QUEUES IN SERIES IN LIGHT TRAFFIC

B. Błaszczyszyn and T. Rolski

Mathematical Institute, University of Wrocław

Summary

We study light traffic approximations for queues in series with renewal arrivals and i.i.d. service time vectors. Formulae for limits of functions of the waiting time at different stations based on single customer effect are obtained for two approaches: dilation and thinning of the arrival process. Inter-departure times from a station possess a one-dependence property in light traffic. This paper complements previous studies of Daley & Rolski (1984, 1991, 1992) and also Asmussen's (1991) approach to light traffic limits applied to the cases considered.

AMS 1980 Subject Classifications: Primary: 60 K 25

Key words and phrases: $GI/G/1 \to \cdots \to G/1$ queue, light traffic, waiting time, inter-departure times, γ-dilation, π-thinning, asymptotically conditionally equivalent families, single customer effect, asymptotically one-dependent family.

Version accepted by the Annals of Applied Probability

Appeared in 1993, Vol. 3, No. 3, 881–896

1. Introduction

A series of queues is the simplest example of a queueing network. In most cases the performance characteristics in such queues are analytically intractable. One possible approach proposed recently for overcoming this problem is to study systems in extreme situations, i.e., in light and heavy traffic conditions. Although this approach does not give explicit formulae, interpolation formulae are possible in some cases as suggested in Reiman & Simon (1988) or Whitt (1989).

We assume a renewal input at the first station and that the vectors consisting of each customer's service times at all stations are independent and identically distributed. Moreover we assume that the service times are independent of the input but allow dependence, for each customer, of service times at different stations. All customers obtain service in the same order, and an unlimited queue may form in front of each server. If the input is Poisson, light traffic results were given by Wolff (1982), Pinedo & Wolff (1982), Greenberg & Wolff (1988). Reiman & Simon (1989) studied open Markovian networks of queues in light traffic conditions and from their paper one can also get results for queues in series. However the techniques used in these papers are not applicable in a non-Markovian situation.

This paper arises out of work on light traffic in single-server queues by Daley & Rolski (1984, 1991, 1993), Asmussen (1991) and Błaszczyszyn (1990), using in particular methods developed in Daley & Rolski (1992) for many-server queues. As in these papers we study stationary waiting times at a station when the light traffic conditions are defined in terms of either a γ-dilation or a π-thinning scheme. We note that there is an essential difference between such characteristics and for example the work-load at a station, because the asymptotic behavior in light traffic of the work-load does not depend on the fine structure of the input process but is a consequence of Campbell's type formulae (see e.g. recent papers by Baccelli & Bremaud (1991) or Sigman (1991) for such results). A new feature studied in this paper concerns inter-departure times from a node in light traffic conditions: these are inter-arrival times at the next station. In light traffic we find asymptotic properties of a marginal inter-departure time from a station, and also show that the sequence of inter-departure times is asymptotically one-dependent. Our asymptotic results are based on the single customer effect which has been observed in other situations, namely, that the waiting time of any customer at the mth station is similar in light traffic to the waiting time of a customer that follows a customer that has found the system empty.

As in previous papers we conclude from our analysis that light traffic limits for queues in series reflect consequences of any local clustering or clumping behavior of the arrival process. For example, this paper considers renewal arrival processes for which

$$\Pr\{\, T \le t\} = c_A t^{\alpha} + o(t^{\alpha}) \quad (t \downarrow 0) \tag{1.1}$$

for some strictly positive constant c_A and nonnegative α; using the definition introduced in Daley &

Rolski (1991) we say that the inter-arrival time d.f. belongs to the class $\mathcal{S}_\alpha = \mathcal{S}_\alpha(c_A)$. Notice that a Poisson arrival process at rate λ belongs to $\mathcal{S}_1(\lambda)$.

2. Preliminaries

The following idea of equivalence of two sequences $\{X(\gamma)\}$, $\{Y(\gamma)\}$ of non-negative random variables that are degenerate at zero for $\gamma \to \infty$, comes from Asmussen (1991); see also Daley & Rolski (1992).

Definition 2.1. *Two families of non-negative random variables $\{X(\gamma)\}$ and $\{Y(\gamma)\}$ are asymptotically conditionally equivalent (ACE) when in the limit $\gamma \to \infty$,*

(a) $\Pr\{X(\gamma) \neq 0\} \to 0, \quad \Pr\{Y(\gamma) \neq 0\} \to 0, \quad \dfrac{\Pr\{X(\gamma) \neq 0\}}{\Pr\{Y(\gamma) \neq 0\}} \to 1$ *and*

(b) $\|\Pr\{X(\gamma) \in \cdot \mid X(\gamma) \neq 0\} - \Pr\{Y(\gamma) \in \cdot \mid Y(\gamma) \neq 0\}\| \to 0.$

Suppose that a model is parameterized by $\gamma \geq 0$ and has a characteristic of interest $Y(\gamma)$ for which $\Pr\{Y(\gamma) > 0\} \to 0$ for $\gamma \to \infty$. The following technique was successfully applied to many-server queues in Daley & Rolski (1992) to characterize a family $\{Y(\gamma)\}$ in "light traffic conditions". It is supposed that there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\Pr})$, and families $\{\bar{X}(\gamma)\}$, $\{\bar{Y}(\gamma)\}$ such that for each $\gamma \geq 0$, $\bar{X}(\gamma) \leq \bar{Y}(\gamma)$ a.s., where $\bar{Y}(\gamma) =_{\mathrm{d}} Y(\gamma)$ and a family of nonnegative random variables $\bar{U}(\gamma)$ are such that $\bar{X}(\gamma) = \bar{Y}(\gamma)$ on $\{\bar{U}(\gamma) = 0\}$ The following proposition is extracted from the proof of Theorem 4 from Daley & Rolski (1992).

Proposition 2.2. *If for each γ the random variable $\bar{U}(\gamma)$ is independent of $\bar{X}(\gamma)$, and if as $\gamma \to \infty$*

$$\bar{\Pr}\{\bar{U}(\gamma) = 0\} \to 1 \qquad \textit{and} \qquad \frac{\bar{\Pr}\{\bar{X}(\gamma) > 0\}}{\bar{\Pr}\{\bar{Y}(\gamma) > 0\}} \to 1,$$

then the families $\{\bar{X}(\gamma)\}$ and $\{\bar{Y}(\gamma)\}$ are ACE.

The following lemma, proved by Daley & Rolski (1992), is frequently used in the paper.

Lemma 2.3 (Multivariate Abelian Lemma). *Let $f : \mathbb{R}_+^j \to \mathbb{R}_+$ be a componentwise non-increasing function with $f(\mathbf{0})$ finite and $f(\mathbf{t}) \downarrow 0$ for $\max_{1\leq i\leq j} t_i \to \infty$. If the d.f. A is in $\mathcal{S}_\alpha(c_A)$ for some $0 < \alpha < \infty$, then as $\gamma \to \infty$*

$$\gamma^{j\alpha} \int_{\mathbb{R}_+^j}\!\!\cdots\!\int f(\gamma\mathbf{t})dA(t_1)\cdots dA(t_j) \to (\alpha c_A)^j \int_{\mathbb{R}_+^j}\!\!\cdots\!\int (t_1\cdots t_j)^{\alpha-1} f(\mathbf{t})dt_1\cdots dt_j \tag{2.1}$$

whenever the integral on the RHS is finite.

