Tail asymptotics for a Lévy-driven tandem queue with an intermediate input

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Abstract

We consider a Lévy-driven tandem queue with an intermediate input assuming that its buffer content process obtained by a reflection mapping has the stationary distribution. For this queue, no closed form formula is known, not only for its distribution but also for the corresponding transform. In this paper we consider only light-tailed inputs. For the Brownian input case, we derive exact tail asymptotics for the marginal stationary distribution of the second buffer content, while weaker asymptotic results are obtained for the general Lévy input case. The results generalize those of Lieshout and Mandjes from the recent paper [16, 17] for the corresponding tandem queue without an intermediate input.

1 Introduction

Brownian fluid networks have been studied for many years, and they are well understood as reflected Brownian motions (see, e..g., [5, 11, 12]). They can be easily extended to spectrally positive Lévy inputs. These extended networks are referred here to as Lévy driven network queues. In applications of those networks, it may be interesting to look for their stationary behavior, provided the existence of their stationary distributions are assumed. Since it is hard to get the stationary distributions except for special cases, asymptotic tail behavior of those distributions is interesting. However, it is still hard to get the asymptotics such as tail decay rates from primitive modeling data even for two node fluid networks although there are some notable results for rough asymptotics on the two dimensional reflected Brownian motions (see, e.g., [2]). This is contrasted with their discrete time version, which has been well studied. For example, the study on rough asymptotics can be dated back to the work on intree networks by Chang et al. [6, 7]. Furthermore, sharper asymptotic behavior has been obtained for a two node network which is described by a reflected random walk in a nonnegative two dimensional quadrant (see, e.g., [4, 21] and references therein).

Recently, Lieshout and Mandjes [16, 17] study this asymptotic decay problem for a Lévy-driven two node tandem queue when there is no intermediate input, which means that the second queue has no exogenous input. In [16], the joint stationary distribution function was obtained in closed form for the Brownian input case, and using these closed form expressions, the exact tail asymptotics were obtained in all directions for the Brownian input. In [17], they use the joint Laplace transform due to Dębicki, Dieker and Rolski [9], which has closed form and is obtained for the general Lévy-driven n-node tandem queue. With the help of some sample path large deviations techniques, the rough decay rates were obtained in all directions for the general Lévy input case in [17]. Here, the tail probability is said to have rough decay rate α if its logarithm at level x divided by x converges to $-\alpha$ as $x \to \infty$, and said to have exact asymptotics h for some function h if the ratio of the tail probability at level x and $h(x)$ converges to one.

The results in [16, 17] are very interesting. However, following Abate and Whitt [1], a Tauberian type arguments under the name of Heaviside operational calculus were applied for derivations of the exact asymptotics for the general Lévy input case. Unfortunately as long as we know, rigorously proven theorems require verification of extra conditions and therefore in our view the exact asymptotics results for the general Lévy input case in [17] needs additional justification. We also note that their approach Lieshout and Mandjes used the explicit form of the Laplace transform, which is only available for a pure tandem case. In this paper, we consider the tandem queue with an intermediate input, where up to now there is no formulas, even in the form of a Laplace transform available.

In this paper, we consider these asymptotic decay problems mainly for the marginal stationary distributions. Since the marginal distribution for the first node is obtained in term of Laplace transform as the Pollaczek-Khinchine formula, our main interest is asymptotics of the tail distribution of the second node. We have a complete answer for the Brownian input case while weaker asymptotic results are obtained for the general Lévy input case which extends the corresponding results in [17]. Those asymptotic results exhibit some analogy to the discrete time counterpart, that is, the reflected random walk. Similar exact asymptotics are also reported for hitting probabilities of two dimensional risk processes in [3]. As a by product, we also get some asymptotics for the tail distribution of a convex combination of the two buffer contents in the Brownian input case.

The approach of this paper is purely analytic, and exact asymptotics is studied only for marginal distributions. This enables us to use classical results on one dimensional distributions through their Laplace transforms. In other words, our results may not be so informative on sample path behavior, which has been extensively studied in the large deviations theory. Nevertheless, the results has multidimensional, precisely, two-dimensional, feature. For example, we identify the convergence domain of the moment generation function for the two-dimensional stationary distribution. This may be related to the rate function in the sample path large deviations. We hope the present results would be also useful for the large deviations theory of Levy-driven tandem networks with intermediate input.

This paper is composed of six sections. In Section 2, we first derive stationary equation for the moment generating function of the joint distribution of the steady state buffer content by the use of Kella-Whitt martingale; see [13]. In Sections 3 and 4, we assume that there is no jump input at both nodes, that is, the Brownian case. In Section 3, we identify the domain of the moment generating function of the joint buffer contents. In Section 4, exact asymptotics are obtained. In Section 5 we discuss how results for the Brownian networks can be extended for the general Lévy input case. Unfortunately our exact asymptotic results here can be considered only as conjectures. We finally discuss consistency with existing results and possible extensions of the presented results in Section 6.

2 Stationary equation for moment generating functions (MGFs)

We now formally introduce a Lévy-driven tandem fluid queue with an intermediate input. This tandem queue has two nodes, numbered as 1 and 2. Both nodes has exogenous input processes, and constant processing rates. Outflow from node 1 goes to node 2, and outflow from node 2 leaves the system. As usual, we always assume that all processes are right-continuous and have left-hand limits.

We assume that those exogenous inputs are independent Lévy processes of the form: for node i

$$X_i(t) = a_i t + B_i(t) + J_i(t), \qquad i = 1, 2, \tag{2.1}$$

where a_i is a nonnegative constant, $B_i(t)$ is a Brownian motion with variance σ_i^2 and null drift, and $J_i(t)$ is a pure jump Levy process with positive jumps and spectral measure ν_i fulfilling

$$\int_0^\infty \min(x^2, 1)\nu_i(dx) < \infty, \qquad i = 1, 2,$$

Denote the Lévy exponent of $X_i(t)$ by $\kappa_i(\cdot)$, i.e.

$$E(e^{\theta X_i(t)}) = e^{t\kappa_i(\theta)}, \qquad \theta \leq 0.$$

Clearly, $\kappa_i(\theta)$ is increasing and convex for all $\theta \in \mathbb{R}$ as long as it is well defined. Since we are interested in the light tail behavior, we assume now

(2-i) $\kappa_i(\theta_i^{(0)}) < \infty$ for some $\theta_i^{(0)} > 0$ for $i = 1, 2$.

This implies that $E(X_i(1)) = \kappa_i'(0)$ is positive and finite. Let $\lambda_i = E(X_i(1))$, which is the mean input rate at node i.

Then, according to the decomposition of $X_i(t)$, the exponent $\kappa_i(\cdot)$ can be decomposed as

$$\kappa_i(\theta) = a_i\theta + \frac{1}{2}\sigma_i^2\theta^2 + \kappa_i^J(\theta), \qquad i = 1, 2. \tag{2.2}$$

From the definitions,

$$\lambda_i = a_i + E(J_i(1)), \qquad i = 1, 2,$$

so $\lambda_i = a_i$ if the exogenous input at queue i has no jump.

Denote the processing rate at node i by $c_i > 0$. Let $L_i(t)$ be buffer content at node i at time $t \geq 0$ for $i = 1, 2$, which are formally defined as

$$L_1(t) = L_1(0) + X_1(t) - c_1 t + Y_1(t), \tag{2.3}$$
$$L_2(t) = L_2(0) + X_2(t) + c_1 t - Y_1(t) - c_2 t + Y_2(t), \tag{2.4}$$

where $Y_i(t)$ is a regulator at node i, that is, a minimal nondecreasing process for $L_i(t)$ to be nonnegative. Namely, $(L_1(t), L_2(t))$ is a generated by a reflection mapping from net flow processes $(X_1(t) - c_1 t, X_2(t) + c_1 t - c_2 t)$ with reflection matrix

$$R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

See Section 6.3 for R, and the literature [5, 11, 25] for the reflection mapping. We enrich filtration $\mathcal{F}_t^{\boldsymbol{X}}$ adding events on $L_1(0)$ and $L_2(0)$ to $\mathcal{F}_0^{\boldsymbol{X}}$. Then, $(L_1(t), L_2(t))$ is adapted to this filtration. We refer to the model described by this reflected process as a Lévy-driven tandem queue.

It is easy to see that this tandem queue has the stationary distribution if and only if

(2-ii) $\lambda_1 < c_1$ and $\lambda_1 + \lambda_2 < c_2$.

We assume this stability condition throughout the paper, and denote the stationary distribution by π.

We consider two types of asymptotic tail behavior of π, called rough and exact asymptotics. Let $g(x)$ a positive valued function of $x \in [0, \infty)$. If

$$\alpha = \lim_{x\to\infty} -\frac{1}{x} \log g(x)$$

exists, $g(x)$ is said to have rough decay rate α. On the other hand, if there exists a function h such that

$$\lim_{x\to\infty} \frac{g(x)}{h(x)} = 1,$$

then $g(x)$ is said to have exact asymptotics $h(x)$. Let (L_1, L_2) be a random vector with distribution π. Our main interest is to find exact asymptotics of $P(d_1 L_1 + d_2 L_2 > x)$ for $\boldsymbol{d} \equiv (d_1, d_2) \geq \boldsymbol{0}$. We are particularly interested in exact asymptotics of the marginal distributions of L_1 and L_2, and denote their rough decay rates by α_1 and α_2, respectively.

Those exact asymptotics will be determined by their moment generating functions (see, e.g., [1]). In the sequel we will use

$$\varphi(\theta_1, \theta_2) = E_\pi(e^{\theta_1 L_1 + \theta_2 L_2}),$$
$$\varphi_1(\theta_2) = E_\pi(\int_0^1 e^{\theta_2 L_2(u)} dY_1(u)), \qquad \varphi_2(\theta_1) = E_\pi(\int_0^1 e^{\theta_1 L_1(u)} dY_2(u)),$$

where φ_i for $i = 1, 2$ are not directly related to φ, but will be useful. It will be shown in Proposition 2.1 that $E_\pi(Y_i(1))$ for $i = 1, 2$ are finite. Using this fact, we can see that

$\varphi_i(\theta_i)/\varphi_i(0)$ is the moment generating function of L_{3-i} under Palm distribution concerning the random measure $\int_B dY_i(u)$ for $B \in \mathcal{B}(\mathbb{R})$, where $Y_1(t)$ and $Y_2(t)$ are extended on the whole line $\mathbb{R}$ under the stationary probability measure for the reflected process generated by the initial distribution π (e.g., see [20] for Palm distribution). Intuitively, it may be considered as the conditional moment generating function of L_{3-i} given that $L_i = 0$. However, it should be noted that $Y_i(t)$ increases on the set of Lebesgue measure 0 if the input has a continuous component.

We start with deriving the stationary equation for the moment generating functions of π and its certain marginals. For this, we use so called the Kella-Whitt martingale of [13]. For row vector $\boldsymbol{\theta} \equiv (\theta_1, \theta_2) \le \mathbf{0}$, let

$$\overline{L}(t) = \langle \boldsymbol{\theta}, \boldsymbol{L}(t) \rangle, \qquad S(t) = \langle \boldsymbol{\theta}, \boldsymbol{X}(t) - tR\boldsymbol{c} \rangle, \qquad \overline{Y}(t) = \langle \boldsymbol{\theta}, R\boldsymbol{Y}(t) \rangle,$$

where $\boldsymbol{L}(t) = (L_1(t), L_2(t))^{\mathrm{T}}$, $\boldsymbol{X}(t) = (X_1(t), X_2(t))^{\mathrm{T}}$, $\boldsymbol{Y}(t) = (Y_1(t), Y_2(t))^{\mathrm{T}}$ and $\boldsymbol{c} = (c_1, c_2)^{\mathrm{T}}$. Here, $\boldsymbol{x}^{\mathrm{T}}$ denotes the transpose of vector $\boldsymbol{x}$, and $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ is the inner product of vectors $\boldsymbol{x}, \boldsymbol{y}$. Then, we have, from (2.3) and (2.4),

$$\overline{L}(t) = \overline{L}(0) + S(t) + \overline{Y}(t), \qquad t \ge 0.$$

Since $S(t)$ is a Lévy process and $\overline{Y}(t)$ is a continuous process of bounded variations, it follows from Lemma 1 of [13] that

$$M(t) \equiv \overline{\kappa}(1) \int_0^t e^{\overline{L}(u)} du + e^{\overline{L}(0)} - e^{\overline{L}(t)} + \int_0^t e^{\overline{L}(u)} d\overline{Y}(u) \tag{2.5}$$

is a local martingale, where $\overline{\kappa}(s)$ is the Lévy exponent of $S(t)$ given by

$$\overline{\kappa}(s) = \kappa_1(s\theta_1) + \kappa_2(s\theta_2) - c_1 s\theta_1 - (c_2 - c_1)s\theta_2.$$

Denote $-\overline{\kappa}(1)$ by $\gamma(\theta_1, \theta_2)$. That is,

$$\gamma(\theta_1, \theta_2) = c_1\theta_1 + (c_2 - c_1)\theta_2 - \kappa_1(\theta_1) - \kappa_2(\theta_2). \tag{2.6}$$

We are now ready to prove the following results.

