# Ross type conjectures on monotonicity of queues

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**Abstract**

In his seminal paper from 1978, Ross set up a few conjectures which formalize a common belief that more variable arrival processes lead to a worse performance in queueing systems. We study these types of problems for Cox/GI/1/$\infty$, Cox/GI/$\infty$/$\infty$, and Cox/GI/1/0 systems. Assumptions are stated in terms of $\le_{\rm idcx}$-regularity. For example, in the class of stationary Markov processes, the regularity property holds under doubly stochastic monotonicity assumption. A special case is a result of Daley (1968) on the decreasing covariance function for stochastically monotone stationary Markov processes.

## 1 Introduction

It is a common sense to believe that, in queueing systems, it should be a general tendency that more variable arrival processes lead to a worse performance. In 1978, Sheldon Ross tried to formalize this belief into a set of conjectures for some single server and loss systems and verified these conjectures for a few special cases. In the case of single server queues, the conjecture was first proved by Rolski (1981) and Chang *et al* (1991). In this note, we show a few refinements of old results in this area and a few new ones concerning other types of queueing systems, like infinite server systems or single server loss systems. For this, we study a notion of $\le_{\rm sm}$ or weaker $\le_{\rm idcx}$-regularity for stationary stochastic processes, which the notion, in the case of stationary Markov processes, is fulfilled under double stochastic monotonicity assumption. The concept of stochastically monotone Markov chains was introduced by Daryl Daley (1968). We give a few examples of $\le_{\rm sm}$-regular stationary processes, but we lack nontrivial examples besides Markov processes.

Throughout this note, the input to queues is always a stationary Cox process $A$ which is formally introduced as follows. Let $A$ denote the point process on $\mathbb{R}$ representing the number of arrivals, that is,

$$A(B) = \sum_{n\in\mathbb{Z}} 1_B(T_n), \quad B \in \mathcal{B}(\mathbb{R}),$$

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where $\{T_n\}_{n\in\mathbb{Z}}$ denotes the sequence of arrival epochs, satisfying $\cdots < T_0 \leq 0 < T_1 < \cdots$, and $1_B$ denotes the indicator function for the set $B$. We assume that $A$ is a Cox process with stationary and ergodic intensity process $\{\lambda(t)\}_{t\in\mathbb{R}}$, where the sample paths of $\{\lambda(t)\}_{t\in\mathbb{R}}$ are a.s. Riemann integrable (see Daley and Vere–Jones (1988)). A special case of interest is the stationary Poisson process with intensity $\bar{\lambda}$ and we compare it with Cox processes such that $\bar{\lambda} = \mathrm{E}\lambda(0)$. The sequence of service times $\{S_n\}_{n\in\mathbb{Z}}$ is assumed to be independent from the arrival process $A$ and consists of independent and identically distributed (i.i.d.) nonnegative random variables, where $S_n$ represents the service time of the $n$-th customer. Such systems we denote by Cox/GI/$n/m$, where $1 \leq n \leq \infty$ is a number of channels and $0 \leq m \leq \infty$ is the capacity of the waiting room. In particular, when $A$ is the stationary Poisson process with intensity $\bar{\lambda}$, we then write M/GI/$n/m$. Depending on the system, we will consider a performance characteristic of interest $\Phi = \Phi(\lambda(\cdot))$ for general Coxian input, $\Phi(\bar{\lambda})$ for the Poissonian input with constant intensity $\bar{\lambda} = \mathrm{E}\lambda(0)$ and $\Phi(\lambda^*)$ for mixed Poisson input with fixed but random intensity $\lambda^* = \lambda(0)$. It was an original Ross (1978) idea to compare a system with Coxian arrivals versus the *associated system* with Poissonian arrivals with the same mean arrival rate and the same service process. Another kind of Ross type theorem needs a definition of a class of more variable input. Thus, for a stationary stochastic process $X(t)$ and $c > 0$, we define the stochastic process $\{X_c(t)\}_{t\in\mathbb{R}}$ by $X_c(t) = X(ct)$ for $t \in \mathbb{R}$. Note that $\{X_c(t)\}$ changes faster as $c$ tends large while, under the stationarity assumption, the mean of $X_c(t)$ remains the same. Let $\prec$ be an ordering for random variables. In this note, we will consider the following monotonicity problems: $\Phi(\lambda_c(\cdot))$ is $\prec$-decreasing in $c$. We then have the following two extreme cases: $\Phi(\lambda_c(\cdot)) \xrightarrow{\mathrm{d}} \Phi(\bar{\lambda})$, $c \to \infty$, and $\Phi(\lambda_c(\cdot)) \xrightarrow{\mathrm{d}} \Phi(\lambda^*)$, $c \to 0$, where $\xrightarrow{\mathrm{d}}$ denotes the convergence in distribution. For example, we prove that for Cox/GI/1/$\infty$ with the arrival intensity $\{\lambda_c(t)\}$, if the stationary workload $W(c)$ is considered, it is sufficient to assume that the arrival intensity process is $\leq_{\mathrm{idcx}}$-regular for $\leq_{\mathrm{icx}}$-monotonicity of $W(c)$.

