Stochastic Processes for Insurance and Finance

August 5, 1998

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JOHN WILEY & SONS
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Preface

This book is designed for a beginning or an intermediate graduate course in stochastic modelling. It is intended for a serious student in probability theory, statistics, actuarial sciences or financial mathematics. The overall objective of the manuscript, is to make the basic concepts of stochastic modelling and insurance accessible to students and research workers in a comprehensive manner. Renewal theory, random walks, discrete and continuous-time Markov processes, martingale theory and point processes are among the major subjects treated. The selection of the topics has been largely made on the basis of their relevance within an actuarial or financial context. In this sense, the book is rather special. On the other hand, one could have written a similar textbook but with queueing theory or stochastic networks as scrutinizing subject.

A few words are in order about the selection of topics. Not only space limitations have forced us to make a choice from the union of stochastic processes, actuarial mathematics and mathematics of finance. Each one of the authors, coming from four European countries, had a list of favourite topics when the writing project got in the starting blocks.

One advantage of using insurance questions as guidelines in the selection of the topics, is that the treated subjects gain in coherence. Another facet is that any important actuarial problem is highlighted from a variety of different stochastic angles. A possible disadvantage might be that important subjects are not duly treated. We consider the topics that are covered as the basic intersection of stochastic modelling, insurance mathematics and financial mathematics. As a result, we only give a few elements of branching processes or of jump-diffusion processes. In the same fashion, we do not cover credibility theory, IBNR claims (IBNR = incurred but not reported) or topics from advanced finance. On the other hand, we do treat some uncommon subjects like subexponentiality, phase-type distributions, piecewise deterministic Markov processes, stationary and marked point processes, etc. We really hope that what has been covered will be appetizing enough to instigate the reader to continue his efforts.

Some special features of this book are the following:

1. The first chapter gives the reader a bird's-eye view of the main themes treated in the book. We have made an attempt to introduce the principal concepts from insurance and finance in an intuitive fashion, avoiding rigour. This chapter should however convince the reader about the need of the mathematically sound treatment in the rest of the book.
2. The book is not covering the statistical aspects of stochastic models in insurance and finance. However, to emphasize the relevance of the stochastic models, a fair number of practical illustrations with real life data has been included.
3. In a similar fashion, some numerical and algorithmic procedures have been included since they make up a vital portion of current day practical approaches in insurance.
4. An attempt has been made to make the book self-contained. Only well-known results from analysis, probability and measure theory have been mentioned without proofs. A couple of times, we allude to results that are only treated in a subsequent chapter. This happens most often when the proof of a result depends on more sophisticated material treated in the later part of the book.
5. Notes and comments at the end of each section include references to additional reading material. An extensive list of references is included. This bibliographical material should serve two purposes: helping the probabilistically trained reader to find his way in the actuarial and financial literature, while at the same time informing the practitioner on the sources where to find the mathematical treatment of one or the other useful methodology.

Thanks to the unusual and unifying approach, many of the topics are put in a novel framework. While we do not claim originality, a sizable set of results and proofs appear here for the first time.

The numbering of chapters, sections, subsections, definitions, formulae, lemma's, and theorems is traditional. Chapters are subdivided into sections while sections are further subdivided into subsections. References of the form $X.i.j$ refer to chapter X in roman numerals, section i, serial number j of subsections, definitions, lemma's, and theorems. If we refer to a formula, we write $(X.i.j)$. References of the form $i.j$ and $(i.j)$ respectively refer to things within the same chapter, i.e. the roman numeral of the chapter is omitted in this case. References to the literature are of the form: name(s) of the author(s) plus year; for example Smith (1723) or Kovalyov and Smith (1794).

We emphasize that the book has been conceived as a course text. In an attempt to keep the size of the book at a reasonable level, we decided not to include sections with exercises. In many places, the reader is however asked to provide additional arguments and parts of proofs. This of course is not sufficient. A subject like stochastic modelling does not only require routine-like skills. It demands training and sharpening of accurate probabilistic intuition which can only be achieved by tackling nontrivial exercises. We are convinced that the best way to help students and teachers is by supplementing this textbook with a forthcoming *Teacher's Manual.*

This book would never have been finished without the help of many people. Individually, we take pleasure in thanking our home institutions for the indulgence and logistic support, needed during the many working sessions of the quartet. Each one of us extends his thanks to his students and colleagues who patiently read first, second or n-th drafts of the manuscript and who helped us whenever there were problems with the styling of the manuscript. Jointly, our appreciation goes to a large number of colleagues in academia and in insurance companies who helped us during the selection process of what and what not to include.

In particular we are grateful to Sabine Schlegel for her invaluable help during the whole process of the preparation of the book. Apart from reading the manuscript and helping to eliminate a number of errors, she provided the computer illustrations included in the text. Also, our contacts with J. Wiley & Sons through Helen Ramsey and Sharon Clutton have always been pleasant and constructive.

Further support and help is greatly acknowledged. TR thanks the Department of Stochastics at the University of Ulm for hospitality and for creating optimal working conditions during his extended stays in Ulm. The latter were made possible by the financial support of the Deutsche Forschungsgemeinschaft and of the Institut für Finanz- und Aktuarwissenschaften Ulm. TR and VS want to express their gratitude to the Mathematisches Forschungsinstitut Oberwolfach and to the Volkswagen Stiftung. The programme *Research in Pairs* granted them a very useful five-weeks stay in Oberwolfach. TR and VS also acknowledge the financial support from the Exchange Programme between the Deutsche Forschungsgemeinschaft and the Polish Academy of Sciences.

Our deepest gratitude is reserved for four loving families, who have endured our preoccupation with this project for so long.

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August 1998

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List of Principal Notation

A	event
A^{c}	complement of set A
$\boldsymbol{A}^{\top}$	transposition of matrix $\boldsymbol{A}$
$\mathbb{I}(A)$	indicator of event A
B	Borel set
$\mathcal{B}(\mathbb{R})$, $\mathcal{B}(E)$	Borel σ-algebra on $\mathbb{R}$, E
$\mathbb{C}$	complex plane
E	state space
$\boldsymbol{E}$	squared matrix with all entries equal to 1
$\boldsymbol{e}$	(row) vector with all components equal to 1
$\boldsymbol{e}_i = (0,\ldots,0,1,0,\ldots,0)$	
$\mathbb{E}$	expectation
$\mathbb{E}(X\vert\mathcal{G})$	conditional expectation
Cov	covariance
Var	variance
F, G	distributions; distribution functions
$F * G$	convolution
F^{*k}	k-th convolution of F; $F^{*1} = F$, $F^{*0} = \delta_0$
F^{s}	integrated tail distribution
$\overline{F}(x)$	tail function
$F^{-1}(y)$	generalized inverse function
$f(t)$	density function
$\mathcal{F}, \mathcal{F}_n, \mathcal{G}$	σ-algebras
$\hat{g}(s)$	generating function
$g_1(x) \sim g_2(x)$	asymptotical equivalence as $x \to \infty$
$H(t)$	renewal function
I	index of dispersion
$\boldsymbol{I}$	identity matrix
$\Im z$	imaginary part of z
i	imaginary unit
$L(x)$	slowly varying function as $x \to \infty$
$\hat{L}(s)$	Laplace transform
$\hat{l}(s)$	Laplace-Stieltjes transform

