On the exact asymptotics of the busy period in GI/G/1 queues.

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Abstract

In this paper we study the busy period in GI/G/1 work-conserving queues. We give the exact asymptotics of the tail distribution of the busy period under the light-tailed assumptions. We also study the workload process in the M/G/1 system conditioned to stay positive.

Keywords: GI/G/1 queue, busy period, workload process conditioned to stay positive, exponential change of measure, quasi-stationary distribution, exact asymptotics.

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1 Introduction

The main object studied in this paper is the busy period τ in single server queues operating under a work-conserving discipline (e.g. FCFS). Throughout the paper we assume that the traffic intensity $\rho < 1$, which yields that τ is finite a.s. and that the initial workload is x. Recall that the busy period is greater than t if and only if the workload process in the interval $[0, t]$ is strictly positive.

The case of the M/G/1 queue was considered by Kyprianou (1971), who studied the asymptotics of the tail probability of the busy period, where a service time has a meromorphic moment generating function $\hat{m}_B(\theta)$. Let λ be the arrival rate and $\kappa(s) = \lambda(\hat{m}_B(s) - 1) - s$. Kyprianou (1971) considered the light-tailed case, namely that there exists $\theta_0 > 0$ such that $\kappa^{'}(\theta_0) = 0$, or equivalently $\lambda \hat{m}_B^{'}(\theta_0) = 1$. Using analitical methods he proved that

$$\mathbb{P}_x(\tau > t) \sim h(x) e^{-\gamma t} t^{-3/2}, \tag{1}$$

for $t \to \infty$, where $h(x) = Cxe^{\theta_0 x}$ and

$$\gamma = -\kappa(\theta_0), \qquad C = \frac{1}{-\kappa(\theta_0)\sigma\sqrt{2\pi}}, \qquad \sigma^2 = \lambda \hat{m}_B^{''}(\theta_0). \tag{2}$$

A result of this type for the density function was first given in the Appendix of the book of Cox and Smith (1961), formula (46) on p. 154, and it was proved using the saddle-point method.

The main result of the paper is the exact asymptotics for the tail of the distribution function of the busy period in the GI/G/1 queue. Of course in this case we must first markovize the workload process by attaching the second component of the remaining arrival time process to get a Markovian setting and therefore in Section 2 in Theorem 2.1 we give the asymptotics of $\mathbb{P}_{(x,z)}(\tau > t)$, where x and z are the initial workload and remaining arrival time respectively. In the Corollary 2.1 we state a similar to (1) result for GI/G/1 work-conserving system. Recently a weaker logarithmic asymptotics in GI/G/1 queues was found by Mandjes and Zwart (2006), as a by product of their studies of GI/G/1 processor sharing queues.

In the light of Kyprianou (1971) studies, a naturally related concept is the workload process conditioned to stay positive. This is a counterpart notion of the corresponding ones like the random walk to stay positive (see e.g. Keener (1992) and Bertoin and Doney (1994)) or the Brownian motion to stay positive derived by Knight (1969). In Section 4 we study this concept for M/G/1 queue, completing Kyprionou (1971) studies.

2 Main result

We will introduce queueing notations used throughout the paper. Customers arrive at the instants $T_0, T_0+T_1, T_1+T_2, \ldots$, where $T_0, T_1 \ldots$ are inter-arrival times. If there is an arrival at 0, then we set $T_0 = 0$. The customer arriving at $\varrho_n := T_0 + \ldots + T_n$ brings its service time B_n. Let $S(t) = \sum_{j=1}^{N(t)} B_j$ be the cumulative service up to time t, where $N(t)$ denotes the number of all arrivals (included the one at zero if this is the case) by time t. That is,

$$N(t) = \sum_{j=0}^{\infty} \mathbb{I}(T_0 + \ldots + T_n \leq t).$$

We assume that random variables $T_0, T_1, T_2 \ldots$ are independent and $T_1, T_2, T_3 \ldots$ are i.i.d. random variables. This sequence of variables is independent of an i.i.d. sequence $B_1, B_2, \ldots$ with the non-lattice distribution F_B. Throughout the paper we assume also that generic $T = T_1$ and $B = B_1$ are light-tailed, that is there exists $\delta > 0$ such that

$$E_x e^{\delta T} < \infty, \qquad E_x e^{\delta B} < \infty. \tag{3}$$

Under $\mathbb{P}_x$ we define process $X(t) = x + S(t) - t$. Let $\mathbb{E}_x$ be a corresponding expectation. In any system working under a work-conserving discipline is

$$V(t) = \max_{0 \leq s \leq t} (X(t), X(t) - X(s)).$$

By the busy period we mean the following stopping time:

$$\tau = \min\{t : V(t) = 0\} = \min\{t : x + S(t) - t = 0\}.$$

Our permanent assumption is that τ is a proper random variable, that is $\rho = \mathbb{E}_x B / \mathbb{E}_x T < 1$. Remark that two processes $X(t)$ and $V(t)$ are identical within the first busy period. We denote

$$\lambda = \lim_{t \to \infty} \frac{N(t)}{t} \qquad \text{a.s.}$$

With the above queueing description in continuous time we associate the following random walk. For $\zeta_i = B_i - T_i$ $(i = 1, 2, \ldots)$, let

$$Z_0 = x, \quad Z_n = \sum_{k=1}^{n} \zeta_k, \quad k = 1, 2, \ldots \tag{4}$$

and

$$\nu = \min\{n : Z_n < 0\}. \tag{5}$$

Note that $\mathbb{E}_x \zeta < 0$ since $\rho < 1$. We also consider a Palm version of the problem with probability measure $\mathbb{P}^\circ_x$, under which $T_0 = 0$.

