Generalized Frobenius formula and asymptotics of characters of symmetric groups

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Outline

1. Problem: asymptotics of characters of symmetric groups
2. Generalized Frobenius formula
3. Upper bounds for characters of symmetric groups
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1. Problem: asymptotics of characters of symmetric groups
   - Murnaghan–Nakayama rule
   - Asymptotics of characters

2. Generalized Frobenius formula

3. Upper bounds for characters of symmetric groups
Russian convention for Young diagrams

- English convention
- French convention
- Russian convention
Irreducible representations $\rho^\lambda$ of $S_n$ are indexed by Young diagrams with $n$ boxes. For a given Young diagram $\lambda$ and permutation $\pi \in S_n$, what is the value of the unnormalized character $\text{Tr} \rho^\lambda(\pi)$?

Example:

$$\pi = (1, 2, 3, 4)(5, 6, 7, 8)\times(9, 10, 11, 12)(13, 14, 15, 16, 17) = 4^35^1$$
Let \( l_1, \ldots, l_k \) be the lengths of the cycles of \( \pi \). In order to compute the character \( \text{Tr} \rho^\lambda(\pi) \) we need to consider all decompositions of \( \lambda \) into strips of lengths \( l_1, \ldots, l_k \). For each strip we get a factor \((-1)^{\text{height}} \ldots \)

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\pi = 4^3 5^1
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contribution = \((-1)^2 \times (-1)^1 \times (-1)^1 \times (-1)^2 \)

The character is equal to the sum of the contributions over all decompositions.
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Asymptotic questions

We would like to study asymptotic questions: how big is the character

$$\chi^\lambda(\pi) = \frac{\text{Tr} \rho^\lambda(\pi)}{\text{Tr} \rho^\lambda(e)}$$

of the symmetric group $S_n$ in the limit as $n \to \infty$. Alternatively, how big are normalized characters

$$\sum_{k_1, \ldots, k_l} \frac{\text{Tr} \rho^\lambda(k_1, \ldots, k_l, 1^{n-k_1-\cdots-k_l})}{\text{Tr} \rho^\lambda(e)} (n)_{k_1+\cdots+k_l},$$

where $(k_1, \ldots, k_l, 1^{n-k_1-\cdots-k_l})$ is a permutation with a given cycle structure and $(n)_k = \frac{n!}{(n-k)!} = n(n-1) \cdots (n-k+1)$ denotes the falling power.

Murnaghan–Nakayama rule does not tell us anything useful.
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Murnaghan–Nakayama rule does not tell us anything useful.
Balanced Young diagrams

We assume that $\lambda$ is a balanced diagram, i.e. it has at most $c \sqrt{n}$ rows and columns, where $n$ is the number of boxes.

Kerov, Biane, ... proved that for some constant $d$

$$|\chi^\lambda(\pi)| < \left( \frac{d}{\sqrt{n}} \right)^{|\pi|}$$

if $|\pi|$ is bounded. Is the above inequality true for general $|\pi|$?

Motivations:

- random walks on symmetric group $S_n$ (Diaconis and Shahshahani),
- quantum computations (Moore and Russell).
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2. Generalized Frobenius formula
   - How to encode a Young diagram?
   - Generalized Frobenius formula
3. Upper bounds for characters of symmetric groups
Young diagram can be encoded by the sequences of local minima \((x_1, \ldots, x_s)\) and maxima \((y_1, \ldots, y_{s-1})\). We define a function

\[
H(z) = \frac{(z - x_1) \cdots (z - x_s)}{(z - y_1) \cdots (z - y_{s-1})}.
\]
Why is $H(z)$ so nice?

