Combinatorics of asymptotic representation theory

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\[ \text{shape of Young diagrams} \]

\[ \text{characters} \]

\[ \text{map} \]

\[ \text{Gaussian fluctuations} \]

\[ \text{open problems} \]
representations

shape of Young diagrams

characters

\[ \text{character} \quad \hat{\text{Ch}_5} = R_6 + 15R_4 + 5R_2^2 + 8R_2 \]

maps

Gaussian fluctuations

open problems

?
representation theory: how an abstract group can be concretely realized as a group of matrices?

Example

symmetric group $\mathfrak{S}(3)$ permutations of $\{1, 2, 3\}$

formal definition: representation $\rho$ of a group $G$ is a homomorphism

$$\rho : G \rightarrow M_k$$

from the group to invertible matrices
Example

any rotation of the dodecahedron gives an even permutation of the five cubes, element of the alternating group $\mathcal{A}(5)$

despite this is a bijection

revert the optics: representation of the alternating group $\mathcal{A}(5)$
Representations 2

Example

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?
representation $\rho$ is called **reducible** if can be written as a direct sum of smaller representations:

$$\rho(g) = \begin{bmatrix} \rho_1(g) \\ \rho_2(g) \end{bmatrix}$$

for every $g \in G$;

we are interested in **irreducible representations**

irreducible representation $\rho^\lambda$ of the symmetric group $\mathfrak{S}(n)$ ←→ Young diagram $\lambda$ with $n$ boxes
shape of Young diagram

Young diagram $\lambda$

dilated diagram $2\lambda$

goal for today:

study $\rho^{s\lambda}$ for $s \to \infty$
homogeneous functions

Young diagram $\lambda$

dilated diagram $2\lambda$
homogeneous functions

Young diagram $\lambda$

---

dilated diagram $2\lambda$

we need *nice* functions on the set of Young diagrams which depend only on shape of $\lambda$, not on its size:

\[ f(s\lambda) = f(\lambda) \]
homogeneous functions

Young diagram $\lambda$          dilated diagram $2\lambda$

we need *nice* functions on the set of Young diagrams which depend only on shape of $\lambda$, not on its size:

$f(\lambda) = f(2\lambda)$
homogeneous functions

Young diagram $\lambda$  

dilated diagram $2\lambda$

we need *nice* functions on the set of Young diagrams which
homogeneous functions

Young diagram $\lambda$  
dilated diagram $2\lambda$

we need *nice* functions on the set of Young diagrams which depend nicely on the size of $\lambda$:

$$f(s\lambda) = s^k f(\lambda)$$

homogeneous function of degree $k$
$\text{character} \ \widehat{\text{Ch}}_5 \ = \ R_6 + 15R_4 + 5R_2^2 + 8R_2$

maps

Gaussian fluctuations

open problems
character $\leftrightarrow$ shape

for irreducible representation

$$\rho^\lambda(\pi) \in M_k \quad \text{for } \pi \in S(n)$$

we define irreducible character

$$\chi^\lambda(\pi) := \text{Tr} \rho^\lambda(\pi) \quad \text{for } \pi \in S(n)$$

classical combinatorics:

Murnagahan-Nakayama rule

$$\pi = (2, 7, 9)(1, 10, 8, 3)(4, 6, 5) = 3 \cdot 4 \cdot 3$$

$$\chi^\lambda(\pi) = (-1)^0 \cdot (-1)^1 \cdot (-1)^1 + \cdots$$
representations  shape of Young diagrams  characters

\[ \text{character } \text{Ch}_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2 \]

maps  Gaussian fluctuations  open problems

?
dual combinatorics of the representation theory of $\mathfrak{S}(n)$

<table>
<thead>
<tr>
<th>classical combinatorics</th>
<th>dual combinatorics</th>
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<tbody>
<tr>
<td>$\lambda$ is fixed</td>
<td>conjugacy class is fixed</td>
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<td>character $\chi^\lambda(\pi)$ — function of $\pi$</td>
<td>character $\text{Ch}_k(\lambda)$ — function of $\lambda$</td>
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\[ [5] := \begin{array}{cccccc}
3 & 2 & \downarrow & 1 & 6 & \leftarrow 7 \\
\downarrow & 4 & \nearrow & & \leftarrow \cdots & \nearrow n
\end{array} \]

normalized character:

\[ \text{Ch}_5(\lambda) := \frac{n(n-1) \cdots (n-4)}{5 \text{ factors}} \frac{\text{Tr} \rho^\lambda([5])}{\text{Tr} \rho^\lambda(e)} , \quad n \text{ — the number of boxes of } \lambda \]