# 2 Ordering and monotonicity

A good reference for this section is a recent book by Müller and Stoyan (2002). Throughout this note, we use "increasing" and "decreasing" in the non-strict sense.

## 2.1 Preliminaries

We give definitions of orderings and classes of functions related to orderings.

**Definition 1** (i) A function $f\colon \mathbb{R}^k \to \mathbb{R}$ is said to be *supermodular (sm)* if for any $x$ and $y \in \mathbb{R}^k$,

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y),$$

where the operators $\wedge$ and $\vee$ denote, respectively, coordinate-wise minimum and maximum.

(ii) A function $f$: $\mathbb{R}^k \to \mathbb{R}$ is said to be *directionally convex (dcx)* if for any $x_1$, $x_2$ and $y \in \mathbb{R}^k$ such that $x_1 \le x_2$ and $y \ge 0$,

$$f(x_1 + y) - f(x_1) \le f(x_2 + y) - f(x_2).$$

If sm and dcx functions are increasing, then we write ism and idcx, respectively. Note that a function $f$ is dcx if and only if it is coordinate-wise convex and supermodular (see Shaked and Shanthikumar (1990)).

**Example 1** The following are idcx functions important in the paper.

- $f$: $\mathbb{R}^k \to \mathbb{R}$ defined by $f(x_1, \dots, x_k) = \max(0, x_1 - s_1, x_1 + x_2 - s_2, \dots, x_1 + \dots + x_k - s_k)$, where $s_1, \dots, s_k$ are constants,
- if $f$: $\mathbb{R}^k \to \mathbb{R}$ is idcx and $c_{ij}$ are positive numbers, then $\hat{f}$: $\mathbb{R}^{l \times k} \to \mathbb{R}$ defined by $\hat{f}(x_{11}, \dots, x_{kl}) = f(\sum_{j=1}^{l} c_{1j} x_{1j}, \dots, \sum_{j=1}^{l} c_{kj} x_{kj})$ is idcx,
- if $\phi$: $\mathbb{R}_+^k \to \mathbb{R}$ is idcx, then $\psi$: $\mathbb{R}_+^{k+1} \to \mathbb{R}$ defined by $\psi(x_0, x_2, \dots, x_k) = x_0 \phi(x_1, \dots, x_k)$ is idcx.

Important properties of dcx functions are as follows. For the proof, see Shaked and Shanthikumar (1990), Rolski (1986), and Meester and Shanthikumar (1993).

**Lemma 1** *(i) Let $\{S_n^{(i)}\}_{n \in \mathbb{Z}}$, $i = 1, \dots, k$, denote independent sequences of i.i.d. non-negative random variables. If $f$: $\mathbb{R}^k \to \mathbb{R}$ is idcx, then $\phi$: $\mathbb{Z}_+^k \to \mathbb{R}$ defined by $\phi(n_1, \dots, n_k) = \mathrm{E}\big[f\big(\sum_{j=1}^{n_1} S_j^{(1)}, \dots, \sum_{j=1}^{n_k} S_j^{(k)}\big)\big]$ is also idcx on $\mathbb{Z}_+^k$.*