3. Single server queues in series

We consider a series of m single-server queues GI/G/1 $\to \cdots \to$ G/1 in which the arrivals occur at the epochs of a point process, specifically, the inter-arrival times $\{T_n : n = 0, \pm 1, \ldots\}$

are assumed to constitute a renewal process, independent of the service times $\{\mathbf{S}_n\} \equiv \{(S_n^i, i = 1,\dots,m) : n = 0, \pm 1, \dots\}$ which are assumed to form a sequence of independent identically distributed (i.i.d.) non-negative random vectors. Here S_n^i denotes the service time of the nth customer at the ith station. Recall that we do not require independence between the service time of any given customer at different stations. Customers proceed from one station to the next in sequence and are served at each station in order of arrival. Denote the waiting time of the nth customer at the ith station by W_n^i. For the first station it satisfies the recurrence relationship

$$W_{n+1}^1 = (W_n^1 + S_n^1 - T_n)_+ \,, \tag{3.1}$$

and for other stations we have

$$W_{n+1}^i = \left(\min\left(R_n^i, P_n^i - T_n\right)\right)_+ \,, \tag{3.2}$$

where

$$P_n^i = W_n^1 + S_n^1 + \sum_{k=2}^{i} (W_n^k + S_n^k - S_{n+1}^{k-1}), \tag{3.3}$$

$$R_n^i = \begin{cases} P_n^1 & (i = 1), \\ \min\limits_{2\le j\le i} \left\{ \sum_{k=j}^{i} (W_n^k + S_n^k - S_{n+1}^{k-1}) \right\} & (i = 2,\dots,m). \end{cases} \tag{3.4}$$

Then clearly (3.1) may be written in the form (3.2). For two station queues the above relationship was given by Niu (1980), while longer series of queues were described in a similar manner by Masterson & Sherman (1963). Relations (3.1) and (3.2) can be established by noting that the input at the ith station equals the output from the $(i-1)$th station. Thus if T_n^i denotes the time between departures of the nth and $(n+1)$th customers from the $(i-1)$th station then

$$W_{n+1}^i = \left(W_n^i + S_n^i - T_n^i\right)_+ \,, \tag{3.5}$$

$$T_n^{i+1} = S_{n+1}^i + (T_n^i - W_n^i - S_n^i)_+ \tag{3.6}$$

where for consistency with (3.1) we put $T_n^1 = T_n$.

We assume that

$$0 < \max_{1\le i\le m} \mathrm{E}S^i < \mathrm{E}T < \infty \,. \tag{3.7}$$

Here and in the sequel S^i denotes the generic service-time r.v. at the ith station, and T a r.v. with the marginal distribution of any T_n. It is known (see e.g. Loynes (1962)) that condition (3.7) ensures stability at all stations. It means that there exists a stationary sequence $\{(W_n^1,\dots,W_n^m), n \in \mathbb{Z}\}$ satisfying (3.1)–(3.4). According to the convention introduced above we write $(W^1,\dots,W^m)$ for a random vector distributed like any member of the stationary sequence $\{(W_n^1,\dots,W_n^m), n \in \mathbb{Z}\}$. The process can be described briefly as a stable GI/G/1 $\to \cdots \to$ G/1 queueing system.

We now construct a lower bound for W^i. It is typical for our approach to choose for proofs a suitable probability space. We remark that this does not influence on our results because they are distributional only.

Lemma 3.1. *Let* $\sum_{k=1}^{i} \mathrm{E}S^k < \mathrm{E}T$. *There exists a probability space* $(\Omega, \mathcal{F}, \mathrm{Pr})$ *supporting generic random elements* $\mathbf{V}_n$, $\mathbf{W}_n$, $\mathbf{S}_n$, T_n $(n = 1, 2, \ldots)$ *such that for each* $i = 1, \ldots, m$,

$$\left(\min\left(\underline{R}_n^i, \underline{P}_n^i - T_n\right)\right)_+ \leq W_n^i \tag{3.8}$$

where

$$\underline{P}_n^i = S_n^1 + \sum_{k=2}^{i} (S_n^k - S_{n+1}^{k-1}), \tag{3.9}$$

$$\underline{R}_n^i = \begin{cases} \underline{P}_n^1 & (i = 1) \\ \min\limits_{2 \leq j \leq i} \left\{ \sum_{k=j}^{i} (S_n^k - S_{n+1}^{k-1}) \right\} & (i = 2, \ldots, m); \end{cases} \tag{3.10}$$

further,

$$W_n^i = \left(\min\left(\underline{R}_n^i, \underline{P}_n^i - T_n\right)\right)_+ \quad on \quad \{V_n^i = 0\}, \tag{3.11}$$

and for each n *and* i, *and all* $k \geq n$, *the random elements* V_n^i, T_k *and* $\mathbf{S}_k$ *are independent.*

Proof. By standard arguments we construct a stationary sequence $\{T_n, \mathbf{S}_n, \left(W_n^1, \ldots, W_n^i, V_n^i\right) : n \in \mathbb{Z}\}$ satisfying (3.2)–(3.4) and

$$V_{n+1}^i = \left(V_n^i + \sum_{k=1}^{i} S_n^k - T_n \right)_+ . \tag{3.12}$$

Observe that $\{V_n^i\}$ can be regarded as the total waiting time in the series system restricted to the first i nodes, with a modified rule that allows only one customer to be present at any of the stations at any given moment. It is clear that such a discipline can only increase waiting times so

$$\sum_{k=1}^{i} W_n^k \leq_{\text{a.s.}} V_n^i \tag{3.13}$$

and we have (3.11). By construction of V_n^i and by assumption for the other elements, V_n^i is independent of $(T_k, \mathbf{S}_k : k \geq n)$.

Remark. Note that the LHS of (3.8) is the waiting time of a customer which follows one finding the system empty. Thus we call it a *single customer representation.*

We need the most general version of Lemma 3.1 only in Section 5. Elsewhere it suffices that $n = 1$, in which case we omit the subscript n and write $\hat{\mathbf{S}}$ for $\mathbf{S}_2$. It is convenient to denote the partial sum of consecutive terms $a^i, \ldots, a^j$ of a sequence $\{a^k\}$ of reals by $a^{[i,j]}$. Thus equations

(3.8)–(3.10) can be rewritten as $\underline{P}_n^i = S_n^{[1,i]} - S_{n+1}^{[1,i-1]}$ and $\underline{R}_n^i = \min_{2\le j\le i}\left\{S_n^{[j,i]} - S_{n+1}^{[j-1,i-1]}\right\}$. Similar notation can be used for subscripts.

