - P_s(\widetilde{x}_1 - \widetilde{z}_1)| \, d\widetilde{z}_1 \, ds \\ &+ \int_0^1 \left(\int_{\mathbb{V}_1} |P_s(x_1 + h_1 - z_1) - P_s(x_1 - z_1)|^{q'} \, dz_1 \right)^{1/q'} \|V_1(z_1)\|_{L^q(\mathbb{V}_1)} \, ds \\ &\leq C(|\widetilde{h}_1|^\gamma + |h_1|^\gamma), \end{aligned} \tag{3.2.13}$$

which finishes the proof of (a).

Next we note that

$$K_t(x,y) > 0 \quad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^d. \tag{3.2.14}$$

The proof of (3.2.14) is a straightforward adaptation of the proof of [22, Lemma 2.12]. We omit the details.

Our next task is to establish that there exists $\delta > 0$ such that

$$\omega(x) \geq \delta. \tag{3.2.15}$$

The proof[1] of (3.2.15) we present here is based on the Hölder inequality and goes by induction on m. Assume first that we have only one potential V_1, that is, $m = 1$. Then, $K_t(x,y) = K_t^{\{1\}}(x_1, y_1) P_t(\widetilde{x}_1 - \widetilde{y}_1)$, where $K_t^{\{1\}}(x_1, y_1)$ is the kernel of the semigroup generated by $\boldsymbol{\Delta} - V_1(x_1)$ on $\mathbb{V}_1$ and $P_t(\widetilde{x}_1)$ is the classical heat semigroup on $\mathbb{V}_1^\perp$. Hence $\omega(x) = \omega_0(x_1)$, where $\omega_0(x_1) = \lim_{t\to\infty} \int_{\mathbb{V}_1} K_t^{\{1\}}(x_1, y_1) \, dy_1$. Therefore, there is no loss of generality in proving (3.2.15) if we assume that $\mathbb{V} = \mathbb{R}^d$. If we integrate (2.3.9) over $\mathbb{R}^d$ and take the limit as $t \to \infty$, then we get

$$1 - \omega(x) = \int_{\mathbb{R}^d} V(y) \Gamma_{\mathbf{L}}(x,y) \, dy \leq C \int_{\mathbb{R}^d} V(y) |x - y|^{2-d} \, dy. \tag{3.2.16}$$

By (A_6) and the Hölder inequality we find $t, s > 1$ such that $V \in L^t(\mathbb{R}^d) \cap L^s(\mathbb{R}^d)$, $\chi_{\{|x| \leq 1\}}(x) |x|^{2-d} \in L^{t'}(\mathbb{R}^n)$, and $\chi_{\{|x| > 1\}}(x) |x|^{2-d} \in L^{s'}(\mathbb{R}^d)$. Thus (3.2.16) leads to

$$\lim_{|x|\to\infty} \int_{\mathbb{R}^d} V(y) |x - y|^{2-d} \, dy = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \omega(x) = 1. \tag{3.2.17}$$

The equation (3.1.3) combined with (3.2.14) and (3.2.17) imply that $w(x) > 0$ for every $x \in \mathbb{R}^d$. Since ω is continuous (see (a)) and $\lim_{|x|\to\infty} \omega(x) = 1$, we get (3.2.15).

[1] We would like to note that (3.2.15) can also be deduced from [36]. The author learnt about the results of [36] when the thesis was already written down.

Using induction, we assume that (3.2.15) is true for V being a sum of $m-1$ potentials. Take $V = V_1 + ... + V_m$. As in the case of $m = 1$, we can assume that $\mathrm{lin}\{\mathbb{V}_1, ..., \mathbb{V}_m\} = \mathbb{R}^d$. Consider the semigroup $\{\mathbf{S}_t\}_{t>0}$ generated by $-\boldsymbol{\Delta} + V_2 + ... + V_m$. Let $\omega_1(x) = \lim_{t\to\infty} \int_{\mathbb{R}^d} S_t(x,y)\,dy$. By the inductive assumption $\omega_1(x) \geq \delta_1$. Similarly to (3.2.16), the perturbation formula

$$\mathbf{S}_t = \mathbf{K}_t + \int_0^t \mathbf{S}_{t-s} V_1 \mathbf{K}_s\,ds$$

implies

$$\delta_1 \leq \omega_1(y) \leq \omega(y) + C\int_{\mathbb{R}^d} V_1(z)|z-y|^{2-d}\,dz \leq \omega(y) + C\int_{\mathbb{V}_1} V_1(z_1)|z_1-y_1|^{2-d_1}\,dz_1, \tag{3.2.18}$$

where the last inequality is proved in (3.2.4). If $y_1 \to \infty$ then the integral on the right hand side of (3.2.18) goes to zero. Hence, $\omega(y) > \delta_1/2$ provided $|y_1| > R_1$. We repeat the argument for each V_2, ..., V_d instead of V_1 and deduce that there exists $R, \delta > 0$ such that $\omega(x) > \delta$ for $|x| > R$. Consequently, by using (3.1.3), (3.2.14) and continuity of ω we obtain (3.2.15). □

3.3 Proof of Theorem 3.1.2

By (2.3.9) we get

$$\mathbf{K}_t - \mathbf{P}_t(\mathbf{I} - V\mathbf{L}^{-1}) = \mathbf{Q}_t - \mathbf{W}_t, \tag{3.3.1}$$

where

$$\mathbf{W}_t = \int_0^t (\mathbf{P}_{t-s} - \mathbf{P}_t)\, V\, \mathbf{K}_s\,ds, \qquad \mathbf{Q}_t = \int_t^\infty \mathbf{P}_t\, V\, \mathbf{K}_s\,ds.$$

Let

$$\begin{aligned} W_t(x,y) &= \sum_{i=1}^m W_t^{\langle i\rangle}(x,y) &&= \sum_{i=1}^m \int_0^t \int_{\mathbb{R}^d} (P_{t-s}(x-z) - P_t(x-z))V_i(z)K_s(z,y)\,dz\,ds,\\ Q_t(x,y) &= \sum_{i=1}^m Q_t^{\langle i\rangle}(x,y) &&= \sum_{i=1}^m \int_{\mathbb{R}^d} P_t(x,z)\int_t^\infty V_i(z)K_s(z,y)\,ds\,dz \end{aligned}$$

be the integral kernels of $\mathbf{W}_t$ and $\mathbf{Q}_t$ respectively. In order to prove Theorem 3.1.2 it is sufficient to establish that the maximal operators: $f \mapsto \sup_{t>0} |\mathbf{W}_t f|$ and $f \mapsto \sup_{t>0} |\mathbf{Q}_t f|$ are bounded on $L^1(\mathbb{R}^d)$. The proofs of these facts are presented in the following four lemmas.

Lemma 3.3.2. *The operator $f \mapsto \sup_{t>2} |\mathbf{W}_t f|$ is bounded on $L^1(\mathbb{R}^d)$.*

Proof. It suffices to prove that

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\sup_{t>2}|W_t(x,y)|\,dx<\infty.$$

Without loss of generality we can consider only $W_t^{\langle 1\rangle}(x,y)$. For $0<\beta<1$, which will be fixed later on, we write

$$\begin{aligned}W_t^{\langle 1\rangle}(x,y)&=\int_0^t\int_{\mathbb{R}^d}(P_{t-s}(x-z)-P_t(x-z))V_1(z)K_s(z,y)\,dz\,ds\\&=\int_0^{t^\beta}...+\int_{t^\beta}^t...=F_1(x,y;t)+F_2(x,y;t).\end{aligned}$$

To estimate F_1 observe that for $t>2$ and $s\le t^\beta<t$ there exists $\phi\in\mathcal{S}(\mathbb{R}^d)$ such that

$$|P_{t-s}(x-z)-P_t(x-z)|\le C\frac{s}{t}\phi_t(x-z). \tag{3.3.3}$$

Here and subsequently, till the end of the present chapter, $f_t(x)=t^{-d/2}f(x/\sqrt{t})$. From (3.3.3) and (3.1.1), we get

$$|F_1(x,y;t)|\le Ct^{-1+\beta}\int_{\mathbb{R}^d}\phi_t(x-z)V_1(z)|z-y|^{2-d}dz.$$

Since $\sup_{t>2}t^{-1+\beta}\phi_t(x-z)\le C(1+|x-z|)^{-d-2+2\beta}$, we have that

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\sup_{t>2}|F_1(x,y;t)|dx\le C\int_{\mathbb{R}^d}V_1(z)|z-y|^{2-d}\,dz\le C,$$

where the last inequality comes from Lemma 3.2.1.

To deal with F_2 we write

$$\begin{aligned}F_2(x,y;t)&=\int_{t^\beta}^t\int_{\mathbb{R}^d}P_{t-s}(x-z)V_1(z)K_s(z,y)\,dz\,ds-\int_{t^\beta}^t\int_{\mathbb{R}^d}P_t(x-z)V_1(z)K_s(z,y)\,dz\,ds\\&=F_2'(x,y;t)-F_2''(x,y;t)\end{aligned}$$

Observe that for $s\in[t^\beta,t]$ we have

$$K_s(z,y)\le Ct^{-\beta d/2}\exp\left(-|z-y|^2/4t\right). \tag{3.3.4}$$

Also

$$\int_0^t P_{t-s}(x-z)\,ds=\int_0^t P_s(x-z)\,ds\le C|x-z|^{2-d}\exp\left(-|x-z|^2/ct\right). \tag{3.3.5}$$

As a consequence of (3.3.4)–(3.3.5) we obtain

$$F_2'(x,y;t)\le C\int_{\mathbb{R}^d}t^{-\beta d/2}|x-z|^{2-d}\exp\left(-|x-z|^2/ct\right)V_1(z_1)\exp\left(-|z-y|^2/4t\right)\,dz.$$

Then, for $\varepsilon > 0$,

$$\begin{aligned}\sup_{t>2} t^{-\beta d/2} \exp\left(-|x-z|^2/ct\right) \exp\left(-|z-y|^2/4t\right) \\ \le C \sup_{t>2} t^{-1-\varepsilon} \exp\left(-|x-z|^2/ct\right) \cdot \sup_{t>2} t^{-\beta d/2+1+\varepsilon} \exp\left(-|z-y|^2/4t\right) \\ \le C(1+|x-z|)^{-2-2\varepsilon}|z-y|^{2+2\varepsilon-\beta d}.\end{aligned}$$

Consequently,

$$\begin{aligned}\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d} \sup_{t>2} F_2'(x,y;t)\,dx &\le C \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|x-z|^{2-d}}{(1+|x-z|)^{2+2\varepsilon}}|z-y|^{2+2\varepsilon-\beta d} V_1(z)\,dx\,dz \\ &\le C \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d} |z-y|^{2+2\varepsilon-\beta d} V_1(z_1)\,dz.\end{aligned}$$

If we choose $\beta < 1$ close to 1 and ε small, then we can apply Lemma 3.2.1 and get

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d} \sup_{t>2} F_2'(x,y;t)\,dx \le C.$$

We now turn to estimate $F_2''(x,y;t)$. Observe that for $\varepsilon > 0$ we have

$$\int_{t^\beta}^t K_s(z,y)\,ds \le C\int_{t^\beta}^\infty t^{-\beta\varepsilon} s^{-d/2+\varepsilon} \exp\left(-|z-y|^2/(4s)\right)\,ds \le Ct^{-\beta\varepsilon}|z-y|^{2-d+2\varepsilon}.$$

Then from Lemma 3.2.1 we conclude that

$$\begin{aligned}\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d} \sup_{t>2} F_2''(x,y;t)\,dx &\le C \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sup_{t>2} t^{-\beta\varepsilon} P_t(x-z)V_1(z)|z-y|^{2-d+2\varepsilon}\,dx\,dz \\ &\le C \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (1+|x-z|)^{-d-2\beta\varepsilon} V_1(z)|z-y|^{2-d+2\varepsilon}\,dx\,dz \\ &\le C \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d} V_1(z)|z-y|^{2-d+2\varepsilon}\,dz \le C,\end{aligned}$$

provided $\varepsilon > 0$ is small enough. □

Lemma 3.3.6. *The operator* $f \mapsto \sup_{t\le 2}|\mathbf{W}_t f|$ *is bounded on* $L^1(\mathbb{R}^d)$.

Proof. It is enough to prove that

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d} \sup_{t\le 2} |W_t^{\langle 1\rangle}(x,y)|\,dx < \infty.$$

We have

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} (P_{t-s}(x-z) - P_t(x-z))V_1(z)K_s(z,y)\,dz\,ds &= \int_0^{t/2} \dots + \int_{t/2}^t \dots \\ &= F_3(x,y;t) + F_4(x,y;t).\end{aligned}$$

To deal with F_3 observe that for $t \le 2$, $s \le t/2$ we have

$$|P_{t-s}(x-z) - P_t(x-z)| \le C\phi_t(x-z),$$

where $\phi \in \mathcal{S}(\mathbb{R}^d)$, $\phi \ge 0$. Therefore

$$\sup_{t\le 2} |F_3(x,y;t)| \le C \sup_{t\le 2} \int_{\mathbb{R}^d} \phi_t(x-z)V_1(z)|z-y|^{2-d} dz.$$

Denote by $\mathbf{M}^0_\phi$ the classical local maximal operator associated with ϕ, that is

$$\mathbf{M}^0_\phi(f)(x) = \sup_{t\le 2} |\phi_t * f(x)|.$$

Then

$$\sup_{t\le 2} |F_3(x,y;t)| \le C\mathbf{M}^0_\phi(\xi_y)(x),$$

where $\xi_y(z) = V_1(z)|z-y|^{2-d}$. We claim that

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{t\le 2} |F_3(x,y)| dx \le C \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{M}^0_\phi(\xi_y)(x)\, dx \le C. \tag{3.3.7}$$

To obtain (3.3.7) we write

$$\xi_y(z) = \sum_{k=1}^{\infty} \xi_{y,k}(z),$$

where

$$\xi_{y,1}(z) = V_1(z)|z-y|^{2-d}\chi_{B(y,2)}(z), \quad \xi_{y,k}(z) = V_1(z)|z-y|^{2-d}\chi_{B(y,2^k)\setminus B(y,2^{k-1})}(z), \quad k>1.$$

From Lemma 3.2.1 it follows that there exists $s > 1$ such that

$$\operatorname{supp} \xi_{y,1} \subseteq B(y,2) \text{ and } \|\xi_{y,1}\|_{L^s(\mathbb{R}^d)} \le C \le C|B(y,2)|^{-1+1/s}. \tag{3.3.8}$$

Consider $\xi_{y,k}$ for $k > 1$. Set $q < d_1/2$ such that $V_1 \in L^q(\mathbb{V}_1)$. Then

$$\begin{aligned} &\operatorname{supp} \xi_{y,k} \subseteq B(y,2^k). \\ &\|\xi_{y,k}\|_{L^q(\mathbb{R}^d)} \le C2^{k(2-d)}\|V_1\|_{L^q(\mathbb{V}_1)}2^{k(d-d_1)/q} \le C|B(y,2^k)|^{-1+1/q}2^{-\rho k}, \end{aligned} \tag{3.3.9}$$

where $\rho = d_1/q - 2$. Now, our claim (3.3.7) follows from (3.3.8), (3.3.9), and the classical theory of local maximal operators.

It remains to analyze $F_4 = F_5 - F_6$, where

$$F_5(x,y;t) = \int_{t/2}^{t} \int_{\mathbb{R}^d} P_{t-s}(x-z)V_1(z)K_s(z,y)\, dz\, ds,$$

$$F_6(x,y;t) = \int_{t/2}^{t} \int_{\mathbb{R}^d} P_t(x-z)V_1(z)K_s(z,y)\, dz\, ds.$$

Clearly,

$$\sup_{s\in[t/2,t]} K_s(z,y) \le Ct^{-d/2}\exp\left(-|z-y|^2/ct\right).$$

Therefore, for $0<t\le 2$ and $0<\gamma<1$ close to 1 we get

$$\begin{aligned}
F_5(x,y;t) &\le C\int_0^{t/2}\int_{\mathbb{R}^d} t^{-\gamma}P_s(x-z)V_1(z)t^{-d/2+\gamma}\exp\left(-|z-y|^2/ct\right)\,dz\,ds\\
&\le C\int_{\mathbb{R}^d}|x-z|^{2-d}t^{-\gamma}\exp\left(-|x-z|^2/ct\right)V_1(z)|z-y|^{-d+2\gamma}\,dz\\
&\le C\int_{\mathbb{R}^d}|x-z|^{2-d-2\gamma}\exp\left(-|x-z|^2/c'\right)V_1(z)|z-y|^{-d+2\gamma}\,dz.
\end{aligned}$$

Thus, by using Lemma 3.2.1, we get

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\sup_{0<t\le 2} F_5(x,y;t)dx \le C.$$

To deal with F_6 we observe that for $0<t\le 2$ and $0<\gamma<1$ close to 1 we have

$$\begin{aligned}
F_6(x,y;t) &\le C\int_{\mathbb{R}^d} tP_t(x-z)V_1(z_1)t^{-d/2}\exp\left(-|z-y|^2/ct\right)\,dz\\
&\le \int_{\mathbb{R}^d}|x-z|^{2-d-2\gamma}\exp\left(-|x-z|^2/c'\right)V_1(z)|z-y|^{-d+2\gamma}\,dz
\end{aligned}$$

and, consequently,

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\sup_{t<2} F_6(x,y;t)\,dx \le C.$$

□

Lemma 3.3.10. *The operator* $f\mapsto \sup_{t>2}|\mathbf{Q}_t f|$ *is bounded on* $L^1(\mathbb{R}^d)$.

Proof. Notice that for $\varepsilon>0$ and $t>2$ we have

$$\int_t^\infty K_s(z,y)\,ds \le C\int_t^\infty s^{-\varepsilon}s^{-d/2+\varepsilon}\exp\left(-\frac{|y-z|^2}{4s}\right)\,ds \le Ct^{-\varepsilon}|y-z|^{2-d+2\varepsilon}. \quad (3.3.11)$$

It causes no loss of generality to consider only $Q_t^{\langle 1\rangle}(x,y)$. If $t>2$, then

$$0\le Q_t^{\langle 1\rangle}(x,y)\le C\int_{\mathbb{R}^d}P_t(x-z)V_1(z)t^{-\varepsilon}|y-z|^{2-d+2\varepsilon}\,dz.$$

Since $\sup_{t>2}t^{-\varepsilon}P_t(x-z)\le C(1+|x-z|)^{-d-2\varepsilon}$, we find that

$$\begin{aligned}
\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\sup_{t>2}Q_t^{\langle 1\rangle}(x,y)\,dx &\le C\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(1+|x-z|)^{-d-2\varepsilon}V_1(z)|y-z|^{2-d+2\varepsilon}\,dz\,dx\\
&\le C\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}V_1(z)|y-z|^{2-d+2\varepsilon}\,dz\le C.
\end{aligned} \quad (3.3.12)$$

The last inequality follows from Lemma 3.2.1. □

Lemma 3.3.13. *The operator* $f \mapsto \sup_{t\le 2} |\mathbf{Q}_t f|$ *is bounded on* $L^1(\mathbb{R}^d)$.

Proof. The estimate $\int_t^\infty K_s(z,y)\,ds \le C|z-y|^{2-d}$ implies

$$\sup_{t\le 2} Q_t(x,y) \le C \sup_{t\le 2} \int_{\mathbb{R}^d} P_t(x-z)V(z)|z-y|^{2-d}\,dz.$$

We claim that for fixed $y \in \mathbb{R}^d$ the foregoing function (of variable x) belongs to $L^1(\mathbb{R}^d)$ and

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{t\le 2} Q_t(x,y)\,dx < \infty.$$

The claim follows by arguments identical to that we used to prove (3.3.7). □

Now, Theorem 3.1.2 follows directly from Lemmas 3.3.2, 3.3.6, 3.3.10, 3.3.13.

3.4 Proof of Theorem 3.1.4

Proof. Thanks to (3.2.16) and Proposition 3.2.10, for $g \in L^1(\mathbb{R}^d)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} (\mathbf{I} - V\mathbf{L}^{-1})(g/\omega)(x)\,dx &= \int_{\mathbb{R}^d} \frac{g(x)}{\omega(x)}\,dx - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V(x)\Gamma_{\mathbf{L}}(x,y)\frac{g(y)}{\omega(y)}\,dy\,dx \\ &= \int_{\mathbb{R}^d} \frac{g(x)}{\omega(x)}\,dx - \Big(\int_{\mathbb{R}^d} \frac{g(y)}{\omega(y)}\,dy - w(y)\frac{g(y)}{\omega(y)}\,dy\Big) \\ &= \int_{\mathbb{R}^d} g(y)\,dy.\end{aligned} \tag{3.4.1}$$

First, we are going to prove that

$$\|\omega f\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)} \le \|f\|_{H^1_{\mathbf{L},max}}. \tag{3.4.2}$$

Theorem 3.1.2 combined with (3.2.7) implies that (3.4.2) is equivalent to

$$\|\omega(\mathbf{I} - V\mathbf{\Delta}^{-1})f\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)} \le C\|f\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)}. \tag{3.4.3}$$

Assume that a is a classical $(1,\infty)$−atom associated with $B = B(y_0, r)$, i.e.

$$\operatorname{supp} a \subseteq B, \quad \|a\|_\infty \le |B|^{-1}, \quad \int_B a(x)\,dx = 0. \tag{3.4.4}$$

By the atomic characterization of $H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$ the inequality (3.4.3) will be obtained, if we have established that $b = \omega(\mathbf{I} - V\mathbf{\Delta}^{-1})a \in H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$ and

$$\|b\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)} \le C \tag{3.4.5}$$

with a constant $C > 0$ independent of a.