*(ii) Let $N_i(\lambda_i)$, $i = 1, \dots, k$, be mutually independent Poisson random variables, where each $N_i(\lambda_i)$ has mean $\lambda_i$. If $\phi$: $\mathbb{Z}_+^k \to \mathbb{R}$ is idcx, then $g$: $\mathbb{R}_+^k \to \mathbb{R}$ defined by $g(\lambda_1, \dots, \lambda_k) = \mathrm{E}\big[\phi\big(N_1(\lambda_1), \dots, N_k(\lambda_k)\big)\big]$ is also idcx.*

We next define some important stochastic orderings. A good reference to the theory is Müller and Stoyan (2002).

**Definition 2** (i) For two $\mathbb{R}$-valued random variables $X$ and $Y$, we say that $X$ is smaller than $Y$ in the *increasing convex (icx) ordering* and write $X \le_{\mathrm{icx}} Y$ if $\mathrm{E}[f(X)] \le \mathrm{E}[f(Y)]$ for any increasing and convex function $f$: $\mathbb{R} \to \mathbb{R}$, provided the expectations exist.

(ii) For two $\mathbb{R}^k$-valued random vectors $X$ and $Y$, we say that $X$ is smaller than $Y$ in the *supermodular (sm) ordering* and write $X \le_{\mathrm{sm}} Y$ if for any supermodular function $f$: $\mathbb{R}^k \to \mathbb{R}$,

$$\mathrm{E}[f(X)] \le \mathrm{E}[f(Y)], \tag{1}$$

provided the expectations exist.

(iii) For two $\mathbb{R}^k$-valued random vectors $X$ and $Y$, we say that $X$ is smaller than $Y$ in the *idcx ordering* and write $X \le_{\mathrm{idcx}} Y$ if (1) holds for any idcx function $f$: $\mathbb{R}^k \to \mathbb{R}$, provided the expectations exist.

(iv) Let $T = \mathbb{R}$ or $\mathbb{Z}$. For two $\mathbb{R}$-valued stochastic processes $\{X(t)\}_{t \in T}$ and $\{Y(t)\}_{t \in T}$, we say that $\{X(t)\}_{t \in T}$ is smaller than $\{Y(t)\}_{t \in T}$ in the supermodular [idcx resp.] ordering and write $\{X(t)\}_{t \in T} \le_{\mathrm{sm}}$ [$\le_{\mathrm{idcx}}$ resp.] $\{Y(t)\}_{t \in T}$ if for any positive integer $k$ and any $-\infty < t_1 \le \dots \le t_k < +\infty$, $\big(X(t_1), \dots, X(t_k)\big) \le_{\mathrm{sm}}$ [$\le_{\mathrm{idcx}}$ resp.] $\big(Y(t_1), \dots, Y(t_k)\big)$.

Since each idcx function is an sm function, we have that $X \leq_{\mathrm{sm}} Y$ implies $X \leq_{\mathrm{idcx}} Y$. Both the sm and idcx orderings are known as comparing the strength of positive dependency in random vectors.

A family of stochastic processes $\{X_c(t)\}_{t\in T}$ is said to be $\prec$-decreasing in $c$, if for any $k$ and $-\infty < t_1 \leq \ldots \leq t_k < \infty$ the family of vectors $(X_c(t_1), \ldots, X_c(t_k))$ is $\prec$-decreasing in $c$.

## 2.2 $\leq_{\mathrm{sm}}$-regular stochastic processes

We now study a regularity property for stationary stochastic processes, which will serve as the theorem hypothesis for Ross type results.

**Definition 3** We say that a stationary stochastic process $\{Y_t\}_{t\in T}$, where $T = \mathbb{Z}$ or $\mathbb{R}$, is $\leq_{\mathrm{sm}}$-regular [$\leq_{\mathrm{idcx}}$-regular resp.] if for each $k$ and all supermodular [idcx resp.] functions $f : T^{k+1} \to \mathbb{R}$, function $\phi$ defined by

$$\phi(t_1, \ldots, t_k) = \mathrm{E}f(Y_0, Y_{t_1}, Y_{t_1+t_2}, \ldots, Y_{t_1+\ldots+t_k})$$

is decreasing in $t_1, t_2, \ldots, t_k \in T$.