4. Waiting time in light traffic

Except for section 6 the rest of this paper considers light traffic conditions defined via γ-dilation discussed in Daley & Rolski (1984, 1991). It means that each inter-arrival time is multiplied by some $\gamma > 0$ and we take the limit $\gamma \to \infty$. We write $(W^1(\gamma), \dots, W^m(\gamma))$ and $V^m(\gamma)$ for appropriate waiting times when the input is rescaled by γ, i.e., each inter-arrival time is multiplied by this constant. We consider renewal arrival processes for which the inter-arrival time d.f. belongs to $\mathcal{S}_\alpha(c_A)$ (see (1.1)) for some strictly positive constant α.

When $\alpha > 0$ the inter-arrival times are a.s. strictly positive, in which case, in view of (3.13), it follows from the result (2.3a) of Daley & Rolski (1991) that the vectors $(W^1(\gamma), \dots, W^m(\gamma))$ decrease stochastically to $(0, \dots, 0)$. (A trickier part of checking this assertion concerns monotonicity; this is done in Błaszczyszyn (1990).) The case $\alpha = 0$, i.e., when $\Pr\{T = 0\} > 0$, is different, the single customer effect fails (see Corollary 2 below for definition), there is a positive waiting time vector which, however, remains difficult to compute except the case $m = 2$; see also Błaszczyszyn (1990).

In this section we show that for queues in series the families $\{W^i(\gamma)\}$ and $\{(\min(\underline{R}^i, \underline{P}^i - \gamma T))_+\}$ are ACE. For convenience we introduce the notation $p^i = (\underline{P}^i)_+$ and $r^i = (\underline{R}^i)_+$.

The results of this section are based on the following theorem.

Theorem 4.1. *In a GI/G/1 $\to \cdots \to$ G/1 queueing system for which the inter-arrival time d.f. is in $\mathcal{S}_\alpha(c_A)$ for some $0 < \alpha < \infty$,*

$$\lim_{\gamma\to\infty} \gamma^\alpha \mathrm{E} g(W^1(\gamma), \dots, W^m(\gamma)) = c_A \sum_{i=1}^m \mathrm{E}\left[\int_0^{r^i} (p^i - t)^\alpha g(0, \dots, 0, dt, r^{i+1}, \dots, r^m)\right], \tag{4.1}$$

where $g : \mathbb{R}^m \to \mathbb{R}$ is a coordinatewise monotonic, nondecreasing function for which $g(0, \dots, 0) = 0$, provided that for some $\gamma_0 > 0$ and each $i = 1, \dots, m$,

$$\mathrm{E}\left[\int_0^{\bar r^i} (\bar p^i - t)^\alpha g(0, \dots, 0, dt, \bar r^{i+1}, \dots, \bar r^m)\right] < \infty, \tag{4.2}$$

where

$$\bar P^i = \bar P^i(\gamma_0) = W^1(\gamma_0) + S^1 + \sum_{k=2}^i (W^k(\gamma_0) S^k - \hat S^{k-1}), \tag{4.3}$$

$$\bar R^i = \bar R^i(\gamma_0) = \begin{cases} \bar P^1 & (i = 1), \\ \min\limits_{2\le j\le i}\left\{\sum_{k=j}^i (W^k(\gamma_0) + S^k - \hat S^{k-1})\right\} & (i = 2, \dots, m), \end{cases} \tag{4.4}$$

and $\bar p^i = (\bar P^i)_+$, $\bar r^i = (\bar R^i)_+$.

Before the proof we give the following lemma which states the result for single customer representation in light traffic.

Lemma 4.2. *If the inter-arrival time d.f. is in $\mathcal{S}_\alpha(c_A)$ for some $0 < \alpha < \infty$ then*

$$\lim_{\gamma\to\infty} \gamma^\alpha \mathrm{E} g\Big((\min(\underline{R}^1, \underline{P}^1 - \gamma T))_+, \ldots, (\min(\underline{R}^m, \underline{P}^m - \gamma T))_+\Big)$$
$$= c_A \sum_{i=1}^m \mathrm{E}\Big[\int_0^{r^i} (p^i - t)^\alpha g(0, \ldots, 0, dt, r^{i+1}, \ldots, r^m)\Big], \tag{4.5}$$

where $g : \mathbb{R}^m \to \mathbb{R}$ is a coordinatewise monotonic, nondecreasing function for which $g(0, \ldots, 0) = 0$ and such that the integrals at the RHS of (4.5) exist.

Proof of Lemma 4.2. By an Abelian lemma (see e.g. below (2.1) or Daley & Rolski (1992))

$$\lim_{\gamma\to\infty} \gamma^\alpha \mathrm{E}\Big[g\Big((\min(\underline{R}^1, \underline{P}^1 - \gamma T))_+, \ldots, (\min(\underline{R}^m, \underline{P}^m - \gamma T))_+\Big)\Big]$$
$$= \alpha c_A \int_0^\infty t^{\alpha-1} \mathrm{E} g\Big((\min(\underline{R}^1, \underline{P}^1 - t))_+, \ldots, (\min(\underline{R}^m, \underline{P}^m - t))_+\Big)\, dt. \tag{4.6}$$

To work out the RHS of (4.6), observe that by (3.9)–(3.10), $0 = \underline{P}^1 - \underline{R}^1 \le \ldots \le \underline{P}^m - \underline{R}^m$, and for $i = 1, \ldots, m-1$, $\underline{P}^i \le \underline{P}^{i+1} - \underline{R}^{i+1}$. Hence the right hand side of (4.6) can be rewritten as

$$\alpha c_A \sum_{i=1}^m \mathrm{E}\Big[\int_{\underline{P}^i - \underline{R}^i}^{\underline{P}^{i+1} - \underline{R}^{i+1}} t^{\alpha-1} g\Big(0, \ldots, 0, (\underline{P}^i - t)_+, r^{i+1}, \ldots, r^m\Big)\, dt\Big],$$

where $\underline{P}^{m+1} - \underline{R}^{m+1} = \infty$. Integration by parts yields the RHS of (4.5).

Proof of Theorem 4.1. Fix $\gamma_0 > 0$. We have for $\gamma > \gamma_0$

$$\limsup_{\gamma\to\infty} \gamma^\alpha \mathrm{E}\big[g(W^1(\gamma), \ldots, W^m(\gamma))\big]$$
$$\le \lim_{\gamma\to\infty} \gamma^\alpha \mathrm{E} g\Big((\min(\bar{r}^1, \bar{p}^1 - \gamma T))_+, \ldots, (\min(\bar{r}^m, \bar{p}^m - \gamma T))_+\Big). \tag{4.7}$$

Much as in Lemma 4.2, apply the Abelian lemma (see e.g. below (2.1) or Daley & Rolski (1992)) to the RHS of (4.7) to show that it equals

$$\alpha c_A \int_0^\infty t^{\alpha-1} \mathrm{E} g\Big((\min(\bar{r}^1, \bar{p}^1 - t))_+, \ldots, (\min(\bar{r}^m, \bar{p}^m - t))_+\Big)\, dt. \tag{4.8}$$

By a similar argument to that of the proof of Lemma 4.2, (4.8) equals

$$c_A \sum_{i=1}^m \mathrm{E}\Big[\int_0^{\bar{r}^i} (\bar{p}^i - t)^\alpha g(0, \ldots, 0, dt, \bar{r}^{i+1}, \ldots, \bar{r}^m)\Big]. \tag{4.9}$$

A monotone convergence argument applied to the limit $\gamma_0 \to 0$ in (4.9), provided (4.2) holds, coupled with the result of Lemma 4.2 proves the theorem.