$\hat{m}(s)$	moment generating function
N	$\mathbb{N}$-valued random variable
$\{N(t)\}$	counting process
$\mathbb{N}$	set of nonnegative integers
$\emptyset$	empty set
$\mathbb{P}$	probability measure
$Q(y)$	quantile function
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	real line
$\mathbb{R}_+$	nonnegative reals
$\Re z$	real part of z
S, T, U, V, W, X, Y, Z	random variables
$\mathcal{S}$	family of subexponential distributions
T_n	inter-occurrence times
$U_{(i)}, X_{(i)}$	i-th order statistic
$\{X(t)\}$	stochastic process
$\lfloor x \rfloor$	largest integer $\le x$
$\lceil x \rceil$	smallest integer $\ge x$
$x \wedge y = \min\{x, y\}$	
$x \vee y = \max\{x, y\}$	
$\mathbb{Z}$	set of integers
β	premium rate
γ	adjustment coefficient
δ_a	Dirac measure
$\delta,\ \delta(t)$	force of interest
η	relative safety loading
λ	claim occurence rate
$\mu^{(n)}$	n-th moment
$\nu^-,\ \nu^+$	ladder epochs
$\Pi(X)$	premium for risk X
ρ	expected aggregate claim amount
τ	stopping time
$\tau_{\mathrm{d}}(u), \tau(u)$	ruin times
$\hat{\varphi}(t)$	characteristic function
$\psi(u),\ \psi(u; x)$	ruin probabilities
σ_n	arrival epoch
Ω	basic space
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
$\le_{\mathrm{sl}}$	stop-loss order
$\le_{\mathrm{st}}$	stochastic order
$\sum_{i=1}^{j} a_i = 0$ if $j = 0$	
$\prod_{i=1}^{j} a_i = 1$ if $j = 0$	
$\inf \emptyset = \infty$	

I

Concepts from Insurance and Finance

1 Introduction

Let us start out with a concrete example. Consider all policies covering fire of apartments in a suburb of a major city that are underwritten with a specific insurance company. The insured goods have a comparable value and the chances for fire are probably not much different from one building to another. Such a set of policies makes up a homogeneous portfolio.

Most client-related insurance business uses such portfolios as basic building blocks. Properly compiled they make up branches within the insurance firm, like fire, automobile, theft, property, life, health, pension, etc. The branch *fire* contains many portfolios dealing with different types of risk. Single standing houses, terraced houses, apartments, apartment buildings, shops, supermarkets, business premises and industrial sites constitute an incomplete but already varied set of *insurance risks* for which different premiums have to be designed. Indeed, for each of the mentioned portfolios, the probabilities for a fire might depend on the portfolio. Moreover, the resulting claim sizes may very well be incomparable.

In the chapters dealing with insurance aspects, we will restrict our attention to one specific portfolio. Such a portfolio is characterized by a number of ingredients of both a deterministic and a stochastic nature.

Among the first we mention the *starting position* and a *time period*. Usually, data referring to an insurance portfolio refer to a time span of one year in accordance with the bookkeeping of the company. Far more important is the *initial reserve* or *initial capital*. One interpretation of the latter is the amount of capital set aside to cover costs, occurring during the initial period of the portfolio when the company has not yet received the yearly premiums. In the sequel the initial reserve will be denoted by u.

Among the elements that usually have a stochastic nature are the following:

- The *epochs of the claims*; denote them by $\sigma_1, \sigma_2, \ldots$. In some cases we consider an additional claim arrival epoch at time zero denoted by $\sigma_0 = 0$. Apart from the fact that the epochs form a nondecreasing sequence we do not in general assume anything specific about their interdependence. The random variables defined by $T_0 = 0$ and $T_n = \sigma_n - \sigma_{n-1}$, $n \geq 1$, are called the *inter-occurrence times* in between successive claims.

- The *number of claims* up to time t is denoted by $N(t)$ where $N(t) = \sup\{n : \sigma_n \leq t\}$. The intrinsic relation between the sequence of claim arrivals $\{\sigma_0, \sigma_1, \sigma_2, \ldots\}$ and the counting process $\{N(t), t \geq 0\}$ is given by $\{N(t) = n\} = \{\sigma_n \leq t < \sigma_{n+1}\}$.
- The claim occurring at time σ_n has size U_n. The sequence $\{U_n, n = 1, 2, \ldots\}$ of consecutive *claim sizes* is often assumed to consist of independent and identically distributed random variables. However, other possibilities will show up in the text as well.
- The *aggregate claim amount* up to time t is given by $X(t) = \sum_{i=1}^{N(t)} U_i$ while $X(t) = 0$ if $N(t) = 0$. By its very definition, the aggregate claim amount is in general a *random sum* of random variables.
- The *premium income*. In the course of time 0 to t we assume that a total of $\Pi(t)$ has been received through premiums.
- The *risk reserve* at time t is then $R(t) = u + \Pi(t) - X(t)$.

The above set up allows flexibility in that an individual claim may mean a claim from an individual customer (e.g. third-liability insurance) or a claim caused by a single event (e.g. windstorm insurance).

In the next sections we will give some more details on the concepts introduced above and on actuarial quantities linked to them. We also show by practical illustrations how the stochastic character of these elements can be formalized. This approach will then automatically serve as an invitation for a thorough probabilistic treatment by concrete stochastic processes later on in the book.

2 The Claim Number Process

Let us start out by considering in some more detail the claim number process $\{N(t), t \geq 0\}$, built on the claim epochs. We will always assume that the claim number process is a *counting process*. This means that we require the process $\{N(t), t \geq 0\}$ to satisfy the following three conditions: For all $t, h \geq 0$

- $N(0) = 0$,
- $N(t) \in \mathbb{N}$,
- $N(t) \leq N(t+h)$,

$N(t+h) - N(t)$ models the number of claims occurring in the time interval $(t, t+h]$. We note that realizations of a counting process (also called the *sample* *paths* or *trajectories*) are monotonically nondecreasing and right-continuous functions. Usually one assumes that the jumps are of size one so that multiple claim arrivals are excluded. But there are cases where this assumption is not valid. For example there is a vast proportion of road accidents where more than one person is injured. It is then useful to associate a *mark* to each jump epoch. The resulting generalization naturally leads to the notion of a *marked counting process* treated in Chapter XII.

Let us write $\{p_k(t), k = 0, 1, \ldots\}$ with

$$p_k(t) = \mathbb{P}(N(t) = k) = \mathbb{P}\left(\sum_{i=1}^{k} T_i \leq t < \sum_{i=1}^{k+1} T_i\right)$$

for the probability function of the counting variable $N(t)$. In some cases, time t does not play a significant role; if so, the t-dependence will be dropped from the notation. This happens for example when the insurer is interested in the number of claims he experiences in successive one-year periods.

A general computational expression for the probabilities $p_k(t)$ is impossible because we have not specified the interdependence between the epochs of the claims. We indicate some possible choices.

2.1 Renewal Processes

Depending on the type of portfolio, the insurer can make a variety of different assumptions on the sequence of inter-occurrence times $\{T_n, n \geq 1\}$. In some particular cases it might be useful to assume that this sequence is generated by a *renewal process* $\{\sigma_n, n \geq 1\}$ of claim arrival epochs, i.e. the random variables T_n are nonnegative, independent and have the same distribution as a *generic random variable* T. The distribution function of the inter-occurrence time is then denoted by $F_T(x) = \mathbb{P}(T \leq x)$.

Because this model appears as a natural candidate for the underlying stochastic mechanism in a wealth of processes, we will spend substantial space on a thorough discussion of renewal processes in Chapter VI. One might think that a renewal process is a rather simple type of process. However, the reader may be surprised seeing some of the highly nontrivial results that form the backbone of all applications of renewal theory. In Chapter VI we will deal in more detail with the *Sparre Andersen model* in which the claim number process is a (general) renewal process.

