Life II Exercises 2.

(1) [Exercise in notation; TKskrypt] By making use of the actuarial notations show that:

$$\begin{aligned}
{}_{s+t}p_x &= {}_tp_x \cdot {}_sp_{x+t}, \\
f(t) &= {}_tp_0 \cdot \mu(t), \\
f_x(t) &= {}_tp_{x+t} \cdot \mu(x+t), \\
{}_tp_x &= \exp\left(-\int_x^{x+t} \mu(s)\, ds\right), \\
{}_tq_x &= \int_0^t {}_sp_x \cdot \mu(x+s)\, ds.
\end{aligned}$$

(2) (i) Suppose that $s(t) = \exp(-\mu t)$, $t \geq 0$ for some positive constant μ. Show that $\mu(t) \equiv \mu$.
(ii) Suppose that T is uniform on the interval $[0, \omega]$. Show that $\mu(t) = 1/(\omega - t)$, for $0 \leq t < \omega$ (and zero otherwise).
(iii) Suppose that $s(t) = t^{-\alpha}$, for $t \geq 1$, for some positive α. Show that $\mu(t) = \alpha/t$, for $t \geq 1$.

(3) Show that, among positive continuous random variables, only the exponential distribution has the memoryless property. Also show that the memoryless property can be written as ${}_tp_s = {}_tp_0$ for all $s, t \geq 1$. Hint. Recall the memoryless property says that

$$P(T - t > s | T > s) = P(T > t), \qquad s, t > 0. .$$

(4) It is said that the lifetime T fulfils hypothesis HU if its survival function $s(x)$ is continuous and linear in each interval $(n, n+1)$, $n = 0, 1, \ldots$. Show that then the force of mortality is

$$\mu(n+u) = \frac{q_n}{1 - uq_n}, \qquad n = 0, 1, \ldots, \quad 0 \leq u < 1.$$

Write the force of mortality for T_x.

(5) It is said that the lifetime T fulfils hypothesis HCFM if

$$\mu(x+u) = \mu(x), \qquad x = 0, 1, \ldots, \quad 0 \leq u < 1 .$$

Show that

$${}_{n+u}p_0 = {}_np_0 (p_n)^u, \qquad n = 0, 1 \ldots, quad 0 \leq u < 1 .$$

Write ${}_{n+u}p_x$, for x integer. Furthermore show that The HCFM assumption is equivalent to

$$\mu(n+t) = -\log(1-q_n), \qquad n = 0, 1, \ldots, \quad 0 \le t < 1,$$

and to

$${}_tq_x = P(T_x > t) = (1-q_x)t.$$

(6) It is said that the lifetime T fulfils hypothesis HB if

$${}_{1-u}q_{n+u} = (1-u)q_n, \qquad n = 0, 1, \ldots \quad 0 \le u < 1.$$

Show that

$$\begin{aligned} {}_{n+u}p_0 &= {}_np_0 \frac{p_n}{u + (1-u)p_n} \\ &= \frac{{}_{n+1}p_0}{1-(1-u)q_n}, \\ \mu(n+u) &= \frac{q_n}{1-(1-u)q_n}. \end{aligned}$$

Write the corresponding quantities for T_x

(7) Discuss the uses of the assumptions regarding interpolation, noting that in each interval $(n, n+1)$: HU implies increasing mortality rate, HB (Balducci assumption): decreasing mortality rate, HCFM: constant mortality rate.

(8) Consider two independent, positive, absolutely continuous random variables T_1 , T_2 , with force of mortalities $\mu_1(t)$ and $\mu_2(t)$ respectively. Let $\tau_1 = 1, \tau_2 = T_1 + T_2$. Consider the point process $\{\tau_1, \tau_2\}$. Prove that the compensator of its counting process $N(t) = 1(\tau_1 \le t) + 1(\tau_2 \le t)$ is

$$\Lambda(t) = \int_0^{t\wedge\tau_1} \mu_1(s)ds + \int_{\tau_1}^{t\wedge\tau_2} \mu_2(s-\tau_1)\, ds.$$

(9) Wprowadza się *centralne natężenie śmiertelności* lub *roczny współczynnik natężenia zgonów dla wieku x* wzorem

$$m_x = \frac{\int_0^1 l(x+t)\mu(x+t), dt}{\int_0^1 l(x+t)\, dt} = \frac{d_x}{L_x},$$

gdzie $L_x = \int_0^1 l(x+t)\, dt$. Udowodnić, że przy HU

$$m_x = \frac{q_x}{1-q_x/2}.$$

Zauważyć, że wtedy $m_x = \mu(x+1/2)$