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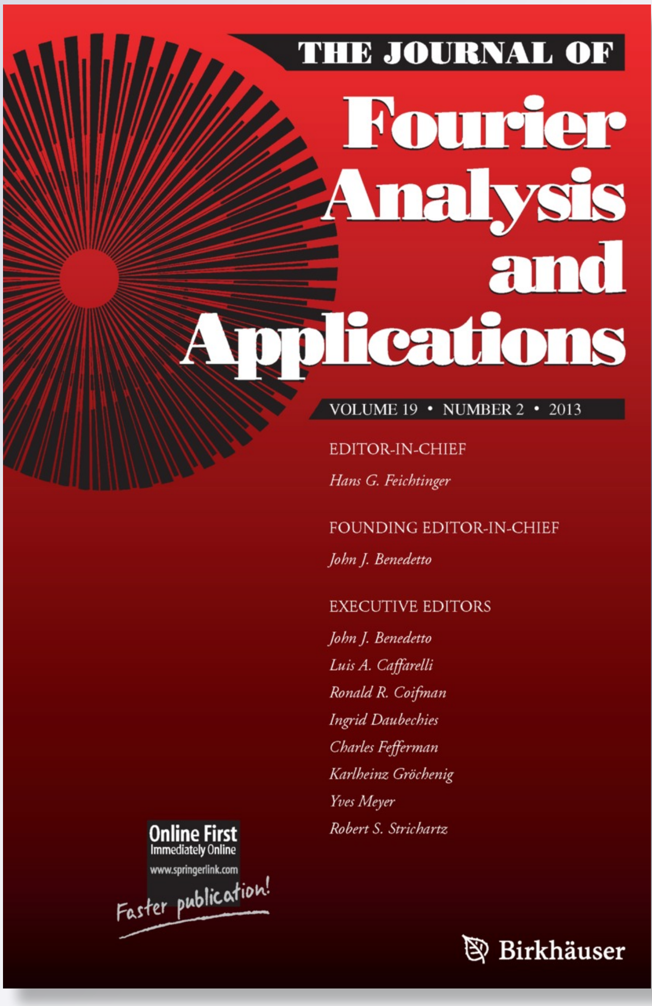

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Multivariate Hörmander-Type Multiplier Theorem for the Hankel transform

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Abstract Let $\mathcal{H}(f)(x) = \int_{(0,\infty)^d} f(\lambda) E_x(\lambda) d\nu(\lambda)$, be the multivariate Hankel transform, where $E_x(\lambda) = \prod_{k=1}^{d} (x_k\lambda_k)^{-\alpha_k+1/2} J_{\alpha_k-1/2}(x_k\lambda_k)$, with $d\nu(\lambda) = \lambda^{2\alpha} d\lambda$, $\alpha = (\alpha_1, \dots, \alpha_d)$. We give sufficient conditions on a bounded function $m(\lambda)$ which guarantee that the operator $\mathcal{H}(m\mathcal{H}f)$ is bounded on $L^p(d\nu)$ and of weak-type (1,1), or bounded on the Hardy space $H^1((0,\infty)^d, d\nu)$ in the sense of Coifman-Weiss.

Keywords Spectral multiplier · Bessel operator · Hankel transform · Hardy space

Mathematics Subject Classification Primary 42B15 · Secondary 42B20 · 42B30

1 Introduction and Preliminaries

For a multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_k > -1/2$, we consider the measure space $X = ((0,\infty)^d, d\nu(x))$, where $d\nu(x) = d\nu_1(x_1)\cdots d\nu_d(x_d)$, $d\nu_k(x_k) = x_k^{2\alpha_k} dx_k$, $k = 1, \dots, d$. The space X equipped with the Euclidean distance is a space of homogeneous type in the sense of Coifman-Weiss.

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For a function $f \in L^1(X)$ the (modified) Hankel transform $\mathcal{H}f$ is defined by

$$\mathcal{H}(f)(x) = \int_{(0,\infty)^d} f(\lambda) E_x(\lambda)\, d\nu(\lambda),$$

where

$$E_x(\lambda) = \prod_{k=1}^{d} (x_k\lambda_k)^{-\alpha_k+1/2} J_{\alpha_k-1/2}(x_k\lambda_k) = \prod_{k=1}^{d} E_{x_k}(\lambda_k).$$

Here J_ν is the Bessel function of the first kind of order ν, see [16, Chap. 5]. The system $\{E_x\}_{x\in(0,\infty)^d}$ consists of the eigenvectors of the Bessel operator

$$L = -\Delta - \sum_{k=1}^{d} \frac{2\alpha_k}{\lambda_k} \frac{\partial}{\partial\lambda_k};$$

that is, $L(E_x) = |x|^2 E_x$. Also, the functions E_{x_k}, $k = 1, \dots, d$, are eigenfunctions of the one-dimensional Bessel operators

$$L_k = -\frac{\partial^2}{\partial{\lambda_k}^2} - \frac{2\alpha_k}{\lambda_k} \frac{\partial}{\partial\lambda_k},$$

namely, $L_k(E_{x_k}) = x_k^2 E_{x_k}$.

It is known that $\mathcal{H}$ is an isometry on $L^2(X)$ that satisfies $\mathcal{H}^{-1} = \mathcal{H}$ (see, e.g., [23, Chap. 8]). Moreover, for $f \in L^2(X)$, we have

$$L_k(f) = \mathcal{H}\big(\lambda_k^2 \mathcal{H}f\big). \tag{1.1}$$

For $y \in X$ let τ^y be the d-dimensional generalized Hankel translation given by

$$\mathcal{H}\big(\tau^y f\big)(x) = E_y(x)\mathcal{H}f(x).$$

Clearly, $\tau^y f(x) = \tau^{y_1} \cdots \tau^{y_d} f(x)$, where for each $k = 1, \dots, d$, the operator τ^{y_k} is the one-dimensional Hankel translation acting on a function f as a function of the x_k variable with the other variables fixed. It is also known that, if $\alpha_k > 0$ for $k = 1, \dots, d$, then τ^y is a contraction on all $L^p(X)$ spaces, $1 \le p \le \infty$, and that

$$\tau^y f(x) = \tau^x f(y).$$

For two reasonable functions f and g define their Hankel convolution as

$$f \natural g(x) = \int_X \tau^x f(y) g(y)\, d\nu(y).$$

It is not hard to check that $f \natural g = g \natural f$ and

$$\mathcal{H}(f \natural g)(x) = \mathcal{H}f(x)\mathcal{H}g(x). \tag{1.2}$$

As a consequence of the contractivity of τ^y, for $\alpha \in (0,\infty)^d$, we also have

$$\|f \natural g\|_{L^1(X)} \le \|f\|_{L^1(X)} \|g\|_{L^1(X)}, \quad f \in L^1(X), \quad g \in L^1(X). \tag{1.3}$$

For details concerning translation, convolution, and transform in the Hankel setting we refer the reader to, e.g., [13, 23], and [26].

For a function $f \in L^1(X)$ and $t > 0$ let f_t denote the $L^1(X)$-dilation of f given by

$$(f_t)(x) = t^Q f(tx),$$

where $Q = \sum_{k=1}^d (2\alpha_k + 1)$. Then we have:

$$\mathcal{H}(f_t)(x) = \mathcal{H}f\big(t^{-1}x\big), \tag{1.4}$$

$$\tau^y(f_t)(x) = \big(\tau^{ty} f\big)_t(x). \tag{1.5}$$

Notice that Q represents the dimension of X at infinity, that is, $\nu(B(x,r)) \sim r^Q$ for large r.

Let $m : X \to \mathbb{C}$ be a bounded measurable function. Define the multiplier operator $\mathcal{T}_m$ by

$$\mathcal{T}_m(f) = \mathcal{H}(m\mathcal{H}f). \tag{1.6}$$

Clearly, $\mathcal{T}_m$ is bounded on $L^2(X)$. Also note that if $m(\lambda_1,\dots,\lambda_d) = n(\lambda_1^2,\dots,\lambda_d^2)$, for some bounded, measurable function n on $\mathbb{R}^d$, then from (1.1) it can be deduced that the Hankel multiplier operator defined by (1.6) coincides with the joint spectral multiplier operator $n(L_1,\dots,L_d)$. The smoothness requirements on m that guarantee the boundedness of $\mathcal{T}_m$ on, e.g., $L^p(X)$ will be stated in terms of appropriate Sobolev space norms.

For $z \in \mathbb{C}$, $\operatorname{Re} z > 0$, let

$$G_z(x) = \Gamma(z/2)^{-1} \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/4t} e^{-t} t^{z/2} \frac{dt}{t}$$

be the kernels of the Bessel potentials. Then

$$\|G_z\|_{L^1(\mathbb{R}^d)} \le \Gamma(\operatorname{Re} z/2)\big|\Gamma(z/2)\big|^{-1} \quad \text{and} \quad \mathcal{F}G_z(\xi) = \big(1+|\xi|^2\big)^{-z/2}, \tag{1.7}$$

where $\mathcal{F}G_z(\xi) = \int_{\mathbb{R}^d} G_z(x) e^{-i<x,\xi>} dx$ is the Fourier transform.