We need the following integral of the beta density

$$\mathrm{B}(x_1, x_2, a, b) = \int_{x_1}^{x_2} (1 - x)^{a-1} x^{b-1}\, dx.$$

Corollary 1. *Let $x \geq 0$. If* $\mathrm{E}\left(S^k\right)^{\alpha+1} < \infty$; $(k = 1, \ldots, m)$ *then*

$$\lim_{\gamma \to \infty} \gamma^\alpha \Pr\{W^m(\gamma) > x\} = c_A \mathrm{E}\left[(p^m - x)^\alpha ; \underline{R}^m > x\right] \tag{4.10}$$

and if $\mathrm{E}\left(S^k\right)^{\alpha+\beta+1} < \infty$; $(k = 1, \ldots, m)$ *then*

$$\lim_{\gamma \to \infty} \gamma^\alpha \mathrm{E}\left(W^m(\gamma)\right)^\beta = \beta c_A \mathrm{E}\left[(p^m)^{\alpha+\beta} \mathrm{B}\left(0, \frac{r^m}{p^m}, \alpha+1, \beta\right)\right]. \tag{4.11}$$

Proof. Equations (4.10) and (4.11) follow from (4.1) after substituting $g(\mathbf{x}) = f(x_m)$ with appropriate function f; all we need to do is to check that the moment conditions on $\mathbf{S}$ ensure finiteness in (4.2). Observe from (3.9)–(3.10), (3.13) and (4.3)–(4.4) that

$$\mathrm{E}\left[\int_0^{\bar{r}^m} (\bar{p}^m - t)^\alpha \, f(dt)\right] \leq \mathrm{E}\left(p^m + V^m(\gamma_0)\right)^\alpha f\left(r^m + V^m(\gamma_0)\right),$$

so for $f(y) = \mathrm{I}(y > x)$, finiteness is ensured when $\mathrm{E}\left(p^m + V^m(\gamma_0)\right)^\alpha < \infty$, for which it is enough by Kiefer & Wolfowitz (1956) that $\mathrm{E}\left(S^k\right)^{\alpha+1} < \infty$ (k=1,2,...,i). Similarly for $f(y) = y^\beta$, by Hölder's inequality, it suffices that $\mathrm{E}\left(p^m + V^m(\gamma_0)\right)^{\alpha+\beta} < \infty$ and $\mathrm{E}\left(r^m + V^m(\gamma_0)\right)^{\alpha+\beta} < \infty$, for which $\mathrm{E}\left(S^k\right)^{\alpha+\beta+1} < \infty$ (k=1,2,...,i) is enough.

The proof of Lemma 4.2 with Corollary 1 and Theorem 4.1 yield the following result.

Corollary 2. *If* $\mathrm{E}\left(S^k\right)^{\alpha+1} < \infty$ $(k = 1, \ldots, m)$ *then the families of r.v.s* $\left\{(\min(\underline{R}^m, \underline{P}^m - \gamma T))_+\right\}$, $\{W^m(\gamma)\}$ *are ACE so we conclude that the single customer effect holds.*

As a result of Theorem 4.1 in its general version we have e.g. the following fact which can be justified in details in the similar manner.

Corollary 3. *If* $\mathrm{E}\left(S^i\right)^{\alpha+3} < \infty$ $(i = 1, \ldots, l \leq m)$ *than for* $k < l$

$$\mathrm{cov}\left(W^k(\gamma), W^l(\gamma)\right) = \gamma^{-\alpha} c_A \mathrm{e}(k, l) + o(\gamma^{-\alpha}),$$

where

$$\mathrm{e}(k,l) = \mathrm{E}\left[r^l \int_0^{r^k} (p^k - t)^\alpha \, dt\right] = \frac{1}{\alpha+1} \mathrm{E}\left[r^l (p^k)^{\alpha+1} - r^l \left(p^k - r^k\right)^{\alpha+1}\right].$$

5. Inter-departure times in light traffic

In this section we study the stationary ergodic sequence $\{T_i^m, i \in \mathbb{Z}\}$ of inter-arrival times at the m-th station (or equivalently inter-departure times from the $(m-1)$th station) in light traffic; such a sequence constitutes the inter-arrival time process at the mth station. More specifically we are going to study the expected value

$$\mathrm{E} g\left(T_1^m(\gamma), \ldots, T_n^m(\gamma)\right) \quad \text{when } \gamma \to \infty, \tag{5.1}$$

for a coordinatewise nonincreasing function g. From this we prove as the main corollary of this section that the arrival process at the mth station is in light traffic asymptotically one-dependent. This one-dependent property can be defined as follows.

Definition 5.1. *It is said that a family of stationary sequences* $\{X_n(\gamma) : n \in \mathbb{Z}\}$ $(\gamma \geq 0)$ *is asymptotically one-dependent (AOD) if for each pair of monotone nonincreasing indicator functions* $g_j : \mathbb{R}^{n_j} \to \mathbb{R}_+$ $(j = 1, 2)$ *such that* $g_j(\mathbf{x}) \downarrow 0$ *when* $\max_{1\leq i\leq n_j} x_i \to \infty$, *and* $d \geq 2$

$$\lim_{\gamma\to\infty} \frac{\mathrm{E}g_1\left(X_1(\gamma), \ldots, X_{n_1}(\gamma)\right) g_2\left(X_{n_1+d}(\gamma), \ldots, X_{n_1+d+n_2}(\gamma)\right)}{\mathrm{E}g_1\left(X_1(\gamma), \ldots, X_{n_1}(\gamma)\right)\ \mathrm{E}g_2\left(X_{n_1+d}(\gamma), \ldots, X_{n_1+d+n_2}(\gamma)\right)} \to 1. \tag{5.2}$$

As before we consider here only the case when $\Pr\{T = 0\} = 0$. We can prove by (3.1)–(3.6) that

$$T_l^m = W_l^m + S_l^m - \min\left(R_l^m, P_l^m - T_l\right). \tag{5.3}$$

We note that in general the departure process it is not a renewal one. Moreover, while for the first station $T_l^1(\gamma) = \gamma T_l$, it is not true that inter-arrival times at the mth station satisfy $T_l^m(\gamma) = \gamma T_l^m$, although it is true that for $\gamma \to \infty$

$$T_l^m(\gamma) - \gamma T_l \to \sum_{k=1}^{m-1} (S_{l+1}^k - S_l^k)$$

from which we obtain

$$\gamma^{-1} T_l^m(\gamma) \to T_l \quad (\gamma \to \infty).$$

To see this it suffices to apply the sample path argument to (5.3), keeping in mind that $W_l^k(\gamma) \to 0$.