Proposition 2.1 Under the conditions (2-i) and (2-ii),

$$E_\pi(Y_1(1)) = c_1 - \lambda_1, \tag{2.7}$$

$$E_\pi(Y_2(1)) = c_2 - (\lambda_1 + \lambda_2), \tag{2.8}$$

and, for $\theta_1, \theta_2 \le 0$,

$$\gamma(\theta_1, \theta_2)\varphi(\theta_1, \theta_2) = (\theta_1 - \theta_2)\varphi_1(\theta_2) + \theta_2\varphi_2(\theta_1). \tag{2.9}$$

Remark 2.1 One may think that (2.7) and (2.8) are immediate from (2.3) and (2.4) by taking the expectation under π. However, this requires the finiteness of $E_\pi(L_1)$ and $E_\pi(L_2)$, which we cannot use at this stage.

PROOF. We first show that $M(t)$ is a martingale by verifying that

$$E_\pi \sup_{0 \le s \le t} |M(s)| < \infty.$$

Since $M(t)$ is a local martingale, there exists an increasing sequence of stopping times $\{\tau_n; n = 1, 2, \ldots\}$ such that $M(\tau_n \wedge t)$ is a martingale for each n and $\tau_n \uparrow \infty$ with probability one. Since $e^{\overline{L}(t)} \le 1$, by taking the expectation under π,

$$E_\pi(M(\tau_n \wedge t)) = E_\pi(M(0)) = 0. \tag{2.10}$$

Hence

$$E_\pi \int_0^{t \wedge \tau_n} e^{\overline{L}(u)} d\overline{Y}(u) \le 2 + \overline{\kappa}(1)t < \infty$$

and now passing with $n \to \infty$

$$E_\pi \int_0^t e^{\overline{L}(u)} d\overline{Y}(u) \le 2 + \overline{\kappa}(1)t < \infty.$$

Furthermore, again from (2.10),

$$E_\pi \sup_{0 \le s \le t} |M(s)| \le \overline{\kappa}(1)t + 2 + E_\pi \int_0^t e^{\overline{L}(u)} d\overline{Y}(u) < \infty$$

which yields that $M(t)$ is a martingale. Now setting $t = 1$ and taking the expectation of both the sides of (2.5) we obtain (2.9) since $L_1(t) = 0$ when $Y_1(t)$ is increasing and $E_\pi(e^{\theta_1 L_1(u)})$ is independent of u. Let $t = 1$ and $\theta_2 = 0$ in (2.9), then we have

$$\gamma(\theta_1, 0)\varphi(\theta_1, 0) = \theta_1 E_\pi(Y_1(1)).$$

Dividing both the sides of this equation by $\theta_1 < 0$ and letting $\theta_1 \uparrow 0$, we have (2.7). Similarly, letting $t = 1$, $\theta_1 = \theta_2 = \theta$ in (2.9), we have

$$\gamma(\theta, \theta)\varphi(\theta, \theta) = \theta E_\pi(\int_0^1 e^{\theta L_1(u)} dY_2(u)),$$

and dividing both the sides by $\theta < 0$ and letting $\theta \uparrow 0$, we have (2.8). □

In proving Proposition 2.1, we have used the Kella-Whitt martingale. Instead of this, we can use Itô's integration formula (see, e.g., Chapter 26 of [14]). This is more instructive since dynamics of the reflected process is described, but it requires more computations. Its details can be found in [22].

In general, it is hard to get the stationary distribution π or its moment generating function in closed form from (2.9). There is one special case that φ is obtained in closed form. This is the case that $X_2(t) \equiv 0$, that is, there is no intermediate input. This case has been recently studied in [9].

Example 2.1 (Tandem queue without an intermediate input) Suppose $X_2(t) \equiv 0$ in the Lévy-driven tandem queue satisfying the stability condition (2-ii). We assume that $c_1 > c_2$ since $L_2(t) \equiv 0$ otherwise.

Since $L_2(u) = 0$ implies $L_1(u) = 0$, so $L_1(u) = 0$ when $Y_2(u)$ is increasing, we have $\varphi_2(\theta_1) = E_\pi Y_2(1) = c_2 - \lambda_1$ from Proposition 2.1. Hence by (2.9) (with $\lambda_2 = 0$, and $\kappa_2(\theta) = 0$) we have

$$(c_1\theta_1 - (c_1 - c_2)\theta_2 - \kappa_1(\theta_1))\varphi(\theta_1, \theta_2) = (\theta_1 - \theta_2)\varphi_1(\theta_2) + (c_2 - \lambda_1)\theta_2. \tag{2.11}$$

Since $\varphi_1(0) = c_1 - \lambda_1$ by (2.7), letting $\theta_2 = 0$ in (2.11) implies the well known Pollaczek-Khinchine formula:

$$E(e^{\theta_1 L_1}) = \varphi(\theta_1, 0) = \frac{(c_1 - \lambda_1)\theta_1}{c_1\theta_1 - \kappa_1(\theta_1)}, \qquad \theta_1 \leq 0. \tag{2.12}$$

Assume that $c_1\theta_1 - \kappa_1(\theta_1) = 0$ has a positive solution, which must be the rough decay rate α_1. Furthermore, it is not hard to see that $P(L_1 > x)$ has exact asymptotic $ce^{-\alpha_1 x}$ with known constant c.

We next let $\theta_1 = 0$ in (2.11), then we have

$$\varphi_1(\theta_2) = (c_1 - c_2)\varphi(0, \theta_2) + c_2 - \lambda_1. \tag{2.13}$$

Substituting this into (2.11), we have

$$(c_1\theta_1 - (c_1 - c_2)\theta_2 - \kappa_1(\theta_1))\varphi(\theta_1, \theta_2) = (c_1 - c_2)(\theta_1 - \theta_2)\varphi(0, \theta_2) + (c_2 - \lambda_1)\theta_1. \tag{2.14}$$

For each $\theta_2 \leq 0$, let $\xi_1(\theta_2)$ be the smallest solution θ_1 of the equation

$$c_1\theta_1 - \kappa_1(\theta_1) = (c_1 - c_2)\theta_2,$$

which always exists and is negative since $\kappa_1(0) = 0$ and $c_1 - \kappa_1'(0) = c_1 - \lambda_1 > 0$. Then, letting $\theta_1 = \xi_1(\theta_2)$ in (2.14) yields

$$\varphi(0, \theta_2) = \frac{(c_2 - \lambda_1)\xi_1(\theta_2)}{(c_1 - c_2)(\theta_2 - \xi_1(\theta_2))} \tag{2.15}$$

Plugging this into (2.14), we arrive at

$$\varphi(\theta_1, \theta_2) = \frac{(c_2 - \lambda_1)(\theta_1 - \xi_1(\theta_2))\theta_2}{(\theta_2 - \xi_1(\theta_2))(c_1\theta_1 - (c_1 - c_2)\theta_2 - \kappa_1(\theta_1))}. \tag{2.16}$$

This is the formula obtained in [9]. Based on it, the asymptotic behavior of the stationary distribution π is studied in [17]. This idea can be extended to the case of more than two nodes, for which we refer to the draft [22]. □

3 Convergence domain of the MGF

We now consider the Lévy-driven tandem queue with the intermediate input. To avoid complicated presentation, we assume in this and next sections that there are no jump inputs at nodes 1 and 2, that is, $J_1(t) = J_2(t) \equiv 0$. In this case, the tandem queue is referred to as a Brownian tandem queue with an intermediate input. We will include those jump inputs in Section 5. The main purpose of this section is to identify the convergence domain of φ

$$\mathcal{D} = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \varphi(\theta_1, \theta_2) < \infty\}.$$

The knowledge of this domain allows us to study asymptotic decay of some interesting tail distributions. We first note that $\mathcal{D}$ is convex since φ is a convex function, and it obviously includes the set $\{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1, \theta_2 \le 0\}$.

Since there are no jump inputs here, (2.6) is simplified to

$$\gamma(\theta_1, \theta_2) = r_1\theta_1 + r_2\theta_2 - \frac{1}{2}(\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2), \tag{3.1}$$

and condition (2-i) is always satisfied, where

$$r_1 = c_1 - \lambda_1, \qquad r_2 = c_2 - c_1 - \lambda_2.$$

Furthermore, $r_1, r_1 + r_2 > 0$ by the stability condition (2-ii), but r_2 can be negative or positive.

Note that the stationary equation (2.9) of Proposition 2.1 holds as long as $\varphi(\theta_1, \theta_2)$, $\varphi_2(\theta_1)$ and $\varphi_1(\theta_2)$ are finite. Hence, $\gamma(z_1, z_2)\varphi(z_1, z_2)$ is an analytic function of two complex variables z_1, z_2 for $\Re z_1 < 0$ and $\Re z_2 < 0$, and this domain is extendable as long as $\varphi_2(z_1)$ and $\varphi_1(z_2)$ are finite, where a complex valued function of two complex variables is said to be analytic if it is analytic as a one variable function for each fixed other variable (see, e.g., II.15 of [19]). Hence, we have proved the following fact.

Lemma 3.1 If both of $\varphi_2(\theta_1)$ and $\varphi_1(\theta_2)$ are finite, then $\gamma(\theta_1, \theta_2)\varphi(\theta_1, \theta_2)$ is finite. In particular, if $\gamma(\theta_1, \theta_2) \neq 0$ in this case, then $\varphi(\theta_1, \theta_2)$ is finite. Conversely, if $\varphi(\theta_1, \theta_2)$ is finite, then $\varphi_2(\theta_1)$ and $\varphi_1(\theta_2)$ are finite.

Using r_1 and r_2, (2.7) and (2.8) of Proposition 2.1 are written as

$$E_\pi(Y_1(1)) = r_1, \qquad E_\pi(Y_2(1)) = r_1 + r_2. \tag{3.2}$$

Since $\varphi_2(0) = E_\pi(Y_2(1))$, substituting $\theta_1 = 0$ in (2.9) yields

$$\varphi_1(\theta_2) = (\frac{1}{2}\sigma_2^2\theta_2 - r_2)\varphi(0, \theta_2) + r_1 + r_2. \tag{3.3}$$

Note that both sides of (3.3) are simultaneously finite or infinite due to Lemma 3.1. Furthermore, $z_2 = \frac{2r_2}{\sigma_2^2}$ is a removable singular point of $\varphi(0, z)$ since

$$\varphi(0, \theta_2) = \frac{\varphi_1(\theta_2) - \varphi_1(z_2)}{\frac{1}{2}\sigma_2^2\theta_2 - r_2}.$$

Hence, we have

Lemma 3.2 $\varphi_1(z)$ and $\varphi(0, z)$ have the same singularity.

We cannot get a similar direct relation between $\varphi_2(\theta_1)$ and $\varphi(\theta_1, 0)$, but the following result will be sufficient. Recall that α_1 is the rough decay rate of L_1.

Lemma 3.3 $\varphi_2(\theta)$ is finite for $\theta < \alpha_1$, where $\alpha_1 = \frac{2r_1}{\sigma_1^2}$.

Remark 3.1 This result will be sharpened in Corollary 3.1.