→ Kerov & Olshanski
representations  
shape of Young diagrams  
characters

\[
\widehat{\text{Ch}_5} = R_6 + 15 R_4 + 5 R_2^2 + 8 R_2
\]
free cumulants $\leftrightarrow$ shape

$R_3(\lambda)$

$R_4(\lambda)$

$\rightarrow$ Biane,
using random matrix theory / Voiculescu's free probability,
Speicher's free cumulants and non-crossing partitions
free cumulants

\[ s \mapsto \text{Ch}_k(s\lambda) \] is a polynomial of degree \( k + 1 \)

free cumulants \( R_2(\lambda), R_3(\lambda), \ldots \) are top-degree coefficients:

\[
R_{k+1}(\lambda) := \lim_{s \to \infty} \frac{1}{s^{k+1}} \text{Ch}_k(s\lambda)
\]

free cumulant \( R_k \) is homogeneous with degree \( k \):

\[
R_k(s\lambda) = s^k R_k(\lambda)
\]

\[
R_{k+1} \approx \text{Ch}_k
\]
representations \hspace{1cm} \text{shape of Young diagrams} \hspace{1cm} \text{characters}

\[ \text{character} \quad \text{shape} \quad \begin{aligned} \text{Ch}_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2 \\ \end{aligned} \]

maps \hspace{1cm} \text{Gaussian fluctuations} \hspace{1cm} \text{open problems}
Kerov polynomials

\[
\begin{align*}
\text{character} & \quad \text{shape} \\
\widehat{\text{Ch}}_2 & = \widehat{R}_3, \\
\text{Ch}_3 & = R_4 + R_2, \\
\text{Ch}_4 & = R_5 + 5R_3, \\
\text{Ch}_5 & = R_6 + 15R_4 + 5R_2^2 + 8R_2, \\
\text{Ch}_6 & = R_7 + 35R_5 + 35R_3R_2 + 84R_3
\end{align*}
\]

Kerov positivity conjecture:
the coefficients are non-negative integers; 
what is their combinatorial meaning?
$\text{character} \ \widehat{\text{Ch}_5} \quad = \quad \text{shape} \quad R_6 + 15R_4 + 5R_2^2 + 8R_2$
map

- is a graph drawn on an oriented surface,
- bipartite,
- with one face,
- labeled,
- connected
what Kerov polynomials count?

coefficient of $R_{i_1} \cdots R_{i_\ell}$ in $\text{Ch}_k$ counts the number of maps with $k$ edges with black vertices labelled by $R_{i_1}, \ldots, R_{i_\ell}$,

each black vertex $R_i$ produces $i - 1$ units of liquid,

each white vertex demands 1 unit of the liquid,

each edge transports strictly positive amount of liquid from black to white vertex.

→ Féray, Dolęga & Śniady
embedding of a map to a Young diagram

$\Pi \Sigma \ W \ V \ 4 \ 3 \ 2 \ 1 \ \rightarrow \text{Stanley, Féray, Śniady}$

$N_M(\lambda) = \# \text{embeddings of } M \text{ to } \lambda$

$N_M(\lambda)$ is a homogeneous function,

$\deg N_M = k - 1 + \chi(M) = k + 1 - 2 \text{ genus}(M)$

biggest contribution: planar maps
Stanley’s character formula

\[ \Pi \rightarrow_{Stanley, Féray, Śniady} \]

\[ N_M(\lambda) = \# \text{ embeddings of } M \text{ to } \lambda \]
Stanley’s character formula

\[ \text{Stanley, Féray, Šniady} \]

\[ N_M(\lambda) = \# \text{ embeddings of } M \text{ to } \lambda \]

\[ \text{Ch}_k(\lambda) = \sum_M (-1)^{k - \# \text{white vertices}} N_M(\lambda), \]

where the sum runs over maps \( M \) with \( k \) edges.
\[ \text{character} \quad Ch_5 \quad = R_6 + 15R_4 + 5R_2^2 + 8R_2 \]
characters on two cycles

the normalized character $Ch_{k,l}(\lambda)$

$$(1, 2, \ldots, k)(k + 1, k + 2, \ldots, k + l) \in \mathfrak{S}(k + l)$$

---

Kerov polynomials

$Ch_{3,2} = R_3 R_4 - 5R_2 R_3 - 6R_5 - 18R_3$

not nice!