Note that if a process $\{Y_t, t \in T\}$ is $\leq_{\mathrm{sm}}$-regular, it is also $\leq_{\mathrm{idcx}}$-regular.

**Lemma 2** *If a stationary process $\{X(t)\}_{t\in\mathbb{R}}$ is $\leq_{\mathrm{sm}}$-regular [$\leq_{\mathrm{idcx}}$-regular resp.], then $\{X_c(t)\}_{t\in\mathbb{R}}$ is $\leq_{\mathrm{sm}}$-decreasing [$\leq_{\mathrm{idcx}}$-decreasing resp.] in $c > 0$.*

*Proof:* Let $f$ be sm [idcx]. From the stationarity, we have

$$\begin{aligned}&\mathrm{E}f(X(ct_1), X(ct_2), \ldots, X(ct_k))\\ &= \mathrm{E}f(X(0), X(c(t_2 - t_1)), \ldots, X(c(t_k - t_{k-1}) + \cdots + c(t_2 - t_1))),\end{aligned}$$

from which the lemma immediately follows. □

We prove, for completeness, the following lemma from Meester & Shantikumar (1993).

**Lemma 3** *Suppose a stationary intensity process $\lambda(t)$ is $\leq_{\mathrm{idcx}}$-regular with finite mean $\mathrm{E}\lambda(0) < \infty$. Then*

$$\Big(\int_{I_1} \lambda_c(t)\,\mathrm{d}t, \ldots, \int_{I_k} \lambda_c(t)\,\mathrm{d}t\Big)$$

*is $\leq_{\mathrm{idcx}}$-decreasing for $c > 0$, where $I_j = (s_j, t_j]$ with $-\infty < s_1 < t_1 \leq s_2 < \cdots \leq s_k < t_k < \infty$.*

*Proof:* For the proof, we have to use the following facts: $\leq_{\mathrm{idcx}}$ order is generated by continuous idcx functions (Theorem 3.12.9 from Müller and Stoyan (2002)); hence for a continuous idcx function $f$, we have the convergence in distribution of

$$f\Big(\sum_{j:j/n\in I_1} \frac{\lambda(j/n)}{n}, \ldots, \sum_{j:j/n\in I_k} \frac{\lambda(j/n)}{n}\Big) \xrightarrow{\mathrm{d}} f\Big(\int_{I_1} \lambda(s)\,\mathrm{d}s, \ldots, \int_{I_k} \lambda(s)\,\mathrm{d}s\Big).$$

The function $\hat{f}$: $\mathbb{R}^{l_1+\cdots+l_k} \to \mathbb{R}$,

$$\hat{f}(x_{11},\dots,x_{1l_1},\dots,x_{k1},\dots,x_{kl_k}) = f\Big(\sum_{j=1}^{l_1} c_{1j}x_{1j},\dots,\sum_{j=1}^{l_k} c_{kj}x_{kj}\Big),$$

is idcx provided $f : \mathbb{R}^k \to \mathbb{R}$ is idcx and $c_{ij}$ are positive numbers. To complete the proof, we use Theorem 3.12.8 from Müller & Stoyan (2002). □

The next lemma is a modification of Lemma 3, wherein we admit an infinite interval $I$, and therefore we have to assume direct Riemann integrability, for definition see e.g. Daley and Vere-Jones (1988).