By (3.2.7), $a = (\mathbf{I} - V\mathbf{L}^{-1})(b/\omega)$. Hence, using (3.4.1) we get

$$\int_{\mathbb{R}^d} b(x)\, dx = 0. \tag{3.4.6}$$

The proof of (3.4.5) is divided into two cases.

Case 1: $r \geq 1$. Set

$$\begin{aligned} b(x) &= (b(x) - c_1)\chi_{2B}(x) + \sum_{k=2}^{\infty}\Big(b(x)\chi_{2^kB\setminus 2^{k-1}B}(x) + c_{k-1}\chi_{2^{k-1}B}(x) - c_k\chi_{2^kB}(x)\Big) \\ &= \sum_{k=1}^{\infty} b_k(x), \quad \text{where} \quad c_k = -|2^kB|^{-1}\int_{(2^kB)^c} b(x)\, dx, \qquad k = 1, 2, \dots\,. \end{aligned}$$

Here and throughout, $\rho B = B(y_0, \rho r)$ for $B = B(y_0, r)$.

We claim that

$$\sum_{k=1}^{\infty} \|b_k\|_{H^1_{\boldsymbol{\Delta}}(\mathbb{R}^d)} \leq C. \tag{3.4.7}$$

From Lemma 3.2.8 and Proposition 3.2.10 we conclude that there exists $\sigma > 0$ such that

$$\begin{aligned} |c_k| &\leq |2^kB|^{-1}\int_{(2^kB)^c} V(x)|\boldsymbol{\Delta}^{-1}a(x)|\, dx \\ &\leq C|2^kB|^{-1}\int_{(2^kB)^c}\int_B V(x)\,|x-y|^{2-d}|a(y)|\, dy\, dx \\ &\leq C|2^kB|^{-1}\int_B |a(y)|\int_{(2^kB)^c} V(x)\,|x-y_0|^{2-d}\, dx\, dy \leq C|2^kB|^{-1}(2^kr)^{-\sigma}. \end{aligned} \tag{3.4.8}$$

Note that $\operatorname{supp} b_k \subseteq 2^kB$ and $\int_{\mathbb{R}^d} b_k(x)\, dx = 0$. Therefore (3.4.7) follows, if we have verified that there exists $q > 1$ such that

$$\sum_{k=1}^{\infty} \|b_k\|_{L^q(\mathbb{R}^d)}|2^kB|^{1-1/q} \leq C, \tag{3.4.9}$$

where C does not depend on a.

If $k = 1$, then

$$|b_1(x)| \leq |c_1|\chi_{2B}(x) + |a(x)| + V(x)|\boldsymbol{\Delta}^{-1}a(x)|\chi_{2B}(x)$$

and

$$\|b_1\|_{L^q(\mathbb{R}^d)} \leq C|2B|^{-1+1/q} + \left(\int_{2B} V(x)^q|\boldsymbol{\Delta}^{-1}a(x)|^q dx\right)^{1/q}.$$

Notice that

$$\left(\int_{2B} V(x)^q|\boldsymbol{\Delta}^{-1}a(x)|^q dx\right)^{1/q} \leq Cr^2|B|^{-1}\sum_{i=1}^{m}\left(\int_{2B} V_i(x)^q dx\right)^{1/q}.$$

We can consider only the summand with V_1. By the Hölder inequality

$$\begin{aligned} r^2|B|^{-1}\left(\int_{2B} V_1(x)^q dx\right)^{1/q} &\le Cr^2|B|^{-1}r^{\widetilde{d}_1/q}\|V_1\|_{L^{qs}(\mathbb{V}_1)}r^{d_1(1-1/s)/q} \\ &= C|B|^{-1+1/q}r^{2-d_1/(sq)}. \end{aligned}$$

Choosing $q,s>1$ such that $V_1 \in L^{qs}(\mathbb{V}_1)$ and $2-d_1/(qs)<0$ we get

$$\|b_1\|_{L^q(\mathbb{R}^d)} \le C|2B|^{-1+1/q}. \tag{3.4.10}$$

For $k>1$, by the definition of b_k, we get that

$$\|b_k\|_{L^q(\mathbb{R}^d)} \le |c_{k-1}||2^{k-1}B|^{1/q} + |c_k||2^kB|^{1/q} + \|b\|_{L^q(2^kB\setminus 2^{k-1}B)}$$

From (3.4.8) we see that first two summands can be estimated by $C|2^kB|^{-1+1/q}2^{-k\sigma}$. Then it remains to deal with the last summand. By using Lemma 3.2.8 there exists $\sigma'>0$ such that for $q\in(1,1+\varepsilon]$ we have

$$\begin{aligned} \|b\|_{L^q(2^kB\setminus 2^{k-1}B)} &\le C\left(\int_{2^kB\setminus 2^{k-1}B}\left(\int_B V(x)|x-y|^{2-d}|a(y)|\,dy\right)^q dx\right)^{1/q} \\ &\le C\left(\int_{(2^{k-1}B)^c} V(x)^q|x-y_0|^{q(2-d)}dx\right)^{1/q} \le C(2^kr)^{-\sigma'} \\ &= C|2^kB|^{-1+1/q}(2^kr)^{-\sigma'+d-d/q} \le C|2^kB|^{-1+1/q}2^{-k\delta} \end{aligned} \tag{3.4.11}$$

provided that $\delta=-\sigma'+d-d/q>0$.

The estimate (3.4.9) follows from (3.4.10) and (3.4.11). This ends Case 1.

Case 2: $r<1$. Fix $N\in\mathbb{N}\cup\{0\}$ such that $1/2<2^Nr\le 1$. Then

$$\begin{aligned} b(x) =& (a(x)\omega(x)-c_0\chi_B(x)) + \sum_{l=1}^N c_0|B|\left(|2^{l-1}B|^{-1}\chi_{2^{l-1}B}(x) - |2^lB|^{-1}\chi_{2^lB}(x)\right) \\ &+ \left(b(x)-a(x)\omega(x)+c_0|B||2^NB|^{-1}\chi_{2^NB}(x)\right) = d_0(x)+\sum_{l=1}^N d_l(x)+b'(x), \end{aligned}$$

where

$$c_0 = |B|^{-1}\int_B a(x)\omega(x)\,dx.$$

By using $\int_B a=0$ and property (a) from Proposition 3.2.10, we obtain

$$|c_0| \le |B|^{-1}\int_B |a(x)||\omega(x)-\omega(y_0)|dx \le r^\delta|B|^{-1}. \tag{3.4.12}$$

Observe that $\operatorname{supp} d_0\subseteq B$, $\int_B d_0=0$, and $\|d_0\|_\infty\le C|B|^{-1}$. Similarly, for $l=1,...,N$, $\operatorname{supp} d_l\subseteq 2^lB$, $\int d_l=0$ and $\|d_l\|_\infty\le Cr^\delta|2^lB|^{-1}$. Therefore

$$\sum_{l=0}^N \|d_l\|_{H^1_\Delta(\mathbb{R}^d)} \le C+CNr^\delta \le C-Cr^\delta\log_2 r \le C.$$

Denote $B' = 2^N B$. Obviously $|B'| \sim 1$. To deal with $b'(x)$ we apply the method from Case 1 with respect to B', i.e.

$$b' = (b'(x) - c_1')\chi_{2B'}(x) + \sum_{k=2}^{\infty} \Big(b'(x)\chi_{2^kB'\setminus 2^{k-1}B'}(x) + c_{k-1}'\chi_{2^{k-1}B'}(x) - c_k'\chi_{2^kB'}(x) \Big)$$
$$= \sum b_k', \quad \text{where} \quad c_k' = |2^kB'|^{-1} \int_{2^kB'} b'(x)\, dx = -|2^kB'|^{-1} \int_{(2^kB')^c} b'(x)\, dx.$$

The arguments that we used in Case 1 also give

$$|c_k'| \le C|2^kB'|^{-1}2^{-k\sigma} \ \text{ for } k = 1, 2, \ldots \quad \text{and} \quad \sum_{k=2}^{\infty} \|b_k'\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)} \le C. \tag{3.4.13}$$

It remains to obtain that

$$\|b_1'\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)} \le C. \tag{3.4.14}$$

It is immediate that $\operatorname{supp} b_1' \subseteq 2B'$ and $\int_{2B'} b_1' = 0$. Also,

$$\|b_1'\|_{L^q(\mathbb{R}^d)} \le \left(\int_{2B'} V(x)^q |\mathbf{\Delta}^{-1}a(x)|^q \right)^{1/q} + C|c_0||B||2B'|^{-1+1/q} + C|c_1'||2B'|^{1/q}. \tag{3.4.15}$$

By (3.4.12) and (3.4.13) only the first summand needs to be estimated. Observe that

$$|\mathbf{\Delta}^{-1}a(x)| \le \int_B |x-y|^{2-d}|a(y)|\, dy \le \left\{ \begin{array}{ll} Cr^{2-d} & \text{if } |x-y_0| < 2r \\ C|x-y_0|^{2-d} & \text{if } |x-y_0| > 2r \end{array} \right\} \le C|x-y_0|^{2-d}.$$

Therefore, by using Lemma 3.2.1, we get

$$\|b_1'\|_{L^q(\mathbb{R}^d)} \le C$$

and (3.4.14) follows, which finishes off Case 2 and the proof of (3.4.2).

In order to complete the proof of Theorem 3.1.4 it remains to prove that

$$\|f\|_{H^1_{\mathbf{L},max}} \le C\|\omega f\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)}. \tag{3.4.16}$$

In virtue of Theorem 3.1.2 the inequality (3.4.16) is equivalent to

$$\left\| (\mathbf{I} - V\mathbf{L}^{-1})\,(g/\omega)\, \right\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)} \le C\|g\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)}. \tag{3.4.17}$$

Assume that a is an $H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$-atom (see (3.4.4)). Set $b = (\mathbf{I} - V\mathbf{L}^{-1})(a/\omega)$. The proof will be finished if we have obtained that

$$\|b\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)} \le C \tag{3.4.18}$$

with C independent of atom a. By (3.4.1), we have

$$\int_{\mathbb{R}^d} b(x)\, dx = \int_{\mathbb{R}^d} a(x)\, dx = 0.$$

Note that the proof of (3.4.5) only relies on estimates of $|\Gamma_{\mathbf{\Delta}}(x,y)|$ from above by $C|x-y|^{2-d}$. The same estimates hold for $\Gamma_{\mathbf{L}}(x,y)$. Moreover, the weight $1/\omega$ has the same properties as ω, that is, boundedness from above and below by positive constants and the Hölder condition. Therefore the proof of (3.4.18) follows by the same arguments. Details are omitted. □

3.5 Proof of Theorem 3.1.8

By (2.3.9) we get a formula similar to (3.3.1).

$$\mathbf{K}_t - \mathbf{P}_t(\mathbf{I} - V\mathbf{L}^{-1}) = \mathbf{Q}'_t - \mathbf{W}'_t, \tag{3.5.1}$$

where

$$\mathbf{W}'_t = \int_0^t \mathbf{P}_{t-s}\, V\, \mathbf{K}_s\, ds, \qquad \mathbf{Q}'_t = \int_0^\infty \mathbf{P}_t\, V\, \mathbf{K}_s\, ds.$$

Recall that for $j=1,...,d$ we denote by $\frac{\partial}{\partial x_j}$ the derivative in the direction of j-th standard coordinate. For $f \in L^1(\mathbb{R}^d)$ from (3.3.1) and (3.5.1) we get

$$\int_\varepsilon^{\varepsilon^{-1}} \frac{\partial}{\partial x_j}\mathbf{K}_t f \frac{dt}{\sqrt{t}} - \int_\varepsilon^{\varepsilon^{-1}} \frac{\partial}{\partial x_j}\mathbf{P}_t(\mathbf{I}-V\mathbf{L}^{-1}) f \frac{dt}{\sqrt{t}} = \mathcal{W}'_{j,\varepsilon} f + \mathcal{Q}'_{j,\varepsilon} f + \mathcal{W}_{j,\varepsilon} f + \mathcal{Q}_{j,\varepsilon} f, \tag{3.5.2}$$

$$\mathcal{Q}_{j,\varepsilon} = \int_2^{\varepsilon^{-1}} \frac{\partial}{\partial x_j}\mathbf{Q}_t \frac{dt}{\sqrt{t}}, \qquad \mathcal{Q}'_{j,\varepsilon} = \int_\varepsilon^{2} \frac{\partial}{\partial x_j}\mathbf{Q}'_t \frac{dt}{\sqrt{t}},$$

$$\mathcal{W}_{j,\varepsilon} = -\int_2^{\varepsilon^{-1}} \frac{\partial}{\partial x_j}\mathbf{W}_t \frac{dt}{\sqrt{t}}, \qquad \mathcal{W}'_{j,\varepsilon} = -\int_\varepsilon^{2} \frac{\partial}{\partial x_j}\mathbf{W}'_t \frac{dt}{\sqrt{t}}.$$

All the operators above are well-defined and bounded on $L^1(\mathbb{R}^d)$. Recall Theorem 1.1.10, which says that $\mathbf{R}_j^{\mathbf{\Delta}} f = \lim_{\varepsilon\to 0}\int_\varepsilon^{\varepsilon^{-1}} \frac{\partial}{\partial x_j}\mathbf{P}_t f \frac{dt}{\sqrt{t}} \in L^1(\mathbb{R}^d)$ for every $j=1,...,d$, exactly when $f \in H^1_{\mathbf{\Delta}}(\mathbb{R}^d)$ and

$$\|f\|_{H^1_{\mathbf{\Delta}}(\mathbb{R}^d)} \sim \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \|\mathbf{R}_j^{\mathbf{\Delta}} f\|_{L^1(\mathbb{R}^d)}. \tag{3.5.3}$$

The subsequent four lemmas prove that the operators $\mathcal{Q}_{j,\varepsilon}, \mathcal{Q}'_{j,\varepsilon}, \mathcal{W}_{j,\varepsilon}, \mathcal{W}'_{j,\varepsilon}$ converge strongly as $\varepsilon \to 0$ in the space of $L^1(\mathbb{R}^d)$-bounded operators.

Lemma 3.5.4. *For every $j=1,...,d$ the operators $\mathcal{Q}_{j,\varepsilon}$ converge as $\varepsilon \to 0$ in norm-operator topology.*

Proof. The operators $\mathcal{Q}_{j,\varepsilon}$ have the integral kernels

$$\mathcal{Q}_{j,\varepsilon}(x,y) = \int_2^{\varepsilon^{-1}} \int_t^\infty \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} P_t(x-z) V(z) K_s(z,y)\, dz\, ds\, \frac{dt}{\sqrt{t}}.$$

The lemma will be proved when we have obtained

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{Q}_j^{\langle i\rangle}(x,y)dx\le C,$$

where

$$\mathbb{Q}_j^{\langle i\rangle}(x,y)=\int_2^\infty\int_t^\infty\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}P_t(x-z)\right|V_i(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}.$$

Since $|\frac{\partial}{\partial x_j}P_t(x-z)|\le Ct^{-1/2}\phi_t(x-z)$ for some $\phi\in\mathcal{S}(\mathbb{R}^d)$ we get

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{Q}_j^{\langle i\rangle}(x,y)dx &\le C\int_{\mathbb{R}^d}\int_2^\infty\int_t^\infty\int_{\mathbb{R}^d}t^{-1/2}\phi_t(x-z)V_i(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\,dx\\ &\le C\int_2^\infty\int_t^\infty\int_{\mathbb{R}^d}t^{-1}V_i(z)K_s(z,y)\,dz\,ds\,dt\\ &\le C\int_2^\infty\int_t^\infty\int_{\mathbb{R}^d}t^{-1-\varepsilon}V_i(z)s^{-d/2+\varepsilon}\exp(-|z-y|^2/4s)\,dz\,ds\,dt\\ &\le C\left(\int_2^\infty t^{-1-\varepsilon}dt\right)\cdot\left(\int_{\mathbb{R}^d}V_i(z)|z-y|^{2-d+2\varepsilon}\,dz\right)\le C,\end{aligned}\tag{3.5.5}$$

where in the last inequality we use Lemma 3.2.1 and C does not depend on $y\in\mathbb{R}^d$. □

Lemma 3.5.6. *For every $j=1,...,d$ the operators $\mathcal{W}_{j,\varepsilon}$ converge as $\varepsilon\to0$ in norm-operator topology.*

Proof. The operators $\mathcal{W}_{j,\varepsilon}$ have the integral kernels

$$\mathcal{W}_{j,\varepsilon}(x,y)=\int_2^{\varepsilon^{-1}}\int_0^t\int_{\mathbb{R}^d}\frac{\partial}{\partial x_j}\big(P_{t-s}(x-z)-P_t(x-z)\big)V(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}.$$

Set

$$\mathbb{W}_j^{\langle i\rangle}(x,y)=\int_2^\infty\int_0^t\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}\left(P_{t-s}(x-z)-P_t(x-z)\right)\right|V_i(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}.$$

The proof will be completed when we have obtained that

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{W}_j^{\langle i\rangle}(x,y)dx\le C.\tag{3.5.7}$$

For fixed $y\in\mathbb{R}^d$ and $0<\beta<1$, β will be determined later on, we write

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{W}_j^{\langle i\rangle}(x,y)dx &\le \int\int_2^\infty\int_0^t\int\left|\frac{\partial}{\partial x_j}\left(P_{t-s}(x-z)-P_t(x-z)\right)\right|V_i(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\,dx\\ &\le\int_0^{t^\beta}...\,ds+\int_{t^\beta}^t...\,ds=J_1+J_2.\end{aligned}$$

Observe that there exists $\psi\in\mathcal{S}(\mathbb{R}^d)$, $\psi\ge0$ such that for $s\in(0,t^\beta)$ and $t>2$ we have

$$\left|\frac{\partial}{\partial x_j}\left(P_{t-s}(x)-P_t(x)\right)\right|\le st^{-3/2}\psi_t(x).$$

Thus by using Lemma 3.2.1 we get

$$\begin{aligned} J_1 &\le \int_{\mathbb{R}^d}\int_2^\infty\int_0^{t^\beta}\int_{\mathbb{R}^d} st^{-2}\psi_t(x-z)V_i(z)K_s(z,y)\,dz\,ds\,dt\,dx \\ &\le C\int_2^\infty t^{-2+\beta}\,dt\cdot\int_{\mathbb{R}^d} V_i(z)|z-y|^{2-d}dz \le C_1. \end{aligned} \tag{3.5.8}$$

Note that if $t>2$ and $s\in[t^\beta,t]$ then $K_s(z)\le Ct^{-\beta d/2}\exp(-|z|^2/ct)$. Choosing $0<\beta<1$, β close to 1, and applying Lemma 3.2.1 we obtain

$$\begin{aligned} J_2 &\le \int_{\mathbb{R}^d}\int_2^\infty\int_{t^\beta}^t\int_{\mathbb{R}^d}\left(\frac{\psi_{t-s}(x-z)}{\sqrt{t-s}}+\frac{\psi_t(x-z)}{\sqrt{t}}\right)V_i(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\,dx \\ &\le C\int_2^\infty\int_0^t\int_{\mathbb{R}^d}\left(((t-s)t)^{-1/2}+t^{-1}\right)V_i(z)t^{-\beta d/2}\exp(-|z-y|^2/ct)\,dz\,ds\,dt \\ &\le C\int_2^\infty\int_{\mathbb{R}^d}V_i(z)t^{-\beta d/2}\exp(-|z-y|^2/ct)\,dz\,dt \le C\int_{\mathbb{R}^d}V_i(z)|z-y|^{2-\beta d}dz\le C_2. \end{aligned} \tag{3.5.9}$$

Notice that the constants C_1 and C_2 in (3.5.8) and (3.5.9) do not depend on $y\in\mathbb{R}^d$. Thus (3.5.7) follows. □

Lemma 3.5.10. *For $j=1,...,d$ the operators $\mathcal{W}'_{j,\varepsilon}$ converge as $\varepsilon\to 0$ in norm-operator topology.*

Proof. The operators $\mathcal{W}'_{j,\varepsilon}$ have the integral kernel

$$\mathcal{W}'_{j,\varepsilon}(x,y)=\int_\varepsilon^2\int_0^t\int_{\mathbb{R}^d}\frac{\partial}{\partial x_j}P_{t-s}(x-z)V(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}.$$

The lemma will be proved if we have shown that

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{W}_j^{\langle i\rangle\prime}(x,y)\,dx\le C, \tag{3.5.11}$$

where

$$\mathbb{W}_j^{\langle i\rangle\prime}(x,y)=\int_0^2\int_0^t\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}P_{t-s}(x-z)\right|V_i(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}.$$