Mathematically the simplest renewal process is the *Poisson process* introduced in Section V.2 where the generic random variable T is exponentially distributed. Poisson processes have particular properties that distinguish them among other renewal processes. The main reason for this extra structure is provided by the lack of memory property of the exponential distribution. The latter distribution plays a similar crucial role in actuarial applications of stochastic processes as the normal distribution does in statistics. In particular, for the Poisson process we have

$$p_k(t) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^k}{k!}\,, \qquad k = 0, 1, \ldots$$

and $\mathbb{E}\,N(t) = \lambda t$, $\operatorname{Var} N(t) = \lambda t$ for all $t \geq 0$, where $\lambda^{-1} = \mathbb{E}\,T$ is the mean inter-occurrence time. As a side result, the *index of dispersion* $\mathrm{I}(t) = \operatorname{Var} N(t)/\mathbb{E}\,N(t)$ is constant and equal to 1.

It should be mentioned that time is not to be considered as real time, but as *operational time*. Indeed, some seasonal effects might affect the claim reporting process and the portfolio does not have the same size over time. It is therefore advantageous to measure time via the expected mean number of claims. In Sections 2.2 and 2.3 we consider some possible deviations from the constant expected number of claims per unit of operational time.

2.2 Mixed Poisson Processes

As early as 1948 actuaries noticed that the variability in a portfolio, expressed for example by $\mathrm{I}(t)$, was often larger than 1, the value corresponding to the Poisson case. One reason is this. The risk is exposed to some environment, for example weather conditions in motor insurance. This environment is different each year and influences the number of claims from the portfolio. If one would know the environment completely, one would also know the mean number of claims Λ to be expected in a particular year. Because one hardly has any information on how the environment influences the mean number of claims, one can estimate the distribution F_Λ of Λ and model the environment via F_Λ. This observation naturally leads to the following representation for the claim number distribution:

$$p_k(t) = \mathbb{P}(N(t) = k) = \int_0^\infty \frac{\mathrm{e}^{-\lambda t}(\lambda t)^k}{k!}\, dF_\Lambda(\lambda) \tag{2.1}$$

where $F_\Lambda(\lambda) = \mathbb{P}(\Lambda \leq \lambda)$ is the distribution function of the *mixing random variable* Λ.

Counting processes of this type are called *mixed Poisson processes*. They appeared in this general form in the actuarial literature since 1959, see Thyrion (1959). A sound definition can be given using the theory of continuous-time Markov processes, the subject of Section VIII.5. This treatment is inspired by Lundberg (1964). Alternatively, mixed Poisson processes can be introduced via the general theory of doubly stochastic Poisson processes studied in Section XII.2.

The special case where F_Λ is a gamma distribution has already been introduced by Ammeter (1948) and is known as the *Pólya process* or the *Pascal* *process*. Other choices however are possible. For example, the case where F_Λ is an inverse normal distribution has been rather popular in insurance but also in geophysical modelling.

It is easy to prove that for the mixed Poisson model, $\mathrm{I}(t) = 1 + t(\mathrm{Var}\,\Lambda / \mathbb{E}\,\Lambda)$ which equals 1 if and only if Λ is degenerate, the classical case of a homogeneous Poisson process. The above explanation makes it not surprising that the mixed Poisson process has always been very popular among insurance modellers.

2.3 Some Other Models

Of course, both of the above models do not exhaust the possibilities for the claim counting process. Let us just mention a few alternatives that will be discussed in some more detail in the forthcoming chapters.

- *Recursively defined claim number distributions.* Panjer (1980) used a recursion relation for the probability function $\{p_k(t), k = 0, 1, \ldots\}$ of the number of claims $N(t)$ to derive a recursive relation for the distribution of the aggregate claim amount. In the actuarial literature we find an increasing number of papers dealing with variations on the following recursion formula

$$p_k(t) = \left(a + \frac{b}{k}\right) p_{k-1}(t), \qquad k = 2, 3, \ldots \tag{2.2}$$

  where the quantities a and b may depend on the time variable t. The above class has been introduced by Sundt and Jewell (1981) in an attempt to gather a variety of classical claim number distributions under the same umbrella. Later on Willmot (1988) reconsidered equation (2.2) and added a number of overlooked solutions. We will deal with (2.2) in Section IV.3. Most often, time dependence in (2.2) is suppressed. Also, some authors include the value $k = 1$ in (2.2), hence lowering the number of free parameters in the model. Special cases of the above model are (shifted) versions of the Poisson, binomial, negative binomial (or Pascal) and logarithmic distribution. On the other hand, the recursion relation (2.2) can readily be generalized and we find a variety of such extensions in the actuarial literature.
- *Processes with stationary or independent increments.* Another line of thinking is based on the following observations. The distribution of the *increment* $N(t+h) - N(t)$ of the counting process $\{N(t)\}$ may be the same for all $t \geq 0$. This means that, as a function of time, the counting process has *stationary increments*. This model is studied in Chapter XII where $\{N(t)\}$ is said to be the counting process corresponding to a *stationary point process*. For example, the number of industrial fires in a small region might easily satisfy this type of stationarity condition while the number of

car accidents on a road, experiencing different types of weather conditions, may fail.

The counting process $\{N(t)\}$ may also satisfy the *independent increments* condition. This means that increments over disjoint time intervals are independent random variables. For example, car portfolios usually satisfy this condition. However the number of severe accidents on oil platforms probably does not, since imposing stricter safety regulations will change the distribution of the number of similar accidents in later periods.

- *Compound Poisson processes.* In general it is a complicated matter to derive a closed expression for $p_k(t)$. However, in some cases it may be mathematically much more convenient to work with the *probability generating function* given by $\hat{g}_{N(t)}(s) = \mathbb{E}\, s^{N(t)}$. This is the case if claims are arriving with stationary *and* independent increments. It follows from arguments covered in Section V.2.2 that in this case we have the relation

$$\hat{g}_{N(t)}(s) = \mathrm{e}^{-\lambda t(1-\hat{g}_V(s))} \tag{2.3}$$

where $\hat{g}_V(s)$ is itself the probability generating function of a discrete random variable V, concentrated on the strictly positive integers. As such the counting process $\{N(t)\}$ could also be called a *discrete compound Poisson process*. The probabilities $p_k(t)$ are then given in the form

$$p_k(t) = \sum_{j=0}^{\infty} \mathrm{e}^{-\lambda t} \frac{(\lambda t)^j}{j!} p_{V,k}^{*j}$$

where $\{p_{V,k}^{*j}, k = 1, 2, \ldots\}$ is the j-fold convolution of the probability function $\{p_{V,k}, k = 1, 2, \ldots\}$ with probability generating function $\hat{g}_V(s) = \mathbb{E}\, s^V$. Needless to say that the explicit evaluation of the above probabilities is mostly impossible because of the complicated nature of the convolutions. However, approximations and bounds are available. Some of them are discussed in Chapter IV. If the probability generating function $\hat{g}_V(s)$ is known, a numerical method which is based on the *inverse fast Fourier transform*, can be used to compute the $p_k(t)$. This is discussed in Section IV.7.