The time T_l^m which elapses between departures from the $(m-1)$th station of the lth and $(l+1)$th customer is fully determined by $\mathbf{W}_1 = \{W_1^i : i = 1, \ldots, m-1\}$, $(T_k : k = 1, \ldots, l)$ and $(S_k^i : i = 1, \ldots, m-1, k = 1, \ldots, l+1)$ i.e., by the waiting times of the first customer in all $m-1$ stations and by characteristics brought by the first $l+1$ customers. With this in mind we use the representation with explicit dependence on $\mathbf{W}_1$ and $(T_k : k = 1, \ldots, l)$

$$T_l^m = \mathcal{T}_l^m(\omega)(\mathbf{W}_1, T_1, \ldots, T_l), \tag{5.4}$$

where $\mathcal{T}_l^m(\cdot)(\mathbf{w}, t_1, \ldots, t_l)$ is a $\sigma(S_k^i : i = 1, \ldots, m-1;\ k = 1, \ldots, l+1)$-measurable function. The main idea of this is to have $T_1, \ldots, T_l$ independent of other arguments, both explicit and implicit, of the representation $\mathcal{T}_l^m$.

Theorem 5.2. *Let* $g : \mathbb{R}_+^n \to \mathbb{R}_+$ *be a coordinatewise nonincreasing function with* $g(0, \ldots, 0)$ *finite, and such that* $g(\mathbf{t}) \downarrow 0$ *when* $\max_{1\leq i\leq n} t_i \to \infty$. *In a* GI/G/1 $\to \cdots \to$ G/1 *queueing system for which the inter-arrival time d.f. is in* $\mathcal{S}_\alpha(c_A)$ *for some* $0 < \alpha < \infty$,

$$\lim_{\gamma\to\infty} \gamma^{\alpha n} \mathrm{E}g\left(T_1^m(\gamma), \ldots, T_n^m(\gamma)\right) = (\alpha c_A)^n \int\!\!\ldots\!\!\int_{\mathbb{R}_+^n} \mathrm{E}g(\mathcal{T}_l^m(\mathbf{0}, t_1, \ldots, t_l) : l = 1, \ldots, n) \prod_{k=1}^n t_k^{\alpha-1}\, dt_k \tag{5.5}$$

provided that for some $\gamma_0 > 0$

$$\int\limits_{\mathbb{R}_+^n}\!\!\cdots\!\!\int \mathrm{E}g\left(\mathcal{T}_l^m\left(\mathbf{W}_1(\gamma_0), t_1, \ldots, t_l\right) : l = 1, \ldots, n\right) \prod_{k=1}^{n} t_k^{\alpha-1}\, dt_k < \infty. \tag{5.6}$$

Proof. Observe first by (5.3) and (3.3)–(3.6), if $\mathbf{w} \geq \mathbf{w}'$ componentwise then

$$\mathcal{T}_l^m(\mathbf{w}, t_1, \ldots, t_l) \leq \mathcal{T}_l^m(\mathbf{w}', t_1, \ldots, t_l) \qquad \text{a.s.} \tag{5.7}$$

The random function $\mathcal{T}_l^m$ is also monotone nondecreasing with respect to the other part of its explicit arguments and

$$\mathcal{T}_l^m(\mathbf{w}, t_1, \ldots, t_l) \nearrow \infty \qquad \text{a.s. when } t_l \to \infty. \tag{5.8}$$

Now by the monotonicity of g and (5.7) we have

$$\begin{aligned} &\liminf_{\gamma\to\infty} \gamma^{\alpha n} \mathrm{E}g(T_1^m(\gamma), \ldots, T_n^m(\gamma)) \\ &\qquad \geq \lim_{\gamma\to\infty} \gamma^{\alpha n} \int\limits_{\mathbb{R}_+^n}\!\!\cdots\!\!\int \mathrm{E}g\left(\mathcal{T}_l^m\left(\mathbf{0}, \gamma t_1, \ldots, \gamma t_l\right) :\ l = 1, \ldots, n\right) \prod_{k=1}^{n} A(dt_k), \end{aligned} \tag{5.9}$$

which in turn, by (5.8) and the multivariate Abelian lemma (see e.g. below (2.1)), equals the RHS of (5.5) provided (5.6) holds.

On the other hand we fix $\gamma_0 > 0$ and consider $\gamma \geq \gamma_0$. By (5.7)

$$\begin{aligned} &\limsup_{\gamma\to\infty} \gamma^{\alpha n} \mathrm{E}g(T_1^m(\gamma), \ldots, T_n^m(\gamma)) \\ &\qquad \leq \lim_{\gamma\to\infty} \gamma^{\alpha n} \int\limits_{\mathbb{R}_+^n}\!\!\cdots\!\!\int \mathrm{E}g\left(\mathcal{T}_l^m\left(\mathbf{W_1}(\gamma_0), \gamma t_1, \ldots, \gamma t_l\right) :\ l = 1, \ldots, n\right) \prod_{k=1}^{n} A(dt_k). \end{aligned} \tag{5.10}$$

Now again we apply the Abelian Lemma to the RHS of (5.10) and than let $\gamma \to \infty$. This, provided (5.6) holds, and combined with (5.9) gives (5.5).

Remark. It is not easy to verify when condition (5.6) is satisfied. However in view of (3.3)–(3.4), (3.9)–(3.11), (3.13) and (5.1), $T_l^m \geq S_l^m - \min\left(\underline{R}_l^m, \underline{P}_l^m - T_l\right) - V_l^{m-1}$, and by (3.8),

$$V_l^{m-1} = \max\left(0, V_{l-1}^{m-1} + S_{l-1}^{[1,m-1]} - T_{l-1}\right) = \cdots \leq V_1^{m-1} + S_{[1,l-1]}^{[1,m-1]}.$$

So it suffices to check the finiteness of the integral

$$\int\limits_{\mathbb{R}_+^n}\!\!\cdots\!\!\int \mathrm{E}\Big[g\Big(S_l^m - \min(\underline{R}_l^m, \underline{P}_l^m - t_l) - (V_1^{m-1}(\gamma_0) + S_{[1,l-1]}^{[1,m-1]}) : l = 1, \ldots, n\Big)\Big] \prod_{l=1}^{n} t^{\alpha-1}\, dt_l.$$

Applying (5.3) to (5.5) with $n = 1$ and $g(t) = \mathrm{I}(t \le x)$, after some easy transformations including integration by parts, we obtain the following result. The assumption on the moments of $\mathbf{S}$ by Kiefer & Wolfowitz (1955, 1956) provides (5.6).