Fix $y\in\mathbb{R}^d$. Observe that

$$\begin{aligned} \int_{\mathbb{R}^d}\mathbb{W}_j^{\langle i\rangle\prime}(x,y)dx &\le \int_{\mathbb{R}^d}\int_0^2\int_0^t\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}P_{t-s}(x-z)\right|V_i(z)K_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\,dx \\ &\le \int_0^{t/2}...\,ds+\int_{t/2}^t...\,ds=J_3+J_4. \end{aligned}$$

There exists $\psi \in \mathcal{S}(\mathbb{R}^d)$, $\psi \geq 0$, such that

$$\begin{aligned} J_3 &\leq \int_{\mathbb{R}^d} \int_0^2 \int_0^{t/2} \int_{\mathbb{R}^d} (t(t-s))^{-1/2} \psi_{t-s}(x-z) V_i(z) K_s(z,y)\, dz\, ds\, dt\, dx \\ &\leq C \int_0^2 \int_0^t \int_{\mathbb{R}^d} t^{-1} V_i(z) K_s(z,y)\, dz\, ds\, dt \\ &\leq C \int_0^2 \int_{\mathbb{R}^d} t^{-1} V_i(z) |z-y|^{2-d} \exp\left(-|z-y|^2/ct\right) dz\, dt \\ &\leq C \int_{|z-y|>1/2} V_i(z)|z-y|^{2-d} dz + \int_{|z-y|\leq 1/2} V_i(z)|z-y|^{2-d} |\log|z-y|| dz \leq C_3 \end{aligned}$$

and

$$\begin{aligned} J_4 &\leq C \int_{\mathbb{R}^d} \int_0^2 \int_{t/2}^t \int_{\mathbb{R}^d} (t(t-s))^{-1/2} \psi_{t-s}(x-z) V_i(z) t^{-d/2} \exp\left(-\frac{|z-y|^2}{ct}\right) dz\, ds\, dt\, dx \\ &\leq C \int_0^2 \int_0^{t/2} \int_{\mathbb{R}^d} (ts)^{-1/2} V_i(z) t^{-d/2} \exp\left(-\frac{|z-y|^2}{ct}\right) dz\, ds\, dt \\ &\leq C \int_{\mathbb{R}^d} V_i(z) \int_0^\infty t^{-d/2} \exp\left(-\frac{|z-y|^2}{ct}\right) dt\, dz \leq C \int_{\mathbb{R}^d} V_i(z)|z-y|^{2-d} dz \leq C_4 \end{aligned}$$

with constants C_3 and C_4 independent of $y \in \mathbb{R}^d$. So we have obtained (3.5.11). □

Lemma 3.5.12. *For $j = 1, ..., d$ the operators $\mathcal{Q}'_{j,\varepsilon}$ converge strongly as $\varepsilon \to 0$.*

Proof. The kernels of $\mathcal{Q}'_{j,\varepsilon}$ are given by

$$\mathcal{Q}'_{j,\varepsilon}(x,y) = \int_\varepsilon^2 \int_0^\infty \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} P_t(x-z) V(z) K_s(z,y)\, dz\, ds\, \frac{dt}{\sqrt{t}}.$$

For $f \in L^1(\mathbb{R}^d)$ we have

$$\mathcal{Q}'_{j,\varepsilon} f(x) = \int_{\mathbb{R}^d} \mathcal{Q}'_{j,\varepsilon}(x,y) f(y)\, dy.$$

Note that $\mathcal{Q}'_{j,\varepsilon}(x,y) = \mathbf{H}_{j,\varepsilon} * \phi_y(x)$, where $\phi_y(z) = V(z)\Gamma_{\mathbf{L}}(z,y)$, $\mathbf{H}_{j,\varepsilon}(x) = \int_\varepsilon^2 \frac{\partial}{\partial x_j} P_t(x) \frac{dt}{\sqrt{t}}$.

It follows from the theory of singular integrals operators (see [37, Chapter II]) that for $g \in L^r(\mathbb{R}^d)$, $r > 1$, the limits $\lim_{\varepsilon\to 0} \mathbf{H}_{j,\varepsilon} * g(x) = \mathbf{H}_j g(x)$ exist for a.e. x and in $L^r(\mathbb{R}^d)$ norm. Obviously, $\mathbf{H}_j$ are $L^r(\mathbb{R}^d)$-bounded operators. Moreover,

$$\left\| \sup_{0<\varepsilon<2} |\mathbf{H}_{j,\varepsilon} * g| \right\|_{L^r(\mathbb{R}^d)} \leq C\|g\|_{L^r(\mathbb{R}^d)}. \tag{3.5.13}$$

Notice that for $|z| > 1/2$ we have

$$\sup_{0<\varepsilon<2} |\mathbf{H}_{j,\varepsilon}(z)| \leq C_N |z|^{-N}. \tag{3.5.14}$$

From (3.5.13) and (3.5.14) we deduce that if a is a function supported in a ball $B(y_0, R)$, $R > 1/2$, and $\|a\|_{L^r(\mathbb{R}^d)} \leq \tau|B|^{-1+1/r}$, $r > 1$, then

$$\left\| \sup_{0<\varepsilon<2} |\mathbf{H}_{j,\varepsilon} * a| \right\|_{L^1(\mathbb{R}^d)} \leq C\tau. \tag{3.5.15}$$

Using Lemma 3.2.1 we get that for every $y \in \mathbb{R}^d$ the limit $\lim_{\varepsilon\to 0} Q'_{j,\varepsilon}(x,y) = Q'_j(x,y)$ exists for a.e. $x \in \mathbb{R}^d$. The lemma will be proved by using the Lebesgue dominated convergence theorem if we have established that:

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{0<\varepsilon<2} |\mathcal{Q}'_{j,\varepsilon}(x,y)|\, dx \le C \quad \text{and} \tag{3.5.16}$$

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} |\mathcal{Q}'_{j,\varepsilon}(x,y) - \mathcal{Q}'_j(x,y)|\, dx = 0 \qquad \text{for every } y. \tag{3.5.17}$$

For fixed $y \in \mathbb{R}^d$ let

$$\phi_1(z) = \phi_y(z)\chi_{B(y,2)}(z), \qquad \phi_k(z) = \phi_y(z)\chi_{B(y,2^k)\setminus B(y,2^{k-1})}(z),\ \ k \ge 2.$$

Then $\phi_y = \sum_{k=1}^\infty \phi_k$, where the series converges in $L^1(\mathbb{R}^d)$ and $L^r(\mathbb{R}^d)$ norm for r slightly bigger than 1. Notice that $\operatorname{supp}\phi_k \subseteq B(y,2^k)$, $\|\phi_1\|_{L^r(\mathbb{R}^d)} \le C$, and

$$\begin{aligned}\|\phi_k\|^r_{L^r(\mathbb{R}^d)} &= \int_{B(y,2^k)\setminus B(y,2^{k-1})} V_1(z)^r |z-y|^{(2-d)r}\, dz \le 2^{k(2-d)r} \int_{B(y,2^k)} V_1(z)^r\, dz \\ &\le C 2^{k(2-d)r} 2^{k(d-d_1)} \|V_1\|^r_{L^{rq}(\mathbb{V}_1)} 2^{kd_1/q'} = C(2^k)^{-dr+d+2r-d_1/q}. \end{aligned} \tag{3.5.18}$$

Therefore, for $q < d_1/2r$ such that $V_1 \in L^{rq}(\mathbb{V}_1)$, we get

$$\|\phi_k\|_{L^r(\mathbb{R}^d)} \le C|B(y,2^k)|^{-1+1/r} 2^{-\sigma k}, \tag{3.5.19}$$

where $\sigma = d_1/(qr) - 2 > 0$. By using (3.5.15) combined with (3.5.19) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \sup_{0<\varepsilon<2} |\mathcal{Q}'_{j,\varepsilon}(x,y)|\, dx &= \int_{\mathbb{R}^d} \sup_{0<\varepsilon<2} |\mathbf{H}_{j,\varepsilon} * \phi_y(x)|\, dx \\ &\le \sum_{k=1}^\infty \int_{\mathbb{R}^d} \sup_{0<\varepsilon<2} |\mathbf{H}_{j,\varepsilon} * \phi_k(x)|\, dx \\ &\le C \sum_{k=1}^\infty 2^{-\sigma k} \le C, \end{aligned} \tag{3.5.20}$$

which implies (3.5.16), since the last constant C does not depend of y. Additionally (3.5.17) is a consequence of (3.5.16) and the Lebesgue dominated convergence theorem. □

Now, Theorem 3.1.8 follows directly by applying (3.5.2), (3.5.3), and Theorem 3.1.2. Note that the existence of the limits (3.1.7) has been shown parallel.

3.6 Appendix

3.6.1 Proof of Corollary 3.2.6

As it was mentioned after Corollary 3.2.6, the aim of this subsection is to prove the formula (3.2.7). Recall that $d \ge 3$ and that $\mathbf{L}^{-1}$ and $\mathbf{\Delta}^{-1}$ were initially given by the integral kernels

$\Gamma_{\mathbf{L}}$ and $\Gamma_{\mathbf{\Delta}}$ respectively (see Section 3.1). Lemma 3.2.1, in its simple form, asserts that $V\mathbf{L}^{-1}$ and $V\mathbf{\Delta}^{-1}$ are bounded on $L^1(\mathbb{R}^d)$, since

$$\int_{\mathbb{R}^d} V(z)\Gamma_{\mathbf{L}}^{-1}(z,y)\,dz \leq \int_{\mathbb{R}^d} V(z)|\Gamma_{\mathbf{\Delta}}^{-1}(z,y)|\,dz = C\int_{\mathbb{R}^d} V(z)|z-y|^{2-d}\,dz \leq C'$$

independently of $y \in \mathbb{R}^d$. This estimate justify many of the following calculations and will be used without referring to it. Notice that for $\psi \in C_c^\infty(\mathbb{R}^d)$, $f \in L^1(\mathbb{R}^d)$ we have

$$|L^{-1}\psi(x)| + |\Delta^{-1}\psi(x)| \leq C(1+|x|)^{2-d} \qquad \Delta^{-1}f, L^{-1}f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$$

Since in (3.2.7) we only deal with $L^1(\mathbb{R}^d)$-bounded operators, we can only consider a dense subspace of $L^1(\mathbb{R}^d)$, namely we shall prove (3.2.7) for $f \in C_c^\infty(\mathbb{R}^d)$.

Recall that, for given semigroup $\mathbf{T}_t$ on $L^1(\mathbb{R}^d)$, the domain of its infinitesimal generator $\mathbf{A}$ ($\mathbf{A} = \mathbf{\Delta}$ or $\mathbf{A} = -\mathbf{L}$) is given by

$$\mathrm{Dom}(\mathbf{A}) = \{f \in L^1(\mathbb{R}^d) \mid \mathbf{A}f = \lim_{t\to 0} t^{-1}(\mathbf{T}_t f - f) \text{ exists in the } L^1(\mathbb{R}^d)\text{-norm}\}. \quad (3.6.1)$$

Let us distinguish the infinitesimal generators $-\mathbf{L}$ and $\mathbf{\Delta}$ from the differential operators $-\mathcal{L}$ and Δ, where

$$-\mathcal{L}f = \Delta f - V\cdot f, \qquad \Delta f = \frac{\partial^2}{\partial x_1^2}f + \ldots + \frac{\partial^2}{\partial x_d^2}f \quad (3.6.2)$$

for $f \in C^2(\mathbb{R}^d)$.

It is well known that $C_c^\infty(\mathbb{R}^d) \subseteq \mathrm{Dom}(\mathbf{\Delta})$ and for $\psi \in C_c^\infty(\mathbb{R}^d)$ we have $\mathbf{\Delta}\psi = \Delta\psi$. Now we check that the same holds for $\mathbf{L}$.

Lemma 3.6.3. *The class $C_c^\infty(\mathbb{R}^d)$ is contained in* $\mathrm{Dom}(-\mathbf{L})$ *and for $\psi \in C_c^\infty(\mathbb{R}^d)$ we have*

$$\mathbf{L}\psi = \mathcal{L}\psi \quad (3.6.4)$$

Proof. Denote: $V_n = \min(V,n)$, $\mathbf{L}_n = -\mathbf{\Delta} + V_n$, $\mathbf{K}_t^n$ - the semigroup generated by $-\mathbf{L}_n$ on $L^1(\mathbb{R}^d)$. Since V_n is bounded, $-\mathbf{L}_n$ is a perturbation of $\mathbf{\Delta}$ by the operator bounded on $L^1(\mathbb{R}^d)$. Thus $\mathrm{Dom}(-\mathbf{L}_n) = \mathrm{Dom}(\mathbf{\Delta}) \supseteq C_c^\infty(\mathbb{R}^d)$ and for $\phi \in C_c^\infty(\mathbb{R}^d)$ we have $\mathbf{L}_n\psi = \mathcal{L}_n\psi = -\Delta\psi + V\psi$ (see, e.g. [34, Chapter 3]). For $f \in L^1(\mathbb{R}^d)$ and $\psi \in C_c^\infty(\mathbb{R}^d)$ we obtain the following convergences in $L^1(\mathbb{R}^d)$:

$$\lim_{n\to\infty} \mathbf{K}_t^n f = \mathbf{K}_t f, \qquad \lim_{t\to 0} t^{-1}(\mathbf{K}_t^n\psi - \psi) = -\mathbf{L}_n\psi, \quad (3.6.5)$$

where the first limit follows from (2.1.3).

Next,

$$\lim_{n\to\infty} \mathbf{L}_n\psi = \mathcal{L}\psi \tag{3.6.6}$$

in $L^1(\mathbb{R}^d)$, which we have got from the Lebesgue dominated convergence theorem since

$$\int_{L^1(\mathbb{R}^d)} |(L_n - \mathcal{L})\psi| \leq C \int_{\operatorname{supp}\psi} \chi_{\{V(x)>n\}}(x)V(x)\,dx \to 0,$$

as $n \to \infty$. Let us consider

$$t^{-1}(\psi - \mathbf{K}_t^n\psi) = \mathbf{G}_t^n = t^{-1}\int_0^t \mathbf{K}_s^n\mathbf{L}_n\psi\,ds. \tag{3.6.7}$$

Considering the left-hand side of (3.6.7) we obtain the following $L^1(\mathbb{R}^d)$-convergence.

$$\lim_{n\to\infty} \mathbf{G}_t^n = t^{-1}(\psi - \mathbf{K}_t\psi) \tag{3.6.8}$$

By using the right-hand side of (3.6.7) we write

$$\mathbf{G}_t^n = t^{-1}\left(\int_0^t \mathbf{K}_s^n(\mathbf{L}_n - \mathcal{L})\psi\,ds + \int_0^t (\mathbf{K}_s^n - \mathbf{K}_s)\mathcal{L}\psi,ds + \int_0^t \mathbf{K}_s\mathcal{L}\psi\,ds\right). \tag{3.6.9}$$

Since $\mathbf{K}_s^n$ are contractions on $L^1(\mathbb{R}^d)$, by (3.6.6) we see that the first summand tends to 0 in $L^1(\mathbb{R}^d)$ as $n \to \infty$. We claim that the same holds for the second summand. To see this we first note that

$$\left\|\int_0^t (\mathbf{K}_s^n - \mathbf{K}_s)\mathcal{L}\psi\,ds\right\|_{L^1(\mathbb{R}^d)} \leq \int_0^t \|(\mathbf{K}_s^n - \mathbf{K}_s)\mathcal{L}\psi\|_{L^1(\mathbb{R}^d)}\,ds$$

and

$$\|(\mathbf{K}_s^n - \mathbf{K}_s)\mathcal{L}\psi\|_{L^1(\mathbb{R}^d)} \leq 2\|\mathbf{P}_s(|\mathcal{L}\psi|)\|_{L^1(\mathbb{R}^d)}. \tag{3.6.10}$$

Now, applying (3.6.5) and the Lebesgue dominated convergence theorem we obtain the claim. Summarizing,

$$\lim_{n\to\infty} \mathbf{G}_t^n = t^{-1}\int_0^t \mathbf{K}_s\mathcal{L}\psi\,ds \tag{3.6.11}$$

in $L^1(\mathbb{R}^d)$. Joining together (3.6.8) and (3.6.11), and using the strong continuity of the semigroup $\{\mathbf{K}_t\}_{t>0}$, we have

$$\lim_{t\to 0} t^{-1}(\mathbf{K}_t\psi - \psi) = -\lim_{t\to 0} t^{-1}\int_0^t \mathbf{K}_s\mathcal{L}\psi\,ds = -\mathcal{L}\psi, \tag{3.6.12}$$

which finishes the proof of the lemma. □

Lemma 3.6.13. *Let* $\mathbf{A}$ *denotes either* $-\mathbf{L}$ *or* $\mathbf{\Delta}$. *For* $f \in L^1(\mathbb{R}^d)$, $\psi \in C_c^\infty(\mathbb{R}^d)$ *we have:*

$$\langle \mathbf{A}^{-1}f, \mathbf{A}\psi\rangle = \langle f, \mathbf{A}^{-1}\mathbf{A}\psi\rangle = \langle f, \psi\rangle \tag{3.6.14}$$

$$\Delta\mathbf{\Delta}^{-1}\psi(x) = \mathbf{\Delta}^{-1}\Delta\psi(x) = \psi(x) \qquad \text{for } x \in \mathbb{R}^d, \tag{3.6.15}$$

$$\mathbf{L}^{-1}\mathcal{L}\psi = \psi \qquad \text{for a.e. } x \in \mathbb{R}^d \tag{3.6.16}$$

Proof. Denote by $\mathbf{T}_t$ the semigroup generated by $\mathbf{A}$. We already know that $C_c^\infty(\mathbb{R}^d) \subseteq \mathrm{Dom}(\mathbf{A})$. Let us consider

$$\langle \mathbf{T}_\varepsilon f - \mathbf{T}_{\varepsilon^{-1}} f, \psi\rangle = \langle f, \mathbf{T}_\varepsilon \psi - \mathbf{T}_{\varepsilon^{-1}}\psi\rangle = -\langle f, \int_\varepsilon^{\varepsilon^{-1}} \mathbf{T}_t \mathbf{A}\psi \, dt\rangle. \tag{3.6.17}$$

Passing to the limit with $\varepsilon \to 0$ we prove (3.6.14). Here we have used that

$$\sup_{y\in\mathbb{R}^d} \int_{L^1(\mathbb{R}^d)} (|\Delta\psi(x)| + V(x)|\psi(x)|)|x-y|^{2-d}\, dx < \infty. \tag{3.6.18}$$

The formula (3.6.16) is a direct consequence of (3.6.14). Also, (3.6.15) holds, due to the fact that $\psi, \mathbf{\Delta}^{-1}\Delta\psi, \Delta\mathbf{\Delta}^{-1}\psi$ are continuous (or even smooth) and taking $f = \phi \in C_c^\infty(\mathbb{R}^d)$ in (3.6.14) we have $\langle \mathbf{\Delta}^{-1}\psi, \Delta\phi\rangle = \langle\psi,\phi\rangle$. □

Fix $\xi_n(x) = \xi(x/n)$, where $\xi \in C_c^\infty(B(0,2))$, $0 \le \xi \le 1$, $\xi(x) = 1$ for $x \in B(0,1)$. Then $\|\nabla\xi_n\|_\infty \le C/n$ and $\|\Delta\xi_n\|_\infty \le C/n^2$. Fix also $\delta \in C_c^\infty(B(0,1))$ such that $\int\delta = 1$, $\delta \ge 0$, $\delta(x) = \delta(-x)$, and denote $\delta_t(x) = t^{-d}\delta(x/t)$.

The following obvious estimates are needed in the proceeding.

Lemma 3.6.19. *For $\phi,\psi \in C_c^\infty(\mathbb{R}^d)$ we have:*

(a) $|\nabla\cdot\mathbf{\Delta}^{-1}\psi(x)| \le C(1+|x|)^{1-d}$,

(b) $|\mathbf{\Delta}^{-1}\psi(x)| + |\mathbf{L}^{-1}\psi| \le C(1+|x|)^{2-d}$,

(c) $\sup_n |\nabla\xi_n(x)| \le C(1+|x|)^{-1}$,

(d) $\sup_n |\Delta\xi_n(x)| \le C(1+|x|)^{-2}$,

(e) $|\Delta(\delta_t * (\mathbf{\Delta}^{-1}\mathcal{L}\phi))(x)| = |\delta_t * \mathcal{L}\phi(x)| \le C_t(1+|x|)^{-d}$,

(f) $|\nabla(\delta_t * (\mathbf{\Delta}^{-1}\mathcal{L}\phi))(x)| \le C_t(1+|x|)^{1-d}$,

(g) $|\delta_t * (\mathbf{\Delta}^{-1}\mathcal{L}\phi)(x)| \le C_t(1+|x|)^{2-d}$.