Kupper (1963) gives a nice application of discrete compound Poisson processes to claim counts. Assume that the number of accidents that have happened in a factory up to time t is denoted by $N'(t)$. Assume that $\{N'(t)\}$ follows a Poisson process with parameter λ as defined before. In the n-th accident the number of casualties is equal to V_n. The sequence $\{V_n, n \geq 1\}$ is assumed to be a sequence of independent and identically distributed discrete random variables with probability generating function $\hat{g}_V(s) = \sum_{k=1}^{\infty} p_{V,k} s^k$. Note that the inclusion of $p_{V,0}$ would hardly make sense. If the sequence $\{V_n, n \geq 1\}$ is independent of the process $\{N'(t), t \geq 0\}$, then

the probability generating function of $N(t) = \sum_{i=1}^{N'(t)} V_i$, the total number of claims up to time t, is of the form (2.3). A proper choice of $\hat{g}_V(s)$ gives a variety of possible processes. Note in particular that this claim number process allows multiple jumps.
The compound Poisson process will constantly appear as a backbone model for the claims arrival process. We will recognize it in its own right in Chapters IV and V. Later on it will reappear as a special type of continuous-time Markov processes in Section VIII.3.1, as a piecewise deterministic Markov process in Section XI.3 and again as a particular point process in Chapter XII.

- *Claim number distributions related to Markov chains.* Another model for the claim number process which generalizes the classical Poisson process can be provided by a (nonhomogeneous) *pure birth process* as treated in Section VIII.5.4. Here the probabilities $p_k(t)$ satisfy the *Kolmogorov differential equations*, well-known from the theory of continuous-time Markov processes. Explicitly,

$$\frac{dp_k(t)}{dt} = -q_k(t)p_k(t) + q_{k-1}(t)p_{k-1}(t) \tag{2.4}$$

for some nonnegative functions $q_k(t)$, where we take $p_{-1}(t) = q_{-1}(t) = 0$. A further example of a claim number process related to a continuous-time Markov process is the *Markov-modulated Poisson process* studied in Chapter XII in the general framework of marked point processes. Here, instead of considering a single mixing random variable like in (2.1), we truly investigate more general claim number processes where for example the claim arrival rate can fluctuate in time according to the realizations of a Markov chain.

3 The Claim Size Process

In most chapters of this book, we will assume that the sequence $U_1, U_2, \ldots$ of successive claim sizes consists of independent and identically distributed random variables generated by the distribution F_U of a generic random variable U. The n-th *moment* of the claim size distribution will be denoted by $\mu_U^{(n)} = \mathbb{E}(U^n) = \int_0^\infty x^n \, dF_U(x)$. For $n = 1$ we simply write $\mu_U = \mu_U^{(1)}$. For the variance of the claim sizes we write $\operatorname{Var} U = \mu_U^{(2)} - (\mu_U)^2$.

3.1 Dangerous Risks

A nonnegative random variable or its distribution is frequently called a *risk*. In principle any distribution, concentrated on the nonnegative halfline, can be used as claim size distribution. However we will often make a mental distinction between "well-behaved" distributions and dangerous distributions with a *heavy tail*. Concepts like well-behaved or heavy-tailed distributions belong to the common vocabulary of actuaries. We will make a serious attempt to formalize them in a mathematically sound definition.

Roughly speaking, the class of well-behaved distributions consists of those distributions F with an exponentially bounded tail, i.e. $1 - F(x) \leq c\,\mathrm{e}^{-ax}$ for some positive a and c and all $x \geq 0$. The condition means that large claims are not impossible, but the probability of their occurrence decreases exponentially fast to zero as the threshold x becomes larger and larger. As we will see in later chapters, this condition enhances exponential type behaviour of most important actuarial diagnostics like aggregate claim amount and ruin probabilities.

In Chapter II we will give a somewhat streamlined approach to heavy-tailed distributions. For such distributions there is no proper exponential bound and huge claims are getting more likely. A natural nonparametric class of heavy-tailed claim size distributions is the class $\mathcal{S}$ of *subexponential distributions* introduced and studied in Section II.5. The class $\mathcal{S}$ has some extremely neat probabilistic properties that will be highlighted whenever possible. For example, the aggregate claim amount is mainly determined by the largest claim in the portfolio. From the practical point of view, however, the class $\mathcal{S}$ is too wide since it cannot be characterized by parameters having a useful interpretation. For this reason practitioners usually fall back on "weakly-parametrized" subexponential distributions. For example, the *lognormal distribution* belongs to $\mathcal{S}$ and is extremely popular in modelling motor insurance claim data. However, for the case of fire or catastrophic event insurance, the *Pareto distribution* F with $1 - F(x) = (c/x)^\alpha$, for $x \geq c$, seems to be sufficiently flexible to cope with most practical examples. It is fortunate that, over the past decades, the asymptotical and statistical properties of subexponential distributions have received considerable attention. In this textbook we will however mainly deal with asymptotic properties and only touch upon some of the resulting statistical issues.

3.2 The Aggregate Claim Amount

The *aggregate claim amount* at time t given by $X(t) = \sum_{i=1}^{N(t)} U_i$ has the distribution function

$$F_{X(t)}(x) = \mathbb{P}(X(t) \leq x) = \mathbb{P}\Big(\sum_{i=1}^{N(t)} U_i \leq x\Big) \,.$$

Sometimes the reference to $X(t)$ is omitted and then we write $F(x)$ for $F_{X(t)}(x)$. The derivation of an explicit formula for $F_{X(t)}(x)$ is almost always impossible. In any case, more specific knowledge on the interdependencies between and within the two processes $\{N(t)\}$ and $\{U_n\}$ is needed.

It is most often assumed that the processes $\{N(t)\}$ and $\{U_n\}$ are stochastically independent and we usually will follow this practice. However, it is easy to imagine situations where the processes $\{N(t)\}$ and $\{U_n\}$ will not be independent. Suppose that we are considering a portfolio of road accidents. In winter time there might be a large number of claims due to poor road conditions. However most of these claims will be small since the weather conditions prevent high speeds. A similar kind of thinking applies to the summer period where the number of accidents will usually be rather small but some of them will result in severely large claims due to reckless driving at high speeds.

As another example, take the portfolio of accidents on oil platforms of a certain type. As soon as a severe accident happens on one of these platforms, improved safety measures will try to prevent similar accidents in the future. This action hopefully results in a subsequent drop in the number of accidents.

In Chapter IV we will cover a variety of different approaches to determine the aggregate claim amount $X(t)$. They all have in common that they are geared to investigate the behaviour of $F_{X(t)}(x)$ for a fixed t, which henceforth will be mostly omitted.

As said, the most studied case however is that where the two processes $\{N(t)\}$ and $\{U_n\}$ are independent. Even then, calculation of $F_{X(t)}(x)$ remains a formidable undertaking. By an application of the law of total probability it is easy to see that we have the following fundamental relation

$$F_{X(t)}(x) = \mathbb{P}(X(t) \le x) = \sum_{k=0}^{\infty} p_k(t) F_U^{*k}(x)\,, \tag{3.1}$$

where F_U^{*k} refers to the k-th convolution of F_U with itself, i.e. $F_U^{*k}(x) = \mathbb{P}(U_1 + U_2 + \ldots + U_k \le x)$, the distribution function of the sum on k independent random variables with the same distribution as U. One of the unfortunate aspects in the applications of formula (3.1) is that k-fold convolutions are seldom calculable in closed form. One often needs to rely on approximations, expansions and/or numerical algorithms. In Section IV.4 we deal with recursive and numerical algorithms while in Section IV.6 we look for approximations by easier compounds.