**Lemma 4** *Suppose that a stationary intensity process $\lambda(t)$ is $\leq_{\rm idcx}$-regular with finite mean $\mathrm{E}\lambda(0) < \infty$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-negative function such that a.s. paths $f(t)\lambda(ct)$ are directly Riemann integrable, then*

$$\int_0^\infty f(t)\lambda_c(t)\,\mathrm{d}t$$

*is $\leq_{\rm icx}$–decreasing.*

*Proof:* We proceed similarly to the proof of Lemma 3 with a small modifications. For all $h > 0$ we have that $h^{-1}\sum_{j=1}^\infty f(hj)\lambda_c(hj)$ is $\leq_{\rm icx}$-decreasing in $c > 0$ because $h^{-1}\sum_{j=1}^n f(hj)\lambda_c(hj)$ is $\leq_{\rm icx}$-decreasing in $c$ and the icx order in the limit is preserved since the moments are convergent (see Theorem 3.4.6 from Müller and Stoyan (2002)). We now have to pass with $h \downarrow 0$. □

**Lemma 5** *If a stationary process $\{X(t)\}$ is $\leq_{\rm idcx}$-regular, then $\{X(-t)\}$ is $\leq_{\rm idcx}$-regular too.*

*Proof:* From stationarity, we have

$$\begin{aligned}&\mathrm{E}f(X(0),X(-t_1),\dots,X(-t_1-\cdots-t_k))\\&\quad= \mathrm{E}f(X(t_1+\cdots+t_k),X(t_2+\cdots+t_k),\dots,X(t_k),X(0)).\end{aligned}$$

To complete the proof, we have to note that if $f$ is idcx and $\pi$ is a permutation, then $f\circ\pi$ is idcx. □

**Example 2** Consider a stationary centered Gaussian process $\{X(t)\}$ with the covariance function $R(t) = \mathrm{E}X(0)X(t)$. Then $\{X(t)\}$ is $\leq_{\rm sm}$-regular if and only if $R(t)$ is decreasing. This can be shown as follows. Gaussian vector $(X(0),X(t_1),\dots,X(t_1+\ldots+t_n))$ has mean zero and the covariance matrix

$$\begin{aligned}&\mathbf{\Sigma}(t_1,\dots,t_n) = (\sigma_{ij}(t_1,\dots,t_n))\\&= \begin{pmatrix} R(0) & R(t_1) & \cdots & R(t_1+\ldots+t_n)\\ R(t_1) & R(0) & \cdots & R(t_1+\ldots+t_{n-1})\\ \vdots & \vdots & \ddots & \vdots\\ R(t_1+\ldots+t_n) & R(t_1+\ldots+t_{n-1}) & \cdots & R(0)\end{pmatrix}.\end{aligned}$$

Suppose that $R(t)$ is a decreasing function. Then $(t_1, \ldots, t_n) \leq (t_1^{'}, \ldots, t_n^{'})$ if and only if $\sigma_{ij}(t_1, \ldots, t_n) \geq \sigma_{ij}(t_1^{'}, \ldots, t_n^{'})$ for all $i, j = 1, \ldots, n$. By Theorem 3.13.5 from Müller and Stoyan (2002), we have

$$(X(0), X(t_1^{'}), \ldots, X(t_1^{'} + \ldots + t_n^{'})) \leq_{\mathrm{sm}} (X(0), X(t_1), \ldots, X(t_1 + \ldots + t_n)),$$

which is equivalent to that $\{X(t)\}_{t\in\mathbb{R}}$ is $\leq_{\mathrm{sm}}$-regular.

We now consider the Palm version $\{\lambda^\circ(t)\}_{t\in\mathbb{R}}$ of the stationary process $\{\lambda(t)\}_{t\in\mathbb{R}}$. Recall (see Grandell (1976), page 60) that the Palm version is defined as the stochastic process with the following finite dimensional distributions

$$\mathrm{E}f(\lambda^\circ(t_1), \ldots, \lambda^\circ(t_1 + \ldots + t_n)) = \frac{1}{\bar{\lambda}} \mathrm{E}\lambda(0) f(\lambda(t_1), \ldots, \lambda(t_1 + \ldots + t_n)). \tag{2}$$