Corollary 3.6.20. *For $\psi,\phi \in C_c^\infty(\mathbb{R}^d)$ it holds:*

$$\langle \mathcal{L}\mathbf{\Delta}^{-1}\psi, \mathbf{L}^{-1}\Delta\phi\rangle = \langle\psi,\phi\rangle, \tag{3.6.21}$$

$$\langle \mathbf{L}^{-1}\psi, V\mathbf{\Delta}^{-1}\mathcal{L}\phi\rangle = \langle\psi,\phi\rangle + \langle\psi, \mathbf{\Delta}^{-1}\mathcal{L}\phi\rangle. \tag{3.6.22}$$

Proof. Let us consider $W_1 = \langle \mathcal{L}(\xi_n\mathbf{\Delta}^{-1}\psi), \mathbf{L}^{-1}\Delta\phi\rangle$. By using Lemma 3.6.13 twice we obtain

$$\lim_{n\to\infty} W_1 = \lim_{n\to\infty} \langle \mathcal{L}(\xi_n\mathbf{\Delta}^{-1}\psi), \mathbf{L}^{-1}\Delta\phi\rangle = \lim_{n\to\infty}\langle \xi_n\mathbf{\Delta}^{-1}\psi, \Delta\phi\rangle = \langle\mathbf{\Delta}^{-1}\psi,\Delta\phi\rangle = \langle\psi,\phi\rangle. \tag{3.6.23}$$

On the other hand

$$\begin{aligned} W_1 &= \langle \mathcal{L}(\xi_n \mathbf{\Delta}^{-1}\psi), \mathbf{L}^{-1}\Delta\phi\rangle \\ &= \langle \xi_n \mathcal{L}\mathbf{\Delta}^{-1}\psi, \mathbf{L}^{-1}\Delta\phi\rangle - \langle 2\nabla\xi_n \cdot \nabla(\mathbf{\Delta}^{-1}\psi), \mathbf{L}^{-1}\Delta\phi\rangle - \langle (\Delta\xi_n)(\mathbf{\Delta}^{-1}\psi), \mathbf{L}^{-1}\Delta\phi\rangle. \end{aligned} \tag{3.6.24}$$

The first summand tends to $\langle \mathcal{L}\mathbf{\Delta}^{-1}\psi, \mathbf{L}^{-1}\Delta\phi\rangle$ as $n\to\infty$, because $\mathcal{L}\mathbf{\Delta}^{-1}\psi = -\psi + V\mathbf{\Delta}^{-1}\psi \in L^1(\mathbb{R}^d)$ and $\mathbf{L}^{-1}\Delta\phi \in L^\infty(\mathbb{R}^d)$. The second and third summands tend to 0 as $n\to\infty$ by the Lebesgue dominated convergence theorem, i.e. the integrants tend pointwise to 0 and there are appropriate estimates (see $(a)-(d)$ from Lemma 3.6.19). In this way, the formula (3.6.21) is proved.

Now, we turn to (3.6.22). Consider

$$\begin{aligned} W_2 =& \langle \mathbf{L}^{-1}\psi, \Delta(\xi_n\cdot(\delta_t * \mathbf{\Delta}^{-1}\mathcal{L}\phi))\rangle \\ =& \langle \mathbf{L}^{-1}\psi, \xi_n\cdot\Delta(\delta_t * \mathbf{\Delta}^{-1}\mathcal{L}\phi)\rangle + \langle \mathbf{L}^{-1}\psi, 2\nabla\xi_n\cdot\nabla(\delta_t * \mathbf{\Delta}^{-1}\mathcal{L}\phi)\rangle \\ &+ \langle \mathbf{L}^{-1}\psi, \Delta\xi_n\cdot(\delta_t * \mathbf{\Delta}^{-1}\mathcal{L}\phi)\rangle, \end{aligned} \tag{3.6.25}$$

Fix $t>0$. Thanks to $(b)-(g)$ of Lemma 3.6.19, the second and third summands tend to 0 as $n\to\infty$. Note that the function $\mathcal{L}\phi = -\Delta\phi + V\phi$ belongs to $L^1(\mathbb{R}^d)$ and has a compact support. Thus from (3.6.14) we obtain that $\Delta(\delta_t * \mathbf{\Delta}^{-1}\mathcal{L}\phi) = \delta_t * \mathcal{L}\phi$. It follows that

$$\begin{aligned} \lim_{t\to0}\lim_{n\to\infty} W_2 &= \lim_{t\to0}\lim_{n\to\infty}\langle \mathbf{L}^{-1}\psi, \xi_n\cdot(\delta_t*\mathcal{L}\phi)\rangle = \lim_{t\to0}\langle \mathbf{L}^{-1}\psi, \delta_t*\mathcal{L}\phi\rangle \\ &= \langle \mathbf{L}^{-1}\psi, \mathcal{L}\phi\rangle = \langle\psi,\phi\rangle, \end{aligned} \tag{3.6.26}$$

where in the last equality we have used (3.6.16).

Let us focus attention on the expression

$$W_3 = \langle \mathbf{L}^{-1}\psi, \mathcal{L}(\xi_n\cdot(\delta_t*\mathbf{\Delta}^{-1}\mathcal{L}\phi))\rangle = \langle \psi, \xi_n\cdot(\delta_t*\mathbf{\Delta}^{-1}\mathcal{L}\phi)\rangle.$$

It is easily seen that

$$\lim_{t\to0}\lim_{n\to\infty} W_3 = \lim_{t\to0}\langle\psi, \delta_t*\mathbf{\Delta}^{-1}\mathcal{L}\phi\rangle = \lim_{t\to0}\langle \delta_t*\psi, \mathbf{\Delta}^{-1}\mathcal{L}\phi\rangle = \langle\psi, \mathbf{\Delta}^{-1}\mathcal{L}\phi\rangle. \tag{3.6.27}$$

Moreover,

$$\begin{aligned} \lim_{t\to0}\lim_{n\to\infty}(W_2+W_3) &= \lim_{t\to0}\lim_{n\to\infty}\langle V\mathbf{L}^{-1}\psi, \xi_n(\delta_t*(\mathbf{\Delta}^{-1}\mathcal{L}\phi))\rangle = \lim_{t\to0}\langle V\mathbf{L}^{-1}\psi, \delta_t*(\mathbf{\Delta}^{-1}\mathcal{L}\phi)\rangle \\ &= \lim_{t\to0}\langle\widetilde{\delta_t}*V\mathbf{L}^{-1}\psi, \mathbf{\Delta}^{-1}\mathcal{L}\phi\rangle = \langle V\mathbf{L}^{-1}\psi, \mathbf{\Delta}^{-1}\mathcal{L}\phi\rangle. \end{aligned} \tag{3.6.28}$$

Finally, (3.6.22) is a direct consequence of (3.6.26), (3.6.27), and (3.6.28). □

Proof of Corollary 3.2.6. It is enough to see that for $\psi, \phi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\langle (\mathbf{I} - V\mathbf{L}^{-1})(\mathbf{I} - V\mathbf{\Delta}^{-1})\psi, \phi\rangle = \langle \psi, \phi\rangle, \tag{3.6.29}$$

$$\langle (\mathbf{I} - V\mathbf{\Delta}^{-1})(\mathbf{I} - V\mathbf{L}^{-1})\psi, \phi\rangle = \langle \psi, \phi\rangle, \tag{3.6.30}$$

as it was mentioned at the beginning of Subsection 3.6.1.

Let $g = (\mathbf{I} - V\mathbf{\Delta}^{-1})\psi \in L^1(\mathbb{R}^d)$. From (3.6.14) we obtain

$$\begin{aligned}\langle (\mathbf{I} - V\mathbf{L}^{-1})g, \phi\rangle &= \langle g, \phi\rangle - \langle g, \mathbf{L}^{-1}(\Delta + \mathcal{L})\phi\rangle \\ &= \langle g, \phi\rangle - \langle g, \mathbf{L}^{-1}\Delta\phi\rangle - \langle g, \mathbf{L}^{-1}\mathcal{L}\phi\rangle = -\langle g, \mathbf{L}^{-1}\Delta\phi\rangle.\end{aligned}$$

Further,

$$\begin{aligned}\langle g, \mathbf{L}^{-1}\Delta\phi\rangle &= \langle (\mathbf{I} - V\mathbf{\Delta}^{-1})\psi, \mathbf{L}^{-1}\Delta\phi\rangle \\ &= \langle \psi, \mathbf{L}^{-1}\Delta\phi\rangle - \langle (\mathcal{L} + \Delta)\mathbf{\Delta}^{-1}\psi, \mathbf{L}^{-1}\Delta\phi\rangle \\ &= \langle \psi, \mathbf{L}^{-1}\Delta\phi\rangle - \langle \mathcal{L}\mathbf{\Delta}^{-1}\psi, \mathbf{L}^{-1}\Delta\phi\rangle - \langle \Delta\mathbf{\Delta}^{-1}\psi, \mathbf{L}^{-1}\Delta\phi\rangle \\ &= -\langle \mathcal{L}\mathbf{\Delta}^{-1}\psi, \mathbf{L}^{-1}\Delta\phi\rangle = -\langle \psi, \phi\rangle,\end{aligned}$$

where the last inequality follows from (3.6.21). The proof of (3.6.29) is finished.

We turn to prove (3.6.30). Using (3.6.15) we see that

$$\begin{aligned}\langle (\mathbf{I} - V\mathbf{\Delta}^{-1})(\mathbf{I} - V\mathbf{L}^{-1})\psi, \phi\rangle &= \langle (\mathbf{I} - V\mathbf{L}^{-1})\psi, (\mathbf{I} - \mathbf{\Delta}^{-1}V)\phi\rangle \\ &= \langle \psi - V\mathbf{L}^{-1}\psi, \phi - \mathbf{\Delta}^{-1}(\Delta + \mathcal{L})\phi\rangle \\ &= -\langle \psi, \mathbf{\Delta}^{-1}\mathcal{L}\phi\rangle + \langle \mathbf{L}^{-1}\psi, V\mathbf{\Delta}^{-1}\mathcal{L}\phi\rangle.\end{aligned}$$

In view of (3.6.22) the proof is finished. □

4. CHAPTER 4: HARDY SPACES RELATED TO THE LAGUERRE OPERATOR

4.1 Background and main result

In this chapter X denotes the half-line $(0,\infty)$ with the measure $d\mu(x) = x^\alpha dx$, where $\alpha > 0$ is fixed. The space X equipped with the Euclidean distance $d(x,y) = |x-y|$ is a space of homogeneous type in the sense of Coifman-Weiss [8], namely it satisfies the doubling condition, i.e. there exists $C > 0$ such that for all $x \in X$ and $r > 0$ we have that

$$\mu(B(x,2r)) \leq C\mu(B(x,r)). \tag{4.1.1}$$

On $L^2(X)$ we consider the orthogonal system of the Laguerre functions $\{\psi_k^{(\alpha-1)/2}(x)\}_{k=0}^\infty$,

$$\psi_k^{(\alpha-1)/2}(x) = \left(\frac{2k!}{\Gamma(k+\alpha/2+1/2)}\right)^{1/2} L_k^{(\alpha-1)/2}(x^2)e^{-x^2/2}, \tag{4.1.2}$$

where L_k^α is the k-th Laguerre polynomial (see [28, p.76]) given by

$$L_k^\alpha(x) = e^x \frac{x^{-\alpha}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}), \qquad n = 0,1,\dots\ .$$

Each $\psi_k^{(\alpha-1)/2}$ is an eigenfunction of the Laguerre operator

$$\mathbf{L}f(x) = -\frac{d^2}{dx^2}f(x) - \frac{\alpha}{x}\frac{d}{dx}f(x) + x^2 f(x),$$

where the corresponding eigenvalue is $\beta_k = 4k+\alpha+1$. Let

$$\mathbf{K}_t f = \sum_{k=0}^\infty \exp(-t\beta_k)\langle f, \psi_k^{(\alpha-1)/2}\rangle \psi_k^{(\alpha-1)/2}$$

be the semigroup of the self-adjoint linear operators on $L^2(X)$ generated by $-\mathbf{L}$, where $\mathrm{Dom}(-\mathbf{L}) = \{f \in L^2(X) : \sum_k \beta_k^2 |\langle f, \psi_k^{(\alpha-1)/2}\rangle|^2 < \infty\}$ is the domain of $-\mathbf{L}$.

It is well known (see, e.g. [28], [33]) that $\mathbf{K}_t$ has the integral representation, i.e.

$$\mathbf{K}_t f(x) = \int_0^\infty K_t(x,y) f(y) d\mu(y), \tag{4.1.3}$$

where

$$K_t(x,y) = \frac{2e^{-2t}(xy)^{-(\alpha-1)/2}}{1-e^{-4t}} \exp\left(-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}(x^2+y^2)\right) I_{(\alpha-1)/2}\left(\frac{2e^{-2t}}{1-e^{-4t}}xy\right). \tag{4.1.4}$$

Here I_ν denotes the Bessel function of the second kind. The operators (4.1.3) define strongly continuous semigroups of contractions on every $L^p(X)$, $1 \le p < \infty$.

Through this chapter we shall use the following notation: for an interval $I \subseteq (0,\infty)$ we denote by $|I|$ its Euclidean diameter, $B(x,r) = \{y \in X : |x-y| < r\}$, and χ_A is the characteristic function of the set A. We define the auxiliary function

$$\rho(y) = \chi_{(0,1)}(y) + \frac{1}{y}\chi_{[1,\infty)}(y). \tag{4.1.5}$$

Definition 4.1.6. *A function a is called an $\mathbf{L}-$atom if there exists an interval $I = B(y_0, r) \subseteq (0,\infty)$ such that:*

- $\mathrm{supp}(a) \subseteq I$ *and* $r \le \rho(y_0)$,
- $\|a\|_\infty \le \mu(I)^{-1}$,
- *if $r \le \rho(y_0)/4$, then $\int_0^\infty a(x)d\mu(x) = 0$.*

We define the space $H^1_{\mathbf{L},at}$ and the corresponding norm as it was described in Definition 1.2.5.

Let $\delta = \frac{d}{dx} + x$, $\delta^* = -\frac{d}{dx} + x - \frac{\alpha}{x}$. Then $\mathbf{L} = (\alpha+1)\mathbf{I} + \delta^*\delta$, $\delta\psi_k^{(\alpha-1)/2} = -2\sqrt{k}x\psi_{k-1}^{(\alpha+1)/2}$. The Riesz transform $\widehat{\mathbf{R}}^{\mathbf{L}}$, originally defined on $L^2(X)$ (see, e.g. [32], [33]) by the formula

$$\widehat{\mathbf{R}}^{\mathbf{L}} f = \sqrt{\pi}\delta \mathbf{L}^{-1/2} f = -\sum_{k=1}^{\infty} \Big(\frac{4k\pi}{4k+\alpha+1}\Big)^{1/2} \langle f, \psi_k^{(\alpha-1)/2}\rangle x\psi_{k-1}^{(\alpha+1)/2},$$

turns out to be the principal value singular integral operator

$$\widehat{\mathbf{R}}^{\mathbf{L}} f(x) = \lim_{\varepsilon\to 0} \int_{0,\,|x-y|>\varepsilon}^{\infty} \widehat{R}^{\mathbf{L}}(x,y) f(y) d\mu(y),$$

with the kernel

$$\widehat{R}^{\mathbf{L}}(x,y) = \int_0^\infty \Big(\frac{\partial}{\partial x} + x\Big) K_t(x,y) \frac{dt}{\sqrt{t}}.$$

Since the kernel

$$\Gamma(x,y) = \int_0^\infty x K_t(x,y) \frac{dt}{\sqrt{t}}$$

satisfies $\sup_{y>0} \int |\Gamma(x,y)| d\mu(x) < \infty$, it defines a bounded linear operator on $L^1(X)$. Hence, for our purposes, we restrict our consideration to the Riesz transform $\mathbf{R}^{\mathbf{L}} f = \sqrt{\pi}\frac{d}{dx}\mathbf{L}^{-1/2} f$. Clearly, $\mathbf{R}^{\mathbf{L}}$ is a principal value singular integral operator with the kernel

$$R^{\mathbf{L}}(x,y) = \int_0^\infty \frac{\partial}{\partial x} K_t(x,y) \frac{dt}{\sqrt{t}}. \tag{4.1.7}$$

The action of $\mathbf{R}^{\mathbf{L}}$ on $L^1(X)$-functions is well-defined in the sense of distributions (see Section 4.3 for details).

The main goal of the chapter is to prove the following theorem (see [35]).

Theorem 4.1.8. *A function $f \in L^1(X)$ belongs to the Hardy space $H^1_{\mathbf{L},at}$ if and only if $\mathbf{R}^{\mathbf{L}} f$ belongs to $L^1(X)$. Moreover, the corresponding norms are equivalent, i.e.*

$$C^{-1}\|f\|_{H^1_{\mathbf{L},at}} \leq \|f\|_{L^1(X)} + \|\mathbf{R}^{\mathbf{L}} f\|_{L^1(X)} \leq C\|f\|_{H^1_{\mathbf{L},at}}. \tag{4.1.9}$$

The main idea of the proof is to compare the kernel $R^{\mathbf{L}}(x,y)$ with kernels of appropriately scaled local Riesz transforms related to the Bessel operator $\widetilde{\mathbf{L}}f(x) = -f''(x) - \frac{\alpha}{x}f'(x)$, where the scale of localization is adapted to the auxiliary function $\rho(y)$. To do this we consider the Bessel semigroup:

$$\begin{aligned} \widetilde{\mathbf{K}}_t f(x) &= \int_0^\infty \widetilde{K}_t(x,y) f(y) d\mu(y), \\ \widetilde{K}_t(x,y) &= (2t)^{-1} \exp\left(-\frac{x^2+y^2}{4t}\right) I_{(\alpha-1)/2}\left(\frac{xy}{2t}\right)(xy)^{-(\alpha-1)/2} \end{aligned} \tag{4.1.10}$$

and observe that for small t the kernel (4.1.10) is close to the kernel (4.1.4). Thanks to this, $R^{\mathbf{L}}(x,y)$ is comparable to $R^{\widetilde{\mathbf{L}}}(x,y)$ after some suitable localization defined by the function ρ, where $R^{\widetilde{\mathbf{L}}}(x,y)$ denotes the Riesz transform kernel in the Bessel setting. This requires a precise computation of constants appearing in singular parts of the kernels (see Propositions 4.2.4 and 4.3.1). The next step is to use results of Betancor, Dziubański and Torrea [5], which give characterizations of *a global Hardy space* for the Bessel operator (see Theorem 4.2.2), to define and describe *local Hardy spaces* for $\widetilde{\mathbf{L}}$. Having all these prepared we prove the theorem.

We would like to remark that the Hardy space $H^1_{\mathbf{L},at}$ we consider here is also characterized by means of the maximal function:

$$\mathbf{M}_{\mathbf{L}} f(x) = \sup_{t>0} |\mathbf{K}_t f(x)|,$$

that is, $\|f\|_{H^1_{\mathbf{L},at}}$ is comparable with $\|\mathbf{M}_{\mathbf{L}} f\|_{L^1(X)}$. For details concerning the maximal function characterization of the space $H^1_{\mathbf{L},at}$ we refer the reader to [12].

There are other expansions based on the Laguerre functions for which Hardy spaces were investigated. For example, when $\alpha > -1$ systems $\{\varphi_k^\alpha\}_{k=0}^\infty$ and $\{\mathcal{L}_k^\alpha\}_{k=0}^\infty$, where

$$\varphi_k^\alpha(x) = c_{k,\alpha} e^{-x^2/2} x^{\alpha+1/2} L_k^\alpha(x^2), \qquad \mathcal{L}_k^\alpha(x) = c_{k,\alpha} e^{-x/2} x^{\alpha/2} L_k^\alpha(x),$$

are orthogonal on $L^2((0,\infty), dx)$. These systems are related to the operators

$$\widehat{L}_\alpha = -\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}\left(\alpha^2 - \frac{1}{4}\right), \qquad \overline{L}_\alpha = -x\frac{d^2}{dx^2} - \frac{d}{dx} + \frac{x}{4} + \frac{\alpha^2}{4x},$$

respectively. In [4] and [11] the authors proved that the Hardy spaces associated with $\{\varphi_k^\alpha\}_{k=0}^\infty$ and $\{\mathcal{L}_k^\alpha\}_{k=0}^\infty$ are characterized by: the maximal functions, the Riesz transforms, and certain atomic decompositions. Moreover, in [13] the author obtained an atomic description of the Hardy space originally defined by the maximal function related to the system

$$\ell_k^\alpha(x) = c_{k,\alpha} L_k^\alpha(x) e^{-x/2}, \quad k = 0, 1, ..., \quad \text{on} \quad L^2((0,\infty), x^\alpha dx).$$

The functions ℓ_k^α are eigenfunctions of the operator

$$\mathbb{L}_\alpha = -x\frac{d^2}{dx^2} - (\alpha+1)\frac{d}{dx} + \frac{x}{4}.$$

Finally, we would like to note that the system $\{\psi_k^{(\alpha-1)/2}\}_{k=0}^\infty$ we consider in the present chapter is well-defined and orthogonal on $L^2(X)$ for $\alpha > -1$. However, the case $-1 < \alpha \leq 0$ is not included in our investigations.