By definition, a function $f \in W_2^s(\mathbb{R}^d)$, $s > 0$, if and only if there exists a function $h \in L^2(\mathbb{R}^d)$ such that $f = h \star G_s$, and $\|f\|_{W_2^s(\mathbb{R}^d)} = \|h\|_{L^2(\mathbb{R}^d)}$.

Similarly, a function f belongs to the potential space $\mathcal{L}_s^\infty(\mathbb{R}^d)$, $s > 0$, if there is a function $h \in L^\infty(\mathbb{R}^d)$ such that $f = h \star G_s$ (see [22, Chap. V]). Then $\|f\|_{\mathcal{L}_s^\infty(\mathbb{R}^d)} = \|h\|_{L^\infty(\mathbb{R}^d)}$.

Denote $A_{r,R} = \{x \in \mathbb{R}^d : r \le |x| \le R\}$. The main results of the paper are Theorems 1.1, 1.2, and 1.3.

Theorem 1.1 *Assume that* $\alpha_k \geq 1/2$ *for* $k = 1, \dots, d$. *Let*

$$m(\lambda) = n(\lambda_1^2, \dots, \lambda_d^2), \tag{1.8}$$

where n *is a bounded function on* $\mathbb{R}^d$ *such that, for certain real number* $\beta > Q/2$ *and for some (equivalently, for every) non-zero radial function* $\eta \in C_c^\infty(A_{1/2,2})$, *we have*

$$\sup_{j\in\mathbb{Z}} \|\eta(\cdot)n(2^j\cdot)\|_{W_2^\beta(\mathbb{R}^d)} \leq C_\eta. \tag{1.9}$$

Then the multiplier operator $\mathcal{T}_m$ *is a Calderón-Zygmund operator associated with the kernel*

$$K(x,y) = \sum_{j\in\mathbb{Z}} \tau^y \mathcal{H}\big(\psi\big(2^{-j}(\lambda_1^2, \dots, \lambda_d^2)\big)m(\lambda)\big)(x),$$

where ψ *is a* $C_c^\infty(A_{1/2,2})$ *function such that*

$$\sum_{j\in\mathbb{Z}} \psi(2^{-j}\lambda) = 1, \quad \lambda \in \mathbb{R}^d\backslash\{0\}. \tag{1.10}$$

As a consequence $\mathcal{T}_m$ *extends to the bounded operator from* $L^1(X)$ *to* $L^{1,\infty}(X)$ *and from* $L^p(X)$ *to itself for* $1 < p < \infty$.

We denote by $H^1(X)$ the atomic Hardy space associated with X in the sense of [7]. More precisely, we say that a measurable function a is an $H^1(X)$-atom, if there exists a ball B, such that $\operatorname{supp} a \subset B$, $\|a\|_{L^\infty(X)} \leq 1/\nu(B)$, and $\int_{(0,\infty)^d} a(x)d\nu(x) = 0$. The space $H^1(X)$ is defined as the set of all $f \in L^1(X)$, which can be written as $f = \sum_{j=1}^\infty c_j a_j$, where a_j are atoms and $\sum_{j=1}^\infty |c_j| < \infty$, $c_j \in \mathbb{C}$. We equip $H^1(X)$ with the norm

$$\|f\|_{H^1(X)} = \inf \sum_{j=1}^\infty |c_j|, \tag{1.11}$$

where the infimum is taken over all absolutely summable sequences $\{c_j\}_{j\in\mathbb{N}}$, for which $f = \sum_{j=1}^\infty c_j a_j$, with a_j being $H^1(X)$-atoms.

Theorem 1.2 *Assume that* $\alpha_k \geq 1/2$ *for* $k = 1, \dots, d$. *Let* $m(\lambda) = n(\lambda_1^2, \dots, \lambda_d^2)$, *where* n *is a bounded function on* $\mathbb{R}^d$ *such that, for certain real number* $\beta > Q/2$ *and for some (equivalently, for every) non-zero radial function* $\eta \in C_c^\infty(A_{1/2,2})$, *Eq. (1.9) holds. Then the multiplier operator* $\mathcal{T}_m$ *extends to a bounded operator on the Hardy space* $H^1(X)$.

Theorem 1.3 *If we relax the conditions on* α_k *assuming only that* $\alpha_k > 0$, *then the conclusions of Theorems 1.1 and 1.2 hold provided there is* $\beta > Q/2$ *such that*

$$\sup_{j\in\mathbb{Z}} \|\eta(\cdot)n(2^j\cdot)\|_{\mathcal{L}_\beta^\infty(\mathbb{R}^d)} \leq C_\eta. \tag{1.12}$$

The weak type (1, 1) estimate under assumption (1.12) could be proved by applying a general multiplier theorem of Sikora [20]. However, in the case of the Hankel transform Theorem 1.3 has a simpler proof based on Lemmata 2.1 and 2.3.

Hankel multipliers, mostly in one variable, attracted attention of many authors. Gosselin and Stempak in [10] proved a Hankel multiplier theorem assuming that

$$\left(\int_{R/2}^{R} \left|m^{(s)}(\lambda)\right|^2 d\nu(\lambda)\right)^{1/2} \le B R^{(2\alpha+1)/2-s}$$

for $s = 0, \ldots, k$, where k is the least even integer $> (2\alpha + 1)/2$ and $m \in C^k(0, \infty)$. For other results see, e.g., [3, 5, 9, 11, 12], and references therein.

In [3] the authors considered multidimensional Hankel multipliers m of Laplace transform type, that is,

$$m(y) = |y|^2 \int_0^\infty e^{-t|y|^2} \phi(t)\,dt,$$

where $\phi \in L^\infty(0, \infty)$ (see [21]). To see that $m(\lambda)$ satisfies the assumptions of Theorems 1.1, 1.2, 1.3 we set

$$n(\lambda) = \Xi(\lambda)(\lambda_1 + \cdots + \lambda_d) \int_0^\infty e^{-t(\lambda_1+\cdots+\lambda_d)} \phi(t)\,dt, \tag{1.13}$$

where $\Xi \in C^\infty(\mathbb{R}^d \setminus \{0\})$, $\Xi(t\lambda) = \Xi(\lambda)$ for $t > 0$, $\Xi(\lambda) = 1$ for $\lambda \in (0, \infty)^d$, $\Xi(\lambda) = 0$ for $\lambda_1 + \cdots + \lambda_d < |\lambda|/d$. We easily check that $n(\lambda)$ satisfies the Mihlin condition, that is, $|\lambda|^{|\gamma|}|\partial^\gamma n(\lambda)| \le C_\gamma$ for all multiindeces γ. Hence, (1.9) and (1.12) hold with every $\beta > 0$.

A typical example of multipliers of Laplace transform type are the imaginary powers $m_u(y) = |y|^{2iu}$, $u \in \mathbb{R}$, which correspond to $\phi_u(t) = (\Gamma(1 - iu))^{-1} t^{-iu}$. In this case the resulting operators $\mathcal{T}_{m_u}$ coincide with L^{iu}. It is worth to remark that, for $\alpha \in (0, \infty)^d$, using Theorem 1.3 we can prove substantially better bounds on $L^p(X)$, $1 < p < \infty$, than

$$\left\|L^{iu}\right\|_{L^p(X)\to L^p(X)} \le C_p e^{\pi|u||\frac{1}{2}-\frac{1}{p}|}, \quad u \in \mathbb{R},$$

which were obtained in [3, Corollary 1.2]. Namely, for arbitrary small $\varepsilon > 0$, we have

$$\left\|L^{iu}\right\|_{L^p(X)\to L^p(X)} \le C_{p,\varepsilon}\left(1 + |u|\right)^{(Q+2\varepsilon)|\frac{1}{2}-\frac{1}{p}|}, \quad u \in \mathbb{R}. \tag{1.14}$$

In the Appendix we provide a sketch of the proof of (1.14) based on interpolation arguments.

Let us mention that a different multiplier result considering mixed smoothness Sobolev norms on $m(\lambda_1, \ldots, \lambda_d)$ was obtained by one of the authors in [27, Appendix]. It is valid if all $\alpha_k > 0$, although in that case the resulting operators need not be weak type (1, 1). For other results and references concerning spectral multiplier theorems on L^p spaces the reader is referred to [1, 6, 14, 17–20].

It is perhaps worth to point out that in $d = 1$ the assumptions (1.9) and (1.12) could be given in terms of function m instead of n. However, in the multivariate case we assume in (1.9) a W_2^β-Sobolev regularity of a function $n(\lambda)$ which is related with $m(\lambda)$ by (1.8). In the Appendix, see Example 5.1, it is shown that even for the classical Fourier multipliers supported in $A_{1/2,2}$ the Sobolev norms of $n(\lambda)$ and $m(\lambda)$ are not comparable when we apply the change of variables (1.8). Lastly, let us say that at present we do not know whether the smoothness threshold $\beta > Q/2$ required in (1.9) is optimal.