For a random variable Y we denote by $F_Y(\cdot)$ its distribution and by $\hat{m}_Y(s) = \mathbb{E}\exp(sY)$ its moment generating function. We also use notation $\hat{l}_\xi(s) = \int_0^\infty e^{-sx}\, \xi(dx)$ for the Laplace transform of a measure ξ which is not necessarily finite. We will write $k(t) \sim l(t)$ to denote that $\lim_{t\to\infty} k(t)/l(t) = 1$.

Let $\theta_{\mathrm{r}} = \sup\{\theta : \ \hat{m}_B(\theta) < \infty\} > 0$. The function $\kappa(\theta)$ is the unique solution of

$$\hat{m}_B(\theta)\hat{m}_T(-\kappa(\theta) - \theta) = 1 \tag{6}$$

in the interval $(-\infty, \theta_{\mathrm{r}})$. The function $\kappa(\theta)$ is convex and $\kappa^{'}(0) = \rho - 1$ (see Schmidli (1994); also Rolski *et al* (1999), Lemma 11.5.1). We assume that there exists $\theta_0 < \theta_{\mathrm{r}}$ such that

$$\kappa^{'}(\theta_0) = 0. \tag{7}$$

If the above condition togoether with (3) are fulfilled, then we say that the light-tailed case holds.

Unfortunately neither the process $V(t)$ nor $X(t)$ are Markovian unless the queue is the M/G/1 type. However, with the remaining time to the next arrival process $Z(t)$, where

$$Z(t) = \sum_{j=0}^{N(t)} T_j - t,$$

the process $(X(t), Z(t))$ is Markovian. We also set a pair of initial conditions $X(0) = x, Z(0) = z$, which will be indicated by $\mathbb{P}_{(x,z)}$.

Actually we consider the Markov process $(X(t), Z(t), t)$, which falls into the framework of PDMPs; see for the theory Davis (1993). The full generator has the form

$$\boldsymbol{A} f(x,z,t) = \frac{\partial}{\partial t} f(x,z,t) - \frac{\partial}{\partial x} f(x,z,t) - \frac{\partial}{\partial z} f(x,z,t),$$

where f is an absolute continuous function fulfilling the boundary condition

$$f(x,0,t) = \int_0^\infty \int_0^\infty f(x+y,z,t)\, \mathrm{d}F_B(y)\, \mathrm{d}F_T(z) \tag{8}$$

and that

$$\mathbb{E}\left(\sum_{i \le t} |\overline{f}(X(\tau \wedge \varrho_i)) - \overline{f}(X(\tau \wedge \varrho_i -))| \right) < \infty \tag{9}$$

for each $t \ge 0$.

We first show how to remove the drift. Let $\theta < \theta_r$. Consider a function $g(x,v,t) = k(z)e^{\theta x}e^{-\kappa(\theta)t}$. For g to be a harmonic function, we have to check that $\boldsymbol{A}g = 0$, and that the conditions (8) and (9) hold. This means that $k(v)$ solves $-\kappa(\theta)k(z) - \theta k(z) - k'(z) = 0$ and hence $k(z) = k(0)e^{-(\kappa(\theta)+\theta)z}$. We also find that the boundary condition (8) holds because $\kappa(\theta)$ is the solution of (6). Since $\theta < \theta_r$ we can show (9) following Rolski *et al* (1999), Ch. 11.3.1. Note that $\kappa^{'}(0) = \rho - 1 < 0$. Thus $g(x,z,t) = e^{\theta x - (\kappa(\theta)+\theta)z}e^{-\kappa(\theta)t}$ and the process

$$\mathcal{E}(t) = e^{-\theta x}e^{(\theta+\kappa(\theta))z}e^{\theta X(t)}e^{-(\theta+\kappa(\theta))Z(t)}e^{-\kappa(\theta)t}$$

is a mean-one exponential martingale defined by the function g.

We now introduce the new probability measure $\mathbb{P}_{(x,z);\theta}$ by

$$\mathrm{d}\mathbb{P}_{(x,z);\theta|t} = \mathcal{E}(t)\mathrm{d}\mathbb{P}_{(x,y)|t}, \qquad t \ge 0,$$

where $|t$ denotes the restriction of probability measures to $\mathcal{F}_t^{X,Z}$ – the σ-field generated by the process $(X(t), Z(t), t)$ up to time t. From Theorem 5.3 of Palmowski and Rolski (2002), under the new probability, we obtain that the distributions of T and B are

$$\begin{cases} \mathrm{d}F_T^\theta(v) \;=\; \frac{1}{\hat{m}_T(-\kappa(\theta)-\theta)} e^{-(\kappa(\theta)+\theta)v}\mathrm{d}F_T(v), \\ \\ \mathrm{d}F_B^\theta(v) \;=\; \frac{1}{\hat{m}_B(\theta)} e^{-\theta v}\mathrm{d}F_B(v) \end{cases} \tag{10}$$

and so

$$E_{(x,z);\theta}T = \frac{\hat{m}_T^{'}(-\kappa(\theta)-\theta)}{\hat{m}_T(-\kappa(\theta)-\theta)}, \quad E_{(x,z);\theta}B = \frac{\hat{m}_B^{'}(\theta)}{\hat{m}_B(\theta)}\ . \tag{11}$$

Taking the first derivative with respect to θ in (6) we obtain

$$\hat{m}_B^{'}(\theta)\hat{m}_T(-\kappa(\theta)-\theta) - \hat{m}_B(\theta)\hat{m}_T^{'}(-\kappa(\theta)-\theta) - \kappa^{'}(\theta)\hat{m}_B(\theta)\hat{m}_T^{'}(-\kappa(\theta)-\theta) = 0. \tag{12}$$

Using (11) and (12) we can prove that the workload process has zero drift under $\mathbb{P}_{(x,z);\theta_0}$.