The chapter is organized as follows. In Section 4.2 we present a singular integral characterization of local Hardy spaces associated with the Bessel operator $\widetilde{\mathbf{L}}$. Section 4.3 is devoted to stating detailed estimates for $R^{\mathbf{L}}(x,y)$ and proving some auxiliary results. The proof of Theorem 4.1.8 is given in Section 4.4. In Section 4.5 we provide proofs of estimates of the kernels $R^{\widetilde{\mathbf{L}}}(x,y)$ and $R^{\mathbf{L}}(x,y)$ stated in Propositions 4.2.4 and 4.3.1.

4.2 Hardy spaces in the Bessel setting

4.2.1 Global Hardy space

The Hardy spaces $H^1_{\widetilde{\mathbf{L}}}$ related to the Bessel operators $\widetilde{\mathbf{L}}$ were studied in [5].

Definition 4.2.1. *We call a function a an $\widetilde{\mathbf{L}}$-atom if there is an interval $I \subset (0,\infty)$ such that:*

- $\mathrm{supp}(a) \subseteq I$,
- $\|a\|_\infty \leq \mu(I)^{-1}$,
- $\int_0^\infty a(x)d\mu(x) = 0$.

Using $\widetilde{\mathbf{L}}$-atoms we define the space $H^1_{\widetilde{\mathbf{L}},at}$ (see Definition 1.2.5).

The singular integral kernel of the Riesz transform $\mathbf{R}^{\widetilde{\mathbf{L}}}$ is defined by

$$R^{\widetilde{\mathbf{L}}}(x,y) = \int_0^\infty \frac{\partial}{\partial x}\widetilde{K}_t(x,y)\frac{dt}{\sqrt{t}}, \qquad \text{where} \quad x \neq y.$$

Before giving a distributional sense of $\mathbf{R}^{\widetilde{\mathbf{L}}} f$ for $f \in L^1(X)$ we recall results from [5].

Theorem 4.2.2. *For $f \in L^1(X)$ the following conditions are equivalent:*

(i) $f \in \widetilde{H}^1_{at}(X)$,

(ii) $\mathbf{R}^{\widetilde{\mathbf{L}}} f \in L^1(X)$,

(iii) $\sup_{t>0} |\widetilde{\mathbf{K}}_t f| \in L^1(X)$.

Moreover,

$$\|f\|_{H^1_{\widetilde{\mathbf{L}},at}} \sim \left(\|f\|_{L^1(X)} + \|\mathbf{R}^{\widetilde{\mathbf{L}}} f\|_{L^1(X)}\right) \sim \left\| \sup_{t>0} |\widetilde{\mathbf{K}}_t f| \right\|_{L^1(X)}.$$

In the present chapter we use the following notion of dilations: for a function f defined on $(0, \infty)$ and $y > 0$ we denote $f_y(x) = y^{-\alpha-1} f(x/y)$. Let

$$A = A(\alpha) = -\frac{2\Gamma(1+\alpha/2)}{\Gamma((1+\alpha)/2)} = -\frac{2\gamma_1}{\gamma_2}, \qquad B = B(\alpha) = -\frac{\alpha+1}{\sqrt{\pi}}. \tag{4.2.3}$$

The following proposition (see [35, Proposition 2.3]) will play a crucial role in our investigations.

Proposition 4.2.4. *Let A, B be as in (4.2.3). Then for $x \neq y$ we have*

$$R^{\widetilde{\mathbf{L}}}(x,y) = \frac{A-B}{x^{\alpha+1} + y^{\alpha+1}} + \frac{B}{x^{\alpha+1} - y^{\alpha+1}} + h_y(x),$$

where

$$h \in L^1(X) \quad \text{and} \quad |h(x) + A - 2B| \le Cx \quad \text{for} \quad x \le 1/2. \tag{4.2.5}$$

The proof of Proposition 4.2.4 is postponed until Section 4.5.1. To give a precise definition of $\mathbf{R}^{\widetilde{\mathbf{L}}}$ on $L^1(X)$ we need a suitable space of test functions. One of possible choices is

$$\Omega(X) = \left\{ \xi \in C^1(0,\infty) \;\middle|\; \|\xi\|_\infty, \left\|\frac{\xi(x)}{x}\right\|_{L^1(X)}, \|x\xi'(x)\|_\infty < \infty \right\}$$

with the topology defined by the semi-norms γ_i, $i = 1, 2, 3$, where,

$$\gamma_1(\xi) = \|\xi\|_\infty, \quad \gamma_2(\xi) = \left\|\frac{\xi(x)}{x}\right\|_{L^1(X)}, \quad \gamma_3(\xi) = \|x\xi'(x)\|_\infty.$$

Denote by $\Omega'(X)$ the dual space.

The space $f \in L^1(X)$ is contained in $\Omega'(X)$ in the natural sense, i.e. if $f \in L^1(X)$, then

$$< f, \xi >= \int_0^\infty f\xi \, d\mu, \qquad \xi \in \Omega(X).$$

Next, for $f \in L^1(X)$, $\xi \in \Omega(X)$, we define

$$\langle \mathbf{R}^{\widetilde{\mathbf{L}}} f, \xi \rangle = \langle f, \left(\mathbf{R}^{\widetilde{\mathbf{L}}}\right)^* \xi \rangle, \qquad \left(\mathbf{R}^{\widetilde{\mathbf{L}}}\right)^* \xi(y) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} R^{\widetilde{\mathbf{L}}}(x,y)\xi(x) d\mu(x). \tag{4.2.6}$$

Alternatively, we define the Riesz transform as follows:

$$\langle \underline{\mathbf{R}}^{\widetilde{\mathbf{L}}} f, \xi\rangle = \langle f, \left(\underline{\mathbf{R}}^{\widetilde{\mathbf{L}}}\right)^* \xi\rangle, \qquad \left(\underline{\mathbf{R}}^{\widetilde{\mathbf{L}}}\right)^* \xi(y) = \lim_{\varepsilon\to 0}\int_\varepsilon^{\varepsilon^{-1}}\int_X \frac{\partial}{\partial x}\widetilde{K}_t(x,y)\xi(x)d\mu(x)\frac{dt}{\sqrt{t}}. \tag{4.2.7}$$

Proposition 4.2.8. *For $\xi \in \Omega(X)$ and $y > 0$ we have $\left(\mathbf{R}^{\widetilde{\mathbf{L}}}\right)^* \xi(y) = \left(\underline{\mathbf{R}}^{\widetilde{\mathbf{L}}}\right)^* \xi(y)$. Moreover,*

$$\left\| \left(\mathbf{R}^{\widetilde{\mathbf{L}}}\right)^* \xi\right\|_\infty \leq C\left(\|\xi(x)\|_\infty + \|x\xi'(x)\|_\infty + \left\|\frac{\xi(x)}{x}\right\|_{L^1(X)}\right).$$

The proof can be deduced from (4.1.10) and Proposition 4.2.4. We will not go into details here. However, we would like to notice that from Proposition 4.2.8 it is easily seen, that the Riesz transform $\underline{\mathbf{R}}^{\widetilde{\mathbf{L}}}$ is the same as the one defined by the spectral theorem (see, e.g. [33], [32]).

4.2.2 *Local Hardy spaces*

Fix a non-negative function $\phi \in C_c^\infty(-2,2)$ such that $\phi(x) = 1$ for $|x| \leq 3/2$. Similarly to the classical case, for $m > 0$ we define *scaled local Riesz transforms* $\widetilde{\mathbf{r}}^m$ for $f \in L^1(X)$, $\xi \in \Omega(X)$ as follows:

$$\langle \widetilde{\mathbf{r}}^m f, \xi\rangle = \langle f, (\widetilde{\mathbf{r}}^m)^*\xi\rangle, \quad (\widetilde{\mathbf{r}}^m)^*\xi(y) = \lim_{\varepsilon\to 0}\int_{0,|x-y|>\varepsilon}^\infty R^{\widetilde{\mathbf{L}}}(x,y)\phi\left(\frac{x-y}{m}\right)\xi(x)d\mu(x).$$

As in the global case these operators are well-defined and

$$\|(\widetilde{\mathbf{r}}^m)^*\xi\|_\infty < \infty. \tag{4.2.9}$$

For an interval $I = B(y,r) \subseteq X$ and $k > 0$ let $kI = B(y,kr) \subseteq X$.

Lemma 4.2.10. *The operators $\widetilde{\mathbf{r}}^m$ are bounded on $L^2(X)$ with norm-operator bounds independent of m.*

Proof. Because of the dilatation structure (see (4.5.20)) it is enough to prove the lemma in the case $m = 1$. Assume additionally for the moment that supp$f \subseteq I = B(y_0, 1)$. Then $\widetilde{\mathbf{r}}^1 f(x) = 0$ for $x \notin 3I$. Also

$$\|\widetilde{\mathbf{r}}^1 f\|_{L^2(X\cap 3I)} \leq \|(\widetilde{\mathbf{r}}^1 - \mathbf{R}^{\widetilde{\mathbf{L}}})f\|_{L^2(X\cap 3I)} + \|\mathbf{R}^{\widetilde{\mathbf{L}}} f\|_{L^2(X)}.$$

It is well known that $\|\mathbf{R}^{\widetilde{\mathbf{L}}} f\|_{L^2(X)} \leq C\|f\|_{L^2(X)}$ (see [31]). Moreover,

$$|R^{\widetilde{\mathbf{L}}}(x,y)|\chi_{\{|x-y|>3/2\}} \leq C(xy)^{-\alpha/2} + |h_y(x)| \leq C(xy)^{-\alpha/2} + |h_1(x,y)| + |h_2(x,y)|, \tag{4.2.11}$$

where

$$h_1(x,y) = y^{-\alpha-1}(h - D\chi_{(0,1)})(x/y), \quad h_2(x,y) = D(\chi_{(0,1)})_y(x) = Dy^{-\alpha-1}\chi_{(x,\infty)}(y).$$

Here $D = A - 2B$. We claim that

$$\|(\widetilde{\mathbf{r}}^1 - \mathbf{R}^{\widetilde{\mathbf{L}}})f\|_{L^2(X\cap 3I)} \leq C\|f\|_{L^2(X)}.$$

To prove this we consider the three summands from (4.2.11) separately. By the Cauchy-Schwarz inequality we get

$$\Big\| \int_I (xy)^{-\alpha/2} f(y) d\mu(y) \Big\|^2_{L^2(X\cap 3I)} \leq C \int_{3I} \|f\|^2_{L^2(X)} dx \leq C\|f\|^2_{L^2(X)}.$$

From (4.2.5) we deduce

$$\sup_{y>0} \int_0^\infty |h_1(x,y)| d\mu(x) + \sup_{x>0} \int_0^\infty |h_1(x,y)| d\mu(y) < \infty.$$

Thus the operator with the kernel $h_1(x,y)$ is bounded on every $L^p(X)$, $1 \leq p \leq \infty$. The part which contains h_2 is bounded on $L^2(X)$ due to the Hardy inequality (see, e.g. [2, p. 124]).

To omit the assumption $\mathrm{supp} f \subseteq B(y_0, 1)$ let us notice that

$$\|\widetilde{\mathbf{r}}^1 f\|_{L^2(X)} \leq \sum_{j=1}^\infty \|\widetilde{\mathbf{r}}^1(f\cdot\chi_{(j-1,j)})\|_{L^2(X)} \leq C\sum_{j=1}^\infty \|f\cdot\chi_{(j-1,j)}\|_{L^2(X)} = C\|f\|_{L^2(X)}.$$

□

The local Hardy space $\widetilde{h}^{1,m}(X)$ is a subspace of $L^1(X)$ consisting of functions f for which $\widetilde{\mathbf{r}}^m f \in L^1(X)$. In order to state atomic characterization of $\widetilde{h}^{1,m}(X)$ we define a suitable family of atoms.

Definition 4.2.12. *We call a function a a local $\widetilde{\mathbf{L}}^{\{m\}}$-atom, when there exists an interval $I = B(y_0, r) \subset (0,\infty)$ such that*

(i) $\mathrm{supp}(a) \subseteq I$ *and* $r \leq m$,

(ii) $\|a\|_\infty \leq \mu(I)^{-1}$,

(iii) if $r \leq m/4$, then $\int_0^\infty a(x) d\mu(x) = 0$.

Theorem 4.2.13. *Assume that $f \in L^1(X)$. Then $\widetilde{\mathbf{r}}^m f \in L^1(X)$ if and only if there exist sequences $\lambda_k \in \mathbb{C}$ and local $\widetilde{\mathbf{L}}^{\{m\}}$-atoms a_k, such that $f = \sum_{k=1}^\infty \lambda_k a_k$, where $\sum_{k=1}^\infty |\lambda_k| < \infty$. Moreover, we can choose $\{\lambda_k\}_k, \{a_k\}_k$, such that*

$$C^{-1}\sum_{k=1}^\infty |\lambda_k| \leq \|f\|_{L^1(X)} + \|\widetilde{\mathbf{r}}^m f\|_{L^1(X)} \leq C\sum_{k=1}^\infty |\lambda_k|,$$

where C is independent of $m > 0$.

Remark 4.2.14. *Assume in addition* $\operatorname{supp}(f) \subseteq I = B(y_0, m)$. *Then, in the above decomposition, one can take atoms with supports contained in* $3I$.

Proof. The proof is similar to the classical case. For reader's convenience we provide some details. Without loss of generality we may assume that $m = 1$. The operator $\widetilde{\mathbf{r}}^1$ is continuous from $L^1(X)$ to $\Omega'(X)$ (see (4.2.9)), so the first implication will be proved when we have obtained

$$\|\widetilde{\mathbf{r}}^1 a\|_{L^1(X)} \leq C \tag{4.2.15}$$

for every local $\widetilde{\mathbf{L}}^{\{m\}}$-atom a. Notice, that the weak-type $(1,1)$ bounds of $\widetilde{\mathbf{r}}^1$ also reduces the proof to (4.2.15).

Assume then, that a is an $\widetilde{h}^{1,1}(X)$-atom supported by an interval $I = B(y_0, r)$. Note that $\widetilde{\mathbf{r}}^1 a(x) = 0$ on $(9I)^c$. Consider first the case where $r > 1/4$. Recall that μ has the doubling property (see (4.1.1)). By the Cauchy-Schwarz inequality and Lemma 4.2.10 we arrive at

$$\|\widetilde{\mathbf{r}}^1 a\|_{L^1(X\cap 9I)} \leq \mu(9I)^{1/2} \|\widetilde{\mathbf{r}}^1 a\|_{L^2(X)} \leq C\mu(I)^{1/2}\|a\|_{L^2(X)} \leq C.$$

If $r < 1/4$ then a is an $\widetilde{\mathbf{L}}$-atom, so by Theorem 4.2.2 it follows that $\|\mathbf{R}^{\widetilde{\mathbf{L}}} a\|_{L^1(X)} \leq C$. Therefore $\|\widetilde{\mathbf{r}}^1 a\|_{L^1(X)} \leq C + \|(\mathbf{R}^{\widetilde{\mathbf{L}}} - \widetilde{\mathbf{r}}^1)a\|_{L^1(X)}$. Because of the cancellation condition we have

$$(\mathbf{R}^{\widetilde{\mathbf{L}}} - \widetilde{\mathbf{r}}^1)a(x) = \int \Big(R^{\widetilde{\mathbf{L}}}(x,y)(1-\phi(x-y)) - R^{\widetilde{\mathbf{L}}}(x,y_0)(1-\phi(x-y_0))\Big)\, a(y)d\mu(y).$$

Thus it is enough to verify the estimate

$$\begin{aligned} &\sup_{y\in I}\int_0^\infty \big|R^{\widetilde{\mathbf{L}}}(x,y)(1-\phi(x-y)) - R^{\widetilde{\mathbf{L}}}(x,y_0)(1-\phi(x-y_0))\big| d\mu(x) \\ =&\sup_{y\in I}\int_0^\infty \Xi(x,y)d\mu(x) \leq C. \end{aligned} \tag{4.2.16}$$

Fix $y \in I$. From Proposition 4.2.4 we obtain:

$$\begin{aligned} &\Xi(x,y) = 0 && \text{for } |x-y_0| \in (0,1), \\ &\Xi(x,y) \leq C\, x^{-\alpha} + |h_y(x)| + |h_{y_0}(x)| && \text{for } |x-y_0| \in (1,3), \\ &\Xi(x,y) \leq C\, |x-y_0|^{-2} x^{-\alpha} + |h_y(x)| + |h_{y_0}(x)| && \text{for } |x-y_0| \in (3,\infty), \end{aligned} \tag{4.2.17}$$

where in the last inequality we have used that $\phi(x-y) = \phi(x-y_0) = 0$ and the mean-value theorem. From (4.2.17) we get (4.2.16) and, consequently, $\|(\mathbf{R}^{\widetilde{\mathbf{L}}} - \widetilde{\mathbf{r}}^1)a\|_{L^1(X)} \leq C$. This ends the proof of (4.2.15).

For the converse, assume that $f, \widetilde{\mathbf{r}}^1 f \in L^1(X)$ and, in addition, $\mathrm{supp} f \subseteq I = B(y_0, 1)$. Fix $\tau = \mu(I)^{-1} \int_I f d\mu$, $g = f - \tau\chi_I$. We have

$$\|\mathbf{R}^{\widetilde{\mathbf{L}}} g\|_{L^1(X)} \leq \|\widetilde{\mathbf{r}}^1 f\|_{L^1(X)} + \|\tau\, \widetilde{\mathbf{r}}^1(\chi_I)\|_{L^1(X)} + \|(\mathbf{R}^{\widetilde{\mathbf{L}}} - \widetilde{\mathbf{r}}^1) g\|_{L^1(X)}. \tag{4.2.18}$$

By using the first part of the proof we deduce that $\|\tau\, \widetilde{\mathbf{r}}^1 \chi_I\|_{L^1(X)} \leq C\|f\|_{L^1(X)}$. Note that $\mathrm{supp}\, g \subseteq I$, $\int g\, d\mu = 0$, so (4.2.16) implies $\|(\mathbf{R}^{\widetilde{\mathbf{L}}} - \widetilde{\mathbf{r}}^1) g\|_{L^1(X)} \leq C\|g\|_{L^1(X)} \leq C\|f\|_{L^1(X)}$. Therefore, by Theorem 4.2.2, there exist $\widetilde{\mathbf{L}}$-atoms a_j $(j = 1, \ldots)$ such that

$$f - \tau\, \chi_I = g = \sum_{k=1}^{\infty} \lambda_k a_k.$$

Moreover $\sum_{k=1}^{\infty} |\lambda_k| \leq \|f\|_{L^1(X)} + \|\widetilde{\mathbf{r}}^1 f\|_{L^1(X)}$. Denote $\lambda_0 = \int_I f d\mu$, $a_0 = \mu(I)^{-1}\chi_I$ and fix $\psi_I \in C_c^\infty\left(\frac{4}{3}I\right)$ satisfying $\psi_I \equiv 1$ on I and $\|\psi_I\|_\infty \leq C$. What we have obtained is

$$f = f\psi_I = \sum_{k=0}^{\infty} \lambda_k (\psi_I a_k). \tag{4.2.19}$$

It remains to show that each $\psi_I a_k$ can be written in the form: $\psi_I a_k = \sum_{i=1}^{N_k} \kappa_{i,k} b_{i,k}$, where $b_{i,k}$ are local $\widetilde{\mathbf{L}}^{\{1\}}$-atoms supported in $3I$ and $\sum_{i=1}^{N_k} |\kappa_{i,k}| \leq C$, where $C > 0$ is independent of k. For $k = 0$ the claim is clear. Fix $k \geq 1$ and suppose that $\mathrm{supp}\, a_k \subseteq J = B(z_0, r)$. Obviously, if $(\frac{4}{3}I) \cap J = \emptyset$ then $\psi_I a_k = 0$. Moreover, if $r > 1/4$ then $\psi_I a_k = \kappa b$, where b is a local $\widetilde{\mathbf{L}}^{\{m\}}$-atom and $|\kappa| \leq C$. So, suppose that $(\frac{4}{3}I) \cap J \neq \emptyset$ and $r < 1/4$. Under these assumptions we write

$$\begin{aligned}\psi_I(x)a_k(x) = &\left(\psi_I(x)a_k(x) - \sigma\mu(2J)^{-1}\chi_{2J}(x)\right)\\ &+ \sum_{i=1}^{N-1} \sigma\left((\mu(2^i J))^{-1}\chi_{2^i J}(x) - (\mu(2^{i+1}J))^{-1}\chi_{2^{i+1}J}(x)\right)\\ &+ \sigma(\mu(2^N J))^{-1}\chi_{2^N J}(x),\end{aligned}$$

where $\sigma = \int_0^\infty a_k(z)(\psi_I(z) - \psi_I(z_0))d\mu(z)$ and N is such that $2^{-N-1} \leq r < 2^{-N}$. One can check that this is the required decomposition, since $|\sigma| \leq Cr$. Let us note that we have just proved Remark 4.2.14.