2 Auxiliary Estimates

In this section we prove some basic estimates needed in the sequel. Denote $w^s(x) = (1+|x|)^s$.

Lemma 2.1 *For every $s, \varepsilon > 0$ there exists a constant $C_{s,\varepsilon}$ such that if $m(\lambda) = n(\lambda_1^2, \dots, \lambda_d^2)$, $\operatorname{supp} n \subseteq A_{1/4,4}$, then*

$$\left\| \mathcal{H}(m) w^s \right\|_{L^2(X)} \leq C_{s,\varepsilon} \|n\|_{W_2^{s+d/2+\varepsilon}(\mathbb{R}^d)}. \tag{2.1}$$

Proof Since $m(\lambda) = g(\lambda_1^2, \dots, \lambda_d^2) e^{-|\lambda|^2}$, with $g(\lambda) = n(\lambda) e^{\lambda_1 + \cdots + \lambda_d}$, using the Fourier inversion formula for g, we get

$$\begin{aligned} (2\pi)^d m(\lambda) &= e^{-|\lambda|^2} \int_{\mathbb{R}^d} \mathcal{F}(g)(y) e^{i y_1 \lambda_1^2 + \cdots + i y_d \lambda_d^2} dy \\ &= \int_{\mathbb{R}^d} \mathcal{F}(g)(y) e^{(-1+iy_1)\lambda_1^2 + \cdots + (-1+iy_d)\lambda_d^2} dy. \end{aligned}$$

Applying the Hankel transform and changing the order of integration, we obtain

$$\mathcal{H}(m)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathcal{F}(g)(y) \mathcal{H}(e_{\mathbf{1}-iy})(x) dy, \tag{2.2}$$

where for $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, $e_z(\lambda) = \prod_{k=1}^d e_{z_k}(\lambda_k)$ with $e_{z_k}(\lambda_k) = e^{-z_k \lambda_k^2}$, while $\mathbf{1} = (1, \dots, 1)$. Clearly,

$$\mathcal{H}(e_{\mathbf{1}-iy})(x) = \prod_{k=1}^d \mathcal{H}_k(e_{1-iy_k})(x_k),$$

with $\mathcal{H}_k$ denoting the one-dimensional Hankel transform acting on the k-th variable. It is well known that for $t > 0$, $\mathcal{H}_k(e_t)(x_k) = C t^{-(2\alpha_k+1)/2} \exp(-x_k^2/4t)$, see [16, p. 132]. Moreover, for fixed x_k, the functions

$$z_k \mapsto \mathcal{H}_k(e_{z_k})(x_k) \quad \text{and} \quad z_k \mapsto C z_k^{-(2\alpha_k+1)/2} \exp\left(-\frac{x_k^2}{4 z_k}\right)$$

are holomorphic on $\{z_k \in \mathbb{C} : \operatorname{Re} z_k > 0\}$ (provided we choose an appropriate holomorphic branch of the power function $z_k^{-(2\alpha_k+1)/2}$). Hence, by the uniqueness of the holomorphic extension, we obtain

$$\mathcal{H}_k(e_{1-iy_k})(x_k) = C(1-iy_k)^{-(2\alpha_k+1)/2}\exp\left(-\frac{x_k^2}{4(1-iy_k)}\right).$$

Since $\operatorname{Re} x_k^2/4(1-iy_k) = x_k^2/4(1+y_k^2)$, the change of variable $x_k = (1+y_k^2)^{1/2}u_k$ leads to

$$\int_{(0,\infty)} \left|x_k^s \mathcal{H}(e_{1-iy_k})(x_k)\right|^2 d\nu_k(x_k) \lesssim \left(1+y_k^2\right)^s, \quad s \geq 0. \tag{2.3}$$

Now, observing that $(1+|x|)^{2s} \sim 1 + x_1^{2s} + \cdots + x_d^{2s}$ and using (2.3) we arrive at

$$\left\|\left(1+|\cdot|\right)^s \mathcal{H}(e_{\mathbf{1}-iy})(\cdot)\right\|_{L^2(X)} \lesssim \sum_{k=1}^d \left(1+y_k^2\right)^{s/2} \sim \left(1+|y|\right)^s.$$

The latter bound together with (2.2), Minkowski's integral inequality, and the Schwarz inequality give

$$\begin{aligned}\left\|\mathcal{H}(m)w^s\right\|_{L^2(X)} &\lesssim \int_{\mathbb{R}^d} \left|\mathcal{F}(g)(y)\right|\left(1+|y|\right)^s dy \\ &\lesssim \left(\int_{\mathbb{R}^d} \left|\mathcal{F}(g)(y)\right|^2 \left(1+|y|\right)^{2s+d+2\varepsilon} dy\right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{R}^d} \left(1+|y|\right)^{-d-2\varepsilon} dy\right)^{1/2} \\ &\lesssim \|g\|_{W_2^{s+d/2+\varepsilon}(\mathbb{R}^d)}\end{aligned}$$

for any fixed $\varepsilon > 0$. Since $g(\lambda) = n(\lambda)e^{\lambda_1+\cdots+\lambda_d} = n(\lambda)(e^{\lambda_1+\cdots+\lambda_d}\eta_0(\lambda))$, for some $\eta_0 \in C_c^\infty(A_{1/8,8})$, we see that $\|g\|_{W_2^{s+d/2+\varepsilon}(\mathbb{R}^d)} \leq C\|n\|_{W_2^{s+d/2+\varepsilon}(\mathbb{R}^d)}$, which implies (2.1). □

Remark that a slight modification of the reasoning above shows that if $m(\lambda) = n(\lambda_1^2, \ldots, \lambda_d^2)$, $n \in C_c^\infty(A_{1/2,2})$, then

$$\left|\mathcal{H}(m)(x)\right| \leq C_N \|n\|_{C^{N+d}(A_{1/2,2})} w^{-N}(x), \tag{2.4}$$

where C^N denotes the supremum norm on the space of N-times continuously differentiable functions.

Using ideas of Mauceri-Meda [18] combined with the fact that the Hankel transform is an L^2-isometry we can improve Lemma 2.1 in the following way.

Lemma 2.2 *Assume that $\alpha_k \geq 1/2$ for $k = 1, \ldots, d$. Then for every $s, \varepsilon > 0$, there is a constant $C_{s,\varepsilon}$ such that if $m(\lambda) = n(\lambda_1^2, \ldots, \lambda_d^2)$, $\operatorname{supp} n \subseteq A_{1/2,2}$, then*

$$\left\|\mathcal{H}(m)w^s\right\|_{L^2(X)} \leq C_{s,\varepsilon}\|n\|_{W_2^{s+\varepsilon}(\mathbb{R}^d)}.$$

Proof Let $h \in L^2(\mathbb{R}^d)$ be such that $n = h \star G_{s+\varepsilon}$. Set $s' = (s+\varepsilon)(d+6)/2\varepsilon$, $\theta = 2\varepsilon/(6+d)$. Define n_z by

$$\mathcal{F}(n_z)(\xi) = \mathcal{F}h(\xi)\big(1+|\xi|^2\big)^{-s'z/2}, \quad 0 \le \operatorname{Re} z \le 1.$$

Clearly, $n_z = h \star G_{s'z}$, $\operatorname{Re} z > 0$, and $n = n_\theta$. Let η_0 be a C_c^∞ function supported in $A_{1/4,4}$, equal to 1 on $A_{1/2,2}$, and let $N_z(\lambda) = n_z(\lambda)\eta_0(\lambda)$. Then $\operatorname{supp} N_z \subseteq A_{1/4,4}$ and $\mathcal{F}(N_z) = \mathcal{F}(n_z) \star \mathcal{F}(\eta_0)$. Define

$$m_z(\lambda) = n_z\big(\lambda_1^2, \dots, \lambda_d^2\big) \quad \text{and} \quad M_z(\lambda) = N_z\big(\lambda_1^2, \dots, \lambda_d^2\big).$$

Since $\alpha_k \ge 1/2$ for every $k = 1, \dots, d$, we have that $M_z \in L^2(X)$ and $\|M_z\|_{L^2(X)} \lesssim \|N_z\|_{L^2(\mathbb{R}^d)}$. Let g be an arbitrary $C_c^\infty(X)$ function with $\|g\|_{L^2(X)} = 1$. Set

$$F(z) = \int_X \mathcal{H}(M_z)(x)\big(1+|x|\big)^{(s'-3-d/2)z} g(x)\,d\nu(x). \tag{2.5}$$