Note that the random measure $\Lambda^\circ$ defined by $\Lambda^\circ(\mathrm{d}x) = \lambda^\circ(x)\,\mathrm{d}x$ is a Palm version for the stationary random measure $\Lambda$ with random intensity $\lambda(t)$ (see Grandell (1976) or Geman and Horowitz (1973)). Recall that the Cox process with random measure $\Lambda^\circ$ plus a point added at zero is the Palm version of the stationary Cox process with random measure $\Lambda$, where $\Lambda(\mathrm{d}x) = \lambda(x)\,\mathrm{d}x$. Note that if $\{\lambda(t)\}_{t\in\mathbb{R}}$ is $\leq_{\mathrm{idcx}}$-regular, recalling that $\{\lambda(t)\}_{t\in\mathbb{R}}$ is ergodic, then

$$\mathrm{E}\lambda^\circ(t) = \frac{1}{\bar{\lambda}} \mathrm{E}\lambda(0)\lambda(t) \geq \frac{1}{\bar{\lambda}} \mathrm{E}\lambda(0)\mathrm{E}\lambda(t) = \mathrm{E}\lambda(t).$$

An important property is $\mathrm{E}\Lambda^{\circ -1}(t) = \bar{\lambda}^{-1} t$, where $\Lambda^{\circ -1}(t) = \inf\{s \geq 0 : \Lambda^\circ((0, s]) \geq t\}$. In the next lemma, we deal with the random measure $\Lambda_c^\circ(\mathrm{d}x) = \lambda_c^\circ(x)\,\mathrm{d}x$, where $\lambda_c^\circ(t)$ is defined by the stationary intensity process $\{\lambda_c(t)\}_{t\in\mathbb{R}}$. Observe that $\lambda_c^\circ(t) = \lambda^\circ(ct)$.

**Lemma 6** *If the stationary intensity process $\{\lambda(t)\}_{t\in\mathbb{R}}$ is $\leq_{\mathrm{idcx}}$-regular, then $\{\lambda_c^\circ(t)\}_{t\in\mathbb{R}}$ is $\leq_{\mathrm{idcx}}$-decreasing.*

*Proof:* Let $\phi : \mathbb{R}_+^k \to \mathbb{R}$ be idcx. We have

$$\mathrm{E}\phi(\lambda^\circ(ct_1), \ldots, \lambda^\circ(ct_k)) = \frac{1}{\bar{\lambda}} \mathrm{E}\psi(\lambda(0), \lambda(ct_1), \ldots, \lambda(ct_k)),$$

where

$$\psi(x_0, x_1, \ldots, x_k) = x_0 \phi(x_1, \ldots, x_k).$$

Note that $\psi : \mathbb{R}_+^{k+1} \to \mathbb{R}$ is idcx. □

## 3 Case of Markov Processes

Consider now the case when a stationary process is Markov. In this note, we assume that the state space $E$ is $\mathbb{R}$ or $\{x_1, \ldots, x_d\}$. Let $\{X_n\}_{n\in\mathbb{Z}}$ be a stationary discrete time Markov process (DTMP) on $E$ with transition kernel $P = \{P(x, \mathrm{d}y)\}$. By $\pi(\mathrm{d}x)$ we denote the stationary distribution that is fulfilling $\pi(B) = \int_E \pi(\mathrm{d}x) P(x, B)$. Similarly $\{Y(t)\}_{t\in\mathbb{R}}$ is a continuous time Markov process (CTMP) with probability transition function $\{p_t(x, B)\}$ and stationary distribution $\pi$. We also denote by $P^R$ the transition kernel of the stationary DTMP $\{X_{-n}\}_{n\in\mathbb{Z}}$. Note that both the processes have the same stationary distribution.

**Definition 4** (i) A DTMP $\{X_n\}_{n\in\mathbb{Z}}$ with transition kernel $P$ is said to be *stochastically monotone* if the sequence $\int_E P(x,\mathrm{d}y)f(y)$ is increasing in $x$ for any increasing function $f\colon E\to\mathbb{R}_+$.

(ii) A CTMP $\{Y(t)\}_{t\in\mathbb{R}}$ with probability transition function $p_t(x,B)$ is said to be *stochastically monotone* if for any $t\geq 0$ the sequence $\int_E p_t(x,\mathrm{d}y)f(y)$ is increasing in $x$ for any increasing function $f\colon E\to\mathbb{R}_+$.