PROOF. $\alpha_1 = \frac{2r_1}{\sigma_1^2}$ is immediate from Example 2.1 since the first queue is unchanged by the intermediate input. Let $\Gamma_0 = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \gamma(\theta_1, \theta_2) > 0, \theta_1 < \alpha_1\}$. Since Γ_0 is an open convex set, we can find $(\theta_1^{(\epsilon)}, \theta_2^{(\epsilon)}) \in \Gamma_0$ for any $\epsilon > 0$ such that $\max(0, \alpha_1 - \epsilon) < \theta_1^{(\epsilon)} < \alpha_1$ and $\theta_2^{(\epsilon)} < 0$. Substituting this $(\theta_1^{(\epsilon)}, \theta_2^{(\epsilon)})$ into (2.9), we have

$$-\theta_2^{(\epsilon)}\varphi_1(\theta_2^{(\epsilon)}) + \gamma(\theta_1^{(\epsilon)}, \theta_2^{(\epsilon)})\varphi(\theta_1^{(\epsilon)}, \theta_2^{(\epsilon)}) = (\theta_1^{(\epsilon)} - \theta_2^{(\epsilon)})\varphi_2(\theta_1^{(\epsilon)}).$$

Since all the coefficients of $\varphi(\theta_1^{(\epsilon)}, \theta_2^{(\epsilon)}), \varphi_1(\theta_2^{(\epsilon)})$ and $\varphi_2(\theta_1^{(\epsilon)})$ are positive and $\varphi_1(\theta_2^{(\epsilon)}) < \varphi_1(0) = r_1 < \infty$, $\varphi_2(\theta_1^{(\epsilon)})$ must be finite. This proves the lemma since $\varphi_2(\theta)$ is increasing and ϵ can be arbitrarily small. □

We next rewrite (2.9) as

$$\gamma(\theta_1, \theta_2)\varphi(\theta_1, \theta_2) + (\theta_2 - \theta_1)\varphi_1(\theta_2) = \theta_2\varphi_2(\theta_1). \tag{3.4}$$

Since both sides of (3.4) are simultaneously finite or infinite, similarly to Lemma 3.1, it follows from Lemma 3.3 that $\varphi(\theta_1, \theta_2)$ and $\varphi_1(\theta_2)$ must be positive and finite for (θ_1, θ_2) in the region:

$$\mathcal{D}_+^{(1)} = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1, \gamma(\theta_1, \theta_2) > 0, 0 < \theta_1 < \theta_2\}.$$

Since $\gamma(\theta_1, \theta_2) = 0$ is an ellipse, we let

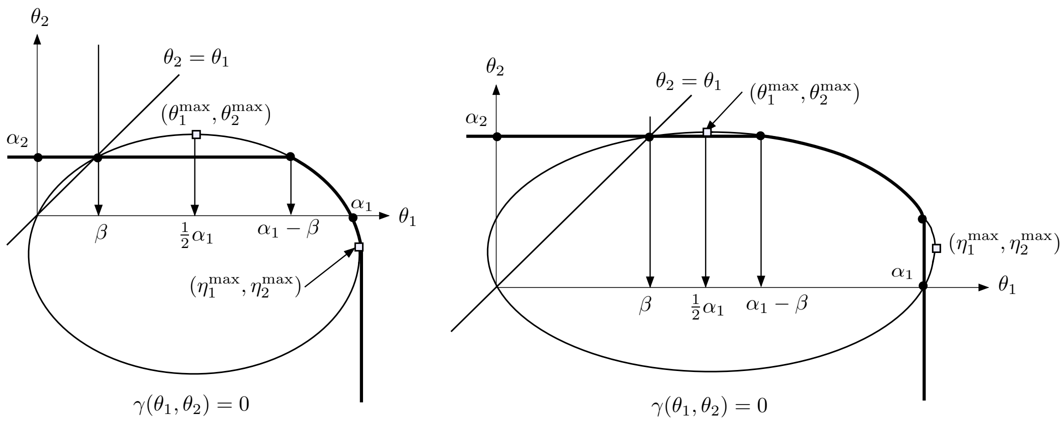
$$(\theta_1^{\max}, \theta_2^{\max}) = \arg\max_{(\theta_1,\theta_2)}\{\theta_2; \gamma(\theta_1, \theta_2) = 0\}, \quad (\theta_1^{\min}, \theta_2^{\min}) = \arg\min_{(\theta_1,\theta_2)}\{\theta_2; \gamma(\theta_1, \theta_2) = 0\},$$
$$(\eta_1^{\max}, \eta_2^{\max}) = \arg\max_{(\theta_1,\theta_2)}\{\theta_1; \gamma(\theta_1, \theta_2) = 0\}, \quad (\eta_1^{\min}, \eta_2^{\min}) = \arg\min_{(\theta_1,\theta_2)}\{\theta_1; \gamma(\theta_1, \theta_2) = 0\},$$
$$\beta = \max\{\theta; \gamma(\theta, \theta) = 0\}.$$

It is easy to compute these values. For example, $\theta_1^{\max} = \frac{1}{2}\alpha_1 < \alpha_1$, and solving $\gamma(\beta, \beta) = 0$ we have

$$\beta = \frac{2(r_1 + r_2)}{\sigma_1^2 + \sigma_2^2}, \tag{3.5}$$

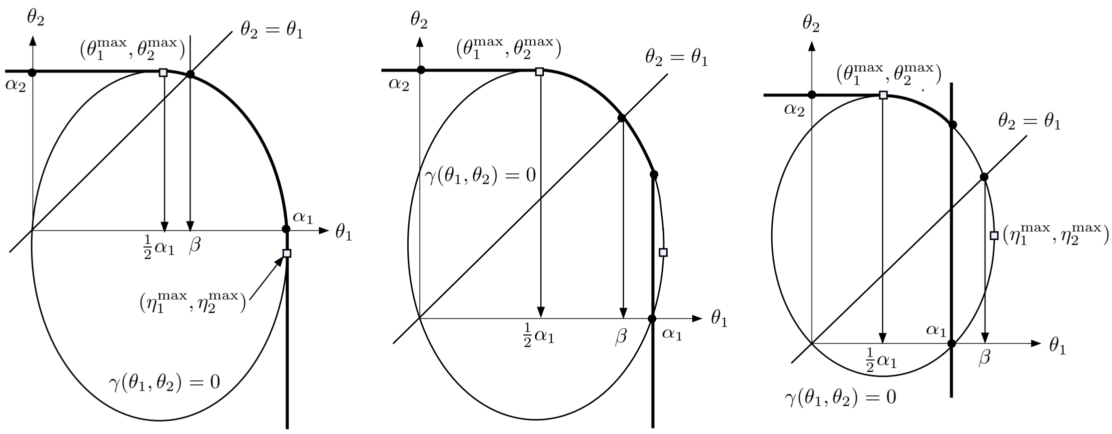
where $\beta > 0$ by the stability condition (2-ii). However, we will not use these specific values as long as possible for applying our arguments to more general Lévy inputs. We next let

$$\alpha_2 = \max\{\theta_2; (\theta_1, \theta_2) \in \mathcal{D}_+^{(1)}\}.$$



(a−) $\beta < \frac{1}{2}\alpha_1$ with $r_2 < 0$ (a+) $\beta < \frac{1}{2}\alpha_1$ with $r_2 > 0$

Figure 1: Typical regions of $\mathcal{D}^{\mathrm{o}}$ for $\beta < \frac{1}{2}\alpha_1$



(b−) $\frac{1}{2}\alpha_1 \le \beta \le \alpha_1$ with $r_2 < 0$ (b+) $\frac{1}{2}\alpha_1 \le \beta \le \alpha_1$ with $r_2 > 0$ (c) $\alpha_1 < \beta$

Figure 2: Typical regions of $\mathcal{D}^{\mathrm{o}}$ for $\frac{1}{2}\alpha_1 \le \beta$

It is easy to see from the definition of $\mathcal{D}_+^{(1)}$ that

$$\alpha_2 = \begin{cases} \beta, & \beta < \frac{1}{2}\alpha_1, \\ \theta_2^{\max}, & \frac{1}{2}\alpha_1 \le \beta, \end{cases} \tag{3.6}$$

and $\varphi_1(\theta)$ is finite for $\theta < \alpha_2$ (see Figures 1 and 2).

Having this α_2 in mind, we define $\mathcal{D}_+^{(2)}$ as

$$\mathcal{D}_+^{(2)} = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1, \theta_2 < \alpha_2, \gamma(\theta_1, \theta_2) > 0, 0 \le \theta_2 \le \theta_1\},$$

and consider (2.9). Since $\varphi_2(\theta_1)$ and $\varphi_1(\theta_2)$ are finite for $(\theta_1, \theta_2) \in \mathcal{D}_+^{(2)}$, the right-hand side of (2.9) is finite and therefore $\varphi(\theta_1, \theta_2)$ must be positive and finite for $(\theta_1, \theta_2) \in \mathcal{D}_+^{(2)}$. Let $\mathcal{D}_+ = \mathcal{D}_+^{(1)} \cup \mathcal{D}_+^{(2)}$. Since $(\theta_1, \theta_2) \in \mathcal{D}_+^{(1)}$ implies $\theta_2 < \alpha_2$, we have

$$\mathcal{D}_+ = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1, 0 \le \theta_2 < \alpha_2, \gamma(\theta_1, \theta_2) > 0\}.$$

We finally rewrite (2.9) as

$$\gamma(\theta_1, \theta_2)\varphi(\theta_1, \theta_2) - \theta_2\varphi_2(\theta_1) = (\theta_1 - \theta_2)\varphi_1(\theta_2), \tag{3.7}$$

and define $\mathcal{D}_-$ as

$$\mathcal{D}_- = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \gamma(\theta_1, \theta_2) > 0, \theta_2 \le \theta_1, \theta_2 < 0\}.$$

Since $\varphi_1(\theta_2)$ is finite for $\theta_2 < 0$, $\varphi(\theta_1, \theta_2)$ must be finite for $(\theta_1, \theta_2) \in \mathcal{D}_-$ similarly to the previous cases.

We are now in a position to identify $\mathcal{D}$ except for its boundary. Let $\xi_1(\theta_2)$ be the minimal solution θ_1 for $\gamma(\theta_1, \theta_2) = 0$ for each θ_2, and let $\xi_2(\theta_1)$ be the maximal solution θ_2 for $\gamma(\theta_1, \theta_2) = 0$ for each θ_1, as long as they exists. Clearly, for $\eta_2^{\min} \le \theta_2 \le \theta_2^{\max}$, $\theta_1 = \xi_1(\theta_2)$ if and only if $\theta_2 = \xi_2(\theta_1)$.

Proposition 3.1 For the Brownian tandem queue with an intermediate input satisfying the condition (2-ii), let $\mathcal{D}^{\mathrm{o}}$ be interior of $\mathcal{D}$. Then,

$$\mathcal{D}^{\mathrm{o}} = \{(\theta_1, \theta_2) \in \mathbb{R}^2; (\theta_1, \theta_2) < (\theta_1', \theta_2') \text{ for some } (\theta_1', \theta_2') \in \mathcal{D}_+ \cup \mathcal{D}_-\}. \tag{3.8}$$

Proof. Denote the right-hand side of (3.8) by $\mathcal{A}$. We have already proved that $\mathcal{D}_+ \cup \mathcal{D}_- \subset \mathcal{D}$, which implies $\mathcal{A} \subset \mathcal{D}$. So, we only need to prove that $\varphi(\theta_1, \theta_2) = \infty$ if $(\theta_1, \theta_2) \notin \overline{\mathcal{A}}$, where $\overline{\mathcal{A}}$ is the closure of $\mathcal{A}$. We consider the following three cases separately.

If $\beta < \frac{1}{2}\alpha_1$, then $(\beta, \beta) \in \overline{\mathcal{A}}$ (see Figure 1). Hence, $\varphi(\theta_1, \theta_2) < \infty$ for $\theta_1, \theta_2 < \beta$, and

$$\varphi_1(\theta_2) = \frac{\theta_2 \varphi_2(\xi_1(\theta_2))}{\theta_2 - \xi_1(\theta_2)}, \qquad 0 < \theta_2 < \beta. \tag{3.9}$$

Since $\beta = \xi_1(\beta) < \alpha_1$, $\varphi_1(z)$ has a simple pole at $z = \beta$. By Lemma 3.2, $\varphi(0, z)$ has the same pole at $z = \beta$. Hence, $\theta_2 > \beta$ and $\theta_1 \ge 0$ imply $\varphi(\theta_1, \theta_2) = \infty$.

If $\frac{1}{2}\alpha_1 \le \beta$, then $(\theta_1^{\max}, \theta_2^{\max}) \in \overline{\mathcal{A}}$ (see Figure 2). We rearrange (3.4) as

$$(\theta_2 - \theta_1)\varphi_1(\theta_2) = -\gamma(\theta_1, \theta_2)\varphi(\theta_1, \theta_2) + \theta_2 \varphi_2(\theta_1).$$

Choose any point $(\theta_1, \theta_2) \in \overline{\mathcal{A}}^c \cap \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1\}$, and assume that $\varphi(\theta_1, \theta_2) < \infty$. Since $\varphi_2(\theta_1)$ is finite, $\varphi_1(\theta_2)$ must be finite, which is proved by partial differentiation with respect to θ_1. Since the left-hand side is always finite for any θ_1, we let θ_1 go to θ_2 or increase to α_1. Both leads to contradiction, and we have $\varphi(\theta_1, \theta_2) = \infty$. If $\beta < \frac{1}{2}\alpha_1$, we can similarly prove that $(\theta_1, \theta_2) \in \overline{\mathcal{A}}^c \cap \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1\}$ implies $\varphi(\theta_1, \theta_2) = \infty$.