Then F is holomorphic in the strip $S = \{z : 0 < \operatorname{Re} z < 1\}$ and also continuous and bounded on its closure $\bar{S}$. Using Parseval's equality and the facts that $\operatorname{supp} N_z \subseteq A_{1/4,4}$ and $\mathcal{F}(\eta_0) \in \mathcal{S}(\mathbb{R}^d)$, for $\operatorname{Re} z = 0$, we get

$$\begin{aligned}\big|F(z)\big| &\le \big\|\mathcal{H}(M_z)\big\|_{L^2(X)} = \|M_z\|_{L^2(X)} \le C\|N_z\|_{L^2(\mathbb{R}^d)} \sim \|\mathcal{F}N_z\|_{L^2(\mathbb{R}^d)} \\ &\le C_{\eta_0,s',\theta}\|n\|_{W_2^{s+\varepsilon}(\mathbb{R}^d)}.\end{aligned}$$

If $\operatorname{Re} z = 1$, then applying in addition Lemma 2.1, we obtain

$$\begin{aligned}\big|F(z)\big| &\le \big\|\mathcal{H}(M_z)w^{s'-3-d/2}\big\|_{L^2(X)} \le C\|N_z\|_{W_2^{s'}(\mathbb{R}^d)} \\ &\le C_{\eta_0}\|n_z\|_{W_2^{s'}(\mathbb{R}^d)} = C\|h\|_{L^2(\mathbb{R}^d)} = C\|n\|_{W_2^{s+\varepsilon}(\mathbb{R}^d)}.\end{aligned}$$

From the Phragmén-Lindelöf principle we get $|F(\theta)| \le C\|n\|_{W_2^{s+\varepsilon}(\mathbb{R}^d)}$. Taking the supremum over all such g we arrive at

$$\big\|\mathcal{H}(M_\theta)w^{(s'-3-d/2)\theta}\big\|_{L^2(X)} \le C\|n\|_{W_2^{s+\varepsilon}(\mathbb{R}^d)}.$$

Recall that $n = n_\theta = N_\theta$, so that also $m = m_\theta = M_\theta$, hence we get the desired conclusion. □

Notice, that the assumption $\alpha_k \ge 1/2$ was crucial to get $\|M_z\|_{L^2(X)} \le C\|N_z\|_{L^2(\mathbb{R}^d)}$ in the proof of Lemma 2.2. This was due to the fact that, in the case $d > 1$, if $\lambda \in \operatorname{supp} N_z \subseteq A_{1/4,4}$, then it can happen that $\lambda_i = 0$ for some $i = 1, \dots, d$. In the full range of α's we can overcome this difficulty, but with different Sobolev condition.

Lemma 2.3 *If we relax the conditions on α_k in Lemma 2.2 by assuming that $\alpha_k > -\frac{1}{2}$ for $k = 1, \dots, d$, then*

$$\big\|\mathcal{H}(m)w^s\big\|_{L^2(X)} \le C_{s,\varepsilon}\|n\|_{\mathcal{L}^\infty_{s+\varepsilon}(\mathbb{R}^d)}.$$

Proof We argue similarly to the proof of Lemma 2.2. Indeed, write $n = h \star G_{s+\varepsilon}$, where $h \in L^\infty(\mathbb{R}^d)$. Since $\operatorname{supp} n \subset A_{1/2,2}$, one can prove that $h \in L^2(\mathbb{R}^d)$ and $\|h\|_{L^2(\mathbb{R}^d)} \le C_{s,\varepsilon}\|n\|_{\mathcal{L}^\infty_{s+\varepsilon}}$.

Set $s' = (2s+\varepsilon)(6+d)/2\varepsilon$, $\theta = \varepsilon/(6+d)$ and define

$$N_z(\lambda) = \eta_0(\lambda) h \star G_{s'z+\varepsilon/2}(\lambda), \quad \lambda \in \mathbb{R}^d, \quad 0 \le \operatorname{Re} z \le 1.$$

Then for every $z \in \bar{S}$ the function $N_z(\lambda)$ is continuous and supported in $A_{1/4,4}$. Let $M_z(\lambda) = N_z(\lambda_1^2, \dots, \lambda_d^2)$. Clearly, $M_\theta = m$. Moreover, by (1.7),

$$\|M_z\|_{L^2(X)} \le C\|M_z\|_{L^\infty(X)} \le C\|N_z\|_{L^\infty(\mathbb{R}^d)} \le C_{s,\varepsilon}\|h\|_{L^\infty(\mathbb{R}^d)} = C_{s,\varepsilon}\|n\|_{\mathcal{L}^\infty_{s+\varepsilon}(\mathbb{R}^d)}.$$

We now use the new functions M_z to define a bounded holomorphic function $F(z)$ by the formula (2.5). Obviously $|F(z)| \le C_{s,\varepsilon}\|n\|_{\mathcal{L}^\infty_{s+\varepsilon}}$ for $\operatorname{Re} z = 0$. To estimate $F(z)$ for $\operatorname{Re} z = 1$ we utilize Lemma 2.1 and obtain

$$\begin{aligned} |F(z)| &\le \|\mathcal{H}(M_z) w^{s'-3-d/2}\|_{L^2(X)} \le C\|N_z\|_{W_2^{s'}(\mathbb{R}^d)} \\ &\le C_{\eta_0,s,\varepsilon}\|h\|_{L^2(\mathbb{R}^d)} \le C_{s,\varepsilon}\|n\|_{\mathcal{L}^\infty_{s+\varepsilon}(\mathbb{R}^d)}. \end{aligned}$$

An application of the Phragmén-Lindelöf principle for $z = \theta$ finishes the proof. □

We will also need the following off-diagonal estimate (see [9, Lemma 2.7]).

Lemma 2.4 *Assume that $\alpha_k > 0$ for $k = 1, \dots, d$. Let $\delta > 0$. Then there is $C > 0$ such that for every $y \in X$ and $r, t > 0$, we have*

$$\int_{|x-y|>r} |\tau^y(f_t)(x)| d\nu(x) \le C(rt)^{-\delta}\|f\|_{L^1(X, w^\delta(x)d\nu(x))}.$$

Proof By homogeneity it suffices to prove the lemma for $t = 1$. Let B be the left-hand side of the inequality from the lemma. If $|x-y| > r$ then there is $k \in \{1, \dots, d\}$ such that $|x_k - y_k| > r/\sqrt{d}$. Hence,

$$B \le \sum_{k=1}^d \int_{|x_k-y_k|>r/\sqrt{d}} |\tau^y(f)(x)| d\nu(x) = \sum_{k=1}^d B_k.$$

It is known that, for $\alpha \in (0,\infty)^d$, the generalized translations can be also expressed as

$$\tau^y f(x) = \int_{|x_1-y_1|}^{x_1+y_1} \cdots \int_{|x_d-y_d|}^{x_d+y_d} f(z_1, \dots, z_d) dW_{x_1,y_1}(z_1) \cdots dW_{x_d,y_d}(z_d), \tag{2.6}$$

with W_{x_k,y_k} being a probability measure supported in $[|x_k-y_k|, x_k+y_k]$ (see [13]). Thus,

$$B_k = \int_{|x_k-y_k|>r/\sqrt{d}} \left| \int_{|x_1-y_1|}^{x_1+y_1} \cdots \int_{|x_d-y_d|}^{x_d+y_d} f(z_1, \dots, z_d) dW_{x_1,y_1}(z_1) \cdots dW_{x_d,y_d}(z_d) \right| d\nu(x).$$

Introducing the factor $z_k^{\delta} z_k^{-\delta}$ to the inner integral in the above formula and denoting $g(x) = |f(x)| x_k^{\delta}$, we see that

$$B_k \leq C r^{-\delta} \int_X \int_{|x_1-y_1|}^{x_1+y_1} \cdots \int_{|x_d-y_d|}^{x_d+y_d} g(z) dW_{x_1,y_1}(z_1) \cdots dW_{x_d,y_d}(z_d) d\nu(x)$$
$$\leq C r^{-\delta} \| \tau^y g \|_{L^1(X)} \leq C r^{-\delta} \| f \|_{L^1(X, w^{\delta} d\nu)},$$

where in the last inequality we have used the fact that τ^y is a contraction on $L^1(X)$. □

Let $T_t(x, y) = \tau^y \mathcal{H}(e^{-t|\lambda|^2})(x)$ be the integral kernels of the heat semigroup corresponding to L. Clearly,

$$T_t(x, y) = T_t^{(1)}(x_1, y_1) \cdots T_t^{(d)}(x_d, y_d),$$

where $T_t^{(k)}(x_k, y_k)$ is the one-dimensional heat kernel associated with the operator L_k.

Lemma 2.5 *Assume that $\alpha_k > 0$ for $k = 1, \dots, d$. Then there is a constant $C > 0$ such that*

$$\int_X \left| T_1(x, y) - T_1\left(x, y'\right) \right| d\nu(x) \leq C |y - y'|, \quad y, y' \in X.$$

Proof The proof is a direct consequence of the one-dimensional result, see [10, Theorem 2.1], together with the equality

$$\int_0^{\infty} T_1^{(k)}(x_k, y_k) d\nu_k(x_k) = 1, \quad k = 1, 2, \dots, d.$$ □

In the proof of Theorem 1.2 the following version of [9, Lemma 2.5] will be used.