There exists a function $h(x,u)$ increasing of $u$ $(x,\in E, u\in[0,1])$ such that $P(x,B) = \Pr(h(x,U)\in B)$, where $U$ is uniformly distributed. For the discrete state space, one can see a formula for $h$, for example, in Bäuerle and Rolski (1998) and, for $E=\mathbb{R}$, we may choose $h(x,u)=\inf\{y : P(x,(-\infty,y])\geq u\}$. Similarly, we have such the function $h^R$ for the kernel $P^R$. Denote $\circ$–superposition by $h_1\circ h_2(x,z_1,z_2) = h_2(h_1(x,z_1),z_2)$. We now define $h^n(x,u_1,\ldots,u_n)$ as the $n$-fold $\circ$–superposition of $h$ for $n$ positive and of $h^R$ for $n$ negative. Moreover, if the transition kernel $P$ $[P^R$ resp.] is stochastically monotone, then $h$ $[h^R$ resp.] can be chosen increasing with respect to variable $x$. Thus, if both $P$ and $P^R$ are stochastically monotone (and such the DTMPs or respectively $P$ we call by *double stochastically monotone*), then we may choose functions $h^n(x,u_1,\ldots,u_n)$ to be increasing with respect to the $x$ variable too for all $n\in\mathbb{Z}$ ($h^0$ is the identity). Let $\{U_i\}_{i\in\mathbb{Z}}$ be a double-ended sequence of independent uniformly distributed random variables. Without loss of generality, we can assume that $\{X_n\}_{n\in\mathbb{Z}}$ with transition kernel $P$ has the following structure: $X_j = h^{j-i}(X_i,U_i,\ldots,U_{j-1})$ for $i<j$ and $h^{j-i}(X_i,U_{i-1},\ldots,U_j)$ for $i>j$ and $X_i$ is independent from $U_i,\ldots,U_{j\pm1}$ respectively. The above structure allows us the following *dimension reduction.* Let $k>1$ and $f:\mathbb{R}^{k+1}\to\mathbb{R}$. With the convention $m_j = n_1+\ldots+n_j$, define $\hat{f}_l:\mathbb{R}^2\to\mathbb{R}$ by

$$\begin{aligned}\hat{f}_l(x,y) = f\big(&h^{-m_l}(x,U_{m_l-1},\ldots,U_0),\ldots,h^{m_{l-1}-m_l}(x,U_{m_l-1},\ldots,U_{m_{l-1}}),x,y,\\ &h^{m_{l+2}-m_{l+1}}(y,U_{m_{l+1}},\ldots,U_{m_{l+2}-1}),\ldots,h^{m_k-m_{l+1}}(y,U_{m_{l+1}},\ldots,U_{m_k-1})\big)\end{aligned}$$

for $l=1,\ldots,k$, and

$$\hat{f}_0(x,y) = f\big(x,y,h^{m_2-m_1}(y,U_{m_1},\ldots,U_{m_2-1}),\ldots,h^{m_k-m_1}(y,U_{m_1},\ldots,U_{m_k-1})\big).$$

Then,

$$\mathrm{E}f(X_0,X_{m_1},\ldots,X_{m_k}) = \begin{cases}\mathrm{E}\hat{f}_0(X_0,X_{n_1}),\\ \mathrm{E}\hat{f}_l(X_{m_l},X_{m_{l+1}}) = \mathrm{E}\hat{f}_l(X_0,X_{n_{l+1}}), & \text{for } l>0.\end{cases}$$

Now notice that if $f$ is sm and $\{X_n\}_{n\in\mathbb{Z}}$ is double stochastically monotone, then $\hat{f}_l$ is sm too.

We first recall that from the Lorenz inequality (see e.g. Müller & Stoyan (2002)) we have:

**Lemma 7** *If* $Z_0 =_{\mathrm{st}} Z_1 =_{\mathrm{st}} \ldots =_{\mathrm{st}} Z_n$*, then* $(Z_0,\ldots,Z_k)\leq_{\mathrm{sm}}(Z_0\ldots,Z_0)$.

A detailed proof for a general case stated in Lemma 7 can be found in the doctoral thesis of Kulik (2002).

The following lemma is an extension of Daley's (1968) result on the monotonicity of covariance functions and in this form appeared in Müller and Stoyan (2002).