To deal with the general case we take a smooth partition of unity $\{\psi_j\}_{j=1}^\infty \subseteq C^\infty(0, \infty)$, i.e.

$$\sum_{j=1}^{\infty} \psi_j(x) = \chi_{(0,\infty)}(x), \quad 0 \leq \psi_j \leq 1, \quad \mathrm{supp}\, \psi_j \subseteq I_j = B\left(y_j, 1\right), \quad \sup_{j\in\mathbb{N}} \|\psi_j'\|_\infty \leq C.$$

Consider

$$g_j = \widetilde{\mathbf{r}}^1(\psi_j f) - \psi_j \widetilde{\mathbf{r}}^1(f).$$

Obviously, $\operatorname{supp} g_j \subseteq 3I_j$ and for $x \in 3I_j$ we have

$$\begin{aligned} |g_j(x)| &= \Big| \int_0^\infty R^{\widetilde{\mathbf{L}}}(x,y)\phi(x-y)f(y)(\psi_j(y)-\psi_j(x))d\mu(y)\Big| \\ &\le C\int_0^\infty |R^{\widetilde{\mathbf{L}}}(x,y)|\chi_{\{|x-y|\le 2\}}|f(y)||x-y|d\mu(y). \end{aligned} \tag{4.2.20}$$

Moreover, from Proposition 4.2.4 we have

$$\sup_{y>0}\int_{|x-y|\le 2} |R^{\widetilde{\mathbf{L}}}(x,y)||x-y|d\mu(x) \le C. \tag{4.2.21}$$

From (4.2.20) and (4.2.21) we deduce that

$$\|g_j\|_{L^1(X)} \le \|f\|_{L^1(X\cap 5I_j)}.$$

Therefore

$$\sum_{j=1}^\infty \|\widetilde{\mathbf{r}}^1(\psi_j f)\|_{L^1(X)} \le \sum_{j=1}^\infty \left(\|\psi_j \widetilde{\mathbf{r}}^1(f)\|_{L^1(X)} + \|g_j\|_{L^1(X)}\right) \le C\left(\|f\|_{L^1(X)} + \|\widetilde{\mathbf{r}}^1(f)\|_{L^1(X)}\right). \tag{4.2.22}$$

By using (4.2.19) and the subsequent remark for each $\psi_j f$ we get the decomposition $\psi_j f = \sum_k \lambda_k^j a_k^j$, where a_k^j are local $\widetilde{\mathbf{L}}^{\{1\}}$-atoms and

$$\sum_k |\lambda_k^j| \le C\left(\|\psi_j f\|_{L^1(X)} + \|\widetilde{\mathbf{r}}^1(\psi_j f)\|_{L^1(X)}\right). \tag{4.2.23}$$

The proof is completed by noticing that

$$f = \sum_{j,k} \lambda_k^j a_k^j,$$

where

$$\sum_{j,k} |\lambda_k^j| \le C\left(\|f\|_{L^1(X)} + \|\widetilde{\mathbf{r}}^1 f\|_{L^1(X)}\right)$$

is guaranteed by (4.2.22) and (4.2.23). □

4.3 The Riesz transform in the Laguerre setting

Let ϕ be the function defined in Section 4.2.2 and ρ be as in (4.1.5). The following proposition (see [35, Proposition 3.1]) gives an essential information about the kernel of the Riesz transform associated with the Laguerre expansion.

Proposition 4.3.1. *Let A and B be as in (4.2.3). The kernel $R(x,y)$ can be written in the form*

$$R(x,y) = \phi\left(\frac{x-y}{\rho(y)}\right)\left(\frac{B}{x^{\alpha+1}-y^{\alpha+1}} + \frac{A-B}{x^{\alpha+1}+y^{\alpha+1}}\right) + g(x,y),$$

where

$$\sup_{y>0} \int_0^\infty |g(x,y)| d\mu(x) < \infty. \tag{4.3.2}$$

The proof of Proposition 4.3.1 is a quite lengthy analysis. We provide details in Section 4.5.2.

For $f \in L^1(X)$, $\xi \in \Omega(X)$, we define the Riesz transform $\mathbf{R^L} f$ as follows

$$\langle \mathbf{R^L} f, \xi \rangle = \langle f, (\mathbf{R^L})^* \xi \rangle, \qquad (\mathbf{R^L})^* \xi(y) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} R^{\mathbf{L}}(x,y) \xi(x) d\mu(x),$$

One can easily check using Proposition 4.3.1 that this limit exists and

$$\| (\mathbf{R^L})^* \xi \|_\infty \le C \left(\|\xi(x)\|_\infty + \|x\xi'(x)\|_\infty + \left\| \frac{\xi(x)}{x} \right\|_{L^1(X)} \right). \tag{4.3.3}$$

Denote by $\mathbf{G}$ the operator with the kernel $g(x,y)$. Obviously, by (4.3.2), $\mathbf{G}$ is bounded on $L^1(X)$. In the proof of Theorem 4.1.8 we will need the following lemma.

Lemma 4.3.4. *Let $z \in (0,\infty)$, $f \in L^1(X)$, $I = B(z, \rho(z))$, and $\eta \in C^\infty(0,\infty)$ satisfies $0 \le \eta \le 1$, $\operatorname{supp} \eta \subset I$, $\|\eta'\|_\infty \le C_1 \rho(z)^{-1}$. Then*

$$\|\mathbf{R^L}(\eta f) - \eta((\mathbf{R^L} - \mathbf{G})f)\|_{L^1(X)} \le C \|f\|_{L^1(X \cap 4I)},$$

with a constant C which depends on C_1, but it is independent of $z \in (0,\infty)$ and $f \in L^1(X)$.

Proof. Note that

$$\begin{aligned} \mathbf{R^L}(\eta f)(x) - \eta(x)(\mathbf{R^L} - \mathbf{G})f(x) &= \int (R^{\mathbf{L}}(x,y) - g(x,y))(\eta(y) - \eta(x)) f(y) d\mu(y) \\ &\quad + \int g(x,y) \eta(y) f(y) d\mu(y) \\ &= \int W_1(x,y) d\mu(y) + \int W_2(x,y) d\mu(y). \end{aligned}$$

Applying (4.3.2) we easily estimate the summand that contains W_2. The function $W_1(x,y)$ vanishes if either $|x-y| > 2\rho(y)$ or $x, y \in I^c$. Therefore it can be verified that $W_1(x,y) = 0$, if either $x \notin 4I$ or $y \notin 4I$. Thus Lemma 4.3.4 follows by

$$\begin{aligned} \int_{4I} \left| \int_{4I} W_1(x,y) d\mu(y) \right| d\mu(x) &\le C \int_{4I} |f(y)| \left(\int_{4I} \left| \frac{1}{x^{\alpha+1} - y^{\alpha+1}} \right| \frac{|x-y|}{\rho(z)} d\mu(x) \right) d\mu(y) \\ &\le C \int_{4I} |f(y)| d\mu(y). \end{aligned}$$

□

4.4 Proof of Theorem 4.1.8

Before proving the main theorem we state a crucial consequence of Propositions 4.2.4 and 4.3.1.

Lemma 4.4.1. *For $y_0 > 0$ we have*

$$\sup_{y\in B(y_0,\rho(y_0))}\int_0^\infty \left|R^{\mathbf{L}}(x,y)-\widetilde{r}^{\rho(y_0)}(x,y)\right|d\mu(x)\leq C, \tag{4.4.2}$$

Proof. By (4.2.5) and (4.3.2) we only need to establish that

$$\sup_{y\in B(y_0,\rho(y_0))}\int_0^\infty \left|\phi\left(\frac{x-y}{\rho(y)}\right)-\phi\left(\frac{x-y}{\rho(y_0)}\right)\right|\left|\frac{B}{x^{\alpha+1}-y^{a+1}}+\frac{A-B}{x^{\alpha+1}+y^{a+1}}\right|d\mu(x)\leq C.$$

In fact we will prove a stronger estimate, namely,

$$\sup_{y\in B(y_0,\rho(y_0))}\int_0^\infty \left|\phi\left(\frac{x-y}{\rho(y)}\right)-\phi\left(\frac{x-y}{\rho(y_0)}\right)\right|\cdot\frac{1}{|x^{\alpha+1}-y^{a+1}|}d\mu(x)\leq C. \tag{4.4.3}$$

Consider the case $y > y_0$ (if $y < y_0$ we use the same type of arguments). The integrant in (4.4.3) is non-zero only when $3/2\,\rho(y) < |x-y| < 2\rho(y_0)$. But always $\rho(y_0) < 2\rho(y)$ if $y \in B(y_0, \rho(y_0))$. Now, one can check that

$$\sup_{y>0}\int_0^\infty \chi_{\{3/2\,\rho(y)<|x-y|<4\rho(y)\}}\frac{1}{|x^{\alpha+1}-y^{\alpha+1}|}d\mu(x)\leq C,$$

which implies (4.4.3). □

Proof of Theorem 4.1.8. Assume $f \in H^1_{\mathbf{L},at}$. The operator $\mathbf{R}^{\mathbf{L}} : L^1(X) \to \Omega'(X)$ is continuous (see (4.3.3)), so the first implication will be proved if we have established that there exists $C > 0$ such that

$$\|\mathbf{R}^{\mathbf{L}}a\|_{L^1(X)}\leq C$$

for any $\mathbf{L}-$atom a. Suppose a is associated with $I = B(y_0, r)$ (recall that $r \leq \rho(y_0)$). We have that

$$\mathbf{R}^{\mathbf{L}}a=(\mathbf{R}^{\mathbf{L}}a-\widetilde{\mathbf{r}}^{\rho(y_0)}a)+\widetilde{\mathbf{r}}^{\rho(y_0)}a.$$

The $L^1(X)$-norm of the function $\widetilde{\mathbf{r}}^{\rho(y_0)}a$ is bounded by a constant independent of a, because a is also a local $\widetilde{\mathbf{L}}^{\{\rho(y_0)\}}$-atom (see Theorem 4.2.13). Therefore, the first part of the proof is finished by (4.4.2).

To prove the converse assume that $f, \mathbf{R}^{\mathbf{L}}f \in L^1(X)$. Introduce a family of intervals $\mathcal{I} = \{I_n = B(z_n, \rho(z_n))\}_{n=1}^\infty$ such that $X = \bigcup_{n=1}^\infty I_n$ and $\mathcal{I}^* = \{4I : I \in \mathcal{I}\}$ has bounded overlap. Denote by η_n a smooth partition of unity associated with the family $\mathcal{I}$, i.e.

$$\eta_n\in C^\infty(0,\infty),\ \ \operatorname{supp}\eta_n\subset I_n,\ \ 0\leq\eta_n\leq 1,\ \ \sum_{n=1}^\infty\eta_n(x)=\chi_{(0,\infty)}(x),\ \ |\eta_n'(x)|\leq C\rho(z_n)^{-1}.$$

We are going to prove an atomic decomposition of $f = \sum_{n=1}^{\infty} \eta_n f$. Note that

$$\widetilde{\mathbf{r}}^{\rho(z_n)}(f\eta_n) = \Big((\widetilde{\mathbf{r}}^{\rho(z_n)} - \mathbf{R}^{\mathbf{L}})(f\eta_n)\Big) + \big(\mathbf{R}^{\mathbf{L}}(f\eta_n) - \eta_n(\mathbf{R}^{\mathbf{L}} - \mathbf{G})(f)\big) - \eta_n \cdot \mathbf{G}(f) + \eta_n \cdot \mathbf{R}^{\mathbf{L}}(f).$$

By using (4.4.2), Lemma 4.3.4, and (4.3.2) we get

$$\begin{aligned}\sum_{n=1}^{\infty} \|\widetilde{\mathbf{r}}^{\rho(z_n)}(f\eta_n)\|_{L^1(X)} &\leq C \sum_{n=1}^{\infty} \big(\|\eta_n f\|_{L^1(X)} + \|\chi_{4I_n} f\|_{L^1(X)} \\ &\qquad + \|\eta_n \mathbf{G} f\|_{L^1(X)} + \|\eta_n \mathbf{R}^{\mathbf{L}} f\|_{L^1(X)}\big) \\ &\leq C \left(\|f\|_{L^1(X)} + \|\mathbf{R}^{\mathbf{L}} f\|_{L^1(X)}\right).\end{aligned} \tag{4.4.4}$$

Applying Theorem 4.2.13, we arrive at

$$\eta_n \cdot f = \sum_{j=1}^{\infty} \lambda_{n,j} a_{n,j}, \quad \text{where} \quad \sum_{j=1}^{\infty} |\lambda_{n,j}| \leq \|\widetilde{\mathbf{r}}^{\rho(z_n)}(f \cdot \eta_n)\|_{L^1(X)}, \tag{4.4.5}$$

and $a_{n,j}$ are local $\widetilde{\mathbf{L}}^{\{\rho(z_n)\}}$-atoms. From (4.4.4) and (4.4.5) we have obtained

$$f = \sum_{n,j=1}^{\infty} \lambda_{n,j} a_{n,j} \quad \text{with} \quad \sum_{n,j=1}^{\infty} |\lambda_{n,j}| \leq C \left(\|f\|_{L^1(X)} + \|\mathbf{R}^{\mathbf{L}} f\|_{L^1(X)}\right). \tag{4.4.6}$$

Remark 4.2.14 states that $\operatorname{supp} a_{n,j} \subseteq 3I_n$ for $j \geq 0$. Notice that for $y \in 3\ I_n$ there exists $C > 0$ such that

$$\rho(z_n)/C \leq \rho(y) \leq C\rho(z_n) \qquad \text{for all } n \geq 1 \text{ and } y \in I_n. \tag{4.4.7}$$

Because of this, each $a_{n,j}$ can be decomposed into a sum of at most N **L**-atoms (where the number N depends only on α and the constant C from (4.4.7)). Finally, Theorem 4.1.8 follows by applying (4.4.6).

□

4.5 *Proofs of Propositions 4.2.4 and 4.3.1*

This section is devoted to proving Propositions 4.2.4 and 4.3.1. The letters c, C, N, M will denote positive constants (N, M are arbitrarily large). We also make the convention that $\int_p^q \dots = 0$, whenever $p \geq q$. For further references we figure out some properties of the Bessel function I_ν ($\nu > 0$) (see, e.g. [41]):

$$\frac{\partial}{\partial x}\left(x^{-\nu} I_\nu(x)\right) = x^{-\nu} I_{\nu+1}(x) \qquad \text{for } x > 0, \tag{4.5.1}$$

$$0 < I_\nu(x) = 2^{-\nu}\Gamma(\nu+1)^{-1} x^\nu + O(x^{\nu+2}) \qquad \text{for } 0 < x < C, \tag{4.5.2}$$

$$U_\nu(x) = (2\pi)^{-1/2} + O(x^{-1}) \qquad \text{for } x > C, \tag{4.5.3}$$

where

$$U_\nu(x) = I_\nu(x) e^{-x} \sqrt{x}.$$

4.5.1 Proof of Proposition 4.2.4

Proof. Assume $y = 1$. By using (4.1.10) and (4.5.1) we get

$$\begin{aligned} R^{\widetilde{\mathbf{L}}}(x,1) &= -\int_0^\infty (2t)^{-2} \exp\left(-\frac{x^2+1}{4t}\right) x^{-(\alpha-3)/2} I_{(\alpha-1)/2}\left(\frac{x}{2t}\right)\frac{dt}{\sqrt{t}} \\ &\quad + \int_0^\infty (2t)^{-2} \exp\left(-\frac{x^2+1}{4t}\right) x^{-(\alpha-1)/2} I_{(\alpha+1)/2}\left(\frac{x}{2t}\right)\frac{dt}{\sqrt{t}} \\ &= \int_0^\infty \widetilde{Q}_1(x;t)dt + \int_0^\infty \widetilde{Q}_2(x;t)dt. \end{aligned} \tag{4.5.4}$$

In calculations below we will often use the following formula:

$$\exp\left(-\frac{x^2+1}{4t}\right) I_\nu\left(\frac{x}{2t}\right)\sqrt{\frac{x}{2t}} = \exp\left(-\frac{(x-1)^2}{4t}\right) U_\nu\left(\frac{x}{2t}\right).$$

Define

$$h(x) = R^{\widetilde{\mathbf{L}}}(x,1) - \frac{A-B}{x^{\alpha+1}+1} - \frac{B}{x^{\alpha+1}-1}.$$

To prove (4.2.5) we consider three cases.

Case 1: $x > 3/2$.

Under this assumption $(x-1) \sim x$. Then we get estimates:

$$\begin{aligned} \int_0^x |\widetilde{Q}_2(x;t)|dt &\le C\int_0^x t^{-2}\exp\left(-\frac{(x-1)^2}{4t}\right) U_{(\alpha+1)/2}\left(\frac{x}{2t}\right) x^{-\alpha/2}dt \\ &\le C\int_0^x t^{-2}\left(\frac{t}{x^2}\right)^N x^{-\alpha/2}dt \le Cx^{-M}, \\ \int_x^{x^2} |\widetilde{Q}_2(x;t)|dt &\le C\int_x^{x^2} t^{-2}\left(\frac{t}{x^2}\right)^N x^{\frac{1-\alpha}{2}}\left(\frac{x}{t}\right)^{\frac{\alpha+1}{2}}\frac{dt}{\sqrt{t}} \le C\int_0^{x^2}\frac{t^{N-3-\frac{\alpha}{2}}}{x^{2N-1}}dt \le Cx^{-\alpha-3}, \\ \int_{x^2}^\infty |\widetilde{Q}_2(x;t)|dt &\le C\int_{x^2}^\infty t^{-2}x^{-(\alpha-1)/2}\left(\frac{x}{t}\right)^{(\alpha+1)/2}\frac{dt}{\sqrt{t}} \le C\int_{x^2}^\infty xt^{-3-\alpha/2}dt \le Cx^{-\alpha-3}, \end{aligned}$$

which imply

$$\int_0^\infty |\widetilde{Q}_2(x;t)|dt \le C\, x^{-\alpha-3}. \tag{4.5.5}$$

Our next task is to obtain

$$\left|\int_0^\infty \widetilde{Q}_1(x;t)dt - \frac{A-B}{x^{\alpha+1}+1} - \frac{B}{x^{\alpha+1}-1}\right| \le Cx^{-\alpha-2}. \tag{4.5.6}$$

By using the same methods as we have utilized to estimate the integral $\int_0^x |\widetilde{Q}_2(x;t)|dt$ we deduce

$$\int_0^x |\widetilde{Q}_1(x;t)|dt \le Cx^{-M}. \tag{4.5.7}$$

Moreover,

$$\begin{aligned}
\Big|\int_x^\infty \widetilde{Q}_1(x;t)dt - \frac{A}{x^{\alpha+1}}\Big| &= \Big| - \int_x^\infty (2t)^{-2}\exp\left(-\frac{x^2+1}{4t}\right) x^{-\frac{\alpha-3}{2}} I_{(\alpha-1)/2}\left(\frac{x}{2t}\right)\frac{dt}{\sqrt{t}} \\
&+ \int_0^\infty (2t)^{-2}\exp\left(-\frac{x^2}{4t}\right) x^{-\frac{\alpha-3}{2}} \frac{2^{-(\alpha-1)/2}}{\Gamma\left(\frac{\alpha+1}{2}\right)}\left(\frac{x}{2t}\right)^{\frac{\alpha-1}{2}}\frac{dt}{\sqrt{t}}\Big| \\
&\le \Big|\int_x^\infty (2t)^{-2}\left(\exp\left(-\frac{x^2+1}{4t}\right) - \exp\left(-\frac{x^2}{4t}\right)\right) x^{-\frac{\alpha-3}{2}} I_{(\alpha-1)/2}\left(\frac{x}{2t}\right)\frac{dt}{\sqrt{t}}\Big| \\
&+ \Big|\int_x^\infty (2t)^{-2}\exp\left(-\frac{x^2}{4t}\right) x^{-\frac{\alpha-3}{2}}\left(I_{(\alpha-1)/2}\left(\frac{x}{2t}\right) - \frac{2^{-(\alpha-1)/2}}{\Gamma\left(\frac{\alpha+1}{2}\right)}\left(\frac{x}{2t}\right)^{\frac{\alpha-1}{2}}\right)\frac{dt}{\sqrt{t}}\Big| \\
&+ \Big|\int_0^x (2t)^{-2}\exp\left(-\frac{x^2}{4t}\right) x^{-\frac{\alpha-3}{2}} \frac{2^{-(\alpha-1)/2}}{\Gamma\left(\frac{\alpha+1}{2}\right)}\left(\frac{x}{2t}\right)^{\frac{\alpha-1}{2}}\frac{dt}{\sqrt{t}}\Big|
\end{aligned} \tag{4.5.8}$$

and

$$\Big|\frac{A}{x^{\alpha+1}} - \frac{A-B}{x^{\alpha+1}+1} - \frac{B}{x^{\alpha+1}-1}\Big| \le Cx^{-2\alpha-2} \qquad \text{for } x > 3/2. \tag{4.5.9}$$

Applying (4.5.2) and the mean-value theorem to (4.5.8), we get

$$\Big|\int_x^\infty \widetilde{Q}_1(x;t)dt - \frac{A}{x^{\alpha+1}}\Big| \le Cx^{-\alpha-3}. \tag{4.5.10}$$

Now (4.5.6) is a consequence of (4.5.7), (4.5.9), and (4.5.10). From (4.5.5)–(4.5.6) we conclude that

$$\int_{3/2}^\infty |h(x)|d\mu(x) = \int_{3/2}^\infty \Big|R^{\widetilde{\mathbf{L}}}(x,1) - \frac{A-B}{x^{\alpha+1}+1} - \frac{B}{x^{\alpha+1}-1}\Big| d\mu(x) \le C. \tag{4.5.11}$$

Case 2: $x < 1/2$.