Corollary 1. *If* $\mathrm{E}\left(S^i\right)^{\alpha+1} < \infty$ *for* $i = 1, \ldots, m$ *then*

$$\lim_{\gamma\to\infty} \gamma^\alpha \Pr\{T^m(\gamma) \le x\} = c_A \mathrm{E}\left[(x - S^m + \underline{P}^m)^\alpha ; (x \ge S^m - \underline{R}^m)\right].$$

Now we study the structure of the departure process.

Corollary 2. *In a* GI/G/1 $\to \cdots \to$ G/1 *queueing system for which the inter-arrival time d.f. is in* $\mathcal{S}_\alpha(c_A)$ *for some* $0 < \alpha < \infty$, *and the service times have finite moments of all orders the arrival process* $\{T_n^m(\gamma) : n \in \mathbb{Z}\}$ *at the m station is AOD when* $\gamma \to \infty$.

Proof. Take g_1, g_2 as in Definition 5.1. For $\{T_l^m : 1 \le l \le n_1\}$ we use the representation (5.4), while for $n_1 + d \le l \le n_1 + d + n_2$ we write also $T_l^m = \bar{\mathcal{T}}_l^m(\omega)(\mathbf{W}_{n_1+d}, T_{n_1+d}, \ldots, T_{n_1+d+n_2})$, where $\bar{\mathcal{T}}_l^m(\cdot)(\mathbf{w}, t_{n_1+d}, \ldots, t_{n_1+d+n_2})$ is a $\sigma(S_k^i : i = 1, \ldots, m-1;\ k = n_1 + d, \ldots, l+1)$-measurable function, and put $\mathbf{W}_{n_1+d} = \{W_{n_1+d}^i : i = 1, \ldots, m-1\}$. We prove that

$$\lim_{\gamma\to\infty} \gamma^{\alpha(n_1+n_2)} \mathrm{E} g_1\left(T_1(\gamma), \ldots, T_{n_1}(\gamma)\right) g_2\left(T_{n_1+d}(\gamma), \ldots, T_{n_1+d+n_2}(\gamma)\right) =$$

$$(\alpha c_A)^{n_1+n_2} \int_{\mathbb{R}_+^{n_1+n_2}}\!\!\!\ldots\!\int \mathrm{E} g_1(\mathcal{T}_l^m(\mathbf{0}, t_1, \ldots, t_l) : l = 1, \ldots, n) \prod_{k=1}^{n_1} t_k^{\alpha-1}\, dt_k \tag{5.11}$$

$$g_2(\bar{\mathcal{T}}_l^m(\mathbf{0}, t_{n_1+d}, \ldots, t_l) : l = n_1 + d, \ldots, n_1 + d + n_2) \prod_{k=n_1+d}^{n_1+d+n_2} t_k^{\alpha-1}\, dt_k.$$

Now, since $\{\mathcal{T}_l^m, l = 1, \ldots, n_1\}$ and $\{\bar{\mathcal{T}}_l^m, l = n_1 + d, \ldots, n_1 + d + n_2\}$ are independent, we can factorize the RHS of (5.11) and conclude by Theorem 5.2 that the numerator and denominator of (5.2) are of the same order of magnitude with the same limit constant. To obtain (5.11) we cannot apply Theorem 5.2 directly but, much as in the proof of (5.5), we observe that in (5.11) the LHS is not less then the RHS, and that the LHS of (5.11) is not greater then

$$(\alpha c_A)^{n_1+n_2} \int_{\mathbb{R}_+^{n_1+n_2}}\!\!\!\ldots\!\int \mathrm{E} g_1(\mathcal{T}_l^m(\mathbf{W}_1(\gamma_0), t_1, \ldots, t_l) : l = 1, \ldots, n) \tag{5.12}$$

$$g_2(\mathcal{T}_l^m(\mathbf{W}_1(\gamma_0), t_1, \ldots, t_{n_1}, \gamma_0 T_{n_1+1}, \ldots, \gamma_0 T_{n_1+d-1}, t_{n_1+d}, \ldots, t_l) :$$
$$l = n_1 + d, \ldots, n_1 + d + n_2) \prod_{k=1}^{n_1} t_k^{\alpha-1}\, dt_k \prod_{k=n_1+d}^{n_1+d+n_2} t_k^{\alpha-1}\, dt_k.$$

Letting $\gamma \to \infty$ in (5.12), provided it is finite, we obtain again the RHS of (5.11). To ensure finiteness of (5.12) as well as (5.6) for g_1 and g_2 separately it suffices to have finite moments of

$\mathbf{S}$ of order $\alpha(n_1+n_2)+1$. In formal proof we use the Remark after Theorem 5.2 and Kiefer & Wolfowitz (1955, 1956).

6. Light traffic via thinning

Another possible approach to light traffic limits proceeds via a thinning operation (called π-thinning in Daley & Rolski (1991); see also Daley & Rolski (1992) and Asmussen (1991)). By π-thinning we mean that arrivals of customers into the system can be regarded as the result of subjecting the process of potential arrivals to independent thinning with common retention probability π. We approach light traffic conditions by allowing $\pi \to 0$. In this section we discuss briefly the consequence of taking such limits for GI/G/1 $\to \cdots \to$ G/1 systems.

Note first that the intervals of a renewal process subject to independent thinning can be represented as

$$T_n(\pi) = \sum_{i=1}^{\nu_n} T_{in} \tag{6.1}$$

where $\{T_{in} : i = 1,2,\ldots;\ n = 1,2,\ldots\}$ is a doubly infinitely indexed sequence of i.i.d. r.v.s, each $T_{in} =_{\mathrm{d}} T$ for some generic inter-arrival time r.v. T of the unthinned potential arrival process, and $\{\nu_i\}$ is a sequence of i.i.d. geometric r.v.s with

$$\Pr\{\nu_n = r\} = \pi(1-\pi)^{r-1} \quad (r = 1,2,\ldots).$$

Then for each π, $(T_n(\pi), n \in \mathbb{Z})$ is a sequence of i.i.d. r.v.s. with the common distribution function $A^{(\pi)}$ and for each n the family of sequences $\{(T_n(\pi)),\quad 0<\pi<1\}$ is a stochastically monotone $\mathbb{R}_+^\infty$-valued random process, because $T(\pi) \le_{\mathrm{d}} T(\pi')$, when $\pi' < \pi$. Moreover by a standard argument we can assume that it is sample path monotone decreasing random process; (take $\Omega = [0,1]^{\mathbb{Z}}$, $\Pr = \ldots \times dx \times dx \times \ldots$ and $T_n(\pi)(\omega) = (A^{(\pi)})^{-1}(\omega_i)$, where $\omega = (\ldots,\omega_0,\omega_1,\ldots)$ and the inverse function $A^{(-1)}$ is defined as in Stoyan (1983), Section 1.2.) Note that, irrespective of whether or not $T >_{\mathrm{a.s.}} 0$, each $T_n(\pi) \to_{\mathrm{a.s.}} \infty$ as $\pi \to 0$. In result the presented argument justifies the existence of a probability space $(\Omega, \mathcal{F}, \Pr)$ (we do not introduce separate notations here although formally this is a new space), a sequence $\{\mathbf{S}_n\}$ of independent and identically random vectors, all distributed as the generic service times vector $\mathbf{S}$, which are independent of the process $\{(T_n(\pi)),\quad 0<\pi<1\}$. Again as in Section 4 we can prove that the stationary waiting time $(W^m)(\pi)$ at the mth station decreases stochastically to 0.