An alternative way of writing the above formula is given in terms of the *Laplace-Stieltjes transform* $\hat{l}_{X(t)}(s) = \mathbb{E}\,\mathrm{e}^{-sX(t)}$. Namely,

$$\hat{l}_{X(t)}(s) = \hat{g}_{N(t)}(\hat{l}_U(s))\,, \tag{3.2}$$

where $\hat{g}_{N(t)}(s)$ is the probability generating function of the number of claims $N(t)$ up to time t and $\hat{l}_U(s)$ is the Laplace-Stieltjes transform of U. But then inversion techniques as in Section IV.7 or even more specifically as in Section V.5 may be useful.

If the number of claims is very large, then one can forecast that a central limit effect will be dominant. A *large-scale approximation* like

$$F_{X(t)}(x) \approx \Phi\left(\frac{x - \mathbb{E}\,X(t)}{\sqrt{\operatorname{Var} X(t)}}\right),$$

where $\Phi(x)$ denotes the standard normal distribution function, might be possible. In more concrete situations one can even replace $\mathbb{E}\,X(t)$ and $\operatorname{Var} X(t)$ by subsequent approximations. For example, if $\{N(t)\}$ is a renewal counting process, these approximations follow from the basic renewal theorems as will be shown in Section VI.2.3.

On the other hand the above large-scale approximation has often shown to be rather unreliable. Practitioners have tried to remedy this shortcoming by using refined versions of the central limit theorem. For example *Edgeworth expansions* and *Gram-Charlier series* resulted in significant but still not always satisfying improvements. Other approaches like the *normal-power approximations* or approximations using gamma-distributions have been tried out.

The main reason why results inspired by the central limit theorem often give poor results is that the number of those claims that *de facto* determine the entire portfolio is actually (and fortunately) very small. This is particularly true in the case of large claims. There results that a genuine centralization of the claims is totally absent and any central limit approximation is meaningless.

3.3 Comparison of Risks

We have already alluded to the difference between light-tailed and heavy-tailed claim size distributions and we have tried to grasp this intuitive idea into a mathematically sound definition. On the more methological level, one might like to compare risks and even order them according to some ranking criterion. A first and general attempt to *ordering of risks* is made in Section III.2. On further locations, we investigate how this ordering translates into a subsequent ordering of compounds in Section IV.2.4 and of ruin functions in Section V.4.5.

However, running along sample paths, we can even compare full processes and decide which one is most apt to incorporate essential features of a risk situation. In Section VII.4.2 we compare discrete-time Markov chains while Section VIII.1.4 treats a similar comparison in the continuous-time set up. Another illustration of a comparison of entire processes is given in Section XII.2.4 where Markov modulated Poisson processes, Poisson cluster processes, mixed Poisson processes, and merging renewal processes are put in proper perspective to homogeneous Poisson processes with the same claim arrival intensity.

4 Solvability of the Portfolio

4.1 Premiums

We have used the abbreviation $\Pi(t)$ to denote the totality of premiums collected from the policyholders within the portfolio up to time t. Usually premiums are individually collected once a year, but the insurer can safely assume that premium income is evenly spread over the year.

The determination of a functional expression for $\Pi(t)$ is one of the few things where the insurer can intervene in the overall process. The function $\Pi(t)$ should be determined in such a way that the solvability of the portfolio can be guaranteed. This requires $\Pi(t)$ to increase fast enough to cope with incoming claims. On the other hand, a very high value of $\Pi(t)$ may be rather undesirable since then rival insurance companies might attract clients by offering lower premiums while covering the same risk.

The whole area of determining the specific shape of the function $\Pi(t)$ is called *premium calculation* and constitutes an essential part of the actuarial know-how. For an exhaustive treatment, see Goovaerts, de Vylder and Haezendonck (1984). In Chapter III we cover a few and isolated *premium principles* underlying the general thinking behind premium calculations. From these general principles a wide set of possible candidates emerges. Note that most often these premiums are *nonrandom* even if their calculation involves information on the stochastic elements within the portfolio.

The most popular form of the premium function $\Pi(t)$ is

$$\Pi(t) = (1+\eta)\mathbb{E}\,N(t)\mathbb{E}\,U\,. \tag{4.1}$$

where $\mathbb{E}\,N(t)$ is the expected number of claims up to time t while $\mathbb{E}\,U$ is the mean claim size. The constant η is the *safety loading* which has to take care, not only of the administrative costs from handling the portfolio, but also of the necessary gain that the company wants to make ultimately. Moreover, it is clear that η will be portfolio dependent, as for example the higher the risk, the higher η has to be. Remark that in the case of a Poisson process $\{N(t)\}$ the premium function $\Pi(t)$ given in (4.1) is of the form $\Pi(t) = \beta t$ for some constant $\beta > 0$.

The premium principle of the form (4.1) with η independent of the portfolio is called the *expected value principle*. Of course, the expected value principle does not keep track of the variability in the portfolio and so alternative principles include for example the variances of $N(t)$ and U. A few other possibilities are also covered in Chapter III.

Another aspect of premium calculation is that not all individual policies in a portfolio must be charged with the same premium. For example, the by now classical *bonus-malus* premium calculation principle is widely spread in car insurance. Depending on the past history of a policyholder, the insurer ranks the client in a certain *state* and charges an amount typical for that state. In the course of time the customer will move from one state to another, depending on his claim record. If we denote by X_n the state of a client at the beginning of year n, then the process $\{X_n, n \geq 0\}$ describes a jump process adequately modelled by a *discrete-time Markov chain*. No course on stochastic processes can be considered complete if it would lack a serious treatment of this important building block of stochastic modelling. We will deal with discrete-time Markov chains in general in Chapter VII and with bonus-malus in Section VII.1.4 in particular. In Chapter VIII we will then continue this discussion by changing from discrete time to continuous time.

4.2 The Risk Reserve

Recall from the introduction that the risk reserve at time t is given by $R(t) = u + \Pi(t) - X(t)$, where u is the initial reserve. The quantity $R(t)$ is of course random. The ultimate hope would be to get some information on the distribution of $R(t)$ or, if this is impossible, on the first few moments of $R(t)$. As has already been pointed out, the distribution of $X(t)$ is very complicated. If we superimpose a possibly random character of the premium income, then the overall situation will become even worse.

There is one particular situation however where we can give a full treatment of the stochastic nature of the risk reserve. Assume that time is measured in integers. We then add all premiums collected in period n in one single number $\Pi_n = \Pi(n) - \Pi(n-1)$ which can even be assumed to be random. Similarly we add all claims arriving in period n and call this amount X_n where $X_n = X(n) - X(n-1)$. The risk reserve after period n is then equal to

$$R_n = R(n) = u + \Pi(n) - X(n) = u - \sum_{i=1}^{n} (X_i - \Pi_i) \ .$$

The resulting stochastic process $\{S_n, n \geq 0\}$ of partial sums $S_n = \sum_{i=1}^{n} Y_i$ is called the (discrete-time) *claim surplus process*. In particular, $\{S_n\}$ is a *random walk* if we assume that the sequence $\{Y_n, n \geq 1\}$ with $Y_n = X_n - \Pi_n$ consists of independent and identically distributed random variables.

The definition of a random walk is very similar to that of a renewal process, the only difference being that now the generic random variable is not longer concentrated on the nonnegative halfline. The theory of random walks makes up a substantial chapter in a traditional course on stochastic modelling. We will cover the most important aspects of random walk theory in Chapter VI.

If premiums are nonrandom we can write

$$\mathbb{P}(R(t) \geq x) = \mathbb{P}(X(t) \leq u + \Pi(t) - x) = F_{X(t)}(u + \Pi(t) - x)$$

which shows how important it is to have workable expressions for the distribution of the aggregate claim amount.