Lemma 2.1 *The drift under the new probability*

$$E_{(x,z);\theta}B - E_{(x,z);\theta}T = \frac{\hat{m}_B^{'}(\theta)\hat{m}_T(-\kappa(\theta)-\theta) - \hat{m}_B(\theta)\hat{m}_T^{'}(-\kappa(\theta)-\theta)}{\hat{m}_B(\theta)\hat{m}_T(-\kappa(\theta)-\theta)}$$

is 0 if and only if $\kappa^{'}(\theta) = 0$.

Denote $\beta = -(\theta_0 + \kappa(\theta_0))$, which is negative by (6). Consider the random walk Z_n under the new probability $\tilde{\mathbb{P}}_x$, such that the distribution of the generic increment ζ is

$$\tilde{\mathbb{P}}_x(\zeta \in \Lambda) = r^{-1}\mathbb{E}^{\circ}_{x;\theta_0}\left[e^{\beta\zeta}; \zeta \in \Lambda\right], \tag{13}$$

where $r = \mathbb{E}^{\circ}_{x;\theta_0}e^{\beta\zeta}$. Note that $\tilde{\mathbb{E}}_x\zeta < 0$, since $\mathbb{E}^{\circ}_{x;\theta_0}Te^{-\beta T} > \mathbb{E}^{\circ}_{x;\theta_0}Be^{\beta B}$. Let $\tilde{\kappa}(\theta) = \log \tilde{\mathbb{E}}_x e^{\theta\zeta}$ and $\kappa_{\theta_0}(\theta) = \log \mathbb{E}_{x;\theta_0}e^{\theta\zeta}$. We have $\tilde{\kappa}(\theta) = \kappa_{\theta_0}(\theta+\beta) - \kappa_{\theta_0}(\beta)$ and $\tilde{\kappa}'(-\beta) = \kappa'_{\theta_0}(0) = 0$. Moreover,

$$r = e^{\kappa_{\theta_0}(\beta)} = e^{-\tilde{\kappa}(-\beta)} > 1.$$

Denote

$$H(x) = D\sum_{n=0}^{\infty}\mathbb{P}^{\circ}_{x;\theta_0}(L_n = n, Z_n \geq 0), \tag{14}$$

where

$$L_n = \min\{r \geq 0 : Z_r = \min_{k\leq n} Z_k\}$$

is a decreasing ladder epoch and

$$D = \frac{1}{-\beta\sqrt{2\pi\kappa^{''}_{\theta_0}(0)}}\frac{\tilde{\mathbb{E}}(r^{\nu}-1)}{r-1}. \tag{15}$$

Denote $\lambda(\theta) = (\mathbb{E}_{x;\theta}T)^{-1}$. We can now state the main result, whose the proof will be presented in the Section 3.

Theorem 2.1 *For all* $x \geq z \geq 0$, *as* $t \to \infty$,

$$\mathbb{P}_{(x,z)}(\tau > t) \sim \frac{(r-1)}{-\kappa(\theta_0)}e^{\theta_0 x}e^{-(\theta_0+\kappa(\theta_0))z}H(x-z)e^{\kappa(\theta_0)t}\frac{1}{\sqrt{\lambda(\theta_0)}t^{3/2}}\ .$$

In the following corollary after Theorem 2.1. we write $\mathbb{P}_{x;\theta}(\cdot)$ as the short for $\mathbb{P}_{x,z;\theta}(\cdot)\ \mathrm{d}F_{Z(0)}(z)$.

Corollary 2.1 *For all* $x > 0$,

$$\mathbb{P}_x(\tau > t) \sim h(x)e^{\kappa(\theta_0)t}t^{-3/2}, \tag{16}$$

as $t \to \infty$, where

$$h(x) = \frac{(r-1)}{-\kappa(\theta_0)} \frac{1}{\sqrt{\lambda(\theta_0)}} e^{\theta_0 x} \int_0^x e^{-(\theta_0+\kappa(\theta_0))z} H(x-z)\,\mathrm{d}F_{Z(0)}(z)\,. \tag{17}$$

We complete the section comparing (16) with (1) for the M/M/1 queue. In this case $F_{Z(0)}(\cdot)$ is exponentially distribution with parameter λ. The following argument demonstrates that indeed $h(x) = Ce^{\theta_0 x}x$, where C is defined in (2). Suppose that $B \stackrel{\mathcal{D}}{=} \mathrm{Exp}(\varphi)$ and $T \stackrel{\mathcal{D}}{=} \mathrm{Exp}(\lambda)$. Then $\theta_0 = \sqrt{\varphi}(\sqrt{\varphi}-\sqrt{\lambda})$ and $\kappa(\theta_0) = \varphi-\lambda-2\theta_0$. Hence we compute $\beta = -(\theta_0+\kappa(\theta_0)) = \sqrt{\lambda}(\sqrt{\lambda}-\sqrt{\varphi})$. Note that $\lambda-\beta = \sqrt{\varphi\lambda}$. Moreover, we have $B \stackrel{\mathcal{D}}{=} T \stackrel{\mathcal{D}}{=} \mathrm{Exp}(\sqrt{\varphi\lambda})$ under $\mathbb{P}_{x;\theta_0}$. Thus from Feller (1972), p. 405, we obtain $H(x) = C(1+\sqrt{\varphi\lambda}x)$ because, under $\mathbb{P}_{x;\theta_0}$, the downward ladder epoch is exponentially distributed with parameter $\sqrt{\lambda\varphi}$. Now straightforward calculations show that