Lemma 2.6 *Assume that $\alpha_k > 0$ for $k = 1, \dots, d$. Let $f, g \in L^1((0,\infty)^d, w^{\delta} d\nu)$, with certain $\delta > 0$. Then*:

$$\| f \natural g \|_{L^1((0,\infty)^d, w^{\delta} d\nu)} \leq \| f \|_{L^1((0,\infty)^d, w^{\delta} d\nu)} \| g \|_{L^1((0,\infty)^d, w^{\delta} d\nu)}.$$

Proof After recalling the representation (2.6) the proof is analogous to the proof of [9, Lemma 2.5]. □

3 Proof of Theorem 1.1

The scheme of the proof takes ideas from [15]. Assume that (1.9) holds for some $\beta > Q/2$. Fix $\psi \in C_c^{\infty}(A_{1/2,2})$ satisfying (1.10). Let

$$K(x, y) = \sum_{j \in \mathbb{Z}} K_j(x, y) = \sum_{j \in \mathbb{Z}} \tau^y \mathcal{H}(m_j)(x),$$

where $m_j(\lambda) = \psi(2^{-j}(\lambda_1^2, \ldots, \lambda_d^2))m(\lambda) = (\psi(2^{-j}\cdot)n(\cdot))(\lambda_1^2, \ldots, \lambda_d^2)$. To prove that $\mathcal{T}_m$ is indeed a Calderón-Zygmund operator associated with the kernel $K(x, y)$ we need to verify that it satisfies the Hörmander integral condition, i.e.,

$$\int_{|x-y|>2|y-y'|} \left|K(x, y) - K\left(x, y'\right)\right| d\nu(x) \le C \tag{3.1}$$

for $y, y' \in X$, and the association condition

$$\mathcal{T}_m f(x) = \int_X K(x, y) f(y) d\nu(y) \tag{3.2}$$

for compactly supported $f \in L^\infty(X)$ such that $x \notin \operatorname{supp} f$. We start by proving (3.1). It suffices to show that

$$D_j\left(y, y'\right) = \int_{|x-y|>2|y-y'|} \left|K_j(x, y) - K_j\left(x, y'\right)\right| d\nu(x) \le C_j, \quad \text{with} \sum_{j\in\mathbb{Z}} C_j < \infty.$$

Let $r = 2|y - y'|$ and assume first $j > -2\log_2 r$. Let

$$\tilde{m}_j(\lambda) = m_j\left(2^{j/2}\lambda\right) = \left(\psi(\cdot)n\left(2^j\cdot\right)\right)\left(\lambda_1^2, \ldots, \lambda_d^2\right).$$

Note that $\operatorname{supp}(\psi(\cdot)n(2^j\cdot)) \subseteq A_{1/2,2}$. From (1.4) we see that

$$\mathcal{H}(m_j)(x) = 2^{jQ/2}\mathcal{H}(\tilde{m}_j)\left(2^{j/2}x\right) = \left(\mathcal{H}(\tilde{m}_j)\right)_{2^{j/2}}(x).$$

From the Schwarz inequality, Lemma 2.2, and the assumption (1.9) we get

$$\begin{aligned} \int_X \left|\mathcal{H}(\tilde{m}_j)\right| w^\delta d\nu &\le \left(\int_X \left|\mathcal{H}(\tilde{m}_j)\right|^2 w^{Q+4\delta} d\nu\right)^{1/2} \left(\int_X w^{-Q-2\delta} d\nu\right)^{1/2} \\ &\le C_\delta \left\|\psi(\cdot)n\left(2^j\cdot\right)\right\|_{W_2^\beta(\mathbb{R}^d)} \le C_\delta, \end{aligned} \tag{3.3}$$

for sufficiently small $\delta > 0$. Consequently, from Lemma 2.4 it follows that

$$\begin{aligned} D_j\left(y, y'\right) &\lesssim \int_{|x-y|>r} \left|\tau^y\left(\mathcal{H}(\tilde{m}_j)\right)_{2^{j/2}}(x)\right| d\nu(x) \\ &\quad + \int_{|x-y'|>r/2} \left|\tau^{y'}\left(\mathcal{H}(\tilde{m}_j)\right)_{2^{j/2}}(x)\right| d\nu(x) \\ &\lesssim \left(2^{j/2}r\right)^{-\delta} \int_X \left|\mathcal{H}(\tilde{m}_j)\right| w^\delta d\nu \le C_\delta\left(2^{j/2}r\right)^{-\delta}, \end{aligned}$$

so that $\sum_{j>-2\log_2 r} D_j(y, y') \le C$.

Assume now $j \le -2\log_2 r$. Decompose $\tilde{m}_j(\lambda) = \tilde{\theta}_j(\lambda)e^{-|\lambda|^2}$, so that we have $\tilde{\theta}_j(\lambda) = (\psi(\cdot)\exp(\cdot_1 + \cdots + \cdot_d)n(2^j\cdot))(\lambda_1^2, \ldots, \lambda_d^2)$. Clearly, $\psi(\lambda)e^{\lambda_1+\cdots+\lambda_d}$ is a C_c^∞ function supported in $A_{1/2,2}$. Denote $\tilde{\Theta}_j(x) = \mathcal{H}(\tilde{\theta}_j)(x)$. Since $\mathcal{H}(m_j) =$

$(\mathcal{H}(\tilde{m}_j))_{2^{j/2}}$ and $\mathcal{H}(\tilde{m}_j)=\tilde{\Theta}_j\natural\mathcal{H}(e^{-|\lambda|^2})$ (which is a consequence of (1.2)), by using (1.5), we get

$$K_j(x,y)-K_j(x,y')=\left(\tau^{2^{j/2}y}\mathcal{H}(\tilde{m}_j)\right)_{2^{j/2}}(x)-\left(\tau^{2^{j/2}y'}\mathcal{H}(\tilde{m}_j)\right)_{2^{j/2}}(x)$$
$$=\left(\tilde{\Theta}_j\natural\left(T_1\left(\cdot,2^{j/2}y\right)-T_1\left(\cdot,2^{j/2}y'\right)\right)\right)_{2^{j/2}}(x).$$

Proving (3.3) with $\tilde{m}_j$ replaced by $\tilde{\theta}_j$ and $\delta=0$ poses no difficulty. Hence, from Lemma 2.5 and (1.3) we obtain

$$D_j\left(y,y'\right)\le\|\tilde{\Theta}_j\|_{L^1(X)}\left\|T_1\left(\cdot,2^{j/2}y\right)-T_1\left(\cdot,2^{j/2}y'\right)\right\|_{L^1(X)}\le C2^{j/2}|y-y'|.$$

Consequently, $\sum_{j\le-2\log_2 r}D_j(y,y')\le C$ and the proof of (3.1) is finished.

Now we turn to the proof of (3.2). Let $f\in L^\infty(X)$ be a compactly supported function and $x\notin\operatorname{supp} f$. Then, there are $R>r>0$ such that

$$\int_X K_j(x,y)f(y)d\nu(y)=\int_{R>|x-y|>r}K_j(x,y)f(y)d\nu(y).$$

Since $\tau^y(\mathcal{H}(m_j))(x)=\tau^x(\mathcal{H}(m_j))(y)$, proceeding as in the first part of the proof of (3.1) we can easily check that $\sum_{j>-2\log_2 r}|K_j(x,y)|$ is integrable over $\{y\in X: |x-y|>r\}$. Hence, using the dominated convergence theorem (recall that $f\in L^\infty$),

$$\sum_{j>-2\log_2 r}\int_X K_j(x,y)f(y)d\nu(y)=\int_X\sum_{j>-2\log_2 r}K_j(x,y)f(y)d\nu(y). \tag{3.4}$$

From (1.2) it follows that

$$\mathcal{T}_{m_j}f(x)=H(m_j)\natural f(x)=\int_X K_j(x,y)f(y)d\nu(y), \tag{3.5}$$

with $\mathcal{T}_{m_j}$ defined as in (1.6). Since the Hankel transform is an $L^2(X)$-isometry, from the dominated convergence theorem we conclude that $\sum_{j>-2\log_2 r}\mathcal{T}_{m_j}f=\mathcal{T}_{m^{[\infty]}}f$, where the sum converges in $L^2(X)$ and $m^{[\infty]}=\sum_{j>-2\log_2 r}m_j$. Hence, combining (3.4) and (3.5), we obtain

$$\mathcal{T}_{m^{[\infty]}}f(x)=\int_X\sum_{j>-2\log_2 r}K_j(x,y)f(y)d\nu(y),$$

for a.e.x outside supp f. The function $m^{[0]}=m-m^{[\infty]}$ is bounded and compactly supported. Consequently, from (1.2) we get $\mathcal{T}_{m^{[0]}}f(x)=\mathcal{H}(m^{[0]})\natural f(x)$. Moreover, we see that $\sum_{j\le-2\log_2 r}|m_j(\lambda)|\le C|m(\lambda)|\le C$. Hence, from (2.6) we conclude

$$\tau^y\left(\mathcal{H}m^{[0]}\right)(x)=\sum_{j\le-2\log_2 r}\tau^y(\mathcal{H}m_j)(x),$$

so that

$$\mathcal{T}_{m^{[0]}} f(x) = \int_X \sum_{j \le -2\log_2 r} K_j(x,y) f(y) d\nu(y).$$

Then $\mathcal{T}_m f(x) = \mathcal{T}_{m^{[0]}} f(x) + \mathcal{T}_{m^{[\infty]}} f(x) = \int_X K(x,y) f(y) d\nu(y)$, as desired. □

Let us finally comment that the proof of Theorem 1.3 goes in the same way as that of Theorem 1.1. The only difference is that we use Lemma 2.3 instead of Lemma 2.2.