Obviously, if $\theta_1 > \alpha_1$ and $\theta_2 \ge 0$, then $\varphi(\theta_1, \theta_2) = \infty$ while, if $\theta_1 \le \alpha_1$ and $\theta_2 \le 0$, then $\varphi(\theta_1, \theta_2) < \infty$. Furthermore, $\theta_1 < \eta_1^{\max}$ and $\theta_2 < \eta_2^{\max}$ imply $\varphi(\theta_1, \theta_2) < \infty$ from the definition of $\mathcal{A}$. Thus, it remains to prove that $(\theta_1, \theta_2) \in \overline{\mathcal{A}}^c \cap \{(\theta_1, \theta_2) \in \mathbb{R}^2; \alpha_1 < \theta_1, \eta_2^{\max} < \theta_2 < 0\}$ implies $\varphi(\theta_1, \theta_2) = \infty$. For this (θ_1, θ_2), assume that $\varphi(\theta_1, \theta_2) < \infty$. From the definition of $\mathcal{A}$, $\gamma(\theta_1, \theta_2) < 0$, and $\varphi_1(\theta_2) < \infty$ for $\theta_2 < 0$. Hence, rearranging (3.4) as

$$-\gamma(\theta_1, \theta_2)\varphi(\theta_1, \theta_2) + (\theta_1 - \theta_2)\varphi_1(\theta_2) = -\theta_2 \varphi_2(\theta_1),$$

we can see that $\varphi_2(\theta_1)$ must be finite. Let negative θ_2 goes to zero, then the left-hand side must be positive while the right hand side vanishes. This implies that $\varphi_2(\theta_1)$ cannot be finite. Thus, we have a contradiction, and the proof is completed. □

The following corollary sharpens Lemma 3.3.

Corollary 3.1 Let $\alpha_1^{\max} = \sup\{\theta \geq 0; \varphi_2(\theta) < \infty\}$. Then,

$$\alpha_1^{\max} = \begin{cases} \alpha_1, & \eta_2^{\max} \geq 0, \\ \eta_1^{\max}, & \eta_2^{\max} < 0, \end{cases} \tag{3.10}$$

PROOF. From (2.9), it is seen that, if $\theta_2 < \alpha_2$ and if $\gamma(\theta_1, \theta_2) \neq 0$, then $\varphi(\theta_1, \theta_2) < \infty$ if and only if $\varphi_2(\theta_1) < \infty$. Since $(\theta_1, \theta_2) \in \mathcal{D}^{\mathrm{o}}$ implies $\theta_2 < \alpha_2$, $\alpha_1^{\max} = \sup\{\theta_1; (\theta_1, \theta_2) \in \mathcal{D}^{\mathrm{o}}\}$. Hence, Proposition 3.1 concludes (3.10). □

We can write (3.8) in a more explicit form. For this, let

$$\mathcal{D}_1 = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1^{\max}, \theta_2 < \alpha_2\}, \tag{3.11}$$

$$\mathcal{D}_2 = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \theta_1', \theta_2 < \theta_2', \text{ for some } (\theta_1', \theta_2') \text{ such that } \gamma(\theta_1', \theta_2') \leq 0\}, \tag{3.12}$$

then it is easy to get the following corollary from Proposition 3.1 and Corollary 3.1.

Corollary 3.2 $\mathcal{D}^{\mathrm{o}} = \mathcal{D}_1 \cap \mathcal{D}_2$, and $\mathcal{D}^{\mathrm{o}}$ is a convex set.

The open domain $\mathcal{D}^{\mathrm{o}}$ is typical for the discrete-time two dimensional reflected process on the quadrant, but does not cover all the cases because of the structure of a tandem queue. In the terminology of the sample path large deviations, it is important to find the optimal path for the rate function in each direction. For the direction to increase L_2, this path goes up along the 2nd coordinate in Figure 1 while it straightly moves inside the quadrant in cases (b+), (b-) and (c). So, we do not have the case that the optimal path firstly goes along the 1st coordinate, then straightly move inside the quadrant.

4 Exact asymptotic behavior; the Brownian case

In this section, we first derive the exact asymptotics of the tail probability of L_2. For this, we will study the type of singularity of $\varphi(0, \theta)$ at the boundary of $\mathcal{D}$. In a similar way, we work out the exact asymptotics of the tail distribution function of $d_1 L_1 + d_2 L_2$ for $d_1, d_2 > 0$. Let F be the distribution function of L_2 and $\overline{F}$ be its complement. That is,

$$\overline{F}(x) = P(L_2 > x), \qquad x \geq 0.$$

It is known that F is absolutely continuous with respect to Lebesgue measure on the real line (see, e.g., [12]), so $\overline{F}(x)$ is continuous in x. Let

$$\psi(\theta) = \int_0^\infty e^{\theta x} \overline{F}(x)\, dx,$$

and let α be the rightmost point such that $\psi(\theta)$ is finite for real number $\theta < \alpha$. Clearly, $\psi(z)$ is analytic for $\Re z < \alpha$. In what follows, we use the same ψ for its analytic extension. For this ψ, we will apply results in Doetsch [8], whose basic idea is to extract a principal term of an analytic function using a counter integral around a singular point. The first result (S1) below is for the case that the rightmost singular point is a simple pole. This is a special case of Theorems 35.1 of Doetsch [8], but slightly relaxes the required region of the analytic function, limiting to a single pole.

(S1) If the following conditions are satisfied for α, positive integer k and some p, q such that $p < \alpha < q$:

(S1a) $\psi(z)$ is analytic for $p \le \Re z \le q$ except for $z = \alpha$.

(S1b) $\psi(z)$ uniformly converges to 0 as $z \to \infty$ for $p \le \Re z \le q$, and the integral

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-ixy} \psi(q + iy) dy$$

uniformly converges for $x > T$,

(S1c) for some constant $C'_1 > 0$, $\lim_{z\to\alpha}(\alpha - z)^k \psi(z) = C'_1$,

then

$$\overline{F}(x) = \frac{C'_1}{\Gamma(k)} x^{k-1} e^{-\alpha x}(1 + o(1)),$$

where $\Gamma(z)$ is the gamma function.

Remark 4.1 There are some remarks for specializing Theorem 35.1 of [8] to (S1). First, Laplace transforms are used instead of moment generating functions in [8], so its results should be appropriately converted. For example, we put $a = -p, \alpha_0 = -\alpha, \beta_0 = -q$ in Theorem 35.1 of [8]. Secondly, $\psi(z)$ need not to be analytic for $z \ge q$ since we are only concerned with the single pole at $z = \alpha < q$. Thirdly, the Fourier inversion formula on $\mathcal{B}(F)$ in Theorem 35.1 is always satisfied for $a > \Re\alpha_0$, which is equivalent to $p < \alpha$ in our formulation since $\psi(z)$ is the moment generating function of an absolutely integrable function for $\Re z < \alpha$ (see Theorem 24.4 of [8]).

Remark 4.2 In (S1), if the condition $p < \alpha < q$ is replaced by $p < q < \alpha$ and if (S1c) is dropped, then $\overline{F}(x) = o(e^{-qx})$. This fact is easily seen from the proof of Theorem 35.1 of [8] (see pages 236 and 237). Thus, if we can find q to be arbitrarily close to α, the rough decay rate $\overline{F}(x)$ is not less than α.

If the rightmost singular point is not a simple pole, we use Theorem 37.1 of Doetsch [8]. We specialize it in the following way.

(S2) If the following three conditions hold for some $\alpha > 0$ and some $\delta \in [0, \frac{\pi}{2})$:

(S2a) $\psi(z)$ is analytic in the region:
$\mathcal{G}_\alpha(\delta) \equiv \{z \in \mathbb{C}; \Re z > 0, z \ne \alpha, |\arg(z - \alpha)| > \delta\}$, where $\arg z$ is the principal part of the argument of complex number z,

(S2b) $\psi(z) \to 0$ as $|z| \to \infty$ for $z \in \mathcal{G}_\alpha(\delta)$,

(S2c) for some constant K and non integer real number s

$$\psi(z) = K\ 1(s > 0) - C'_2(\alpha - z)^s + o((\alpha - z)^s), \tag{4.1}$$

for $\mathcal{G}_\alpha(\delta) \ni z \to \alpha$, then

$$\overline{F}(x) = \frac{C_2'}{\Gamma(-s)} x^{-s-1} e^{-\alpha x}(1 + o(1)) \ ,$$

where K must be $\psi(\alpha)$ if $s > 0$.

Remark 4.3 Constant K in (4.1) can be replaced by any analytic function on $\mathcal{G}_\alpha(\delta)$, but we do not need this generality here. Abate and Whitt [1] refer to asymptotic results like (S2) as "the Heaviside operational principle", and they suggest to check the weaker conditions than (S2a) and (S2b) from Sutton [24]. However in [1], no conditions except for (S3c) are verified. There are corresponding results for the case of a generating function, which are referred to as "Darboux theorem" (e.g., see [15]).

Remark 4.4 In applications of (S1) and (S2), the verification of other conditions than (S1c) and (S2c) are often ignored. For $\alpha = 0$, this causes no problem for (S1) since $e^{\alpha x}\bar{F}(x)$ is ultimately monotone and Tauberian theorem can be applied. However, for $\alpha > 0$, this monotone property is hard to check.

Theorem 4.1 For the Brownian tandem queue satisfying the stability condition (2-ii), $P(L_2 > x)$ has the exact asymptotics $h(x)$ of the following type.

(4a) If $\beta < \frac{1}{2}\alpha_1$, then $h(x) = C_1 e^{-\beta x}$.

(4b) If $\frac{1}{2}\alpha_1 = \beta$, then $h(x) = C_2 x^{-\frac{1}{2}} e^{-\theta_2^{\max} x}$.

(4c) If $\frac{1}{2}\alpha_1 < \beta$, then $h(x) = C_3 x^{-\frac{3}{2}} e^{-\theta_2^{\max} x}$.

Constants C_1, C_2 and C_3 are given in the proof. Hence, the rough decay rate of $P(L_2 > x)$ is α_2 of (3.6).

Remark 4.5 Quantities α_1, β and $\theta_2^{\max}$ are defined in Section 3. Specifically they are given by

$$\alpha_1 = \frac{2r_1}{\sigma_1^2}, \qquad \beta = \frac{2(r_1 + r_2)}{\sigma_1^2 + \sigma_2^2}, \qquad \theta_2^{\max} = \frac{1}{\sigma_2^2}\left(r_2 + \sqrt{r_2^2 + r_1^2 \frac{\sigma_2^2}{\sigma_1^2}}\right).$$

Remark 4.6 Chang [7] derives the rough decay rates for a more general intree network under discrete time setting. Those results are less explicit since everything is given in terms of rate functions, but it is not hard to see that Theorem 1.2 of [7] yields the exactly corresponding rough decay rate for a two node tandem queue with intermediate arrivals in discrete time.

For the proof of Theorem 4.1, we consider $\psi(\theta) = \int_0^\infty e^{\theta x} P(L_2 > x)\,dx$. Since $\varphi(0,\theta) = 1 + \theta\psi(\theta)$, it follows from (3.3), (3.9) and Proposition 3.1 that, for $\theta_2 < \alpha_2$,

$$\begin{aligned} \psi(\theta_2) &= \frac{\varphi(0,\theta_2) - 1}{\theta_2} \\ &= \frac{\varphi_1(\theta_2) - (r_1 + r_2)}{\theta_2 f(\theta_2)} - \frac{1}{\theta_2} \\ &= \frac{\varphi_2(\xi_1(\theta_2))}{f(\theta_2)(\theta_2 - \xi_1(\theta_2))} - \frac{\frac{1}{2}\sigma_2^2\theta_2 + r_2}{\theta_2 f(\theta_2)}, \end{aligned} \tag{4.2}$$

where $f(\theta) = \frac{1}{2}\sigma_2^2\theta - r_2$.

We first consider the analytic extension of $\psi(z)$, which is obviously analytic for $\Re z < \alpha_2$. In what follows, we let $\chi_0 = \frac{2r_2}{\sigma_2^2}$, which implies that $f(\theta) = \frac{1}{2}(\theta - \chi_0)$. Note that $z = 0, \chi_0$ are removable singular points of $\psi(z)$. The proof of the theorem is preceded by few lemmas.

Lemma 4.1 $\psi(z)$ is analytic on the set $\mathbb{C} \setminus (\{\beta\} \cup [\theta_2^{\max}, +\infty)\})$.