$$\int_0^x e^{-(\theta_0+\kappa(\theta_0))z} H(x-z)\,\mathrm{d}F_{Z(0)}(z) = \lambda \int_0^x e^{-(\lambda-\beta)z} H(x-z)\,\mathrm{d}z = D\lambda x\,,$$

where constant D is given in (15). Thus it suffices to demonstrate that

$$\frac{(r-1)}{-\kappa(\theta_0)} \frac{1}{\sqrt{\lambda(\theta_0)}} \lambda D = C$$

which is equivalent to

$$\frac{\tilde{\mathbb{E}}(r^\nu - 1)}{-\beta\sqrt{\kappa''_{\theta_0}(0)}} \frac{\lambda}{\sqrt{\lambda(\theta_0)}} = \frac{1}{\sigma}\,, \tag{18}$$

where σ is defined in (2). Now (10) yields:

$$\mathrm{Var}_{x;\theta_0} S(1) = \lambda(\theta_0)\mathbb{E}_{\theta_0} B^2 = \lambda \hat{m}_B(\theta_0) \frac{\hat{m}''_B(\theta_0)}{\hat{m}_B(\theta_0)} = \sigma^2$$

and hence $\sigma^2 = \lambda(\theta_0)\frac{2}{\lambda\varphi}$. Moreover, by (13) we get:

$$\begin{aligned}
\tilde{\mathbb{E}} r^\nu &= \sum_{n=1}^\infty r^n \tilde{\mathbb{P}}(\nu = n) \\
&= \sum_{n=1}^\infty r^n r^{-n} \mathbb{P}^\circ_{\theta_0}\left[e^{\beta Z_n}; \nu = n\right] = \mathbb{E}^\circ_{\theta_0} e^{\beta Z_\nu} = \\
&= \sqrt{\lambda\varphi} \int_0^\infty e^{-\beta x} e^{-\sqrt{\lambda\varphi}x}\, dx = \frac{\sqrt{\lambda\varphi}}{\sqrt{\lambda\varphi}+\beta} = \sqrt{\frac{\varphi}{\lambda}}\,.
\end{aligned}$$

By inspection we check that $\kappa''_{\theta_0}(0) = \frac{2}{\lambda\varphi}$. This completes the proof of (18).

3 Proof of Theorem 2.1

We will use the following properties of a class of regularly varying discrete sequences whose proofs the reader can find in Chover *et al* (1973).

Lemma 3.1 *Let* $A_i = \sum_{n=1}^{\infty} a_n^{(i)}\,(i = 2, 3)$. *If* $a_n^{(1)} \sim a_1 n^{-1/2}$, $a_n^{(2)} \sim a_2 n^{-3/2}$ *and* $a_n^{(3)} \sim a_3 n^{-3/2}$, *then* $(a^{(1)} * a^{(3)})_n \sim a_1 A_3 n^{-1/2}$ *and* $(a^{(2)} * a^{(3)})_n \sim (a_2 A_3 + a_3 A_2) n^{-3/2}$.

A more detailed analysis that one can find in Doney (1989) and Iglehart (1974) allows to prove the following lemma.

Lemma 3.2 *For some function* $c(x)$,

$$\tilde{\mathbb{P}}_x(\nu > n) = H(x) r^{-n} n^{-3/2} \left(1 + \frac{c(x)}{n} + \mathrm{o}(n^{-1})\right), \tag{19}$$

as $n \to \infty$.

Proof Let S_n be a random walk, where $S_n = -(Z_n - x)$ for Z_n defined in (4). It starts from 0 and drifts to $+\infty$. Let $M_n = \max_{k \le n} S_k$. To prove the lemma we have to generalize Theorem II of Doney (1989) (see also Theorem 2.1 of Iglehart (1974)). Basically we need to generalize Lemma 4(ii) of Doney (1989) and prove that

$$\begin{aligned} a_n(t) &= \frac{1}{n}\left[E(e^{-tS_n}; S_n > 0) + P(S_n \le 0)\right] \\ &= \frac{1}{-\beta\sqrt{2\pi\tilde{\kappa}''(-\beta)}} \frac{t}{t-\beta} e^{\kappa(\theta_0)n} n^{-3/2}\left[1 + \frac{Q(t)}{n} + \mathrm{o}(n^{-1})\right] \end{aligned} \tag{20}$$

for some function $Q(t)$. Indeed, we start from Spitzer's identity:

$$\sum_{n=0}^{\infty} s^n E(e^{-tM_n}) = \exp\left\{\sum_{n=1}^{\infty} s^n a_n(t)\right\}.$$