- $H(z)$ is easily determined by the shape of Young diagram $\lambda$, good for asymptotic questions;
- $H(z)$ is related to the transition measure $\mu^\lambda$ of $\lambda$, namely $G(z) = \frac{1}{H(z)}$ is the Cauchy transform of $\mu^\lambda$;
- the coefficients in the expansion

$$H(z) = z - B_2 z^{-1} - B_3 z^{-2} - \cdots$$

have a nice interpretation as Boolean cumulants of $\mu^\lambda$. Boolean cumulants describe nicely the shape of $\lambda$. 
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Generalized Frobenius formula and asymptotics of characters
The usual Frobenius formula

\[ -k \sum_k = [z^{-1}] \left[ H(z)H(z-1) \cdots H(z-k+1) \right]. \]
Theorem (Generalized Frobenius formula, simplest case)

\[ k_1 k_2 \sum_{k_1, k_2} = \left[ \frac{1}{z_1} \right] \left[ \frac{1}{z_2} \right] \left( H(z_1) H(z_1 - 1) \cdots H(z_1 - k_1 + 1) \times \right. \]

\[ H(z_2) H(z_2 - 1) \cdots H(z_2 - k_2 + 1) \times \]

\[ \frac{(z_1 - z_2)(z_1 - z_2 + k_2 - k_1)}{(z_1 - z_2 - k_1)(z_1 - z_2 + k_2)} \right] . \]
Theorem (Generalized Frobenius formula)

\[
(-1)^l k_1 \cdots k_l \sum_{k_1, \ldots, k_l} = \\
[z_1^{-1}] \cdots [z_l^{-1}] \left[ \prod_{1 \leq r \leq l} H(z_r) H(z_r - 1) \cdots H(z_r - k_r + 1) \right] \\
\prod_{1 \leq s < t \leq l} \frac{(z_s - z_t)(z_s - z_t + k_t - k_s)}{(z_s - z_t - k_s)(z_s - z_t + k_t)}.
\]

Main advantage: direct expression for characters in terms of Boolean cumulants.

Idea of the proof: encrypted Murnaghan–Nakayama rule.
Theorem (Generalized Frobenius formula)

\[\sum_{k_1,\ldots,k_l = 1} (-1)^l k_1 \cdots k_l \sum_{k_1,\ldots,k_l} = \prod_{1 \leq r \leq l} [z_r]^{-1} \cdots [z_l]^{-1} \left[ \prod_{1 \leq r \leq l} H(z_r) H(z_r - 1) \cdots H(z_r - k_r + 1) \right] \prod_{1 \leq s < t \leq l} \left( \frac{(z_s - z_t)(z_s - z_t + k_t - k_s)}{(z_s - z_t - k_s)(z_s - z_t + k_t)} \right).\]

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Let us fix some constant $\zeta$. The coefficients of the expansion

$$H(z + \zeta) = z + \zeta + \tilde{B}_1 + \tilde{B}_2 z^{-1} + \tilde{B}_3 z^{-2} + \cdots$$

are called *shifted Boolean cumulants*. For $\zeta = 0$ they coincide (up to the sign change) with the usual Boolean cumulants.
Shifted Boolean cumulants

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Positivity of character polynomials

**Theorem**

Let integers $1 \leq k_1, \ldots, k_l \leq \zeta$ be given. Then the normalized character $(-1)^l \sum_{k_1, \ldots, k_l}$ is a polynomial in shifted Boolean cumulants $\tilde{B}_2, \tilde{B}_3, \ldots$ with non-negative coefficients.

Looks like Kerov conjecture for free cumulants.
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Looks like Kerov conjecture for free cumulants.
Corollary

If $\lambda$ and $\nu$ are Young diagrams such that $|\tilde{B}_i^\lambda| < \tilde{B}_i^\nu$ then

$$|\Sigma_{k_1,\ldots,k_l}^\lambda| < |\Sigma_{k_1,\ldots,k_l}^\nu|.$$ 

Now if we want to prove upper bounds for characters it is enough to prove them for some nice Young diagram $\nu$. For example, for $\nu$ we may take rectangular Young diagrams for which characters were calculated by Stanley.
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The main inequality

**Theorem**

*For every $c$ there exists a constant $d$ such that if a Young diagram with $n$ boxes has at most $c\sqrt{n}$ rows and columns then*

\[ |\chi^\lambda(\pi)| < \left(\frac{d}{\sqrt{n}}\right)^{|\pi|}. \]

This talk was about an application of power series to representation theory. More such applications are around!

Amarpreet Rattan, Piotr Śniady.

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