From (4.5.2)–(4.5.3) it follows:

$$\begin{aligned}
\int_0^x |\widetilde{Q}_1(x;t)|dt &\le C\int_0^x \frac{x^{1-\frac{\alpha}{2}}}{t^2}\exp\left(-\frac{(x-1)^2}{4t}\right) U_{\frac{\alpha-1}{2}}\left(\frac{x}{2t}\right)dt \le Cx^{1-\frac{\alpha}{2}}\int_0^x t^{N-2}dt \le Cx, \\
\int_x^1 |\widetilde{Q}_1(x;t)|dt &\le C\int_x^1 t^{-2}\exp\left(-\frac{x^2+1}{4t}\right) x^{-\frac{\alpha-3}{2}}\left(\frac{x}{t}\right)^{(\alpha-1)/2}\frac{dt}{\sqrt{t}} \le Cx\int_0^1 t^M dt \le Cx, \\
\int_1^\infty |\widetilde{Q}_1(x;t)|dt &\le C\int_1^\infty t^{-2}\exp\left(-\frac{x^2+1}{4t}\right) x^{\frac{3-\alpha}{2}}\left(\frac{x}{t}\right)^{\frac{\alpha-1}{2}}\frac{dt}{\sqrt{t}} \le Cx\int_1^\infty t^{-2-\frac{\alpha}{2}}dt \le Cx,
\end{aligned}$$

Thus $\int_0^\infty |\widetilde{Q}_1(x;t)|dt \le Cx$. By the same arguments we also obtain $\int_0^\infty |\widetilde{Q}_2(x;t)|dt \le Cx$. Hence, $|R^{\widetilde{\mathbf{L}}}(x,1)| \le Cx$. As a consequence, for $x < 1/2$, we have

$$|h(x) + A - 2B| = \Big|R^{\widetilde{\mathbf{L}}}(x,1) - \frac{A-B}{x^{\alpha+1}+1} - \frac{B}{x^{\alpha+1}-1} + A - 2B\Big| \le Cx, \tag{4.5.12}$$

$$\int_0^{1/2} |h(x)|d\mu(x) \le \int_0^{1/2}\left(|R^{\widetilde{\mathbf{L}}}(x,1)| + \Big|\frac{A-B}{x^{\alpha+1}+1} + \frac{B}{x^{\alpha+1}-1}\Big|\right)d\mu(x) \le C. \tag{4.5.13}$$

Case 3: $1/2 < x < 3/2$.

In this case a slightly different form of (4.5.4) is needed, i.e.

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial x}\widetilde{K}_t(x,1)\frac{dt}{\sqrt{t}} &= -(x-1)\int_0^\infty (2t)^{-2}\exp\left(-\frac{x^2+1}{4t}\right)x^{-(\alpha-1)/2}I_{(\alpha-1)/2}\left(\frac{x}{2t}\right)\frac{dt}{\sqrt{t}}\\ &+\int_0^\infty (2t)^{-2}\exp\left(-\frac{x^2+1}{4t}\right)x^{-(\alpha-1)/2}\left(I_{(\alpha+1)/2}\left(\frac{x}{2t}\right)-I_{(\alpha-1)/2}\left(\frac{x}{2t}\right)\right)\frac{dt}{\sqrt{t}}\\ &= \int_0^\infty \widetilde{Q}_3(x;t)dt + \int_0^\infty \widetilde{Q}_4(x;t)dt.\end{aligned}$$

We claim that

$$\int_0^\infty |\widetilde{Q}_4(x;t)|dt \le C|x-1|^{-1/2}. \tag{4.5.14}$$

Indeed, by using (4.5.2) and (4.5.3) we get

$$\begin{aligned}\int_0^1 \left|\widetilde{Q}_4(x;t)\right|dt &\le C\int_0^1 t^{-2}\exp\left(-\frac{(x-1)^2}{4t}\right)x^{-\alpha/2}\left|U_{(\alpha+1)/2}\left(\frac{x}{2t}\right)-U_{(\alpha-1)/2}\left(\frac{x}{2t}\right)\right|dt\\ &\le C\int_0^1 t^{-2}\left(\frac{t}{(x-1)^2}\right)^{1/4}\frac{t}{x}dt \le C|x-1|^{-1/2},\\ \int_1^\infty \left|\widetilde{Q}_4(x;t)\right|dt &\le C\int_1^\infty t^{-2}x^{\frac{1-\alpha}{2}}\left(\left(\frac{x}{t}\right)^{\frac{\alpha+1}{2}}+\left(\frac{x}{t}\right)^{(\alpha-1)/2}\right)\frac{dt}{\sqrt{t}} \le C\int_1^\infty t^{-2-\alpha/2}dt \le C.\end{aligned}$$

Next, observe that

$$\int_1^\infty \left|\widetilde{Q}_3(x;t)\right|dt \le C|x-1|\int_1^\infty t^{-2}x^{-(\alpha-1)/2}\left(\frac{x}{t}\right)^{(\alpha-1)/2}\frac{dt}{\sqrt{t}} \le C. \tag{4.5.15}$$

Moreover,

$$\begin{aligned}\left|\int_0^1 \widetilde{Q}_3(x;t)dt - \frac{B(\alpha+1)^{-1}}{x^{\alpha/2}(x-1)}\right| &\le \left|\int_1^\infty \frac{\sqrt{2}(x-1)}{4t}\exp\left(-\frac{(x-1)^2}{4t}\right)x^{-\alpha/2}\frac{1}{\sqrt{2\pi}}\frac{dt}{t}\right|\\ &+\left|\int_0^1 \frac{\sqrt{2}(x-1)}{4t}\exp\left(-\frac{(x-1)^2}{4t}\right)x^{-\alpha/2}\left(U_{(\alpha-1)/2}\left(\frac{x}{2t}\right)-\frac{1}{\sqrt{2\pi}}\right)\frac{dt}{t}\right|.\end{aligned} \tag{4.5.16}$$

Applying (4.5.3) to (4.5.16) we deduce

$$\left|\int_0^1 \widetilde{Q}_3(x;t)dt - \frac{B(\alpha+1)^{-1}}{x^{\alpha/2}(x-1)}\right| \le C. \tag{4.5.17}$$

One can easily check that

$$\left|\frac{B(\alpha+1)^{-1}}{x^{\alpha/2}(x-1)} - \frac{B}{x^{\alpha+1}-1} - \frac{A-B}{x^{\alpha+1}+1}\right| \le C. \tag{4.5.18}$$

From (4.5.14), (4.5.15), (4.5.17), and (4.5.18) we conclude

$$\int_{1/2}^{3/2} |h(x)|d\mu(x) = \int_{1/2}^{3/2}\left|R^{\widetilde{\mathbf{L}}}(x,1) - \frac{B}{x^{\alpha+1}-1} - \frac{A-B}{x^{\alpha+1}+1}\right|d\mu(x) \le C. \tag{4.5.19}$$

Finally, as a consequence of (4.5.11)–(4.5.13), and (4.5.19) we obtain that h satisfies desired properties (4.2.5). The proposition in the general case of $y > 0$ follows by applying the homogeneity

$$R^{\widetilde{\mathbf{L}}}(x,y) = y^{-\alpha-1} R^{\widetilde{\mathbf{L}}}\left(\frac{x}{y}, 1\right). \tag{4.5.20}$$

□

4.5.2 Proof of Proposition 4.3.1

Proof. Let us set

$$g(x,y) = R^{\mathbf{L}}(x,y) - \phi\left(\frac{x-y}{\rho(y)}\right)\left(\frac{B}{x^{\alpha+1}-y^{\alpha+1}} - \frac{A-B}{x^{\alpha+1}+y^{\alpha+1}}\right). \tag{4.5.21}$$

We will prove that (4.3.2) is satisfied. By using (4.1.4) and (4.5.1) we get

$$R^{\mathbf{L}}(x,y) = \int_0^\infty K_t^{[1]}(x,y)\frac{dt}{\sqrt{t}} + \int_0^\infty K_t^{[2]}(x,y)\frac{dt}{\sqrt{t}} = \int_0^\infty K_t^{[3]}(x,y)\frac{dt}{\sqrt{t}} + \int_0^\infty K_t^{[4]}(x,y)\frac{dt}{\sqrt{t}}, \tag{4.5.22}$$

where

$$\begin{aligned}
K_t^{[1]}(x,y) &= \left(\frac{2e^{-2t}}{1-e^{-4t}}\right)^2 y(xy)^{-\frac{\alpha-1}{2}} \exp\left(-\frac{1+e^{-4t}}{1-e^{-4t}}\frac{x^2+y^2}{2}\right) I_{\frac{\alpha+1}{2}}\left(\frac{2e^{-2t}}{1-e^{-4t}}xy\right),\\
K_t^{[2]}(x,y) &= -\frac{2e^{-2t}(1+e^{-4t})}{(1-e^{-4t})^2} x(xy)^{-\frac{\alpha-1}{2}} \exp\left(-\frac{1+e^{-4t}}{1-e^{-4t}}\frac{x^2+y^2}{2}\right) I_{\frac{\alpha-1}{2}}\left(\frac{2e^{-2t}}{1-e^{-4t}}xy\right),\\
K_t^{[3]}(x,y) &= -\frac{2e^{-2t}(1+e^{-4t})}{(1-e^{-4t})^2} (xy)^{-\frac{\alpha-1}{2}}(x-y) \exp\left(-\frac{1+e^{-4t}}{1-e^{-4t}}\frac{x^2+y^2}{2}\right)\\
&\quad \times I_{\frac{\alpha-1}{2}}\left(\frac{2e^{-2t}xy}{1-e^{-4t}}\right),\\
K_t^{[4]}(x,y) &= \frac{2e^{-2t}}{1-e^{-4t}} y(xy)^{-\frac{\alpha-1}{2}} \exp\left(-\frac{1+e^{-4t}}{1-e^{-4t}}\frac{x^2+y^2}{2}\right)\\
&\quad \times\left(\frac{2e^{-2t}}{1-e^{-4t}} I_{\frac{\alpha+1}{2}}\left(\frac{2e^{-2t}}{1-e^{-4t}}xy\right) - \frac{1+e^{-4t}}{1-e^{-4t}} I_{\frac{\alpha-1}{2}}\left(\frac{2e^{-2t}}{1-e^{-4t}}xy\right)\right).
\end{aligned}$$

Note that

$$\begin{aligned}
&\exp\left(-\frac{1+e^{-4t}}{1-e^{-4t}}\frac{x^2+y^2}{2}\right) I_\mu\left(\frac{2e^{-2t}}{1-e^{-4t}}xy\right)\left(\frac{2e^{-2t}xy}{1-e^{-4t}}\right)^{1/2} \\
&= \exp\left(-\frac{1+e^{-4t}}{1-e^{-4t}}\frac{(x-y)^2}{2}\right)\exp\left(-\frac{(1-e^{-2t})^2}{1-e^{-4t}}xy\right) U_\mu\left(\frac{2e^{-2t}xy}{1-e^{-4t}}\right).
\end{aligned} \tag{4.5.23}$$

The formula (4.5.23) will be frequently used, without additional comments, when we deal with $I_\mu(\theta)$ for $\theta > C$.

We provide the proof in six cases as it is shown in Figure 1 on page 66. The grey part denotes the support of $\phi\left((x-y)/\rho(y)\right)$. Moreover, the dark grey color means that $\phi\left((x-y)/\rho(y)\right) = 1$.

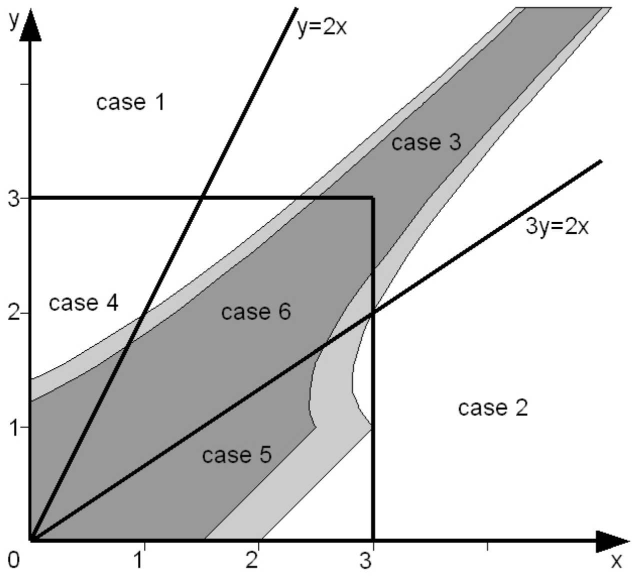

Fig. 4.1: Partition of $X \times X$

In Cases 1, 2, 4, 5 we will use the decomposition (4.5.22) that contains $K_t^{[1]}$ and $K_t^{[2]}$.

Case 1: $y > 3$**,** $x < y/2$**.**

At the beginning we consider $K_t^{[1]}$ and $t < 1$. Under additional assumption $xy < 1$ we get

$$\begin{aligned}
\textstyle\int_{xy}^1 |K_t^{[1]}(x,y)|\frac{dt}{\sqrt{t}} &\leq C\int_{xy}^1 t^{-2}y(xy)^{-\frac{\alpha-1}{2}}\exp\left(-\frac{y^2}{ct}\right)\left(\frac{xy}{t}\right)^{\frac{\alpha+1}{2}}\frac{dt}{\sqrt{t}} \leq Cy^{-M},\\
\textstyle\int_0^{xy} |K_t^{[1]}(x,y)|\frac{dt}{\sqrt{t}} &\leq C\int_0^{xy} \frac{y}{t^2}(xy)^{-\frac{\alpha-1}{2}}\exp\left(-\frac{y^2}{ct}\right)e^{-ctxy}U_{\frac{\alpha+1}{2}}\left(\frac{2e^{-2t}xy}{1-e^{-4t}}\right)\sqrt{\frac{t}{xy}}\frac{dt}{\sqrt{t}}\\
&\leq C\int_0^{xy} t^{-2}y(xy)^{-\frac{\alpha}{2}}\left(\frac{t}{y^2}\right)^N dt \leq C\left(\frac{x}{y}\right)^N x^{-\alpha-1}.
\end{aligned}$$

In the last line we have used (4.5.23) and (4.5.3). If $xy > 1$ we similarly get

$$\int_0^1 |K_t^{[1]}(x,y)|\frac{dt}{\sqrt{t}} \leq Cy^{-M}.$$

Next, we deal with $K_t^{[1]}$ and $t > 1$. If $xy > e^2$ then

$$\begin{aligned}
\int_1^{\log\sqrt{xy}} |K_t^{[1]}(x,y)|\frac{dt}{\sqrt{t}} &\leq C\int_1^{\log\sqrt{xy}} e^{-4t}y(xy)^{-\frac{\alpha-1}{2}}\exp(-cy^2)(e^{-2t}xy)^{-\frac{1}{2}}\frac{dt}{\sqrt{t}} \leq Cy^{-M},\\
\int_{\log\sqrt{xy}}^\infty |K_t^{[1]}(x,y)|\frac{dt}{\sqrt{t}} &\leq C\int_{\log\sqrt{xy}}^\infty e^{-4t}y(xy)^{-\frac{\alpha-1}{2}}e^{-cy^2}(e^{-2t}xy)^{\frac{\alpha+1}{2}}\frac{dt}{\sqrt{t}} \leq Cy^{-M}.
\end{aligned}$$

Identically, when $xy < e^2$ we have $\int_1^\infty |K_t^{[1]}(x,y)|\frac{dt}{\sqrt{t}} \leq Cy^{-M}$.

We can write the same estimates for $K_t^{[2]}$. Thus we get

$$\int_0^\infty \left(|K_t^{[1]}(x,y)| + |K_t^{[2]}(x,y)|\right)\frac{dt}{\sqrt{t}} \leq Cy^{-M}\max(1, x^{M-\alpha-1}).$$

Observe that $g(x,y) = R^{\mathbf{L}}(x,y)$ (see Figure 1), so the last estimate implies

$$\sup_{y>3} \int_0^{y/2} |g(x,y)|d\mu(x) < \infty. \tag{4.5.24}$$

Case 2: $x > 3$, $y < 2x/3$.

We proceed very similarly to Case 1 and obtain

$$\int_0^\infty \left(|K_t^{[1]}(x,y)| + |K_t^{[2]}(x,y)|\right) \frac{dt}{\sqrt{t}} \leq Cx^{-M} \max(1, y^{M-\alpha-1}).$$

We have $g(x,y) = R^{\mathbf{L}}(x,y)$ (see Figure 1). Hence

$$\sup_{y<2} \int_3^\infty |g(x,y)|d\mu(x) < \infty, \qquad \sup_{y\geq 2} \int_{3y/2}^\infty |g(x,y)|d\mu(x) < \infty. \tag{4.5.25}$$

Case 3: ($x > 3$ or $y > 3$) and $|x-y| < y/2$.

Notice that

$$\begin{aligned} K_t^{[4]}(x,y) = &\left(\frac{2e^{-2t}}{1-e^{-4t}}\right)^{1/2} y(xy)^{-\alpha/2} \exp\left(-\frac{1+e^{-4t}}{1-e^{-4t}}\frac{(x-y)^2}{2}\right) \\ &\times \exp\left(-\frac{(1-e^{-2t})^2}{1-e^{-4t}}xy\right)\left(V_t^{[4']}(x,y) + V_t^{[4'']}(x,y) + V_t^{[4''']}(x,y)\right), \end{aligned} \tag{4.5.26}$$

where

$$\begin{aligned} V_t^{[4']}(x,y) &= \frac{2e^{-2t}}{1-e^{-4t}}\left(U_{(\alpha+1)/2}\left(\frac{2e^{-2t}}{1-e^{-4t}}xy\right) - \frac{1}{\sqrt{2\pi}}\right), \\ V_t^{[4'']}(x,y) &= \frac{1}{\sqrt{2\pi}}\left(\frac{2e^{-2t}}{1-e^{-4t}} - \frac{1+e^{-4t}}{1-e^{-4t}}\right) = -\frac{(1-e^{-2t})^2}{\sqrt{2\pi}(1-e^{-4t})}, \\ V_t^{[4''']}(x,y) &= -\frac{1+e^{-4t}}{1-e^{-4t}}\left(U_{(\alpha-1)/2}\left(\frac{2e^{-2t}}{1-e^{-4t}}xy\right) - \frac{1}{\sqrt{2\pi}}\right). \end{aligned}$$

By using (4.5.26) and (4.5.3) one obtains

$$\int_0^1 |K_t^{[4]}(x,y)|\frac{dt}{\sqrt{t}} \leq C|x-y|^{-1/2}x^{-\alpha-1}. \tag{4.5.27}$$

Also, as in Case 1, we get

$$\int_1^\infty \left(|K_t^{[3]}(x,y)| + |K_t^{[4]}(x,y)|\right) \frac{dt}{\sqrt{t}} \leq Cx^{-M}. \tag{4.5.28}$$

Next,

$$\int_0^1 K_t^{[3]}(x,y)\frac{dt}{\sqrt{t}} - \chi_{\{y|x-y|<1\}}\frac{B(xy)^{-\alpha/2}}{(\alpha+1)(x-y)} = D_1 - D_6 = \sum_{j=1}^5 (D_j - D_{j+1}), \tag{4.5.29}$$

where

$$D_2 = -\int_0^1 \frac{e^{-t}(1+e^{-4t})}{\sqrt{\pi}(1-e^{-4t})^{3/2}}\frac{x-y}{(xy)^{\frac{\alpha}{2}}} \exp\left(-\frac{1+e^{-4t}}{1-e^{-4t}}\frac{(x-y)^2}{2}\right) \exp\left(-\frac{(1-e^{-2t})^2}{1-e^{-4t}}xy\right)\frac{dt}{\sqrt{t}},$$

$$\begin{aligned}
D_3 &= -\int_0^1 \frac{1}{4t\sqrt{\pi}}(xy)^{-\alpha/2}(x-y)\exp\left(-\frac{(x-y)^2}{4t}\right)\exp\left(-txy\right)\frac{dt}{t},\\
D_4 &= -\int_0^{y^{-2}/4} \frac{1}{4t\sqrt{\pi}}(xy)^{-\alpha/2}(x-y)\exp\left(-\frac{(x-y)^2}{4t}\right)\exp\left(-txy\right)\frac{dt}{t},\\
D_5 &= -\int_0^{y^{-2}/4} \frac{1}{4t\sqrt{\pi}}(xy)^{-\alpha/2}(x-y)\exp\left(-\frac{(x-y)^2}{4t}\right)\frac{dt}{t},\\
D_6 &= -\chi_{\{y|x-y|<1\}}\int_0^{\infty} \frac{1}{4t\sqrt{\pi}}\frac{x-y}{(xy)^{\frac{\alpha}{2}}}\exp\left(-\frac{(x-y)^2}{4t}\right)\frac{dt}{t} = -\chi_{\{y|x-y|<1\}}\frac{(xy)^{-\alpha/2}}{\sqrt{\pi}(x-y)}.
\end{aligned}$$