Using Lemma 3.1 and following ideas from the proof of Theorem 2.1 of Iglehart (1974) (see also Lemma 1 of Doney (1989)) in the next step we obtain

$$E(e^{-tM_n}) = \exp\left\{\sum_{n=1}^{\infty} e^{-\kappa(\theta_0)n} a_n(t)\right\} \frac{1}{-\beta\sqrt{2\pi\tilde{\kappa}''(-\beta)}} \frac{t}{t-\beta} e^{\kappa(\theta_0)n} n^{-3/2}\left[1 + \frac{Q_1(t)}{n} + \mathrm{o}(n^{-1})\right],$$

where $Q_1(t)$ is a certain function. Then similarly as in the proof of Theorem II of Doney (1989) we derive (19) with a function $c(x)$ of the form uninteresting for us and

$$H(x) = De^{-\beta x}\sum_{n=0}^{\infty} r^n \tilde{\mathbb{E}}_0\left[e^{\beta S_n}; L_n^S = n, 0 \le S_n \le x\right], \tag{21}$$

where L_n^S are increasing ladder epochs of the random walk S_n and in calculating constant D we use the identity $\tilde{\kappa}''(-\beta) = \kappa_{\theta_0}''(0)$. The function $H(x)$ given in (21) can be, after some work, transformed to

$$H(x) = D\sum_{n=0}^{\infty} r^n \tilde{\mathbb{E}}_x\left(e^{-\beta Z_n}; L_n = n, Z_n \ge 0\right).$$

Next using (13), we can write the function in the form (14). Finally, to demonstrate (20) we follow the proof of Lemma 4(ii) of Doney (1989). However we use the asymptotics:

$$P(S_n \le x) = \frac{1}{\theta_1\sqrt{2\pi\tilde{\kappa}''(\theta_1)}} e^{x\theta_1} e^{\kappa(\theta_1)n} n^{-1/2}\left(1 + d(x)\frac{1}{n} + \mathrm{o}(n^{-1})\right) \tag{22}$$

which holds uniformly in $x \le \frac{1}{2}nES_1$, instead of (2.8) of Doney (1989), where θ_1 is a solution of $\kappa'(\theta) = x/n$ and $d(x)$ is a certain function. The proof of (22) follows from the proof of Theorem 1 of Petrov (1965) using Theorem VI.3.4 of Petrov (1972) instead of (4.12) therein. This completes the proof of lemma. $\square$

Since under the Palm probability we have $Z(0) = 0$, we may write $\mathbb{P}^{\circ}_{(x,0);\theta_0} = \mathbb{P}^{\circ}_{x;\theta_0}$. Recall that under $\mathbb{P}^{\circ}_{x;\theta_0}$ random walk Z_n defined in (4) oscillates. We have $\tau = \inf\{s \ge 0 : X(s) < 0\} > t$ if and only if $\nu > N(t)$, where $N(t)$ is a number of customers arrived up to time t. By the law of large numbers,

$$\lim_{t\to\infty}\frac{N(t)}{t} = \lambda(\theta_0) \quad \mathbb{P}^{\circ}_{x;\theta_0}\text{-a.s.}$$

Furthermore, by the central limit theorem (see e.g. Asmussen (2003) or Theorem IX.4.28 of Petrov (1972)), for each $\delta > 0$,

$$\mathbb{P}^{\circ}_{x;\theta}\left(\left|\frac{N(t)}{t} - \lambda(\theta_0)\right| > \delta\right) = \mathrm{o}(t^{-3/2})$$

since $Ee^{\varsigma T} < \infty$ for some $\varsigma > 0$. Note also that $Z(\tau) = -Z_\nu$.

Lemma 3.3 *For $t \to \infty$ and $c_1 = r/(2(r-1))$ we have:*

$$\mathbb{E}^{\circ}_{x;\theta_0}\left[e^{-\beta Z(\tau)}; \tau > t\right] = \mathbb{E}^{\circ}_{x;\theta_0}\left[e^{\beta Z_\nu}; \nu > \lambda(\theta_0)t\right] + \mathrm{o}(t^{-3/2})$$

$$= \frac{2H(x)(r-1)}{\sqrt{\lambda(\theta_0)t}}\left(1 + c_1(\lambda(\theta_0)t)^{-1} + \mathrm{o}(t^{-1})\right).$$

Proof We prove the first asymptotic equivalence. For the upper bound we have the estimation

$$\begin{aligned}\mathbb{E}^{\circ}_{x;\theta_0}\left[e^{-\beta Z(\tau)};\tau > t\right] &\le \mathbb{E}^{\circ}_{x;\theta_0}\left[e^{-\beta Z(\tau)};\tau > t;\ \left|\frac{N(t)}{t} - \lambda(\theta_0)\right| < \delta\right] + \mathrm{o}(t^{-3/2})\\ &\le \mathbb{E}^{\circ}_{x;\theta_0}\left[e^{\beta Z_\nu};\nu > (\lambda(\theta_0)-\delta)t\right] + \mathrm{o}(t^{-3/2})\ .\end{aligned}$$

For the lower bound we have,

$$\begin{aligned}\mathbb{E}^{\circ}_{x;\theta_0}\left[e^{-\beta Z(\tau)};\tau > t\right] &\ge \mathbb{E}^{\circ}_{x;\theta_0}\left[e^{-\beta Z(\tau)};\tau > t;\ |\frac{N(t)}{t} - \lambda(\theta_0)| < \delta\right] - \mathrm{o}(t^{-3/2})\\ &\ge \mathbb{E}^{\circ}_{x;\theta_0}\left[e^{\beta Z_\nu};\nu > (\lambda(\theta_0)+\delta)t\right] - \mathrm{o}(t^{-3/2})\ .\end{aligned}$$

Passing with $\delta \to 0$ completes the proof of the first part. To prove the second asymptotic equivalence, remark that:

$$\begin{aligned}\mathbb{E}^{\circ}_{x;\theta_0}\left[e^{\beta Z_\nu};\nu > n\right] &= \sum_{k>n}\mathbb{E}^{\circ}_{x;\theta_0}\left[e^{\beta Z_k};\nu = k\right]\\ &= \sum_{k>n} r^k\tilde{\mathbb{P}}_x(\nu = k) = \tilde{\mathbb{E}}_x\left[r^\nu;\nu > n\right]\ .\end{aligned}$$