4.3 Economic Environment

A number of problems in insurance and finance contain an economic element like interest, discounting or indexing. As such, a variety of subjects will be treated within an *economic environment.*

Consider data from car insurance. When calculating the premium, the insurer needs certain characteristics of the claim sizes, such as their mean value and variance. Just using the common estimators for mean and variance would lead to problems. Over the years, cars have become more expensive. Therefore, the claim data from some years ago cannot be compared directly with claim data of today and so need to be adjusted.

Suppose that we consider a time horizon of n years. Then, for each $i = 1, \ldots, n$, an index I_i for year i is defined which is related to the costs of that year, including for instance prices of new cars and repair costs. This index is then used to measure all claims in units of year 0. Denote by X_i the aggregate claim amount of year i. Then adequate estimators for mean and variance of the aggregate claim amount measured in units of year 0, are given by

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i/I_i\,, \qquad \bar{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i/I_i - \bar{X})^2\,.$$

The corresponding estimates for year $n+1$ are then

$$\hat{X}_{n+1} = \hat{I}_{n+1}\bar{X}\,, \qquad \hat{\sigma}_{n+1} = \hat{I}^2_{n+1}\bar{\sigma}^2$$

where $\hat{I}_{n+1}$ is an estimate of the index for year $n+1$.

The indices I_i are often expressed via the *interest rates* $r_i = I_i/I_{i-1} - 1$. This is particularly advantageous in *life insurance*. For example, in the case of a single premium payment, suppose that the dependant of the policyholder will get a predefined lump sum x in the case of death of the policyholder within n years. The insurance company will invest the premium. Thus the value V of the payment will be different depending on the year of death. Let p_i denote the probability that the policyholder dies in year i after issuing of the policy. Then, for a given sequence $r_1, \ldots, r_{n-1}$ of interest rates, the expected value $\mathbb{E}\, V$ of the contract is

$$\mathbb{E}\, V = p_1 x + p_2 x(1+r_1) + \ldots + p_n x(1+r_1)\ldots(1+r_{n-1})\,.$$

The problem with the above expression is that at policy issue, the interest rates r_i are not known in advance. Therefore insurance companies use a *technical interest rate* r instead of the true interest rates r_i. This leads to a technical value V' with expectation

$$\mathbb{E}\, V' = p_1 x + p_2 x(1+r) + \ldots + p_n x(1+r)^{n-1}\,.$$

We will use this kind of economic setting in Section VII.3 where we deal with Markov chains with rewards. In Section XI.4 we deal with risk processes in an economic environment within the framework of piecewise deterministic Markov processes. Questions of insurance and finance connected with stochastic interest and discounting are discussed in Chapter XIII.

5 Reinsurance

Reinsurance is a prime activity of almost all insurers. We first explain why reinsurance may be necessary. Then we give a number of specific forms of reinsurance and point out how each one of them provides further support for the topics that will be treated in later chapters.

5.1 Need for Reinsurance

As phrased by Borch (1960), a company makes arrangements for reinsurance when it seeks to reduce the probability of suffering losses which may jeopardize the position of the company. Among the reasons to ponder reinsurance the insurer can think about the following.

- The appearance of *excessively large claims;* here we think about claims resulting from severe accidents like with nuclear power stations or cases of serious medical maltreatment. In other instances an insurer might be confronted with large claims coming from a policy involving very valuable items such as air carriers, oil tankers, dams, large building complexes.
- An unusually *large number of claims* or *clustering of claims*, whether large or not. Extensive forest fires may temporarily lead to a very large number of more or less large claims. Hurricanes, earthquakes and floods can cause similar explosions of the number of claims.
- Unexpected *changes in premium collection* like in case of a sudden inflation or unforeseen increase in handling costs. Under these circumstances the company actually does not quite receive the premium income it had been expected at the beginning of the book year.
- There are *legal restrictions* forcing the company to have reserves to cover a certain part of future claims. For a smaller company these restrictions would cause a non-competitive premium. Taking reinsurance is a comfortable way for solving that problem.
- If a company can take reinsurance, it also can offer more services to its clients. Reinsurance can therefore be pondered to *increase the capacity of the company.*

There exists a variety of reinsurance forms. What they all have in common is the wish to diminish the impact of the large claims. In what follows we provide the mathematical formulation of most of the currently employed *reinsurance treaties.*

5.2 Types of Reinsurance

Recall that we use $\{U_1, U_2, \ldots, U_{N(t)}\}$ as the sequence of successive claims up to time t when the underlying claim number process is $\{N(t), t \geq 0\}$. Further the aggregate claim amount over that period is $X(t) = \sum_{i=1}^{N(t)} U_i$.

On the basis of the past history of the portfolio or stimulated by one or more arguments under Section 5.1, the insurer will redesign the portfolio in such a way that he himself keeps a certain amount of the aggregate claim amount $X(t)$ while he looks for reinsurance for the remaining part. The amount that he keeps is called the *deductible* and will be denoted by $h(X(t))$ or $h(U_i)$, depending on whether the reinsurance form acts on the whole portfolio or on single claims. The remaining part, i.e. $X(t) - h(X(t))$ or $U_i - h(U_i)$ is the *reinsured part.* Acting like that, the insurer himself is taking an insurance with a reinsurance company and hence becomes himself a client. The *first line insurer* is still drafting the premiums to be asked from his own customers. Part of that premium now has to be transferred to the *second line insurer* who has accepted to cover the risk at a negotiated premium. The second line insurer can then again redivide the risk and go to a third line company, hence building up a *reinsurance chain* where at each step deductibles and corresponding premiums have to be negotiated and transmitted down the chain to the first underwriter.

Here are the most commonly used types of reinsurance. The first form of reinsurance is *proportional* or *quota-share reinsurance* where a certain proportion, say a, of the total portfolio is reinsured. This means that $h(x)$ takes the special form $h(x) = ax$. But then

$$h(X(t)) = aX(t) = a \sum_{i=1}^{N(t)} U_i = \sum_{i=1}^{N(t)} h(U_i)$$

where $0 < a < 1$ is the *proportionality factor.*

This form of reinsurance is very popular in almost all insurance branches presumably because of its conceptual and administrative simplicity. Moreover this kind of reinsurance is often used at the start of smaller companies to broaden their chances in underwriting policies. In general the first line insurance cedes to the reinsurer a similarly determined proportion of the premiums.

From the distributional point of view the basic properties of $h(X(t))$ can be derived from the analogous properties for the case $a = 1$. Indeed, by its definition

$$\mathbb{P}(h(X(t)) \leq x) = \mathbb{P}(X(t) \leq x/a) = F_{X(t)}(x/a)\,.$$

This formula shows again how important it is to get reliable and accurate formulae for the distribution of the aggregate claim amount.

Another important form of reinsurance is *excess-loss* which is determined by a positive number b, called the *retention level*. The reinsured amount is then equal to $\sum_{i=1}^{N(t)}(U_i - b)_+$ where $x_+ = \max\{x, 0\}$. This reinsurance form covers the overshoot over the retention level b for all individual claims whether or not they are considered to be large. It is clear that excess-loss reinsurance limits the liability of the first line insurer. It looks like the underwriter of the policy decides that he himself will cover all claims below the retention b.

Among the insurance branches where this type of reinsurance is used we mention in particular general liability, and to a lesser extent motor-liability and windstorm reinsurance. Excess-loss reinsurance is particularly interesting if a relatively small number of risks is insured, and the individual claim size distributions are heavy-tailed. Because of its very form, all claims have to be checked individually and as such this reinsurance contract leads to an expensive administration.