Proof. We first consider $\xi_1(z)$. By its definition, $\xi_1(z)$ has the following expression for real number $\theta \in [\theta_2^{\min}, \theta_2^{\max}]$,

$$\begin{aligned} \xi_1(\theta) &= \frac{1}{\sigma_1^2}\left(r_1 - \sqrt{r_1^2 - \sigma_1^2(\sigma_2^2\theta^2 - 2r_2\theta)}\right) \\ &= \frac{1}{\sigma_1^2}\left(r_1 - \sigma_1\sigma_2\sqrt{(\theta_2^{\max} - \theta)(\theta - \theta_2^{\min})}\right). \end{aligned} \tag{4.3}$$

Hence, complex variable function $\xi_1(z)$ has only two singular points at $z = \theta_2^{\max} > 0$ and $z = \theta_2^{\min} < 0$, which are branch points. We choose one branch which is identical with $\xi_1(\theta)$ for $z = \theta \in (0, \theta_2^{\max})$ and analytic on

$$\mathcal{G}_{\theta_2^{\max}}(0) = \{z \in \mathbb{C}; 0 < \Re(z), z \notin [\theta_2^{\max}, \infty)\}$$

Denote this branch by the same notation $\xi_1(z)$, which is given by

$$\xi_1(z) = \frac{\alpha_1}{2} - \frac{\sigma_2}{\sigma_1}|(\theta_2^{\max} - z)(z - \theta_2^{\min})|^{\frac{1}{2}}\left(\cos\frac{\omega_- + \omega_+}{2} + i\sin\frac{\omega_- + \omega_+}{2}\right),$$

where $\omega_- = \arg(z - \theta_2^{\min})$ and $\omega_+ = \arg(\theta_2^{\max} - z)$ (see, e.g., Chapter I.11 of [19]). It should be noticed that $-\pi < \omega_1, \omega_2 < \pi$ for $z \neq \Re z$, $\omega_- = \omega_+ = 0$ for real z satisfying $\theta_2^{\min} < z < \theta_2^{\max}$, and ω_1 and ω_+ have different signs. Thus, we have $-\pi < \omega_- + \omega_+ < \pi$, which yields

$$\Re\xi_1(z) \leq \frac{\alpha_1}{2} < \alpha_1, \qquad z \in \mathcal{G}_{\theta_2^{\max}}(0). \tag{4.4}$$

Since $\varphi_2(z)$ is analytic for $\Re z < \alpha_1$, this implies that $\varphi_2(\xi_1(z))$ is analytic for all $z \in \mathcal{G}_{\theta_2^{\max}}(0)$. Thus, $\psi(z)$ is analytic for $z \in \mathcal{G}_{\theta_2^{\max}}(0) \setminus \{\beta, \chi_0\}$ since the denominators in (4.2) only vanish at $z = 0, \beta, \chi_0$, reminding that $z = 0, \chi_0$ are removable singular points of $\psi(z)$. Hence, the lemma is obtained since $\psi(z)$ is analytic for $\Re z < \alpha_1$. □

Lemma 4.2 If $\beta < \frac{1}{2}\alpha_1$, then (S1a) and (S1b) are satisfied for $\alpha = \beta$ and $k = 1$. Otherwise, if $\frac{1}{2}\alpha_1 \leq \beta$, then (S2a) and (S2b) are satisfied for $\alpha = \theta_2^{\max}$ and $\delta = 0$.

PROOF. We first suppose that $\beta < \frac{1}{2}\alpha_1$, which implies that $\beta < \theta_2^{\max}$. Then, (S1a) is immediate from Lemma 4.1. For the uniform convergence of the integral in (S1b), it suffices to verify the uniform convergence of the following two integrals

$$\begin{aligned} I_1(x) &\equiv \int_{-\infty}^{+\infty} \frac{\varphi_2(\xi_1(q+iy))e^{-ixy}}{(q+iy-\chi_0)(q+iy-\xi_1(q+iy))}dy, \\ I_2(x) &\equiv \int_{-\infty}^{+\infty} \frac{(q+iy+\chi_0)e^{-ixy}}{(q+iy)(q+iy-\chi_0)}dy. \end{aligned}$$

Because $q < \theta_2^{\max}$, $q \neq \chi_0$ and $\gamma(z,z) = 0$ has only two solutions $z = 0, \beta$, the denominator of the integrand of $I_1(x)$ is bounded away from zero. Hence, it is not hard to see that the integrand of $I_1(x)$ is absolutely integrable, so the convergence is uniform in x. For $I_2(x)$, we note that

$$\left|\int_{-\infty}^{+\infty} \frac{e^{-ixy}}{q+iy}dy\right| = \left|\left[\frac{-e^{-ixy}}{ix(q+iy)}\right]_{-\infty}^{+\infty} - \frac{1}{x}\int_{-\infty}^{+\infty} \frac{e^{-ixy}}{(q+iy)^2}dy\right| \leq \frac{1}{x}\int_{-\infty}^{+\infty} \frac{1}{q^2+y^2}dy.$$

Using similar arguments, we can see that $I_2(x)$ converges to zero as $x \to \infty$. Thus, (S2b) is verified.

We next suppose that $\frac{1}{2}\alpha_1 \leq \beta$. This and Lemma 4.1 verify (S2a) for $\alpha = \theta_2^{\max}$ and $\delta = 0$. Furthermore, (S2b) obviously holds true since $\varphi_2(z) \to 0$ as $|z| \to \infty$ for $\Re z < \frac{\alpha_1}{2}$. □

Lemma 4.3 For $\theta_2 \uparrow \theta_2^{\max}$,

$$\theta_1^{\max} - \xi_1(\theta_2) = \sqrt{\frac{2(\theta_2^{\max} - \theta_2)}{-\xi_2''(\theta_1^{\max})}} + o(|\theta_2^{\max} - \theta_2|^{\frac{1}{2}}), \tag{4.5}$$

and, particularly if $\beta = \theta_2^{\max}$, then

$$\theta_2 - \xi_1(\theta_2) = \sqrt{\frac{2(\theta_2^{\max} - \theta_2)}{-\xi_2''(\theta_1^{\max})}} + o(|\theta_2^{\max} - \theta_2|^{\frac{1}{2}}), \tag{4.6}$$

where

$$\xi_2''(\theta_1^{\max}) = -\frac{\sigma_1^3}{\sqrt{r_1^2\sigma_1^2 + r_2^2\sigma_2^2}} < 0.$$

PROOF. Since $\xi_2(\theta_1)$ is concave from its definition and $\xi_2'(\theta_1^{\max}) = 0$, its Taylor expansion at $\theta_1 = \theta_1^{\max}$ yields

$$\xi_2(\theta_1) = \xi_2(\theta_1^{\max}) + \frac{1}{2}\xi_2''(\theta_1^{\max})(\theta_1 - \theta_1^{\max})^2 + o((\theta_1 - \theta_1^{\max})^2),$$

which implies, for $\eta_2^{\min} < \theta_2 < \theta_2^{\max}$, or equivalently, $\eta_1^{\min} < \theta_1 < \theta_1^{\max}$,

$$\theta_1^{\max} - \theta_1 = \sqrt{\frac{2(\xi_2(\theta_1^{\max}) - \xi_2(\theta_1))}{-\xi_2''(\theta_1^{\max})}} + o(|\theta_1 - \theta_1^{\max}|). \tag{4.7}$$

Since $\theta_2 = \xi_2(\theta_1)$ is equivalent to $\theta_1 = \xi_1(\theta_2)$ for $\eta_2^{\min} < \theta_2 < \theta_2^{\max}$, this can be written as

$$\theta_1^{\max} - \xi_1(\theta_2) \quad = \quad \sqrt{\frac{2(\theta_2^{\max} - \theta_2)}{-\xi_2''(\theta_1^{\max})}} + o(|\theta_2 - \theta_2^{\max}|^{\frac{1}{2}}).$$

Hence, we have (4.5). If $\beta = \theta_2^{\max}$, then $\theta_1^{\max} = \theta_2^{\max}$, so we have

$$\theta_2 - \xi_1(\theta_2) \quad = \quad \theta_1^{\max} - \xi_1(\theta_2) - (\theta_2^{\max} - \theta_2) + o(|\theta_1 - \theta_1^{\max}|^2).$$

This and (4.5) yield (4.6). It remains to compute $\xi_2''(\theta_1^{\max})$, but this is easily done by differentiating $\gamma(\theta, \theta') = 0$ with respect to θ at $(\theta, \theta') = (\theta_1^{\max}, \theta_2^{\max})$. □

The proof of Theorem 4.1

We prove the three cases separately. For (4a), we assume that $\beta < \frac{1}{2}\alpha_1$, and consider the conditions of (S1). By Lemma 4.2, (S1a) and (S1b) are already verified. It remains to verify (S1c). However, we have already observed in the proof of Proposition 3.1 that $\varphi_1(z)$ has a simple pole at $z = \beta$ (see (3.9)), so $\psi(z)$ also has the same pole since the singularity of $\psi(z)$, which is the same as $\varphi(0, z)$, and therefore the same as $\varphi_1(z)$ by Lemma 3.2. Hence, we have (S1c) with $k = 1$, which leads to (4a). The constant C_1 is computed from (4.2) as,

$$\begin{aligned} C_1 &= \lim_{z\to\beta}(\beta - z)\psi(z) \\ &= \lim_{z\to\beta} \frac{(\beta - z)}{z(z - \xi_1(z))} \frac{z\varphi_2(\xi_1(z)) - (r_1 + r_2)(z - \xi_1(z))}{f(z)} \\ &= \frac{-1}{\beta(1 - \xi_1'(\beta))} \lim_{z\to\beta} \frac{z\varphi_2(\xi_1(z)) - (r_1 + r_2)(z - \xi_1(z))}{f(z)}. \end{aligned}$$

The condition $\beta < \frac{1}{2}\alpha_1$ is equivalent to

$$r_1\sigma_2^2 - r_1\sigma_1^2 - (r_1 + r_2)\sigma_1^2 > 0.$$

This implies that $f(\beta) = \frac{1}{2}\sigma_2^2\beta - r_2 = \frac{r_1\sigma_2^2 - r_2\sigma_1^2}{\sigma_1^2 + \sigma_2^2} > 0$. From (A.1) of Appendix B,

$$\xi_1'(\beta) = 1 + \frac{(r_1 + r_2)(\sigma_1^2 + \sigma_2^2)}{r_1\sigma_2^2 - r_1\sigma_1^2 - (r_1 + r_2)\sigma_1^2} > 1.$$

Hence, we have

$$C_1 = \frac{(r_1\sigma_2^2 - r_1\sigma_1^2 - (r_1 + r_2)\sigma_1^2)\varphi_2(\beta)}{(r_1 + r_2)(r_1\sigma_2^2 - r_2\sigma_1^2)}.$$

For (4b) and (4c), we verify (S2c) only for $z = \theta_2 \in (0, \theta_1^{\max})$ since the other routes for this limit can be similarly verified by putting $z = \theta_2^{\max} + (\theta_2 - \theta_2^{\max})e^{i\omega}$ for $\omega \in (-\pi, \pi)$.

First consider (4b). Note that $f(\theta_2^{\max}) \neq 0$ since $f(\theta_2^{\max}) = 0$ and $\beta = \theta_2^{\max}$ imply the contradiction that $r_1\sigma_1^2 = 0$. Then, we simply apply (4.6) of Lemma 4.3 to (4.2), and get

$$\lim_{\theta_2 \uparrow \theta_2^{\max}} (\theta_2^{\max} - \theta_2)^{\frac{1}{2}} \psi(\theta_2) = \frac{\varphi_2(\theta_1^{\max})}{f(\theta_2^{\max})} \sqrt{\frac{-\xi_2''(\theta_1^{\max})}{2}} \equiv C_1'.$$

Hence, by (S2) with $s = -\frac{1}{2}$, we have (4b), where C_2 is given by

$$C_2 = \frac{C_1'}{\Gamma(\frac{1}{2})} = \frac{\varphi_2(\theta_1^{\max})}{\frac{1}{2}\sigma_2^2\theta_2^{\max} - r_2} \sqrt{\frac{-\xi_2''(\theta_1^{\max})}{2\pi}}.$$

For (4c), we assume that $\frac{1}{2}\alpha_1 < \beta$, which implies $\beta \neq \theta_2^{\max}$. Due to Lemma 4.2, we only need to verify (4.1). We first verify this for real $z = \theta_2 < \alpha \equiv \theta_2^{\max}$. From (4.2),

$$\begin{aligned} \psi(\theta_2) &= \frac{\varphi_2(\xi_1(\theta_2))}{f(\theta_2)(\theta_2 - \theta_1^{\max} + \theta_1^{\max} - \xi_1(\theta_2))} - \frac{\frac{1}{2}\sigma_2^2\theta_2 + r_1}{\theta_2 f(\theta_2)} \\ &= \frac{\varphi_2(\xi_1(\theta_2))(\theta_2 - \theta_1^{\max} - (\theta_1^{\max} - \xi_1(\theta_2)))}{f(\theta_2)((\theta_2 - \theta_1^{\max})^2 - (\theta_1^{\max} - \xi_1(\theta_2))^2)} - \frac{\frac{1}{2}\sigma_2^2\theta_2 + r_1}{\theta_2 f(\theta_2)}. \end{aligned} \tag{4.8}$$