From $\tilde{\mathbb{P}}_x(\nu > n) = \sum_{k>n}\tilde{\mathbb{P}}_x(\nu = k)$ and Lemma 3.2 we conclude that

$$\tilde{\mathbb{P}}_x(\nu = n) = (r-1)H(x)r^{-n}n^{-3/2}\left(1 + c_0 n^{-1} + \mathrm{o}(n^{-1})\right)$$

for $c_0 = 3r/(2(r-1))$. Similarly, since $\tilde{\mathbb{E}}_x\left[r^\nu;\nu > n\right] = \sum_{k>n} r^k\tilde{\mathbb{P}}_x(\nu = k)$, we derive

$$\tilde{\mathbb{E}}_x\left[r^\nu;\nu > n\right] = 2(r-1)H(x)n^{-1/2}\left(1 + c_1 n^{-1} + \mathrm{o}(n^{-1})\right).$$

This completes the proof. □

Lemma 3.4

$$\begin{aligned}\mathbb{P}_{(x,z)}(\tau > t) &= -\kappa(\theta_0)e^{\theta_0 x}e^{-(\theta_0+\kappa(\theta_0))z}e^{\kappa(\theta_0)t} \\ &\times\int_0^\infty e^{\kappa(\theta_0)v}\left(\mathbb{E}_{(x,z);\theta_0}\left[e^{(\theta_0+\kappa(\theta_0))Z(\tau)};\tau > t\right] - \mathbb{E}_{(x,z);\theta_0}\left[e^{(\theta_0+\kappa(\theta_0))Z(\tau)};\tau > t+v\right]\right)dv\ .\end{aligned} \tag{23}$$

Proof The standard arguments based on the likelihood ratio identity like in Lemma 10.2.2 of Rolski *et al* (1999) (see also the section on the change of measure in Asmussen (2003)) yields

$$\mathbb{P}_{(x,z)}(\tau > t) = e^{\theta_0 x}e^{-(\theta_0+\kappa(\theta_0))z}\mathbb{E}_{(x,z);\theta_0}\left[e^{(\theta_0+\kappa(\theta_0))Z(\tau)}e^{\kappa(\theta_0)\tau};\tau > t\right]\ .$$

Integrating by part completes the proof. □

From Lemma 3.4 we immediately obtain a Lundberg type bounds for $\mathbb{P}_{(x,z)}(\tau > t)$.

Corollary 3.1 *For all* $t \geq 0$,

$$\mathbb{P}_{(x,z)}(\tau > t) \leq e^{\theta_0 x} e^{-(\theta_0+\kappa(\theta_0))z} e^{\kappa(\theta_0)t} .$$

Proof Rewrite (23) in the form

$$\mathbb{P}_{(x,z)}(\tau > t) = -\kappa(\theta_0) e^{\theta_0 x} e^{-(\theta_0+\kappa(\theta_0))z} e^{\kappa(\theta_0)t} \\ \times \int_0^\infty e^{\kappa(\theta_0)v} \mathbb{E}_{(x,z);\theta_0} \left(e^{(\theta_0+\kappa(\theta_0))Z(\tau)}; t+v > \tau > t \right) dv$$

and since $\theta_0 + \kappa(\theta_0) > 0$ and $Z(\tau) \leq 0$ the inequality follows. □

In the proof of the main result we will also use the following technical lemma.

Lemma 3.5 *If* $f(t) = A't^{-1/2} + Bt^{-3/2} + o(t^{-3/2})$, *for some* $A' \neq 0$, *then*

$$f(t+v) - f(t) = -v\frac{A'}{2}t^{-3/2} + o(t^{-3/2}).$$

Proof Since $(1+s)^\alpha = 1 + \alpha s + o(s)$ we have

$$\begin{aligned} & A'(t+v)^{-1/2} + B(t+v)^{-3/2} + o(t^{-3/2}) - A't^{-1/2} + Bt^{-3/2} + o(t^{-3/2}) \\ &= A'((t+v)^{-1/2} - t^{-1/2}) + B((t+v)^{-3/2} - t^{-3/2}) + o(t^{-3/2}) \\ &= \frac{-A'v}{2} t^{-3/2} + o(t^{-3/2}) . \end{aligned}$$

□

Proof of Theorem 2.1. In the proof we use Lemmas 3.3, 3.4 and 3.5 and the dominated convergence theorem, which yields:

$$-\kappa(\theta_0) \int_0^\infty e^{\kappa(\theta_0)v} \left(\mathbb{E}_{(x,z);\theta_0} \left[e^{(\theta_0+\kappa(\theta_0))Z(\tau)}; \tau > t \right] - \mathbb{E}_{(x,z);\theta_0} \left[e^{(\theta_0+\kappa(\theta_0))Z(\tau)}; \tau > t+v \right] \right) dv \\ \sim \frac{H(x-z)(r-1)}{\sqrt{\lambda(\theta_0)}} t^{-3/2} (-\kappa(\theta_0)) \int_0^\infty v e^{\kappa(\theta_0)v}\, dv = \frac{H(x-z)(r-1)}{-\kappa(\theta_0)\sqrt{\lambda(\theta_0)}} t^{-3/2}, \qquad \text{for } t \to \infty.$$