Let A denote the distribution of T. We want to point out that in this section any variable changing its values with the retention probability π in marked by (π). d by (π).

Theorem 6.1. *In a* GI/G/1 $\to \cdots \to$ G/1 *queueing system in which the arrival process is subject to π-thinning, for a coordinatewise non-decreasing function $g : \mathbb{R}_+^m \to \mathbb{R}$ with $g(0,\ldots,0) = 0$,*

$$\lim_{\pi\to 0} \pi^{-1} \mathrm{E} g\left(W^1(\pi),\ldots,W^m(\pi)\right) = \sum_{i=1}^m \mathrm{E}\Big[\int_0^{r^i} H(p^i - t) g(0,\ldots,0,dt,r^{i+1},\ldots,r^m)\Big], \tag{6.2}$$

provided that for some π' in $0 < \pi' \le 1$ and each $i = 1, \ldots, m$

$$\mathrm{E}\left[\int_0^{\bar{r}^i} H(\bar{p}^i - t) g(0, \ldots, 0, dt, \bar{r}^{i+1}, \ldots, \bar{r}^m)\right] < \infty, \tag{6.3}$$

where $H = \sum_{i=1}^{\infty} A^{i*}$ is the zero-deleted renewal function,

$$\bar{P}^i = \bar{P}^i(\pi') = W^1(\pi') + S^1 + \sum_{k=2}^{i}(W^k(\pi')S^k - \hat{S}^{k-1}),$$

$$\bar{R}^i = \bar{R}^i(\pi') = \begin{cases} \bar{P}^1 & (i = 1) \\ \min\limits_{2 \le j \le i} \left\{ \sum_{k=j}^{i} (W^k(\pi') + S^k - \hat{S}^{k-1}) \right\} & (i = 2, \ldots, m) \end{cases}$$

$\bar{p}^i = (\bar{P}^i)_+$ and $\bar{r}^i = (\bar{R}^i)_+$.

Proof. Much as in Section 4 we have for $\pi < \pi'$

$$\left(\min\left(\underline{R}^i, \underline{P}^i - T(\pi)\right)\right)_+ \le W^i(\pi) \le \left(\min\left(\bar{R}^i(\pi'), \bar{P}^i(\pi') - T(\pi)\right)\right)_+ .$$

For the lower bound i.e. for the single customer representation, we have

$$\begin{aligned} &\lim_{\pi \to 0} \pi^{-1} \mathrm{E}\, g\left(\left(\min\left(\underline{R}^1, \underline{P}^1 - T(\pi)\right)\right)_+, \ldots, (\min(\underline{R}^m, \underline{P}^m - T(\pi)))_+\right) \\ &= \lim_{\pi \to 0} \sum_{n=1}^{\infty} (1-\pi)^{n-1} \int_0^{\infty} \mathrm{E} g\left(\left(\min\left(\underline{R}^1, \underline{P}^1 - t\right)\right)_+, \ldots, (\min(\underline{R}^m, \underline{P}^m - t))_+\right) A^{n*}(dt) \\ &= \int_0^{\infty} \mathrm{E} g\left(\left(\min\left(\underline{R}^1, \underline{P}^1 - t\right)\right)_+, \ldots, (\min(\underline{R}^m, \underline{P}^m - t))_+\right) H(dt), \end{aligned}$$

and for an upper bound

$$\limsup_{\pi \to 0} \pi^{-1} \mathrm{E} g\left(\left(\min\left(\bar{R}^i(\pi'), \bar{P}^i(\pi') - T(\pi)\right)\right)_+\right) =$$
$$\int_0^{\infty} \mathrm{E} g\left(\min\left(\bar{R}^1\right), \bar{P}^1 - t\right), \ldots, \min\left(\bar{R}^m, \bar{P}^m - t\right)\right) H(dt).$$

Now arguing as in the proof of Theorem 4.1, we let $\pi' \to 0$ to obtain

$$\begin{aligned} &\lim_{\pi \to 0} \pi^{-1} \mathrm{E} g\left(W^1(\pi), \ldots, W^m(\pi)\right) \\ &= \int_0^{\infty} \mathrm{E} g\left(\left(\min\left(\underline{R}^1, \underline{P}^1 - t\right)_+\right), \ldots, (\min(\underline{R}^m, \underline{P}^m - t))_+\right) H(dt). \end{aligned}$$

This completes the proof since the same decomposition and integration techniques as in the analogous Theorem 4.1 may be used to state (6.2) provided (6.3) holds.

As an analogue of Corollary 3 to Theorem 4.1 the following result is established in a straightforward manner; note that the conditions are analogous to those of the case $\alpha = 1$ in that corollary.

Corollary 1. *If* $\mathrm{E}\big[\,(S^k)^2\,\big] < \infty$ $(k = 1, \ldots, m)$ *then the families of r.v.s* $\{\min\,(\underline{R}^m, \underline{P}^m - T^\pi)_+\}$ *and* $\{W^m(\pi)\}$ *are ACE and single customer effect holds.*

Other results analogous to those of Section 4 are omitted apart from the following three concerning the departure-time process.

Theorem 6.2. *Let* $g : \mathbb{R}^n_+ \to \mathbb{R}_+$ *be a coordinatewise nonincreasing function with* $g(0, \ldots, 0)$ *finite and such that* $g(\mathbf{t}) \downarrow 0$ *when* $\max_{1\le i\le n} t_i \to \infty$. *Let* $\mathcal{T}^m_l$ *be as in (5.4). Then*

$$\lim_{\pi\to 0} \pi^{-n}\mathrm{E}g\,(T^m_1(\pi), \ldots, T^m_n(\pi)) = \int\limits_{\mathbb{R}^n_+}\!\!\ldots\!\int \mathrm{E}g\,(\mathcal{T}^m_l\,(\mathbf{0}, t_1, \ldots, t_l) : l = 1, \ldots, n) \prod_{k=1}^n t_k^{\alpha-1}\,H(dt_k) \tag{6.4}$$

provided that for some $\pi' > 0$

$$\int\limits_{\mathbb{R}^n_+}\!\!\ldots\!\int \mathrm{E}g\,(\mathcal{T}^m_l\,(\mathbf{W}_1(\pi'), t_1, \ldots, t_l) : l = 1, \ldots, n) \prod_{k=1}^n t_k^{\alpha-1}\,H(dt_k) < \infty. \tag{6.5}$$