---

(abstract) covariance

$$\text{Cov}(Ch_k, Ch_l) := Ch_{k,l} - Ch_k Ch_l$$

$$\text{Cov}(Ch_3, Ch_2) = -(6R_2 R_3 + 6R_5 + 18R_3)$$

is nice!
surprising cancellations

\[
\begin{align*}
\text{Ch}_2 &= \underbrace{R_3}_\text{degree 3}, \\
\text{Ch}_3 &= \underbrace{R_4}_\text{degree 4} + R_2, \\
\text{Cov}(\text{Ch}_3, \text{Ch}_2) &= -(6 \underbrace{R_2 R_3}_\text{degree only 5} + 6 \underbrace{R_5}_\text{degree only 5} + 18 R_3)
\end{align*}
\]

explanation by Kerov polynomials:
\(\text{Cov}(\text{Ch}_3, \text{Ch}_2)\) counts connected maps with two cells, such that...
Gaussian fluctuations

(abstract) cumulant

\[ k(\text{Ch}_{i_1}, \ldots, \text{Ch}_{i_\ell}) = \text{Ch}_{i_1, \ldots, i_\ell} - \cdots \]

surprising cancellation:

\[ \text{deg } k(\text{Ch}_{i_1}, \ldots, \text{Ch}_{i_\ell}) = \text{deg } \text{Ch}_{i_1} + \cdots + \text{deg } \text{Ch}_{i_\ell} - 2(\ell - 1) \]

\( \text{Ch}_1, \text{Ch}_2, \text{Ch}_3, \ldots \) behave asymptotically as (abstract) Gaussian random variables

---

**Theorem**

for a large class of reducible representations of \( \mathfrak{S}(n) \), if we randomly select an irreducible component \( \rho^\lambda \), for \( n \to \infty \)

\( \lambda \) will concentrate around some limit shape

and the fluctuations are Gaussian

→ Biane

→ Kerov, Śniady
random Young tableaux 1
random Young tableaux

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restriction $\rho^\lambda \downarrow_{\mathfrak{S}(m)}^{\mathfrak{S}(n)}$ to a subgroup
random Young tableaux 2
representations | shape of Young diagrams | characters

\[ \text{character} \quad \overset{\text{Ch}_5}{=} \quad \text{shape} \]

\[ R_6 + 15R_4 + 5R_2^2 + 8R_2 \]

maps | Gaussian fluctuations | open problems

?
open problems

\[ \begin{align*}
\text{Ch}_6 - R_7 &= \frac{35}{4} C_5 + 42C_3, \\
\text{Ch}_7 - R_8 &= 14C_6 + \frac{469}{3} C_4 + \frac{203}{3} C_2^2 + 180C_2. \\
\rightarrow \text{Goulden \& Rattan} \\
positivity?
\end{align*} \]

\[ \begin{align*}
\text{Ch}^{(\gamma)}_3 &= R_4 + 3\gamma R_3 + (1 + 2\gamma^2)R_2, \\
\text{Ch}^{(\gamma)}_4 &= R_5 + 6\gamma R_4 + \gamma R_2^2 + (5 + 11\gamma^2)R_3 + (7\gamma + 6\gamma^3)R_2, \\
\rightarrow \text{Lassalle} \\
positivity?
\end{align*} \]
\[ \text{character} \quad \widehat{\text{Ch}_5} \quad = \quad R_6 + 15R_4 + 5R_2^2 + 8R_2 \]

representations

shape of Young diagrams

characters

maps

Gaussian fluctuations

open problems
further reading

- **Piotr Śniady**
  Combinatorics of asymptotic representation theory.
  Proceedings of 6th European Congress of Mathematics
  arXiv:1203.6509

- **Valentin Féray, Piotr Śniady**
  Asymptotics of characters of symmetric groups related to
  Stanley character formula.

- **Maciej Dołęga, Valentin Féray, Piotr Śniady**
  Explicit combinatorial interpretation of Kerov character
  polynomials as numbers of permutation factorizations.