□

4 Workload of the M/G/1 queue conditioned to stay positiveTR

Assume that $T_0, T_1, T_2, \ldots$ are exponentially distributed with parameter $\lambda > 0$. In this case $\kappa(s) = \lambda(\hat{m}_B(s) - 1) - s$ and throughout this section we assume the light-tailed case. We will study workload process $V(t)$ (or equivalently process $X(t)$) in the case of M/G/1 system conditioned to stay positive. Clearly the process $X(t)$ is Markovian. We denote by $X^\dagger(t)$ the process $X(t)$ killed at the exit from $(0, \infty)$, that is

$$X^\dagger(t) = \begin{cases} X(t), & t < \tau, \\ 0, & t \geq \tau. \end{cases}$$

Lemma 4.1 *Function $h(x) = xe^{\theta_0 x}$ is γ-harmonic for $X^\dagger(t)$, where $\gamma = -\kappa(\theta_0)$.*

Proof Note that $X(t)$ is a spectrally positive Lévy process. Using the Wald martingale $\mathcal{E}(t) = e^{-\theta_0 x} e^{\theta_0 X(t) - \gamma t}$ (see e.g. Kyprianou and Palmowski (2004)), function $h(x) = xe^{\theta_0 x}$ is $(\gamma, \mathbb{P}_x)$-harmonic for $X^\dagger(t)$ if and only if function x is $\mathbb{P}_{x;\theta_0}$-harmonic for $X^\dagger(t)$ if and only if process $X^\dagger(t)$ is a $\mathbb{P}_{x;\theta_0}$-martingale. Note that under $\mathbb{P}_{x;\theta_0}$ process $X(t)$ has zero drift. Moreover, the dual process $\hat{X}(t) = -X(t)$ is upward creeping (has no positive jumps). From Theorem 19, p. 174 of Bertoin (1996) (see also p. 191 there) the renewal function $\hat{\mathcal{V}}(x)$ of $\hat{X}(t)$ is then equal up to the constant to x. To complete the proof it suffices then to demonstrate that $\hat{\mathcal{V}}(X^\dagger(t)) = cX^\dagger(t)$ is a $\mathbb{P}_{x;\theta_0}$-martingale. This is straightforward from Problem 7, p. 184 of Bertoin (1996) since $\hat{\hat{X}}(t) = X(t)$ and the dual process $\hat{X}(t)$ has also zero drift. □

From the above lemma we may define process $X^\uparrow(t)$, which is the process $X(t)$ conditioned to stay positive, by the exponential change of measure using the martingale $h(X^\dagger(t))$ with the good function $h(x) = x\exp(\theta_0 x)$ for $x > 0$ and $h(0) = 0$. The details can be found in Palmowski and Rolski (2004). In the next proposition we identify parameters of $X^\uparrow(t)$.

Proposition 4.1 *An M/G/1 workload process conditioned to stay positive (or equivalently process $X(t)$ to stay positive) is a piecewise deterministic Markov process (PDMP) with the extended generator of the form*

$$\boldsymbol{A}^\uparrow f(x) = -f'(x) + \lambda^\uparrow(x) \int_x^\infty (f(y) - f(x))\, Q^\uparrow(x, dy),$$

where

$$\lambda^{\uparrow}(x) = \frac{\lambda e^{\theta_0 x}[\frac{1}{\lambda} + x\hat{m}_B(\theta_0)]}{xe^{\theta_0 x}} = \frac{1}{x} + \lambda \hat{m}_B(\theta_0),$$

$$Q^{\uparrow}(x, dy) = \frac{ye^{\theta_0 y}}{e^{\theta_0 x}[\frac{1}{\lambda} + x\hat{m}_B(\theta_0)]} F_B(dy - x).$$

Proof Following Theorem 3.1 from Palmowski and Rolski (2004), under the new probability measure, the $M/G/1$ workload process is a PDMP with parameters:

$$\lambda^{\uparrow}(x) = \frac{\lambda \bar{G}(x)}{h(x)} \quad \text{and} \quad Q^{\uparrow}(x, dy) = \frac{h(y)}{\bar{G}(x)} F_B(dy - x),$$

where the normalization function is

$$\begin{aligned}
\bar{G}(x) &= \int_x^{\infty} h(y)\, F_B(dy - x) = \int_x^{\infty} ye^{\theta_0 y}\, F_B(dy - x) \\
&= \int_0^{\infty} (y + x) e^{\theta_0 (y+x)}\, F_B(dy) \\
&= e^{\theta_0 x}[\int_0^{\infty} ye^{\theta_0 y} F_B(dy) + x \int_0^{\infty} e^{\theta_0 y}\, F_B(dy)] \\
&= e^{\theta_0 x}[\hat{m}_B^{'}(\theta_0) + x\hat{m}_B(\theta_0)] \\
&= e^{\theta_0 x}[\frac{1}{\lambda} + x\hat{m}_B(\theta_0)],
\end{aligned}$$

because $\kappa^{'}(\theta_0) = 0$ implies $\hat{m}_B^{'}(\theta_0) = 1/\lambda$. Now the formulas for $\lambda^{\uparrow}(x)$ and $Q^{\uparrow}(x, dy)$ immediately follows. □

Note that $X^{\uparrow}(t)$ is transient. Indeed, for $0 < a < x < b$, denote $\tau_a^- = \inf\{t \geq 0 : X^{\uparrow}(t) < a\}$ $\tau_b^+ = \inf\{t \geq 0 : X^{\uparrow}(t) > b\}$. Then $\mathcal{E}^{\uparrow}(t \wedge \tau_a^- \wedge \tau_b^+)$ is uniformly integrable martingale. Hence using the optional stopping theorem we have

$$\mathbb{P}_x^{\uparrow}(\tau_a^- < \tau_b^+) = \frac{a}{x} e^{\theta_0 (a-x)} \mathbb{P}_x(\tau_a^- < \tau_b^+)\ .$$