Proof. By monotonicity of g we have

$$\begin{aligned}
&\liminf_{\pi\to 0} \pi^{-n}\mathrm{E}g\,(T^m_1(\pi), \ldots, T^m_n(\pi)) \\
&\ge \lim_{\pi\to 0} \sum_{\substack{1\le i_k\le\infty \\ 1\le k\le n}} (1-\pi)^{i_k} \int\limits_{\mathbb{R}^n_+}\!\!\ldots\!\int \mathrm{E}g\,(\mathcal{T}^m_l\,(\mathbf{0}, t_1, \ldots, t_l) :\; l = 1, \ldots, n) \prod_{k=1}^n A^{i_k *}(dt_k),
\end{aligned}$$

which equals to the RHS of (6.4). On the other hand, for $\pi \le \pi'$ we have

$$\begin{aligned}
&\limsup_{\pi\to 0} \pi^{-n}\mathrm{E}g\,(T^m_1(\pi), \ldots, T^m_n(\pi)) \\
&\le \int\limits_{\mathbb{R}^n_+}\!\!\ldots\!\int \mathrm{E}g\,(\mathcal{T}^m_l\,(\mathbf{W}_1(\pi'), t_1, \ldots, t_l) :\; l = 1, \ldots, n) \prod_{k=1}^n H(dt_k) < \infty.
\end{aligned}$$

Now we let $\pi' \to 0$ and obtain, provided (6.5) holds, the RHS of (6.4). This completes the proof.

Corollary 1. *If* $\mathrm{E}\,(S^i)^2 < \infty$ *for* $i = 1, \ldots, m$, *then*

$$\lim_{\pi\to 0} \pi^{-1}\mathrm{Pr}\,\{T^m(\pi) \le x\} = \mathrm{E}\,[H\,(x - S^m + \underline{P}^m)\,; x \ge S^m - \underline{R}^m]\,.$$

Corollary 2. *In a* GI/G/1 $\to \cdots \to$ G/1 *queueing system in which the service times have finite moments of all orders the arrival process* $\{T^m_n(\pi) : n \in \mathbb{Z}\}$ *at the* m *th station is AOD when* $\pi \to 0$.

Concluding remarks

It is useful to point out a few conclusions we can draw from the results presented above. In particular, we have given the order of magnitude in light traffic of various performance characteristics which are functions of the stationary waiting times or the inter-arrival times at nodes of a GI/G/1$\to \cdots \to$ G/1 series queueing system.

One problem which remains concerns the range of validity of the "single customer effect" in a G/G/1 queue (i.e., not necessarily renewal input). There are examples of GI/G/1 queues where the effect breaks down (see e.g. Daley & Rolski, 1984; Asmussen, 1991). Some work on the non-renewal input case is given in Daley & Rolski (1991), but we are still far from a complete understanding of the problem. As another example consider a node in a series of queues with independent service times, and regard it as a G/GI/1 queue in its own right. Inter-departure times $T^m(\gamma)$ from the $(m-1)$th node are inter-arrival times at the mth node. The Corollary 1 after Theorem 5.2 yields for $x \geq 0$

$$\gamma^\alpha A^m_\gamma(x) \to c_A \mathrm{E}\left[(x - S^m + \underline{P}^m)^\alpha ; x \geq S^m - \underline{R}^m\right]$$

where $A^m_\gamma(x) = \Pr\{T^m(\gamma) \leq x\}$. If we suppose that the single customer effect holds then, neglecting mathematical rigor, we would write, for a nondecreasing function g such that $g(0) = 0$,

$$\begin{aligned}\gamma^\alpha \mathrm{E} g\left(W^m(\gamma)\right) &= \gamma^\alpha \int_0^\infty \mathrm{E} g\left(S^m - x\right) A^m_\gamma(dx) \\ &\to c_A \int_0^\infty \mathrm{E}\left[g\left(S^m - x\right)\left(dx - S^m + \underline{P}^m\right)^\alpha ; x \geq S^m - \underline{R}^m\right],\end{aligned}$$

the "justification" here being Theorem 4.1. However, we have not as yet been able to justify the steps above without exploiting the detail of the particular queueing structure.

Acknowledgement

We are grateful to Daryl Daley for his comments on this paper.

References

Asmussen, S. (1991). Light traffic equivalence in single server queues. *Ann. Appl. Probab.* **1** (to appear).

Baccelli, F. & Brémaud, P. (1991). Virtual customers in sensitivity and light traffic analysis via Campbell's formula for point processes. *Adv. Appl. Probab.* (to appear)

Błaszczyszyn, B. (1990). *Szeregowy System Kolejek w Warunkach Słabego Obciążenia. M. Sc. Thesis. Mathematical Institute, Wrocław University.*

Daley, D. J. & Rolski, T. (1984). A light traffic approximation for a single-server queue. *Math. Operat. Res.* **9**, 624–628.

— & — (1991). Light traffic approximations in queues. *Math. Operat. Res.* **16**, 57–71.

— & — (1992). Light traffic approximations in many-server queues. *Adv. Appl. Probab.* **24**, 202–218.

— & — (1993). Light traffic approximations in general stationary single-server queues. *Stochastic Process. Appl.*

GREENBERG, B.S. & WOLFF, R.W. (1988). Optimal order of servers for tandem queues in light traffic. *Management Sci.* **34**, 500–508.

KIEFER, J. & WOLFOWITZ, J. (1955). On the theory of queues with many servers. *Trans. Amer. Math. Soc.* **78**, 1–18.

— & — (1956). On the characteristics of the general queueing process with application to random walks. *Ann. Math. Statist.* **27**, 147–161.

LOYNES, R. M. (1962). The stability of a queue with non-independent inter-arrival and service times. *Proc. Cambridge Philos. Soc.* **58**, 497–520.

MASTERSON, G. E. & SHERMAN, S. (1963). On queues in tandem. *Ann. Math. Statist.* **34**, 300–307.

NIU, S.C. (1980). Bounds for the expected delays in some tandem queues. *J. Appl. Probab.* **17**, 831–838.

PINEDO, M.& WOLFF (1982). A comparison between tandem queues with dependent and independent service times. *Operat. Res.* **30**, 464–479.

REIMAN, M.I. & SIMON, B. (1988). An interpolation approximations for queueing systems with Poisson input. *Operat. Res.* **36**, 454–469.

— & — (1989). Open queueing systems in light traffic. *Math. Operat. Res.* **14**, 26–59.

SIGMAN, K. (1991). Light traffic for work-load in queues. (Manuscript)

STOYAN, D. (1983). *Comparison Methods for Queues and Other Stochastic Processes.* Wiley, Chichester.

WHITT, W. (1989). An interpolation approximation for the mean workload in a GI/G/1 queue. *Operat. Res.* **37**, 936–952.

WOLFF, R. W. (1982). Tandem queues with dependent service times in light traffic. *Operat. Res.* **30**, 619–635.