4 Proof of Theorem 1.2

We shall need the maximal-function characterization of $H^1(X)$. Define the operator $\mathcal{M} f(x) = \sup_{t>0} |T_t f(x)|$, where $T_t f(x) = \int_{(0,\infty)^d} T_t(x,y) f(y) d\nu(y)$. Then we have the following proposition.

Proposition 4.1 *There exists* $C > 0$ *such that*

$$C^{-1} \|f\|_{H^1(X)} \le \|\mathcal{M} f\|_{L^1(X)} \le C \|f\|_{H^1(X)}. \tag{4.1}$$

The reader who is convinced that Proposition 4.1 is true may skip Lemmata 4.2 and 4.3 and continue with the proof of Theorem 1.2 on page 13. To prove the proposition we need two lemmata.

Lemma 4.2 *The heat kernel* $T_t(x,y)$ *satisfies the Gaussian bounds*:

$$0 \le T_t(x,y) \le \frac{C}{\nu(B(x,\sqrt{t}))} \exp\bigl(-c|x-y|^2/t\bigr), \tag{4.2}$$

and the following Lipschitz-type estimates:

$$\bigl|T_t(x,y) - T_t\bigl(x,y'\bigr)\bigr| \le \frac{C|y-y'|}{\sqrt{t}\,\nu(B(x,\sqrt{t}))} \exp\bigl(-c|x-y|^2/t\bigr), \quad 2|y-y'| \le |x-y|, \tag{4.3}$$

$$\bigl|T_t(x,y) - T_t\bigl(x,y'\bigr)\bigr| \le \frac{C|y-y'|}{\sqrt{t}\,\nu(B(x,\sqrt{t}))}. \tag{4.4}$$

Proof Clearly, since the product of Gaussian kernels is Gaussian and ν is a product of doubling measures, it suffices to focus on $d = 1$. It is known that for $\alpha > -1/2$

$$\begin{aligned} T_t(x,y) &= ct^{-1} \exp\bigl(-\bigl(x^2+y^2\bigr)/4t\bigr)(xy)^{-(2\alpha-1)/2} I_{(2\alpha-1)/2}(xy/2t) \\ &= ct^{-1} \exp\bigl(-|x-y|^2/4t\bigr) \exp(-xy/2t)(xy)^{-(2\alpha-1)/2} I_{(2\alpha-1)/2}(xy/2t), \end{aligned}$$

where I_μ is the modified Bessel function of order μ. Recall that

$$I_\mu(x) \sim \begin{cases} \frac{e^x}{\sqrt{2\pi x}} & \text{for } x \ge 1, \\ \frac{1}{\Gamma(\mu+1)}\bigl(\frac{x}{2}\bigr)^\mu & \text{for } 0 < x < 1, \end{cases}$$

(see, e.g., [16]). Hence, it is easy to obtain

$$T_t(x,y) \sim \begin{cases} t^{-(2\alpha+1)/2} \exp(-(x^2+y^2)/4t) & \text{if } xy < t, \\ t^{-1/2}(xy)^{-\alpha} \exp(-|x-y|^2/4t) & \text{if } xy \ge t. \end{cases} \tag{4.5}$$

Moreover,

$$\nu\big(B(x,\sqrt{t})\big) \sim \sqrt{t}(x+\sqrt{t})^{2\alpha}.$$

Now, (4.2) is a consequence of (4.5). To prove (4.3) and (4.4), using the identity $(x^{-\mu}I_\mu(x))' = x^{-\mu}I_{\mu+1}(x)$ and the asymptotics for I_μ we check that

$$\big|\partial_y T_t(x,y)\big| \lesssim \begin{cases} t^{-(2\alpha+3)/2}(x+y)\exp(-(x^2+y^2)/4t) & \text{if } xy < t, \\ \{t^{-3/2}|x-y| + t^{-1/2}y^{-1}\}(xy)^{-\alpha}\exp(-|x-y|^2/4t) & \text{if } xy \ge t. \end{cases}$$

From the above it is not hard to conclude that

$$\big|\nabla_y T_t(x,y)\big| \le \frac{C}{\sqrt{t}} \cdot \frac{1}{\nu(B(x,\sqrt{t}))} \exp\big(-c|x-y|^2/t\big).$$

The latter inequality easily implies (4.3) and (4.4). □

Let $\rho(x,y) = \inf\{\nu(B)\}$, where the infimum is taken over all balls B such that $x, y \in B$. Denote $B_\rho(x,r) = \{y \in X | \rho(x,y) < r\}$. We have:

- $\rho(x,y) \sim \nu(B(x,r_0))$, where $r_0 = |x-y|$,
- $\rho(x,y) \le A(\rho(x,z) + \rho(z,y))$
- $\nu(B_\rho(x,r)) \sim r$,

i.e., the triple $((0,\infty)^d, d\nu, \rho)$ is a space of homogenous type.

Lemma 4.3 *Let $K_r(x,y) = T_{t(x,r)}(x,y)$, where $t = t(x,r)$ is defined by $\nu(B(x,\sqrt{t})) = r$. Then the kernel rK_r satisfies the assumption of Uchiyama's Theorem, see* [25, *Corollary* 1'], *i.e., there are constants $A, \gamma > 0$ such that*

$$K_r(x,x) \ge A^{-1}r^{-1} > 0, \tag{4.6}$$

$$0 \le K_r(x,y) \le Cr^{-1}\left(1 + \frac{\rho(x,y)}{r}\right)^{-1-\gamma}, \tag{4.7}$$

and

$$\big|K_r(x,y) - K_r\big(x,y'\big)\big| \le \frac{C}{r}\left(1 + \frac{\rho(x,y)}{r}\right)^{-1-2\gamma}\left(\frac{\rho(y,y')}{r}\right)^{\gamma},$$

$$\rho\big(y,y'\big) \le \frac{r+\rho(x,y)}{4A}. \tag{4.8}$$

Proof (sketch) The inequality (4.6) is obvious, once we recall (4.5). To prove (4.7) and (4.8) we use Lemma 4.2. From (4.2) we have

$$K_r(x,y) \le Cr^{-1}\exp\big(-c|x-y|^2/t\big).$$

Now, since

$$\left(1+\frac{\rho(x,y)}{r}\right)\leq C\left(1+\frac{\nu(B(x,|x-y|))}{\nu(B(x,\sqrt{t}))}\right)$$
$$\leq C\left(1+\frac{|x-y|}{\sqrt{t}}\right)^n\leq C_\varepsilon \exp\big(\varepsilon|x-y|^2/t\big), \tag{4.9}$$

we get (4.7). Observe that there is $q>0$, such that

$$R^q\nu\big(B(x,t)\big)\leq C\nu\big(B(x,Rt)\big),\quad t>0,\quad R\geq 1. \tag{4.10}$$

Note that we can take $q=1$, if $\alpha_k\geq 0$, $k=1,\dots,d$. The estimate (4.8) for $\rho(y,y')\geq r/(2A)$ is a simple consequence of (4.7). In the opposite case, i.e., $\rho(y,y')<r/(2A)$, we first note that (4.10) implies

$$\frac{\rho(y,y')}{r}\sim\frac{\nu(B(y,|y-y'|)}{\nu(B(x,\sqrt{t}))}=\frac{\nu(B(y,|y-y'|))}{\nu(B(y,\sqrt{t})}\cdot\frac{\nu(B(y,\sqrt{t}))}{\nu(B(x,\sqrt{t}))}$$
$$\gtrsim\left(\frac{|y-y'|}{\sqrt{t}}\right)^\kappa\cdot\frac{\nu(B(y,\sqrt{t}))}{\nu(B(y,\sqrt{t}+|x-y|))}$$
$$\gtrsim\left(\frac{|y-y'|}{\sqrt{t}}\right)^\kappa\cdot\left(\frac{\sqrt{t}}{\sqrt{t}+|x-y|}\right)^{Q+d}, \tag{4.11}$$

where $\kappa=q$, if $|y-y'|\geq\sqrt{t}$, and $\kappa=Q+d$, in the other case. Then (4.8) can be deduced from (4.3), (4.4), and (4.11). □

Proof of Proposition 4.1 Since $\nu(B(x,\sqrt{t}))$ is an increasing continuous function of t taking values in $(0,\infty)$, the maximal function

$$K^*f(x)=\sup_{r>0}\left|\int_{(0,\infty)^d}K_r(x,y)f(y)d\nu(y)\right|$$

coincides with $\mathcal{M}f$. Now, using Lemma 4.3 together with Uchiyama's theorem, [25, Corollary 1'], we obtain a variant of the equivalence (4.1), with respect to atoms corresponding to the metric ρ. A simple observation that

$$B\big(x,\sqrt{t(x,r)}\big)\subset B_\rho(x,r)\subset B\big(x,C\sqrt{t(x,r)}\big),$$

for some $C>0$, finishes the proof. □

The reader interested in more detailed proof of Proposition 4.1 is referred to [4]. Having Proposition 4.1 we turn to prove Theorem 1.2.