Taking $b \to \infty$ and keeping in mind that $\mathbb{P}_x(\tau_a^- < \infty) = 1$, we obtain:

$$\mathbb{P}_x^{\uparrow}(\tau_a^- < \infty) = \frac{a}{x} e^{\theta_0 (a-x)} < 1\ .$$

By Kyprianou (1971) we know that the QS distribution μ exists for the M/G/1 queue with the service time having a meromorphic moment generating function and

$$\hat{m}_{\mu}(t) = \frac{\theta_0 + \lambda(1 - \hat{m}_B(\theta_0))}{\theta_0 - t + \lambda(\hat{m}_B(t) - \hat{m}_B(\theta_0))}.$$

Recently Kyprianou and Palmowski (2006) proved the existence of QS distribution for a more general model than considered here, where $X(t)$ is a Lévy process with light-tailed jumps. Thus, following Proposition 4.1 of Palmowski and Rolski (2004) we know that there exists an invariant measure π (which is not finite in this case) for the transition function $p_t^{\uparrow}(x, B)$ of $X^{\uparrow}(t)$. That is π fulfills

$$\int_A \pi(dx) p_t^{\uparrow}(x, B) = \pi(B) \tag{24}$$

for all bounded Borel subsets B. Furthermore, we have $\int_0^\infty \frac{1}{x} e^{-sx}\ \pi(dx) < \infty$ for $s > 0$. Note that from (24) it follows that $\int_A \pi(dx) \boldsymbol{A}^{\uparrow} f(x) = 0$ for all functions f from the domain of $\boldsymbol{A}^{\uparrow}$ such that the integration has sense and leads to a finite value. In the following corollary we obtain the Laplace-Stieltjes transform $\hat{l}_\pi(s)$ of the invariant measure π.

Corollary 4.1 *Process $X^{\uparrow}(t)$ has invariant measure π given up to the multiplication constant by its Laplace transform:*

$$\hat{l}_\pi(s) = \frac{(\theta_0 + \lambda(1 - \hat{m}_B(\theta_0)))(\lambda \hat{m}_B'(\theta_0 - s) - 1)}{(s + \lambda(\hat{m}_B(\theta_0 - s) - \hat{m}_B(\theta_0)))^2}; \quad s > 0. \tag{25}$$

Proof For $f(x) = e^{-sx}$, $x > 0$, $s > 0$ we compute

$$(\boldsymbol{A}^{\uparrow} f)(x) = se^{-sx} + \lambda \int_0^\infty (e^{(\theta_0 - s)y} - e^{\theta_0 y}) \frac{y + x}{x} e^{-sx} F_B(dy).$$

We now integrate both the sides with respect to $\pi(dx)$ to obtain

$$\begin{aligned}
&\int_0^\infty (\boldsymbol{A}^{\uparrow} f)(x)\, \pi(dx) = \\
&= s\hat{l}_\pi(s) + \lambda \int_0^\infty y(e^{(\theta_0 - s)y} - e^{\theta_0 y}) F_B(dy) \int_0^\infty \frac{1}{x} e^{-sx} \pi(dx) \\
&+ \lambda \int_0^\infty (e^{(\theta_0 - s)y} - e^{\theta_0 y}) F_B(dy) \hat{l}_\pi(s).
\end{aligned}$$

If π is an invariant measure, then $\int_0^\infty (\boldsymbol{A}^{\uparrow} f)(x)\, \pi(dx) = 0$ (see Davis (1993)). Hence

$$s\hat{l}_\pi(s) + \lambda(\hat{m}_B'(\theta_0 - s) - \hat{m}_B'(\theta_0)) \int_0^\infty \frac{1}{x} e^{-sx}\ \pi(dx) + \lambda(\hat{m}_B(\theta_0 - s) - \hat{m}_B(\theta_0))\hat{l}_\pi(s) = 0,$$

for $s > 0$. Using now that

$$\int_0^\infty \frac{1}{x} e^{-sx}\ \pi(dx) = \int_s^{\theta_0} \hat{l}_\pi(t)\, dt + \int_0^\infty \frac{1}{x} e^{-\theta_0 x}\ \pi(dx),$$

which is finite, we have

$$\frac{-\hat{l}_\pi(s)}{\int_s^{\theta_0}\hat{l}_\pi(t)\,dt+\int_0^\infty \frac{1}{x}e^{-\theta_0 x}\,\pi(dx)}=-\frac{1-\lambda\hat{m}_B'(\theta_0-s)}{s+\lambda(\hat{m}(\theta_0-s)-\hat{m}_B(\theta_0))}.$$

Hence we get

$$\int_s^{\theta_0}\hat{l}_\pi(t)\,dt+\int_0^\infty \frac{1}{x}e^{-\theta_0 x}\,\pi(dx)=\frac{c}{s+\lambda(\hat{m}_B(\theta_0-s)-\hat{m}_B(\theta_0))}$$

for some constant c. Choosing the multiplicative constant in π so that

$$\int_0^\infty \frac{1}{x}e^{-\theta_0 x}\,\pi(dx)=1$$

we can derive $c=\theta_0-\lambda\hat{m}_B(\theta_0)+\lambda$, and

$$\int_t^{\theta_0}\hat{l}_\pi(s)\,ds=\frac{\theta_0-\lambda\hat{m}_B(\theta_0)+\lambda}{t+\lambda(\hat{m}_B(\theta_0-t)-\hat{m}_B(\theta_0))}-1.$$

Differentiation completes the proof. □

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