Since $\theta_2^{\max} - \xi_1(\theta_2^{\max}) > 0$, the denominators in (4.8) do not vanish at $\theta_2 = \theta_2^{\max}$. By the Taylor expansion of $\varphi_2(z)$ at $z = \theta_1^{\max}(= \frac{1}{2}\alpha_1)$,

$$\varphi_2(\xi_1(\theta_2)) = \varphi_2(\theta_1^{\max}) + \varphi_2'(\theta_1^{\max})(\xi_1(\theta_2) - \theta_1^{\max}) + o(|\xi_1(\theta_2) - \theta_1^{\max}|),$$

where we have used the fact that $\varphi_2(z)$ is analytic for $\Re z < \alpha_1$ by Lemma 3.3. Hence, (4.8) can be written as

$$\psi(\theta_2) = (\xi_1(\theta_2) - \theta_1^{\max})K_1(\theta_2) + K_2(\theta_2) + o(|\xi_1(\theta_2) - \theta_1^{\max}|),$$

where $K_1(\theta_2)$ and $K_2(\theta_2)$ are given by

$$\begin{aligned} K_1(\theta_2) &= \frac{\varphi_2(\theta_1^{\max}) + (\theta_2 - \theta_1^{\max})\varphi_2'(\theta_1^{\max})}{f(\theta_2)((\theta_2 - \theta_1^{\max})^2 - (\theta_1^{\max} - \xi_1(\theta_2))^2)} \\ K_2(\theta_2) &= \frac{\varphi_2(\theta_1^{\max})(\theta_2 - \theta_1^{\max})}{f(\theta_2)((\theta_2 - \theta_1^{\max})^2 - (\theta_1^{\max} - \xi_1(\theta_2))^2)} - \frac{\sigma_2^2\theta_2 + 2r_1}{\theta_2(\sigma_2^2\theta_2 - 2r_2)}. \end{aligned}$$

Note that $K_1(\theta_2^{\max})$ is a positive constant since $f(\theta_2^{\max}) = \frac{1}{2}\sigma_2^2\theta_2^{\max} - r_2 > 0$ and $\theta_2^{\max} - \theta_1^{\max} > 0$. Thus, by Lemma 4.3, we have

$$\psi(\theta_2) = -(\theta_2^{\max} - \theta_2)^{\frac{1}{2}} K_1(\theta_2^{\max}) \sqrt{\frac{2}{-\xi_2''(\theta_1^{\max})}} + K_2(\theta_2^{\max}) + o((\theta_2^{\max} - \theta_2)^{\frac{1}{2}}). \tag{4.9}$$

Thus, we have (4.1) for real $z = \theta_2 < \alpha = \theta_2^{\max}$. For complex number $z \in \mathcal{G}_\alpha(0)$, we have shown in Appendix A that $\xi_1(z)$ can be chosen as $\Re\xi(z) < \frac{1}{2}\alpha_1$, so $\varphi_2(\xi_1(z))$ is analytic on $\mathcal{G}_\alpha(0)$. The other terms in $\psi(z)$ of (4.8) are similarly analytic on $\mathcal{G}_\alpha(0)$. Hence, (4.9) can be obtained for $z \in \mathcal{G}_\alpha(0)$ in place of θ_2, which implies (4.1) with $s = \frac{1}{2}$ and $C_2' = K_1(\theta_2^{\max})\sqrt{\frac{2}{-\xi_2''(\theta_1^{\max})}}$. Thus, all the conditions of (S2) are verified, which concludes (4c) with

$$C_3 = \frac{1}{\Gamma(-\frac{1}{2})} C_2' = \frac{\varphi_2(\theta_1^{\max}) + (\theta_2^{\max} - \theta_1^{\max})\varphi_2'(\theta_1^{\max})}{(\frac{1}{2}\sigma_2\theta_2^{\max} - r_2)(\theta_2^{\max} - \theta_1^{\max})^2\sqrt{-2\pi\xi_2''(\theta_1^{\max})}}.$$

□

In principle, the rough decay rate α_2 is known for a more general two dimensional reflected Brownian queueing network in the framework of large deviations theory. Namely, the rate function for the sample path large deviations is obtained in [2]. In this paper we sharpen the rough decay rate to exact asymptotics. Nevertheless, the rough decay rate in the present form may be also interesting since it clearly explains how the presence of the exogenous input at node 2 decreases α_2.

We next note exact asymptotics for $L_1 + L_2$. This case may have its own interest for applications. Letting $\theta_1 = \theta_2 = \theta$ in (2.9), we have

$$\gamma(\theta, \theta)\varphi(\theta, \theta) = \theta\varphi_2(\theta).$$

Note that $\varphi(\theta, \theta)$ is the moment generating function of $L_1 + L_2$. Since $\varphi_2(\theta)$ is finite for $\theta < \alpha_1$, the following result is immediate from (S1) and Remark 4.2.

Corollary 4.1 Under the assumptions of exact asymptotic 1, if $\beta < \alpha_1$, then the exact asymptotics of $P(L_1+L_2 > x)$ has the form of $Ce^{-\beta x}$ for some constant $C > 0$. If $\beta \geq \alpha_1$, then the rough decay rate of $P(L_1 + L_2 > x)$ is α_1.

Remark 4.7 The reason that we cannot state a stronger result for $\beta \geq \alpha_1$ as for the other cases is that we do not know the type of the singularity of $\varphi_1(z)$ at $z = \alpha_1$.

This result can be extended for any convex type combination $d_1L_1+d_2L_2$ with $d_1, d_2 > 0$. We give such results in Appendix B. It will be observed that a new prefactor occurs in asymptotic functions, which corresponds to similar results in Corollary 4.4 of [21].

5 The Levy input case

We now extend the Brownian tandem queue to the Lévy-driven tandem queue, provided $X_1(t)$ and $X_2(t)$ are independent with positive jumps. In this case, we have to use (2.6) for γ instead of (3.1), but all the arguments can be straightforwardly extended with some extra light-tail conditions except for verifying conditions in (S1a), (S1b), (S2a) and (S2b). Unfortunately, these conditions are hard to check, so we present only weaker results and state a conjecture for the shape of exact asymptotics.

5.1 Convergence domain

Let us outline the arguments. We first consider the convergence domain of ψ. As mentioned in Example 2.1, we need the following condition for the first queue to have the light-tailed stationary distribution.

(5-i) $c_1\theta = \kappa_1(\theta)$ has a positive solution α_1, and $\kappa_1(\alpha_1 + \epsilon) < \infty$ for some $\epsilon > 0$.

Note that (5-i) implies that $c_1 - \kappa_1'(\theta) = 0$ has a unique positive solution since $\kappa_1(\theta)$ is convex and increasing for $\theta > 0$, and vanishes at $\theta = 0$. This solution θ is identical with $\theta_1^{\max}$ of Section 3, so we continue to use the same notation. From the arguments on the domain of φ in Section 3, we need a positive solution θ for the equation $\gamma(\theta_1^{\max}, \theta) = 0$ for the second queue to be light-tailed. So, we assume

(5-ii) $(c_2 - c_1)\theta - \kappa_2(\theta) = \kappa_1(\theta_1^{\max}) - c_1\theta_1^{\max}$ has a positive solution $\theta_2^{\max}$, and $\kappa_2(\theta_2^{\max} + \epsilon) < \infty$ for some $\epsilon > 0$.

This $\theta_2^{\max}$ also corresponds with that of Section 3. Throughout this section, we assume (5-i) and (5-ii) in addition to the stability condition (2-ii). Note that our first requirement (2-i) is automatically satisfied under these two conditions.

We next show how to extend Lemmas 3.1, 3.2 and 3.3. Obviously, Lemmas 3.1 and 3.3 are still valid since the specific form of γ is not used there. To consider Lemma 3.2, recall that r_1 and r_2 are:

$$r_1 = c_1 - \lambda_1, \qquad r_2 = c_2 - c_1 - \lambda_2,$$

and let $\tilde{\kappa}_i(\theta) = \kappa_i(\theta) - \lambda_i\theta$, that is,

$$\tilde{\kappa}_i(\theta) = \kappa_i(\theta) - \lambda_i\theta, \qquad i = 1, 2.$$

Then, γ of (2.6) can be written as

$$\gamma(\theta_1, \theta_2) = r_1\theta_1 + r_2\theta_2 - \tilde{\kappa}_1(\theta_1) - \tilde{\kappa}_2(\theta_2).$$

Since $\tilde{\kappa}_i(0) = \tilde{\kappa}_i'(0) = 0$, $\tilde{\kappa}_i(\theta_i)$ can play the same role as $\frac{1}{2}\sigma_i^2\theta_i^2$ in (3.1). Thus, we have (3.2), but (3.3) is replaced by

$$\varphi_1(\theta_2) = (\frac{1}{\theta_2}\tilde{\kappa}_2(\theta_2) - r_2)\varphi(0, \theta_2) + r_1 + r_2. \tag{5.1}$$

Hence, Lemma 3.2 is still valid.

We further note that $\gamma(\theta_1, \theta_2)$ is well defined for $\theta_1 \leq \alpha_1, \theta_2 \leq \theta_2^{\max}$. Hence, we can consider the sign of the derivative $\frac{d\theta_2}{d\theta_1}$ as $\theta_1 \uparrow \alpha_1$ when (θ_1, θ_2) moves on the curve $\mathcal{C}$, defined by

$$\mathcal{C} = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \gamma(\theta_1, \theta_2) = 0, \theta_1 < \alpha_1, \theta_2 \ \leq \ \theta_2^{\max}\}.$$

On this curve, we obviously have

$$c_1 - \kappa_1'(\theta_1) + (c_2 - c_1 - \kappa_2'(\theta_2))\frac{d\theta_2}{d\theta_1} = 0. \tag{5.2}$$

This implies that $r_1 + r_2 \left.\frac{d\theta_2}{d\theta_1}\right|_{\theta_1=\theta_2=0} = 0$. Hence, by the stability condition (2-ii) and the convexity of $\mathcal{C}$, we can always find a unique positive solution of the equation $\gamma(\theta, \theta) = 0$. Denote this solution by β. Obviously, this β is the natural extension of the one in Sections 3 and 4.

Similarly, the sign of the derivative is not positive at $(\theta_1, \theta_2) = (\alpha_1, 0)$ if and only if

$$(c_1 - \kappa_1'(\alpha_1))r_2 \geq 0 \tag{5.3}$$

since $\kappa_2'(0) = \lambda_2$, where $\kappa_1'(\alpha_1)$ is defined as the left-hand derivative:

$$\kappa_1'(\alpha_1) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}(\kappa_1(\alpha_1) - \kappa_1(\alpha_1 - \epsilon)).$$

The condition (5.3) corresponds with cases (a-) and (b-) in Figures 1 and 2. Cases (a+), (b+) and (c) of those figures occur if and only if (5.3) does not hold. We also define $\eta_i^{\min}$ and $\xi_i(\theta)$ for $i = 1, 2$ in the exactly same way as in Section 3. That is,

$$\xi_1(\theta) = \min\{\theta_1 \in \mathbb{R}; \gamma(\theta_1, \theta) = 0\}, \quad \xi_2(\theta) = \max\{\theta_2 \in \mathbb{R}; \gamma(\theta, \theta_2) = 0\},$$

as long as they are well defined. Then, for $\eta_2^{\min} \leq \theta_2 \leq \theta_2^{\max}$, $\theta_1 = \xi_1(\theta_2)$ if and only if $\theta_2 = \xi_2(\theta_1)$. Furthermore, (3.9) is valid for $0 < \theta_2 < \min(\beta, \alpha_1)$.

We now have all the materials to get the convergence domain of φ, which is also denoted by $\mathcal{D}$, in the same way as in Section 3. Thus, we get the following results.

Proposition 5.1 For the Lévy driven tandem queue with an intermediate input, if conditions (2-ii), (5-i) and (5-ii) are satisfied, then the interior of $\mathcal{D}$ is given by (3.8), that is, $\mathcal{D}^{\mathrm{o}} = \mathcal{D}_1 \cap \mathcal{D}_2$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are given by (3.11) and (3.12).

Corollary 5.1 Under the same assumptions of Proposition 5.1, we have

$$\limsup_{x \to \infty} \frac{1}{x} P(L_2 > x) \leq \alpha_2. \tag{5.4}$$

5.2 Weak asymptotics and conjecture

We next consider some weaker versions of asymptotics of the stationary distribution of L_2. As aforementioned, it is hard to verify (S1) and (S2) for this case. However, we can check (S1c) and (S2c) for real $z = \theta_2 \in (0, \theta_2^{\max})$, which is performed in a similar way to the proof of Theorem 4.1. Those results can be restated as, for some positive constants C_i' for $i = 1, 2, 3$,

(5a1) If $\beta < \theta_1^{\max}$, then $\lim_{\theta \uparrow \alpha_2} (\alpha_2 - \theta)\psi(\theta) = C_1'$.

(5a2) If $\beta = \theta_1^{\max}$, then $\lim_{\theta \uparrow \alpha_2} (\alpha_2 - \theta)^{\frac{1}{2}}\psi(\theta) = C_2'$.

(5a3) If $\beta > \theta_1^{\max}$, then $\lim_{\theta \uparrow \alpha_2} (\alpha_2 - \theta)^{-\frac{1}{2}}(\psi(\alpha_2) - \psi(\theta)) = C_3'$.