From the distributional angle, the excess-loss amount has a distribution which is completely similar to that of the aggregate claim amount but with a claim size distribution truncated at the retention. Again the importance of good approximations to the distribution $F_{X(t)}(x)$ is apparent.

A reinsurance policy that considers each claim as an integral part of the entire portfolio is determined by the quantity

$$\left(\sum_{i=1}^{N(t)} U_i - b\right)_+ = (X(t) - b)_+$$

where the *retention* b determines the *stop-loss reinsurance*. In this type of reinsurance, also the small claims show their influence on the total amount reinsured. In particular, when the number of claims is very large, the aggregate claim amount is highly depending upon the small claims as well. On the other hand stop-loss reinsurance seems a natural adaptation of the excess-loss treaty but then to the portfolio as a whole.

Stop-loss reinsurance is used in windstorm and hail reinsurance and occasionally in fire insurance. Due to its form, stop-loss reinsurance is very simple to apply and does not require expensive individualized administration. In general one would not use stop-loss reinsurance unless the number of policies is large. Moreover we will show in Chapter III that stop-loss reinsurance has some desirable optimality properties.

From the distributional point of view one notices the necessity to derive compact expressions for the probabilities of overshooting a certain barrier. If one indeed considers the portfolio as a single policy, then again the approximations and bounds for the aggregate claim amount are of prime importance.

There still are other types of reinsurance contracts.

- Thinking especially about coverage against large claims, it seems desirable to look for treaties based on the largest claims in the portfolio. If we denote by $(U_{(1)}, U_{(2)}, \ldots, U_{(N(t))})$ the order statistics of the random vector $(U_1, U_2, \ldots, U_{N(t)})$ of claim sizes then

$$Z(t) = \sum_{i=1}^{r} U_{(N(t)-i+1)} - rU_{(N(t)-r)} = \sum_{i=1}^{N(t)} (U_i - U_{(N(t)-r)})_+$$

  would make up a rather neat reinsurance treaty. It has been introduced by Thépaut (1950) and is called *excédent du coût moyen relatif* or for short *ECOMOR*. The amount $Z(t)$ covers only that part of the r largest claims that overshoots the random retention $U_{(N(t)-r)}$ where one put $U_{(N(t)-r)} = 0$ if $N(t) \leq r$. In some sense the ECOMOR treaty rephrases the excess-loss treaty but with a random retention at a large claim.
  Not much is known about ECOMOR treaties and as such this reinsurance form has been largely neglected by most reinsurers. The main reason is the rather complicated form of $Z(t)$ defying a simple treatment.
- A reinsurance form that is somewhat akin to proportional reinsurance is *surplus reinsurance*. Here the reinsured amount is determined individually and proportionally by the value of the insured object. The insurer is forced to introduce the value of the insured object as an extra unknown and basically random quantity. The overall value of the insured amount is the key factor when choosing this type of reinsurance.
- Of course there are possibilities for *combined reinsurance contracts*. For example, the insurer can first apply an excess-loss retention on the individual claims; at the next step he applies an additional stop-loss cover to the remaining excess over a certain retention.

We will not develop any systematic study of these last three reinsurance treaties and restrict ourselves to the basic theory.

6 Ruin Problems

Ruin theory has always been a vital part of actuarial mathematics. At first glance, some of the theoretically derived results seem to have limited scope in practical situations. Nevertheless calculation of and approximation to ruin probabilities have been a constant source of inspiration and technique development in actuarial mathematics.

Assume an insurance company is willing to risk a certain amount u in a certain branch of insurance, i.e. if claim surplus exceeds the level u some drastic action will have to be taken for that branch. Because in some sense this part of the business starts with the capital u we can safely call u the *initial capital*. The actuary now has to make some decisions, for instance which premium should be charged and which type of reinsurance to take, see Section 5. Often, the premium is determined by company policies and by tariffs of rivals. A possible criterion for optimizing the reinsurance treaty would be to minimize the probability that the claim surplus ever exceeds the level u. To be more specific, consider the risk reserve $R(t) = u + \Pi(t) - X(t)$ and define the random variable $\tau = \inf\{t \geq 0 : R(t) < 0\}$. The instant τ gives us the *ruin time* of the portfolio, where we interpret ruin in a technical sense. Of course, we should allow the possibility that no ruin ever occurs, which means that $\tau = \infty$. We should realize that τ is depending on all the stochastic elements in the risk reserve process $\{R(t)\}$ as well as on the deterministic value u. For this reason one often singles out the latter quantity in the notation for the ruin time by writing $\tau(u)$ for τ. More specifically, the *survival* or *non-ruin probability in finite time* will be defined and denoted by

$$\overline{\psi}(u;x) = \mathbb{P}\big(\inf_{0\leq t\leq x} R(t) \geq 0\big) = \mathbb{P}(\tau(u) > x)$$

when we consider a *finite horizon* $x > 0$. The *survival probability* over an infinite time horizon is defined by the quantity

$$\overline{\psi}(u) = \mathbb{P}\big(\inf_{t\geq 0} R(t) \geq 0\big) = \mathbb{P}(\tau(u) = \infty) \ .$$

Alternative notations that are in constant use refer to the *ruin probabilities* which are defined by the equalities

$$\psi(u;x) = 1 - \overline{\psi}(u;x) \ , \qquad \psi(u) = 1 - \overline{\psi}(u) \, .$$

Each year the insurer of a portfolio has to negotiate a reinsurance contract. While an optimal strategy will depend on $R(t)$, the insurer has to apply the contract at the beginning of the year. Simultaneously, the future policies of the reinsurance companies have to be taken into account. The resulting problem is hard to solve. In one approximation procedure, the premium and the reinsurance treaty is fixed for the whole future and then the optimal reinsurance is chosen, see for instance Dickson and Waters (1997). Then this new reinsurance treaty is chosen as input and the procedure is repeated. To get rid of the dependence on the initial capital, an alternative approach is to consider the *adjustment coefficient* introduced in Chapter V. The adjustment coefficient is some sort of *measure of risk*. Maximizing the adjustment coefficient is in some sense minimizing the risk for large initial capitals. This optimization procedure was for instance considered in Waters (1983).

Ruin theory is often restricted to the classical compound Poisson risk model. The latter model will therefore appear over and over again as a prime example. Unfortunately, because of its intrinsic simplicity, the compound Poisson model does not always give a good description of reality. There are more realistic but still tractable models in the literature, that hopefully will find their way into actuarial applications. We will review a number of these models. In Chapter XI we introduce piecewise deterministic Markov processes to broaden the Markovian treatment of risk analysis. Interest and discounting can then be easily introduced as elements of economic environments. Another prospective area of actuarial and financial applications can be found in the theory of point processes that will be treated in Chapter XII.

As one can expect, ruin probabilities will depend heavily on the claim size distribution. If the latter is well-behaved the ruin probabilities will turn out to be typically exponentially bounded as the initial capital becomes large. However, when the claim size distribution has a heavy tail, then one single large claim may be responsible for the ultimate ruin of the portfolio. The reader will definitely appreciate how the class $\mathcal{S}$ of subexponential distributions provides a beautiful characterization for the heavy-tailed case as shown in Sections V.4.3, VI.5.5 and even more deeply in Sections XII.6 and XIII.2.4.

Phase-type distributions form another and versatile alternative class of distributions for which more or less explicit calculations are possible. They are treated in some detail in Section VIII.2 and applied to ruin calculations later on.