By using the mean-value theorem, (4.5.3), and (4.5.23) one obtains

$$|D_j - D_{j+1}| \leq Cx^{-\alpha-1/2}|x-y|^{-1/2} \quad \text{for } j=1,2. \tag{4.5.30}$$

To deal with $D_j - D_{j+1}$ for $j = 3,4,5$ we consider:

Subcase 1: $y|x-y|<1$**.**

$$\begin{aligned}
|D_3 - D_4| &\leq C\int_{y^{-2}/4}^1 t^{-1}x^{-\alpha}|x-y|\frac{dt}{t} \leq Cx^{-\alpha+2}|x-y|,\\
|D_4 - D_5| &= \int_0^{y^{-2}/4} = \int_0^{(x-y)^2/4} \dots\, dt + \int_{(x-y)^2/4}^{y^{-2}/4} \dots\, dt = Y_1 + Y_2,\\
|Y_1| &\leq C\int_0^{(x-y)^2} t^{N-1}x^{-\alpha+2}|x-y|^{1-2N}dt \leq Cx^{-\alpha+2}|x-y|, \qquad (4.5.31)\\
|Y_2| &\leq C\int_{(x-y)^2/4}^{y^{-2}/4} x^{-\alpha+2}|x-y|\frac{dt}{t} \leq Cx^{-\alpha+2}|x-y|\ln\frac{1}{y|x-y|},\\
|D_5 - D_6| &\leq C\int_{y^{-2}/4}^{\infty} t^{-1}x^{-\alpha}|x-y|\frac{dt}{t} \leq Cx^{-\alpha+2}|x-y|.
\end{aligned}$$

Subcase 2: $y|x-y|>1$**.**

$$\begin{aligned}
|D_3 - D_4| &= \int_{y^{-2}/4}^1 = \int_{y^{-2}/4}^{y^{-1}|x-y|/4} \dots\, dt + \int_{y^{-1}|x-y|/4}^1 \dots\, dt = Y_3 + Y_4,\\
|Y_3| &\leq C\int_{y^{-2}/4}^{y^{-1}|x-y|/4} t^{N-2}x^{-\alpha}|x-y|^{1-2N}dt \leq Cx^{-\alpha+1-N}|x-y|^{-N},\\
|Y_4| &\leq C\int_{y^{-1}|x-y|/4}^1 t^{-2}x^{-\alpha}(txy)^{-N}|x-y|dt \leq Cx^{-\alpha+1-N}|x-y|^{-N}, \qquad (4.5.32)\\
|D_4 - D_5| &\leq C\int_0^{y^{-2}/4} t^{N-1}x^{-\alpha+2}|x-y|^{1-2N}dt \leq Cx^{-\alpha+1-M}|x-y|^{-M},\\
|D_5 - D_6| &= |D_5| = C\int_0^{y^{-2}/4} t^{N-1}x^{-\alpha}|x-y|^{1-2N}\frac{dt}{t} \leq Cx^{-\alpha+1-M}|x-y|^{-M}.
\end{aligned}$$

Reassuming, (4.5.26)–(4.5.32) lead to

$$\sup_{y>2}\int_{y/2}^{3y/2}\left|R^{\mathbf{L}}(x,y) - \chi_{\{y|x-y|<1\}}(x)\frac{B(xy)^{-\alpha/2}}{(\alpha+1)(x-y)}\right|d\mu(x) < \infty. \tag{4.5.33}$$

Moreover,

$$\chi_{\{y|x-y|<2\}}(x)\left|\frac{B(xy)^{-\alpha/2}}{(\alpha+1)(x-y)} - \frac{B}{x^{\alpha+1}-y^{\alpha+1}} - \frac{A-B}{x^{\alpha+1}+y^{\alpha+1}}\right| \leq Cx^{-\alpha-1}. \tag{4.5.34}$$

We claim that

$$\sup_{2<y<3}\int_3^{3y/2}|g(x,y)|d\mu(x)\le C \quad \text{and} \quad \sup_{3<y}\int_{y/2}^{3y/2}|g(x,y)|d\mu(x)\le C. \tag{4.5.35}$$

To prove (4.5.35) we split the area of integration into three parts that correspond to white, light grey, and dark grey regions from Figure 1.

- if $y|x-y|>2$ we have $\phi\left((x-y)/\rho(y)\right)=0$ and we deduce the statement directly from (4.5.33).
- if $1\le y|x-y|\le 2$ then we apply (4.5.33)–(4.5.34), and the inequality
$$\sup_{y>2}\int_{1<y|x-y|<2}\left(\left|\frac{B}{x^{\alpha+1}-y^{\alpha+1}}\right|+\left|\frac{A-B}{x^{\alpha+1}+y^{\alpha+1}}\right|\right)d\mu(x)\le C.$$
- if $y|x-y|<1$ then $\phi\left((x-y)/\rho(y)\right)=1$ and we use again (4.5.33)–(4.5.34).

Case 4: $x,y<3$, $x<y/2$.

By similar analysis to that we have used in Case 1 we obtain

$$\int_0^{xy}\left(|K_t^{[1]}(x,y)|+|K_t^{[2]}(x,y)|\right)\frac{dt}{\sqrt{t}}\le C\left(\frac{x}{y}\right)^M x^{-\alpha-1},$$
$$\int_{xy}^1\left(|K_t^{[1]}(x,y)|+|K_t^{[2]}(x,y)|\right)\frac{dt}{\sqrt{t}}\le C(xy)^{-\alpha/2}x^{-1},$$
$$\int_1^{\infty}\left(|K_t^{[1]}(x,y)|+|K_t^{[2]}(x,y)|\right)\frac{dt}{\sqrt{t}}\le C.$$

Therefore

$$\sup_{y<3}\int_0^{y/2}|R^{\mathbf{L}}(x,y)|d\mu(x)<\infty.$$

and, consequently,

$$\sup_{y<3}\int_0^{y/2}|g(x,y)|d\mu(x)<\infty, \tag{4.5.36}$$

since

$$\sup_{y<3}\int_0^{y/2}\left(\left|\frac{B}{x^{\alpha+1}-y^{\alpha+1}}\right|+\left|\frac{A-B}{x^{\alpha+1}+y^{\alpha+1}}\right|\right)d\mu(x)<\infty.$$

Case 5: $x,y<3$, $y<2x/3$.

By using (4.5.2) and (4.5.3), similarly as in Case 2, one obtains

$$\begin{aligned}
&\int_0^{xy}\left(|K_t^{[1]}(x,y)|+|K_t^{[2]}(x,y)|\right)\frac{dt}{\sqrt{t}}\le C\left(\frac{y}{x}\right)^M x^{-\alpha-1},\\
&\int_{xy}^{x^2}|K_t^{[1]}(x,y)|\frac{dt}{\sqrt{t}}\le C\left(\frac{y}{x}\right)^2 x^{-\alpha-1},\\
&\int_{x^2}^{1}|K_t^{[1]}(x,y)|\frac{dt}{\sqrt{t}}\le C\left(\frac{y}{x}\right)^2 x^{-\alpha-1},\\
&\int_1^{\infty}\left(|K_t^{[1]}(x,y)|+|K_t^{[2]}(x,y)|\right)\frac{dt}{\sqrt{t}}\le C.
\end{aligned} \tag{4.5.37}$$

Recall that $A = -2\gamma_1\gamma_2^{-1}$, where $\gamma_1 = \Gamma(\alpha/2+1)$ and $\gamma_2 = \Gamma((\alpha+1)/2)$. We write

$$\int_{xy}^{1} K_t^{[2]}(x,y)\frac{dt}{\sqrt{t}} - \frac{Ax}{(x^2+y^2)^{\alpha/2+1}} = E_1 - E_4 = \sum_{j=1}^{3}(E_j - E_{j+1}), \tag{4.5.38}$$

where

$$\begin{aligned} E_2 &= -\int_{xy}^{1} \frac{2e^{-2t}(1+e^{-4t})}{(1-e^{-4t})^2} x \exp\left(-\frac{1+e^{-4t}}{1-e^{-4t}}\frac{x^2+y^2}{2}\right)\gamma_2^{-1}\left(\frac{e^{-2t}}{1-e^{-4t}}\right)^{\frac{\alpha-1}{2}} \frac{dt}{\sqrt{t}}, \\ E_3 &= -\int_{xy}^{1} \frac{2}{\gamma_2}(4t)^{-\alpha/2-1} x \exp\left(-\frac{x^2+y^2}{4t}\right)\frac{dt}{t}, \\ E_4 &= -\int_{0}^{\infty} \frac{2}{\gamma_2}(4t)^{-\alpha/2-1} x \exp\left(-\frac{x^2+y^2}{4t}\right)\frac{dt}{t} = -2\frac{\gamma_1}{\gamma_2}\frac{x}{(x^2+y^2)^{\alpha/2+1}}. \end{aligned}$$

Applying (4.5.2) and the mean-value theorem, one gets

$$\begin{aligned} |E_1 - E_2| &\le C\, y^2 x^{-\alpha-3}, \\ |E_2 - E_3| &\le C\, x^{-\alpha+1}, \\ |E_3 - E_4| &\le C \max(1, y^M x^{-\alpha-1-M}). \end{aligned} \tag{4.5.39}$$

Moreover,

$$\left| E_4 - \frac{B}{x^{\alpha+1}-y^{\alpha+1}} + \frac{A-B}{x^{\alpha+1}+y^{\alpha+1}} \right| \le Cyx^{-\alpha-2}. \tag{4.5.40}$$

As a consequence of (4.5.37)–(4.5.40) we get

$$\sup_{y<2}\int_{3y/2}^{3} \left| R^{\mathbf{L}}(x,y) - \frac{B}{x^{\alpha+1}-y^{\alpha+1}} - \frac{A-B}{x^{\alpha+1}+y^{\alpha+1}} \right| d\mu(x) < \infty. \tag{4.5.41}$$

Also,

$$\sup_{y<2}\int_{1}^{3} \chi_{\{y<2x/3\}}\left(\left|\frac{B}{x^{\alpha+1}-y^{\alpha+1}}\right| + \left|\frac{A-B}{x^{\alpha+1}+y^{\alpha+1}}\right| \right) d\mu(x) < \infty. \tag{4.5.42}$$

Observe that if $x<1$ then $\phi((x-y)/\rho(y)) = 1$ (see Figure 1). Therefore (4.5.41)-(4.5.42) lead to

$$\sup_{y<2}\int_{3y/2}^{3} |g(x,y)| d\mu(x) < \infty. \tag{4.5.43}$$

Case 6: $x, y < 3$, $|x-y| < y/2$.

By using the decomposition (4.5.26) one obtains

$$\int_{0}^{xy} |K_t^{[4]}(x,y)|\frac{dt}{\sqrt{t}} \le C|x-y|^{-1/2}x^{-\alpha-1/2}. \tag{4.5.44}$$

In addition

$$\int_{xy}^{1}\left(|K_t^{[3]}(x,y)| + |K_t^{[4]}(x,y)|\right)\frac{dt}{\sqrt{t}} \le Cx^{-\alpha-1}, \quad \int_{1}^{\infty}\left(|K_t^{[3]}(x,y)| + |K_t^{[4]}(x,y)|\right)\frac{dt}{\sqrt{t}} \le C. \tag{4.5.45}$$

Denote

$$\int_0^{xy} K_t^{[3]}(x,y)\frac{dt}{\sqrt{t}} - \left(\frac{B}{x^{\alpha+1}-y^{\alpha+1}} + \frac{A-B}{x^{\alpha+1}+y^{\alpha+1}}\right) = F_1 - F_5 = \sum_{j=1}^{4}(F_j - F_{j+1}),$$

where

$$\begin{aligned} F_2 &= -\int_0^{xy} \frac{\sqrt{2}e^{-t}(1+e^{-4t})}{\sqrt{2\pi}(1-e^{-4t})^{\frac{3}{2}}}(xy)^{-\frac{\alpha}{2}}(x-y)\exp\left(-\frac{1+e^{-4t}}{1-e^{-4t}}\frac{(x-y)^2}{2}\right) \\ &\quad \times \exp\left(-\frac{(1-e^{-2t})^2}{1-e^{-4t}}xy\right)\frac{dt}{\sqrt{t}}, \\ F_3 &= -\int_0^{xy} \frac{1}{4\sqrt{\pi}t^2}(xy)^{-\alpha/2}(x-y)\exp\left(-\frac{(x-y)^2}{4t}\right)dt, \\ F_4 &= -\int_0^{\infty} \frac{1}{4\sqrt{\pi}t^2}(xy)^{-\alpha/2}(x-y)\exp\left(-\frac{(x-y)^2}{4t}\right)dt = -\frac{(xy)^{-\alpha/2}}{\sqrt{\pi}(x-y)}. \end{aligned}$$

Similar analysis to that we did in (4.5.39) leads to

$$|F_i - F_{i+1}| \le Cx^{-\alpha-1}, \; i = 1,\dots,4. \tag{4.5.46}$$

Thanks to (4.5.44)–(4.5.46), we have

$$\sup_{y<3}\int_0^3 \chi_{\{|x-y|<y/2\}}\left|R^{\mathbf{L}}(x,y) - \frac{B}{x^{\alpha+1}-y^{\alpha+1}} - \frac{A-B}{x^{\alpha+1}+y^{\alpha+1}}\right|d\mu(x) < \infty. \tag{4.5.47}$$

Observe that

$$\sup_{y<3}\int_0^3 \chi_{\{|x-y|<y/2\}}\chi_{\{|x-y|>1/2\}}\left(\left|\frac{B}{x^{\alpha+1}-y^{\alpha+1}}\right| + \left|\frac{A-B}{x^{\alpha+1}+y^{\alpha+1}}\right|\right)d\mu(x) < \infty. \tag{4.5.48}$$

Note that if $|x-y| < 1/2$ then $\phi\left((x-y)/\rho(y)\right) = 1$ (see Figure 1). Therefore, it is not difficult to see that (4.5.47)-(4.5.48) imply

$$\sup_{y<3}\int_0^3 \chi_{\{|x-y|<y/2\}}|g(x,y)|\frac{dt}{\sqrt{t}}d\mu(x) < \infty. \tag{4.5.49}$$

Finally, the required estimate (4.3.2) follows directly from (4.5.24), (4.5.25), (4.5.35), (4.5.36), (4.5.43), (4.5.49). □

BIBLIOGRAPHY

[1] P. Auscher, X.T. Duong, A. McIntosh, *Boundedness of Banach space valued singular integral operators and Hardy spaces*. Unpublished preprint (2005).

[2] C. Bennett, R. Sharpley, *Interpolation of operators*. Pure and Applied Mathematics 129, Academic Press, Boston, MA, 1988.

[3] F. Bernicot, J. Zhao, *New abstract Hardy spaces*. J. Funct. Anal. 255 (2008), 1761–1796.

[4] J.J. Betancor, J. Dziubański, G. Garrigos, *Riesz transform characterization of Hardy spaces associated with certain Laguerre expansions*. Tohoku Math. J. 62 (2010), no. 2, 215–231.

[5] J.J. Betancor, J. Dziubański, J.L. Torrea, *On Hardy spaces associated with Bessel operators*. J. Anal. Math. 107 (2009), 195–219.

[6] D.L. Burkholder, R.F. Gundy, M.L. Silverstein, *A maximal function characterisation of the class* H^p. Trans. Amer. Math. Soc. 157 (1971), 137–153.

[7] R.R. Coifman, *A real variable characterisation of* H^p. Studia Math. 51 (1974), 269–274.

[8] R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis*. Bull. Amer. Math. Soc. 83 (1977), 569–645.

[9] W. Czaja, J. Zienkiewicz, *Atomic characterization of the Hardy space* $H^1_L(\mathbb{R})$ *of one-dimensional Schrödinger operators with nonnegative potentials*. Proc. Amer. Math. Soc. 136 (2008), no. 1, 89–94.

[10] J. Duoandikoetxea, *Fourier Analysis*. Graduate Studies in Mathematics 29, Providence, RI, 2001.

[11] J. Dziubański, *Hardy spaces for Laguerre expansions*. Constr. Approx. 27 (2008), no. 3, 269–287.

[12] J. Dziubański, *Hardy spaces associated with semigroups generated by Bessel operators with potentials*. Houston J. Math. 34 (2008), no. 1, 205–234.

[13] J. Dziubański, *Atomic decomposition of Hardy spaces associated with certain Laguerre expansions.* J. Fourier Anal. Appl. 15 (2009), no. 2, 129–152.

[14] J. Dziubański, A. Hulanicki, *On semigroups generated by left-invariant positive differential operators on nilpotent Lie groups.* Studia Math. 94 (1989), 81–95.

[15] J. Dziubański, M. Preisner, *Riesz transform characterization of Hardy spaces associated with Schrdinger operators with compactly supported potentials.* Ark. Mat. 48 (2010), no. 2, 301–310.

[16] J. Dziubański, M. Preisner, *On Riesz transforms characterization of H^1 spaces associated with some Schrdinger operators.* Potential Anal. 35 (2011), no. 1, 39–50.

[17] J. Dziubański, M. Preisner, *Hardy spaces related to Schrdinger operators with potentials which are sums of L^p-functions.* Preprint (2011), arXiv:1109.5626v1.

[18] J. Dziubański, Jacek Zienkiewicz, *Hardy spaces associated with some Schrdinger operators.* Studia Math. 126 (1997), 149–160.

[19] J. Dziubański, J. Zienkiewicz, *Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality.* Rev. Mat. Iberoamericana 15 (1999), no. 2, 279–296.

[20] J. Dziubański, J. Zienkiewicz, *H^p spaces associated with Schrödinger operators with potentials from reverse Hölder classes.* Colloq. Math. 98 (2003), 5–38.

[21] J. Dziubański, J. Zienkiewicz, *Hardy spaces H^1 for Schrödinger operators with certain potentials.* Studia Math. 164 (2004), no. 1, 39–53.

[22] J. Dziubański, J. Zienkiewicz, *Hardy spaces H^1 for Schrödinger operators with compactly supported potentials.* Ann. Mat. Pura Appl. 184 (2005), 315–326.

[23] C. Feferman, E.M. Stein, *H^p spaces of several variables.* Acta Math. 129 (1972), no. 3–4, 137–193.

[24] D. Goldberg, *A local version of real Hardy spaces.* Duke Math. J. 46 (1979), 27–42.

[25] W. Hebisch, *Boundedness of L^1 spectral multipliers for an exponential solvable Lie group.* Colloq. Math. 73 (1997), 155–164.

[26] S. Hofmann, G.Z. Lu, D. Mitrea, M. Mitrea, L.X. Yan, *Hardy spaces associated with non-negative self-adjoint operators satisfying Davies-Gafney estimates.* Memoirs Amer. Math. Soc. 214 (2011), no. 1007.

[27] R.H. Latter, *A decomposition of $H^p(\mathbb{R}^n)$ in terms of atoms.* Studia Math. 62 (1978), no. 1, 93–101.

[28] N.N. Lebedev, *Special functions and their applications.* Dover, New York, 1972.

[29] R.A. Macías, C. Segovia, *A decomposition into atoms of distributions on spaces of homogeneous type.* Adv. in Math. 33 (1979), no. 3, 271-309.

[30] H.P. McKean, $-\Delta$ *plus a bad potential.* J. Math. Phys. 18 (1977), 1277–1279.

[31] B. Muckenhoupt, E. M. Stein, *Classical expansions and their relation to conjugate harmonic functions.* Trans. Amer. Math. Soc. 118 (1965), 17–92.

[32] A. Nowak, K. Stempak, *L^2-theory of Riesz transforms for orthogonal expansions.* J. Fourier Anal. Appl. 12 (2006), no. 6, 675–711.

[33] A. Nowak, K. Stempak, *Riesz transforms for multi-dimensional Laguerre function expansions.* Adv. Math. 215 (2007), no. 2, 642–678.

[34] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Applied Mathematical Sciences 44, Springer-Verlag, New York, 1983.

[35] M. Preisner, *Riesz transform characterization of spaces associated with certain Laguerre expansions.* J. Approx. Theory 164 (2012), no. 2, 229–252.

[36] Yu.A. Semenov, *Stability of L^p-spectrum of generalized Schrödinger operators and equivalence Greens functions.* Int. Math. Research Notices (1997), 573–593.

[37] E.M. Stein, *Singular integrals and differentiability properties of functions.* Princeton University Press, Princeton, NJ, 1970.

[38] E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton University Press, Princeton, NJ, 1993.

[39] E.M. Stein, G. Weiss, *On the theory of harmonic functions of several variables. I. The theory of H^p-spaces.* Acta Math. 103 (1960), 25–62.

[40] A. Uchiyama, *A maximal function characterization of H^p on the spaces of homogeneous type.* Trans. Amer. Math. Soc. 262 (1980), 579–592.

[41] G.N. Watson, *A treatise on the theory of Bessel functions.* Cambrige University Press, Cambridge, 1966.