Proof of Theorem 1.2 The proof follows closely the one-dimensional case, see [9]. Since the operator $\mathcal{T}_m$ maps continuously $H^1(X)$ into $\mathcal{D}'((0,\infty)^d)$, it suffices to prove that there exists a constant $C>0$, such that for every atom $a \in H^1(X)$, we have

$$\left\| \mathcal{M}(\mathcal{T}_m a)\right\|_{L^1(X)} \leq C. \tag{4.12}$$

If a is an atom associated with a ball $B(y_0, r)$, then clearly,

$$\begin{aligned}\left\| \mathcal{M}(\mathcal{T}_m a)\right\|_{L^1(B(y_0,2r),d\nu)} &\leq \nu\big(B(y_0,2r)\big)^{1/2}\left\| \mathcal{M}(\mathcal{T}_m a)\right\|_{L^2(B(y_0,2r),d\nu)} \\ &\leq \nu\big(B(y_0,2r)\big)^{1/2}\|a\|_{L^2(X)} \leq C. \end{aligned}\tag{4.13}$$

Fix a $C_c^\infty(A_{1/2,2})$ function ψ satisfying

$$\sum_{j\in\mathbb{Z}} \psi^2\big(2^{-j}\lambda\big) = 1, \quad \lambda \in \mathbb{R}^d \backslash \{0\}. \tag{4.14}$$

Analogously as in Sect. 3 we define

$$m_j(\lambda) = \psi^2\big(2^{-j}\big(\lambda_1^2,\dots,\lambda_d^2\big)\big)m(\lambda) = \big(\psi^2\big(2^{-j}\cdot\big)n(\cdot)\big)\big(\lambda_1^2,\dots,\lambda_d^2\big).$$

In view of (4.13) it is enough to show that

$$\sum_{j\in\mathbb{Z}} \left\| \mathcal{M}(\mathcal{T}_{m_j} a)\right\|_{L^1((B(y_0,2r))^c,d\nu)} \leq C. \tag{4.15}$$

Let

$$\begin{aligned} m_{(j,t)}(\lambda) &= m_j(\lambda)e^{-t|\lambda|^2}, &\quad \tilde{m}_{(j,t)}(\lambda) &= m_{(j,t)}\big(2^{j/2}\lambda\big), \\ M_{(j,t)}(x) &= \mathcal{H}(m_{(j,t)})(x), &\quad \tilde{M}_{(j,t)}(x) &= \mathcal{H}(\tilde{m}_{(j,t)})(x). \end{aligned}$$

Clearly, $M_{(j,t)}(x,y) = \tau^y M_{(j,t)}(x)$ are the integral kernels of the operators $\mathcal{T}_{e^{-t|\lambda|^2}m_j(\lambda)}$. Also,

$$M_{(j,t)}(x) = (\tilde{M}_{(j,t)})_{2^{j/2}}(x), \qquad M_{(j,t)}(x,y) = 2^{jQ/2}\tilde{M}_{(j,t)}\big(2^{j/2}x, 2^{j/2}y\big). \tag{4.16}$$

The following are the key estimates in the proof of (4.15).

Lemma 4.4 *There exist $\delta>0$ and $C>0$ such that for all $j \in \mathbb{Z}$ and all $r>0$ we have*

$$\int_{|x-y|>r} \sup_{t>0} \left|M_{(j,t)}(x,y)\right| d\nu(x) \leq C\big(2^{j/2}r\big)^{-\delta}, \tag{4.17}$$

$$\int_{(0,\infty)^d} \sup_{t>0} \left|M_{(j,t)}(x,y) - M_{(j,t)}\big(x,y'\big)\right| d\nu(x) \leq C2^{j/2}|y-y'|. \tag{4.18}$$

Proof Denote

$$\psi_{(j,t)}(\lambda) = \psi\big(2^{-j}\big(\lambda_1^2,\dots,\lambda_d^2\big)\big)e^{-t|\lambda|^2}, \qquad \tilde{\psi}_{(j,t)}(\lambda) = \psi_{(j,t)}\big(2^{j/2}\lambda\big),$$

$$\zeta_j(\lambda) = \psi\big(2^{-j}(\lambda_1^2, \ldots, \lambda_d^2)\big)m(\lambda) = \big(\psi(2^{-j}\cdot)n(\cdot)\big)(\lambda_1^2, \ldots, \lambda_d^2),$$
$$\tilde{\zeta}_j(\lambda) = \zeta_j(2^{j/2}\lambda) = \big(\psi(\cdot)n(2^j\cdot)\big)(\lambda_1^2, \ldots, \lambda_d^2).$$

Let $\tilde{Z}_j(x) = \mathcal{H}(\tilde{\zeta}_j)(x)$, $\tilde{\Psi}_{(j,t)}(x) = \mathcal{H}(\tilde{\psi}_{(j,t)})(x)$. Arguing as in (3.3), we see that

$$\sup_{j\in\mathbb{Z}} \|\tilde{Z}_j w^\delta\|_{L^1(X)} \le C, \tag{4.19}$$

for sufficiently small $\delta > 0$. Observe that $\tilde{\psi}_{(j,t)} = n_{(j,t)}$, for some C_c^∞ function $n_{(j,t)}$ with $\operatorname{supp} n_{(j,t)} \subset A_{1/2,2}$. Moreover, we can check that $\sup_{(j,t)} \|n_{(j,t)}\|_{C^N} \le C_N$, for every $N \in \mathbb{N}$. Hence, using (2.4) we see that for every $N > 0$, there exists C_N' such that

$$\sup_{(j,t)} \big|\tilde{\Psi}_{(j,t)}(x)\big| \le C_N' w^{-N}(x).$$

From the above we see that

$$\big|\tilde{M}_{(j,t)}(x)\big| = \big|\tilde{\Psi}_{(j,t)} \natural \tilde{Z}_j(x)\big| \le C_N w^{-N} \natural |\tilde{Z}_j|(x).$$

Hence, using (4.19) and Lemma 2.6 we arrive at

$$\int_{(0,\infty)^d} \sup_{t>0} \big|\tilde{M}_{(j,t)}(x,y)\big| w^\delta d\nu(x) \le C.$$

Combining the above, together with (4.16) and Lemma 2.4, we get (4.17).

We now turn to the proof of (4.18). Let $\tilde{l}_{(j,t)}(\lambda) = e^{-t2^j|\lambda|^2}\psi(\lambda_1^2, \ldots, \lambda_d^2)e^{|\lambda|^2}$ and define $\tilde{L}_{(j,t)}(x) = \mathcal{H}(\tilde{l}_{(j,t)})(x)$. Clearly,

$$\tilde{m}_{(j,t)}(\lambda) = \tilde{l}_{(j,t)}(\lambda)\tilde{\zeta}_j(\lambda)e^{-|\lambda|^2}. \tag{4.20}$$

An argument analogous to the one presented in the previous paragraph shows that

$$\sup_{j\in\mathbb{Z}, t>0} \big|\tilde{L}_{(j,t)}(x)\big| \le C_N' w^{-N}(x).$$

As a consequence, there is $C > 0$, such that for every j

$$\Big\| \sup_{t>0} |\tilde{L}_{(j,t)} \natural \tilde{Z}_j| \Big\|_{L^1(X)} \le C. \tag{4.21}$$

Recalling (4.20), we obtain

$$\begin{aligned}
&\sup_{t>0} \big|\tilde{M}_{(j,t)}(x,y) - \tilde{M}_{(j,t)}(x,y')\big| \\
&\quad = \sup_{t>0} \left| \int_{(0,\infty)^d} \tau^x(\tilde{L}_{(j,t)} \natural \tilde{Z}_j)(z)\big(T_1(z,y) - T_1(z,y')\big)d\nu(z) \right| \\
&\quad \le \int_{(0,\infty)^d} \tau^z\Big(\sup_{t>0} |\tilde{L}_{(j,t)} \natural \tilde{Z}_j|\Big)(x)\big|T_1(z,y) - T_1(z,y')\big|d\nu(z).
\end{aligned} \tag{4.22}$$