The results for ruin probabilities on which we will focus in this book, can be characterized by the following features.

- Only in the easiest cases will we succeed in getting explicit formulae for the ruin probabilities like in Section V.6.
- As soon as we allow more complex models, one way out is to use approximations of *Cramér-Lundberg type* for large initial capital as in Sections V.4, VI.5 or more generally in Chapters XI, XII and XIII. In all these models, a surprisingly different asymptotic behaviour of ruin probabilities is observed depending on whether light-tailed or heavy-tailed claim size distributions are considered.
- An alternative is to employ numerical procedures as in Section V.5.
- A further and even more important theme of the text is the derivation of bounds for the ruin probabilities. A vast number of *Lundberg bounds* have been derived in the text, ranging from the simplest in Section V.4.1 in the compound Poisson model, over Section VI.5.2 in the Sparre Andersen model, to the more general in Chapters XI, XII and XIII. Another type of bounds is obtained in Section XII.2.4 where we are dealing with the comparison of ruin probabilities for risk reserve processes with identical safety loadings but with different volatilities.
- Finally, simulation methods are discussed in Section IX.2.5.

Quite a few of the above results have been derived using *martingale* *techniques.* Martingale theory makes up a vast portion of current day probabilistic modelling and can not be left out of any serious treatment of risk analysis and financial mathematics. In Chapter IX we treat the fairly easy discrete-time situation, making links to random walk theory and life-insurance. The more demanding continuous-time martingale theory is treated in Chapter X. This chapter is also vitally needed in the treatment of financial models; it nevertheless provides some unexpected links to finite-horizon ruin probabilities in Section X.3.

7 Related Financial Topics

7.1 Investment of Surplus

In Section 4 we already mentioned that the calculation of premiums constitutes an essential part of the actuarial know-how. Although these premiums are nonrandom for a certain time horizon, their calculation involves information on the stochastic elements within the portfolio and within the economic environment. However, even once the premiums are fixed, the future premium income of an insurer is not deterministic. For one reason, the number of customers may increase or decrease in time outside the control of the insurer. Since the insurer invests the surplus in financial markets, there still is another source of uncertainty which is caused by the random fluctuations of these markets. Notice that interest and inflation rates usually change in much smaller steps than the risk reserve of an insurance portfolio changes with the arriving claims. This situation is modelled in Section XIII.2 by means of *perturbed risk processes*, which are defined as the sum of a usual risk reserve process $\{R(t)\}$ and a stochastic perturbation process. Typically, perturbed risk processes belong to the class of *jump-diffusion processes.* Their sample paths change discontinuously from time to time since jump epochs and jump sizes are random. In the intervals between the jumps they behave like the sample paths of a diffusion process, see Section 7.2 below. This class of stochastic processes is one of the main subjects of financial mathematics.

Over recent years, a number of textbooks have appeared that provide introductions to the mathematical theory of finance. We refer to the bibliographical notes in Chapter XIII. An inclusion of all the technical material, needed to do full justice to this contemporary subject, would increase the size of our monograph substantially. On the other hand, we felt a need to at least include a streamlined introduction to the subject. We made a special attempt to show the clear links between the material already mentioned above and a subset of topics from the realm of financial stochastics.

7.2 Diffusion Processes

The description of the stochastic properties of stock prices, interest rates, etc. in terms of *diffusion processes* has been one of the great break throughs of stochastic thinking in a real-life context. It is by now generally accepted that an appropriate formulation for the time-evolution of a diffusion process $\{X(t)\}$ should be given in terms of a *stochastic differential equation* of the form

$$dX(t) = a(t, X(t))\,dt + \sigma(t, X(t))\,dW(t)\,.$$

Here, $\{W(t)\}$ is the usual *Brownian motion* which is at the core of stochastic analysis. Further, $a(t,x)$ takes care of the drift of the process while $\sigma(t,x)$ describes the strength of the extraneous fluctuations, caused by the Brownian motion. In Section XIII.1 we give an abridged treatment of stochastic differential equations and we show existing links with martingales and Markov processes.

For specific choices of $a(t,x)$ and $\sigma(t,x)$ we arrive at a wealth of possible models. Here is a more concrete example. A major step forward in the use of stochastic calculus in finance came from an attempt to price options on a security. This led to the popular Black-Scholes formula, for which the Nobel price 1997 was won. In fact, the use of the Black-Scholes model for option pricing, induced a change in the economy so that stock prices in liquid markets became very close to a Black-Scholes model. Up to 1972, the Black-Scholes model for option prices was a bad approximation to real prices. In the above terminology, the price process takes the form $X(t) = \mathrm{e}^{\delta t} X^*(t)$ where δ refers to the *force of interest*, while $\{X^*(t)\}$ satisfies the stochastic differential equation with the choice $a(t,x) = (\mu - \delta)\,x$ and $\sigma(t,x) = \sigma\,x$. The drift is regulated by the difference between the expected *rate of return* μ and the force of interest δ while the fluctuations are modeled by the *volatility* σ.

In Section XIII.3.1 we will explain how trading stategies in a market with two financial goods can be developed once the option is chosen. In particular, we consider the case of a *European call option* where the optionholder has the right (but not the obligation) to buy an asset for a fixed price at a fixed point in time.

7.3 Equity Linked Life Insurance

Section XIII.3.2 provides an inspiring link between the Black-Scholes model and stochastic modelling of life insurance.

From the Middle Ages, states and towns in Europe have been selling annuities. Life insurance mathematics arose in response to the need of evaluation of the price of these annuities. These calculations stimulated developments in different fields, like demography and probability theory. In London, early life tables were proposed by John Graunt in 1662. However it is considered that the first life table in modern sense has been constructed by E. Haley in 1693. He used data from Breslau in Silesia, collected by the pastor and scientist Casper Neumann through 1687-1691. For further historical details, the reader is referred to the survey of Hald (1987).

The subsequent mathematical modelling of life insurance is based on probability theory, since in our modern terminology, lifetimes are random variables. A classical life table contains the expected proportions of that part of a population of age a that reaches a later age c. If we use T_a as the stochastic notation for the remaining lifetime of a policyholder of age a, then the life tables give us the necessary means to estimate (sometimes using interpolations) the distribution $\mathbb{P}(T_a \le y) = \mathbb{P}(T \le y \mid T > a)$, where T is the typical life time of any member of the population.

Under a classical life insurance contract, the insured benefit typically consists of one, single payment - the sum insured. Other useful types of contracts in life insurance are life annuities, consisting of a series of payments which are made during the lifetime of the beneficiary. Furthermore, there are combinations of a life insurance contract with an annuity. The subsequent problem is then to follow such contracts from the instant of policy issue up to i.a. the death of the customer.

Take for instance the case where a customer of age a underwrites a classical *term insurance policy* where he gets the value $\max\{X(T_a), b\}$ at time T_a of his death. Note that this value is similar to the value of a European call option. But the payoff is now paid at a random time. Here, $b > 0$ is a guaranteed lower bound and $\{X(t)\}$ is a stochastic *price process* as described in Section 7.2. With the additional notions of discounting and interest, one is now equipped to develop a coherent theory of *equity linked life insurance* contracts. A detailed treatment of the above life insurance situation is given in Section XIII.3.2 in the case of a constant force of interest $\delta \ge 0$. As there is no obvious reason why the force of interest should be kept constant, the final Sections XIII.3.3 and XIII.4 cover a few possible forms of a stochastic force of interest $\{\delta(t)\}$ via appropriate stochastic differential equations.