Since the analytic properties in (S1a), (S1b), (S2a) and (S2b) are hard to verify, let us consider to apply the Tauberian theorems. To this end, let

$$U(x) = \int_0^x e^{\alpha_2 u} P(L_2 > u) du, \qquad \overline{U}(x) = \int_x^\infty e^{\alpha_2 u} P(L_2 > u) du,$$

where $\overline{U}$ exists only if $\psi(\alpha_2) < \infty$. Then,

$$\int_0^\infty e^{-\theta x} dU(x) = \psi(\alpha_2 - \theta), \qquad \int_0^\infty e^{-\theta x}\overline{U}(x)dx = \frac{\psi(\alpha_2) - \psi(\alpha_2 - \theta)}{\theta}.$$

Hence, we can apply Theorem 2 in Section XIII.5 of [10] to (5a1) and (5a2) and Theorem 4 for ultimately monotone density in the same section to (5a3). Thus, we have the following weak asymptotics.

Proposition 5.2 For the Lévy driven tandem queue satisfying the conditions (2-ii), (5-i) and (5-ii), let $\theta_1^{\max} = \xi_1(\theta_2^{\max})$, and let β be the unique positive solution of $\gamma(\theta,\theta) = 0$. Then we have the following exact asymptotics.

(5b1) If $\beta < \theta_1^{\max}$, then $\displaystyle\int_0^x e^{\beta u} P(L_2 > u)du \sim C_1 x.$

(5b2) If $\beta = \theta_1^{\max}$, then $\displaystyle\int_0^x e^{\theta_2^{\max} u} P(L_2 > u)du \sim 2C_2 x^{\frac{1}{2}}.$

(5b3) If $\beta > \theta_1^{\max}$, then $\displaystyle\int_x^\infty e^{\theta_2^{\max} u} P(L_2 > u)du \sim 2C_3 x^{-\frac{1}{2}}.$

Here, $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$, and C_1, C_2 and C_3 are given by

$$C_1 = \frac{\beta\varphi_2(\beta)}{(\xi_1'(\beta) - 1)(\tilde{\kappa}_2(\beta) - r_2\beta)}, \qquad C_2 = \frac{\theta_2^{\max}\varphi_2(\theta_1^{\max})}{\tilde{\kappa}_2(\theta_2^{\max}) - r_2\theta_2^{\max}}\sqrt{\frac{-\xi_2''(\theta_1^{\max})}{2\pi}},$$

where

$$\xi_1'(\beta) = \frac{\tilde{\kappa}_2'(\beta) - r_2}{r_1 - \tilde{\kappa}_1'(\beta)}, \qquad \xi_2''(\theta_1^{\max}) = \frac{\tilde{\kappa}_1''(\theta_1^{\max})}{r_2 - \tilde{\kappa}_2'(\theta_2^{\max})},$$

and

$$C_3 = \frac{\theta_2^{\max}(\varphi_2(\theta_1^{\max}) + (\theta_2^{\max} - \theta_1^{\max})\varphi_2'(\theta_1^{\max}))}{(\tilde{\kappa}_2(\theta_2^{\max}) - r_2\theta_2^{\max})(\theta_2^{\max} - \theta_1^{\max})^2\sqrt{-2\pi\xi_2''(\theta_1^{\max})}}.$$

Remark 5.1 From (5.2), it is not hard to see that $\beta < (>)\theta_1^{\max}$ holds if and only if $\left.\frac{d\theta_2}{d\theta_1}\right|_{\theta_1=\beta} > (<)0$ on the curve $\mathcal{C}$, which is equivalent to

$$(c_1 - \kappa_1'(\beta))(c_2 - c_1 - \kappa_2'(\beta)) < (>)0.$$

This proposition and Theorem 4.1 strongly suggest the following conjecture, which is particularly true if $P(L_2 > x) \sim Cx^d e^{-\alpha_2 x}$ for some C and d.

Conjecture 5.1 Under the same assumptions of Proposition 5.2, $P(L_2 > x)$ has the following exact asymptotics $h(x)$.

(5c1) If $\beta < \theta_1^{\max}$, then $h(x) = C_1 e^{-\beta x}$.

(5c2) If $\beta = \theta_1^{\max}$. then $h(x) = C_2 x^{-\frac{1}{2}} e^{-\theta_2^{\max} x}$.

(5c3) If $\beta > \theta_1^{\max}$, then $h(x) = C_3 x^{-\frac{3}{2}} e^{-\theta_2^{\max} x}$.

This conjecture corresponds with Theorem 4.3 of [17], which considers the case that there is no intermediate input. However, in the proof of Theorem 4.3 in [17] just (5a1), (5a2), and (5a3) were formally verified, and therefore this result needs additional justification.

5.3 Correlated Lévy inputs

We finally note that Conjecture 5.1 can be extended to the case that the two components $X_1(t)$ and $X_2(t)$ of Lévy process are not independent but with positive jumps only. In this case, the Lévy exponent $\kappa(\theta_1, \theta_2)$ is defined by

$$E(e^{\theta_1 X_1(t) + \theta_2 X_2(t)}) = e^{t\kappa(\theta_1, \theta_2)}.$$

Then, (2.9) still holds, but the γ is changed to

$$\gamma(\theta_1, \theta_2) = c_1\theta_1 + (c_2 - c_1)\theta_2 - \kappa(\theta_1, \theta_2).$$

Although $\kappa_1(\theta_1)$ in (2.12) and $\tilde{\kappa}_2(\theta_2)$ in (5.1) must be changed to $\kappa(\theta_1, 0)$ and $\kappa(0, \theta_2) - \lambda_2\theta_2$, respectively, all the arguments in Section 3 go through under the following conditions corresponding to (5-i) and (5-ii).

(5-ii') $c_1\theta = \kappa(\theta, 0)$ has a positive solution α_1, and $\kappa(\alpha_1 + \epsilon, 0) < \infty$ for some $\epsilon > 0$.

(5-iii') $c_1\theta_1^{\max} + (c_2 - c_1)\theta - \kappa(\theta_1^{\max}, \theta) = 0$ has a positive solution $\theta_2^{\max}$, where $\theta_1^{\max}$ is a positive solution of $\frac{\partial}{\partial\theta_1}\kappa(\theta_1, \theta)|_{\theta_1 = \theta_1^{\max}} = c_1$, and $\kappa(\theta_1^{\max}, \theta_2^{\max} + \epsilon) < \infty$ for some $\epsilon > 0$.

A similar problem discussed in Section 5.2 occurs for the exact asymptotics. Thus, Proposition 5.2 can be extended, but the exact asymptotics are just conjectured.

6 Concluding remarks

In this section, we first examine the present results to be consistent with existing results. We then discuss possible extensions to other performance characteristics or more general models.

6.1 Compatibility to existing results

When there is no intermediate input, the present model is studied in [17]. The rough decay rate is obtained for $P(L_1 > d_1 x, L_2 > d_2 x)$ for $d_i \geq 0$. Theorem 4.3 of [17] claims that the exact asymptotics are obtained for $P(L_2 > x)$. As we discussed in Section 5.2, this claim has not yet been fully proved. In taking this fact into account, we compare the notations of [17] to those of this paper to see how the results are corresponded.

We first note the correspondence:

$$(\mu, \bar{t}, \bar{s}, t_b) \text{ of } [17] \ \Rightarrow (\lambda_1, -\beta, -\theta_1^{\max}, -\theta_2^{\max})$$

Note that Laplace transforms are used in [17] instead of moment generating functions, so the sign of their variables must be changed. This is the reason why the minus signs appears in the above correspondence. In [17], $t_p \equiv \bar{t}$ and $\theta(s) \equiv c_1 s + \kappa_1(-s)$ are also used, but t_p is always replaced by $\bar{t}$. On the other hand,

$$\theta''(\bar{s}) = \kappa_1''(-\bar{s}) = \kappa_1''(\theta_1^{\max}).$$

Since $\kappa_2(\theta_2) = \tilde{\kappa}_2(\theta_2) = 0$, we have, by Conjecture 5.1,

$$\xi''(\theta_2^{\max}) = -\frac{\theta''(\bar{s})}{c_1 - c_2}.$$

Hence, letting $\lambda_2 = 0$ and $\varphi_2(\theta_2) = c_2 - \lambda_1$, we can see that Conjecture 5.1 is indeed identical with Theorem 4.3 of [17].

6.2 Rough asymptotics of the joint tail probability

An interesting characteristic, not considered in this paper, is the joint tail distribution $P(L_1 > d_1 x, L_2 > d_2 x)$. Following Proposition 3.2 of [17], an upper bound in the Chernoff inequality

$$\limsup_{x\to\infty} \frac{1}{x} \log P(L_1 > d_1 x, L_2 > d_2 x) \le -\sup\{d_1\theta_1 + d_2\theta_2; (\theta_1, \theta_2) \in \mathcal{D}^{\rm o}\}. \qquad (6.1)$$

can give the right rough decay rate. Since the set $\mathcal{D}^{\rm o}$ is explicitly given, it is not hard to find the supremum, which is the maximum over the closure $\overline{\mathcal{D}^{\rm o}}$. We conjecture that (6.1) is tight, but this seems to be a hard problem. In [17] a sample path large deviation technique was used. This is left for future research.

6.3 The case of more general networks

Let us consider an extension of our results for the Lévy-driven fluid networks with arbitrary routing or/and more than two nodes, say n nodes. These are challenging problems even for the two node case. For intree networks in discrete time, Theorem 2.1 of [7] answers to their rough decay rates for marginal stationary distributions. This may be a good sign to study such extensions, and our approach may be applied to get exact asymptotics. We here derive the stationary equation in terms of moment generating functions, which is a building block of our approach.

Consider n (infinite-buffer) fluid queues, with exogenous input to buffer j in the time interval $[0, t]$ given by $X_j(t)$, where $X_i(t) = a_i t + B_i(t) + J_i(t)$ and

$$\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))^{\rm T} = \boldsymbol{a}t + \boldsymbol{B}(t) + \boldsymbol{J}(t).$$

We assume that $\boldsymbol{B}(t)$ is a n-dimensional Brownian motion with null drift, $\boldsymbol{J}(t)$ has independent increments which are mutually independent and independent of $\boldsymbol{B}(t)$. We denote the Lévy exponent of $\boldsymbol{X}(t)$ by $\kappa(\boldsymbol{\theta})$.

The buffers are continuously drained at a constant rate as long as it is not empty. These drain rates are given by a vector $\boldsymbol{c}$; for buffer j, the rate is c_j. The interaction between the queues is modeled as follows. A fraction p_{ij} of the output of station i is immediately transferred to station j, while a fraction $1 - \sum_{j \neq i} p_{ij}$ leaves the system. We set $p_{ii} = 0$ for all i, and suppose that $\sum_j p_{ij} \leq 1$. The matrix $P = \{p_{ij} : i, j = 1, \ldots, n\}$ is called the *routing matrix*. Assume that

(6-i) $R \equiv I - P^{\mathrm{T}}$ is nonsingular.

We refer to R as reflection matrix.

For $(\{\boldsymbol{X}(t)\}, \boldsymbol{c}, P)$, the *buffer content* process $\boldsymbol{L}(t)$ is defined by

$$\boldsymbol{L}(t) = \boldsymbol{L}(0) + \boldsymbol{X}(t) - tR\boldsymbol{c} + R\boldsymbol{Y}(t),$$

where $\boldsymbol{Y}(t)$ is a *regulator*, that is, the minimal nonnegative and nondecreasing process such that $Y_i(t)$ can be increased only when $L_i(t) = 0$.

Assume now

(6-ii) $\boldsymbol{L}(t)$ has a stationary distribution.

Denote this stationary distribution by π. As in Proposition 2.1, we first need the finiteness of $E_\pi(Y_i(1))$ for all $i = 1, 2, \ldots, n$. To verify this, define n-dimensional process $\boldsymbol{U}(t)$ as

$$\boldsymbol{U}(t) = R^{-1}\boldsymbol{L}(t).$$

Then,

$$\boldsymbol{U}(t) = \boldsymbol{U}(0) + R^{-1}\boldsymbol{X}(t) - t\boldsymbol{c} + \boldsymbol{Y}(t).$$

By the assumption (6-ii), $\boldsymbol{U}(t)$ has the stationary distribution. Hence, we must have

$$R^{-1}E(\boldsymbol{X}(1)) < \boldsymbol{c}. \tag{6.2}$$

Intuitively, this condition is also sufficient for (6-ii), but its proof seems not easy. So, we keep the assumption (6-ii).

Obviously $\boldsymbol{U}(t)$ is also stationary under π. Then, similarly to our arguments in Section 2, we get the following lemma.

Lemma 6.1 Under conditions (6-i) and (6-ii), $\boldsymbol{L}(t)$ has the stationary distribution π and,

$$E_\pi(\boldsymbol{Y}(1)) = \boldsymbol{c} - R^{-1}E(\boldsymbol{X}(1))$$

is a finite and positive vector.