From (1.3) together with (4.21), (4.22) and Lemma 2.5, we obtain

$$\int_{(0,\infty)^d} \sup_{t>0} \left|\tilde{M}_{(j,t)}(x,y) - \tilde{M}_{(j,t)}\big(x,y'\big)\right| d\nu(x) \le C|y-y'|. \tag{4.23}$$

Now, (4.18) is a consequence of (4.16) and (4.23). □

Using Lemma 4.4 and some standard arguments, as in the final stage of the proof of [9, Eq. (3.3)], we easily justify (4.15). Hence the proof is complete. □

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Appendix

Proof of (1.14) The proof of Theorem 1.3 actually shows that

$$\begin{aligned}&\|\mathcal{T}_m f\|_{L^1(X)\to L^{1,\infty}(X)}\\ &\quad\le C_\varepsilon\Big(1+\|m\|_{L^\infty}^2+\sup_{j\in\mathbb{Z}}\big\|\eta(\cdot)n\big(2^j\cdot\big)\big\|_{\mathcal{L}^\infty_{Q/2+\varepsilon}(\mathbb{R}^d)}\Big)\|f\|_{L^1(X)}.\end{aligned} \tag{5.1}$$

Now, using (1.13) we write $n_u(\lambda) = \Xi(\lambda)(\lambda_1+\cdots+\lambda_d)^{iu}$, so that $n_u(\lambda_1^2,\dots,\lambda_d^2) = m_u(\lambda)$. We claim that

$$\sup_{j\in\mathbb{Z}}\big\|\eta(\cdot)n_u\big(2^j\cdot\big)\big\|_{\mathcal{L}^\infty_s(\mathbb{R}^d)} \le C_s\big(1+|u|\big)^s, \quad s>0. \tag{5.2}$$

Using (5.2) and combining it with (5.1), we get $\|L^{iu}\|_{L^1(X)\to L^{1,\infty}(X)} \le C_\varepsilon(1+|u|)^{Q/2+\varepsilon}$. Since $\|L^{iu}\|_{L^2(X)\to L^2(X)} = 1$, using the Marcinkiewicz interpolation theorem, see, e.g. [8, (2.2) p. 30], together with a duality argument, we obtain (1.14) for all $1<p<\infty$.

Now, we sketch the proof of (5.2). Let $B^s_{p,q}$, $1\le p,q\le\infty$, $s\ge 0$, be the Besov space, as defined in [2, p. 141]. It is known, see [2, Theorem 6.2.4 (10), p.142], that $B^s_{\infty,q}$ is the real interpolation space of the spaces $\mathcal{L}^\infty_s = \mathcal{L}^\infty_s(\mathbb{R}^d)$, precisely

$$\big(\mathcal{L}^\infty_{s_0},\mathcal{L}^\infty_{s_1}\big)_{\theta,q} = B^s_{\infty,q}, \quad s=(1-\theta)s_0+\theta s_1, \quad 0<\theta<1, \quad 1\le q\le\infty. \tag{5.3}$$

Moreover, from [2, Theorem 6.2.4 (9), p.142], we have

$$\|f\|_{\mathcal{L}^\infty_s} \le C_s\|f\|_{B^s_{\infty,1}}, \quad f\in B^s_{\infty,1}, \quad s>0. \tag{5.4}$$

For general Banach spaces X and Y, one has

$$\|x\|_Z \le C_{\theta,q}\|x\|_X^{1-\theta}\|y\|_Y^\theta, \quad Z=(X,Y)_{\theta,q}, \quad 0<\theta<1, \tag{5.5}$$

see [24, Theorem (g), p. 25]. Now, it is straightforward to check that (5.2) is true for $s = 2n$, $n \in \mathbb{N} \cup \{0\}$. Using the latter observation, (5.4), (5.3) with $q = 1$, and (5.5) with $X = \mathcal{L}^\infty_{2n}$, $Y = \mathcal{L}^\infty_{2n+2}$, for $s = (1-\theta)2n + \theta(2n+2)$, $0 < \theta < 1$, we obtain

$$\begin{aligned}\|\eta(\cdot)n_u(2^j\cdot)\|_{\mathcal{L}^\infty_s} &\le C_s\|\eta(\cdot)n_u(2^j\cdot)\|_{B^s_{\infty,1}}\\ &\le C_{\theta,s}\|\eta(\cdot)n_u(2^j\cdot)\|^{1-\theta}_{\mathcal{L}^\infty_{2n}}\|\eta(\cdot)n_u(2^j\cdot)\|^{\theta}_{\mathcal{L}^\infty_{2n+2}}\\ &\le C_{n,\theta}(1+|u|)^{(1-\theta)2n+\theta(2n+2)} = C_{n,\theta}(1+|u|)^s. \qquad \square\end{aligned}$$

The following example shows that in the multivariable case for functions $n(\lambda)$ supported in $A_{1/2,2}$ the Sobolev norms $\|n\|_{W^s_2(\mathbb{R}^d)}$ do not control the Sobolev norms $\|m\|_{W^{s'}_2(\mathbb{R}^d)}$ (even for certain range s' smaller than s) where n and m are related by (1.8).

Example 5.1 Let $F(x, y)$ be a function defined on $\mathbb{R} \times \mathbb{R}^\ell$ and $s > 0$. Observe that

$$\|F\|^2_{W^s_2(\mathbb{R}^{1+\ell})} \sim \|\hat{F}\|^2_{L^2(\mathbb{R}^{1+\ell})} + \int_{\mathbb{R}^\ell}\int_{\mathbb{R}} |\hat{F}(\xi,\eta)|^2(|\xi|^{2s} + |\eta|^{2s})\,d\xi\,d\eta, \tag{5.6}$$

Moreover, it can be shown that for every $r > 0$ there is a constant $C > 0$ such that for f supported in the interval $(\frac{1}{2}, 2)$ one has

$$C^{-1}\|\tilde{f}\|_{W^r_2(\mathbb{R})} \le \|f\|_{W^r_2(\mathbb{R})} \le C\|\tilde{f}\|_{W^r_2(\mathbb{R})}, \tag{5.7}$$

where $\tilde{f}(x) = f(x^2)$.

Let $\varphi \in C^\infty_c(\frac{1}{2}, \frac{3}{2})$ and $\psi \in C^\infty_c(\mathbb{R}^\ell)$, $\psi(y) = 0$ for $|y| > \frac{1}{2}$, $\varphi, \psi \not\equiv 0$. Fix $\varepsilon \in (0, 1)$, $R > 1$ and define the functions $n_R(x, y)$ on $\mathbb{R} \times \mathbb{R}^\ell$ by

$$n(x,y) = n_R(x,y) = \cos(Rx)\varphi(x)\psi(R^{1-\varepsilon}y) = f(x)g(y).$$

The functions $n(x, y)$ are supported in $A_{1/2,2}$, near the vector $\mathbf{e}_1$. Moreover,

$$\hat{f}(\xi) = c(\hat{\varphi}(\xi - R) + \hat{\varphi}(\xi + R)) \quad \text{and} \quad \hat{g}(\eta) = R^{\ell(\varepsilon-1)}\hat{\psi}(R^{\varepsilon-1}\eta). \tag{5.8}$$

From (5.6) and (5.8) we conclude that

$$\|n\|_{W^s_2(\mathbb{R}^{1+\ell})} \le C_s R^{s-(1-\varepsilon)\ell/2}.$$

Set $m(x, y) = m_R(x, y) = n_R(x^2, y_1^2, \dots, y_\ell^2) = f(x^2)g(y_1^2, y_2^2, \dots, y_\ell^2) = \tilde{f}(x)\tilde{g}(y)$. The functions m_R are supported near the vectors $\pm\mathbf{e}_1$. By (5.7) for $s' > 0$ and R large we have

$$\|\tilde{f}\|_{W^{s'}_2(\mathbb{R})} \sim \|f\|_{W^{s'}_2(\mathbb{R})} \sim R^{s'}. \tag{5.9}$$

Clearly,

$$\|\tilde{g}\|_{L^2(\mathbb{R}^\ell)} = \|\hat{\tilde{g}}\|_{L^2(\mathbb{R}^\ell)} = cR^{-(1-\varepsilon)\ell/4}. \tag{5.10}$$

Now, (5.6) combined with (5.9) and (5.10) imply that

$$\|m\|_{W_2^{s'}(\mathbb{R}^{1+\ell})} \geq cR^{s'-(1-\varepsilon)\ell/4} \quad \text{for large } R.$$

Summarizing,

$$\frac{\|m\|_{W_2^{s'}(\mathbb{R}^{1+\ell})}}{\|n\|_{W_2^{s}(\mathbb{R}^{1+\ell})}} \geq cR^{s'-s+(1-\varepsilon)\ell/4},$$

which clearly tends to infinity as $R \to \infty$ provided that $s' > s - (1-\varepsilon)\ell/4$.

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