Thus, $E_\pi(Y_i(1))$ must be finite. Let $\varphi(\boldsymbol{\theta})$ be the moment generating function of π. Similarly, let

$$\varphi_i(\boldsymbol{\theta}_i[0]) = E_\pi \int_0^1 e^{\langle \boldsymbol{\theta}_i[0], \boldsymbol{L}(u)\rangle}\, dY_i(u),$$

where $\boldsymbol{\theta}_i[0]$ be the n-dimensional vector obtained from $\boldsymbol{\theta}$ by replacing θ_i by 0. Denote the column vector whose i-th entry is $\varphi_i(\boldsymbol{\theta}_i[0])$, that is,

$$\underline{\boldsymbol{\varphi}}(\boldsymbol{\theta}) = (\varphi_1(\boldsymbol{\theta}_1[0]), \ldots, \varphi_n(\boldsymbol{\theta}_n[0]))^{\mathrm{T}}$$

Similarly to the two dimensional case, let

$$\gamma(\boldsymbol{\theta}) = \langle \boldsymbol{\theta}, R\boldsymbol{c}\rangle - \kappa(\boldsymbol{\theta}).$$

Then, exactly in the same way as Proposition 2.1, we have the following result (see [22] for its detailed proof).

Proposition 6.1 Under conditions (6-i) and (6-ii), we have, for $\boldsymbol{\theta} \in \mathbb{R}^n$,

$$\gamma(\boldsymbol{\theta})\varphi(\boldsymbol{\theta}) = \langle \boldsymbol{\theta}, R\underline{\boldsymbol{\varphi}}(\boldsymbol{\theta})\rangle, \tag{6.3}$$

as long as $\varphi(\boldsymbol{\theta})$, $\gamma(\boldsymbol{\theta})$ and $\underline{\boldsymbol{\varphi}}(\boldsymbol{\theta})$ are finite, however at least for $\boldsymbol{\theta} \leq \mathbf{0}$.

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A Computation of $\xi_1'(\beta)$

From (4.3) in Appendix A, we have

$$\xi_1'(\theta) = \frac{\sigma_2^2\theta - r_2}{\sqrt{r_1^2 - \sigma_1^2(\sigma_2^2\theta^2 - 2r_2\theta)}},$$

as long as $r_1^2 - \sigma_1^2(\sigma_2^2\theta^2 - 2r_2\theta) > 0$. This is always the case if $\theta \le \beta < \frac{1}{2}$. From $\gamma(\beta, \beta) = 0$, we have $2r_1 - \sigma_1^2\beta = \sigma_2^2\beta - 2r_2$. Hence,

$$r_1^2 - \sigma_1^2(\sigma_2^2\beta^2 - 2r_2\beta) = r_1^2 - \sigma_1^2(2r_1\beta - \sigma_1^2\beta^2) = (r_1 - \sigma_1^2\beta)^2.$$

This implies that

$$\lim_{\theta\to\beta} \xi_1'(\theta) = \frac{\sigma_2^2\beta - r_2}{r_1 - \sigma_1^2\beta}. \tag{A.1}$$

B The asymptotics for $d_1L_1 + d_2L_2$

We consider exact asymptotics for convex type combination $d_1L_1 + d_2L_2$ with $d_1, d_2 > 0$ in Brownian networks as they were considered in Section 4. Technically, they can be obtained by the same method as exact asymptotic 1. However, results themselves may be interesting. So, we present them as a theorem, which includes Corollary 4.1 as a special case.

We introduce some notation first. Let $\delta(d_1, d_2)$ be the non-zero solution θ of $\gamma(d_1\theta, d_2\theta) = 0$. Then, we have

$$\delta(d_1, d_2) = \frac{2(r_1d_1 + r_2d_2)}{(\sigma_1^2d_1^2 + \sigma_2^2d_2^2)}.$$

We next note that

$$\xi_2(\alpha_1) = \frac{2r_2^+}{\sigma_2^2}.$$

Theorem B.1 Under the assumptions of Theorem 4.1, for $d_1, d_2 > 0$, if $\alpha_1 d_2 > \xi_2(\alpha_1) d_1$, then the exact asymptotics of $P(d_1 L_1 + d_2 L_2 > x)$ has the form of $Ch(x)$ for some constant $C > 0$, where $h(x)$ is given by

(A1) If $\beta < \frac{1}{2}\alpha_1$, then

$$h(x) = \begin{cases} e^{-\frac{\beta}{d_2}x}, & \beta d_1 < (\alpha_1 - \beta) d_2, \\ x e^{-\frac{\beta}{d_2}x}, & \beta d_1 = (\alpha_1 - \beta) d_2, \\ e^{-\delta(d_1,d_2)x} & (\alpha_1 - \beta) d_2 < \beta d_1, \xi_2(\alpha_1) d_1 < \alpha_1 d_2. \end{cases}$$

(A2) If $\frac{1}{2}\alpha_1 = \beta$, then

$$h(x) = \begin{cases} x^{-\frac{1}{2}} e^{-\frac{\theta_2^{\max}}{d_2}x}, & d_1 < d_2, \\ e^{-\delta(d_1,d_2)x} & d_1 \geq d_2, \xi_2(\alpha_1) d_1 < \alpha_1 d_2. \end{cases}$$

(A3) If $\frac{1}{2}\alpha_1 < \beta < \alpha_1$, then

$$h(x) = \begin{cases} x^{-\frac{3}{2}} e^{-\frac{\theta_2^{\max}}{d_2}x}, & \theta_2^{\max} d_1 < \frac{1}{2}\alpha_1 d_2, \\ x^{-\frac{1}{2}} e^{-\frac{\theta_2^{\max}}{d_2}x}, & \theta_2^{\max} d_1 = \frac{1}{2}\alpha_1 d_2, \\ e^{-\delta(d_1,d_2)x} & \frac{1}{2}\alpha_1 d_2 < \theta_2^{\max} d_1, \xi_2(\alpha_1) d_1 < \alpha_1 d_2. \end{cases}$$

(A4) If $\alpha_1 \leq \beta$, then

$$h(x) = \begin{cases} x^{-\frac{3}{2}} e^{-\frac{\theta_2^{\max}}{d_2}x}, & \theta_2^{\max} d_1 < \frac{1}{2}\alpha_1 d_2, \\ x^{-\frac{1}{2}} e^{-\frac{\theta_2^{\max}}{d_2}x}, & \theta_2^{\max} d_1 = \frac{1}{2}\alpha_1 d_2, \\ e^{-\delta(d_1,d_2)x}, & \frac{\xi_2(\alpha_1)}{\alpha_1} < \frac{d_2}{d_1} < \frac{2\theta_2^{\max}}{\alpha_1}. \end{cases}$$

Otherwise, that is, if $\alpha_1 d_2 \leq \xi_2(\alpha_1) d_1$, the rough decay rate of $P(L_1 + L_2 > x)$ is α_1.

Remark B.1 In (A1), (A2) and (A3), $\alpha_1 \leq \beta$ cannot occur if $r_2 \leq 0$, equivalently, $\xi_2(\alpha_1) = 0$, but they are always the cases in (A4) since $\alpha_1 \leq \beta$ implies $r_2 > 0$. See Figures 1 and 2 for these facts.

PROOF. The proof is essentially the same proof as of Theorem 4.1, which uses (S1) and (S2). For this, we utilize the condition that $\alpha_1 d_2 > \xi_2(\alpha_1) d_1$, which will be discussed below. Define

$$\psi_{\boldsymbol{d}}(\theta) = \int_0^\infty e^{\theta x} P(d_1 L_1 + d_2 L_2 > x) dx, \qquad \theta \in \mathbb{R}.$$

as long as it exists. Then,

$$\psi_{\boldsymbol{d}}(\theta) = \frac{1}{\theta}(\varphi(d_1\theta, d_2\theta) - 1),$$

To apply (S1) and (S2), we need to consider analytic extension of $\psi_{\boldsymbol{d}}$ and to verify some properties, but they can be similarly done to the proof of Theorem 4.1 once its behavior around the singular point on the real line is identified. So, we here only consider the latter.

We first note that the singularity of $\psi_{\boldsymbol{d}}(\theta)$ is the same as that of $\varphi(d_1\theta, d_2\theta)$. The latter singularity occurs when $(\theta d_1, \theta d_2)$ across the boundary of $\mathcal{D}$, which is obtained in Corollary 3.2. To see what happens at this singular point, we consider the following equation obtained from (2.9).

$$\gamma(d_1\theta, d_2\theta)\varphi(d_1\theta, d_2\theta) = (d_1 - d_2)\theta\varphi_1(d_2\theta) + d_2\theta\varphi_2(d_1\theta). \tag{B.1}$$

From this equation, we observe that at least one of the following conditions causes the singularity of $\varphi(d_1\theta, d_2\theta)$.

(F1) $\gamma(d_1\theta, d_2\theta) = 0$,

(F2) $\varphi_1(d_2\theta)$ is singular, which is the same as $\varphi(0, d_2\theta_2)$,

(F3) $\varphi_2(d_1\theta)$ is singular.

Note that asymptotics of $\varphi_2(\theta_1)$ is not known around its singular point α_1. So, if case (F3) occurs, we can only consider the rough decay rate. Because of this reason, we separately consider the case that $\alpha_1 d_2 \le \xi_2(\alpha_1)d_1$, which causes (F3). W also note that (F1) occurs when $\theta < \min(\frac{\alpha_1}{d_1}, \frac{\alpha_2}{d_2})$. This singularity is a simple pole since the right-hand side of (B.1) is finite for all θ less than $\min(\frac{\alpha_1}{d_1}, \frac{\alpha_2}{d_2})$. On the other hand, if either $\theta = \frac{\alpha_1}{d_1}$ or $\theta = \frac{\alpha_2}{d_2}$ holds, the singularity is also caused by (F2) or (F3). Taking these facts into account, we consider each case separately.

Assume that $\alpha_1 d_2 > \xi_2(\alpha_1)d_1$, and consider case (A1). From our discussions, the first case is obtained from (F2) if $d_1 \neq d_2$, and the third case is obtained from (F1). So, we only need to consider the cases that $d_1 = d_2$ and $\beta d_1 = (\alpha_1 - \beta)d_2$. For the case that $d_1 = d_2$, we have, from (B.1),

$$\gamma(d_1\theta, d_2\theta)\varphi(d_1\theta, d_2\theta) = d_2\theta\varphi_2(d_1\theta).$$

Hence, $\varphi(d_1\theta, d_2\theta)$ has a simple pole at $\theta = \frac{\beta}{d_2}$ since (F1) occurs there. Clearly, this case is included in the first case of $h(x)$ in (A1). Consider the case that $\beta d_1 = (\alpha_1 - \beta)d_2$. In this case, the singularity is caused by (F1) and (F2). From this fact and Theorem 4.1, it is not hard to see that, for some positive constant C,

$$\lim_{\theta \uparrow \frac{\theta_2^{\max}}{d_2}} (d_2\theta - \theta_2^{\max})^2 \varphi(d_1\theta, d_2\theta) = C.$$

Hence, (S1) yields the second case of $h(x)$ in (A1).

Case (A2) is simpler, and similarly proved. For case (A3), the first and third cases are obtained from (F2) and (F1), respectively. Let us consider the remaining case that $\theta_2^{\max} d_1 = \frac{1}{2}\alpha_1 d_2$. In this case, the singularity is caused by (F1) and (F2). By Theorem 4.1, for some positive constants K', C',

$$\varphi_1(\theta) = K' - C'(\theta_2^{\max} - \theta)^{\frac{1}{2}} + o((\theta_2^{\max} - \theta)^{\frac{1}{2}}), \qquad \theta \uparrow \theta_2^{\max}.$$

Furthermore, the right-hand side of (B.1) at $\theta = \frac{\theta_2^{\max}}{d_2}$ becomes

$$
\begin{aligned}
(d_1 - d_2)\frac{\theta_2^{\max}}{d_2}\varphi_1\left(d_2\frac{\theta_2^{\max}}{d_2}\right) &+ d_2\ \frac{\theta_2^{\max}}{d_2}\varphi_2\left(d_1\frac{\theta_2^{\max}}{d_2}\right) \\
&= \frac{1}{d_2}\left((d_1 - d_2)\theta_2^{\max}\varphi_1(\theta_2^{\max}) + d_2\theta_2^{\max}\varphi_2(\theta_1^{\max})\right) = 0,
\end{aligned}
$$

so (B.1) yields, for some positive constant C'',

$$
\lim_{\theta \uparrow \frac{\theta_2^{\max}}{d_2}} (\theta_2^{\max} - d_2\theta)^{\frac{1}{2}}\varphi(d_1\theta, d_2\theta) = C''.
$$

Hence, (S2) with $s = -\frac{1}{2}$ implies the second case of $h(x)$ in (A3). Case (A4) is proved similarly to (A1) and (A2). It remains to consider the case that $\alpha_1 d_2 \leq \xi_2(\alpha_1)d_1$. In this case, we can apply Remark 4.2, and get the rough decay rate. This completes the proof. □