**Lemma 8** *If $\{X_n\}_{n\in\mathbb{Z}}$ is stochastically monotone, then $(X_0, X_{n+1}) \leq_{\rm sm} (X_0, X_n)$ for $n = 0, 1, \ldots$.*

*Proof:* For $n = 0$, it is directly Lorenz inequality. Assuming the result true for $n \geq 0$, we write $\mathrm{E}f(X_0, X_{n+1}) = \mathrm{E}f(X_0, h(X_n, U_n))$, where $f$ is an sm function. Then for some other sm function $\hat{f}$, we have $\mathrm{E}f(X_0, h(X_n, U_n)) = \mathrm{E}\hat{f}(X_0, X_n)$ (here we use that $h(x, u)$ is increasing with respect $x$) which, by the induction hypothesis, is greater than or equal to $\mathrm{E}\hat{f}(X_0, X_{n+1}) = \mathrm{E}f(X_0, X_{n+2})$. □

The following result was first proved by Hu & Pan (2000) but our proof is different.

**Theorem 1** *Let $\{X_n\}$ be a double stochastically monotone DTMP. The mapping*

$$(n_1, \ldots, n_k) \to (X_0, X_{n_1}, \ldots, X_{n_1+\ldots+n_k})$$

*is $\leq_{\rm sm}$-decreasing.*

*Proof:* Let $m_j = n_1 + \cdots + n_j$ and $1 \leq \ell \leq k$. We have to check that

$$(X_0, X_{m_1}, \ldots, X_{m_\ell}, \ldots, X_{m_k}) \geq_{\rm idcx} (X_0, X_{m_1}, \ldots, X_{m_\ell+1}, \ldots, X_{m_k+1}).$$

For this, we have to use the dimension reduction property and Lemma 8. □

**Corollary 1** *Let $\{X(t)\}$ be a stationary double stochastically monotone CTMP. The mapping*

$$(t_1, \ldots, t_k) \to (X(0), X(t_1), \ldots, X(t_1 + \ldots + t_k))$$

*is $\leq_{\rm sm}$-decreasing.*

*Proof:* For each $h > 0$, DTMP $\{X(nh),\ n \in \mathbb{Z}\}$ is stochastically monotone. Therefore,

$$(t_1, \ldots, t_k) \to (X(0), X(t_1), \ldots, X(t_1 + \ldots + t_k))$$

is $\leq_{\rm sm}$-decreasing for $t_1, \ldots, t_k \in \{nh\}$. Due to the continuity which stems from Theorems 3.9.10 and 3.9.11 in Müller and Stoyan (2002), the above mapping is $\leq_{\rm sm}$-decreasing for $t_1, \ldots, t_k \in \mathbb{R}$. □

**Remark 1** Let $\{X_n\}$ be a stationary DTMP. Is it true that if $f(X_0, X_n)$ is $\leq_{\rm sm}$-decreasing for $n = 0, 1, \ldots$, then $P$ is stochastically monotone? Is it true that if $\{X_n\}$ is $\leq_{\rm sm}$-regular, then $P$ is double stochastically monotone?

**Example 3** Consider a stationary birth and death process $\{\eta(t)\}$ with states $\{0, 1, \ldots, i_0, \ldots\}$, which is a stochastically monotone CTMP. Define process $\{Y(t)\}$ as follows: $Y(t) = 0$ if $\eta(t) \leq i_0$ and 1 otherwise. The process is a stationary on-off process and as we show below it is also $\leq_{\rm sm}$-regular. This follows from the fact that for any sm function $f : \{0, 1\}^{k+1} \to \mathbb{R}$ we have $\mathrm{E}f(Y(0), \ldots, Y(t_1 + \ldots + t_k)) = \mathrm{E}f(1(\eta(0) > i_0), \ldots, 1(\eta(t_1 + \ldots + t_k) > i_0))$ and that $f(1(x_1 > i_0), \ldots, 1(x_{k+1} > i_0))$ is an sm function. It is an open problem under what conditions on the distributions of *on* and *off* times the on-off process is $\leq_{\rm sm}$-regular. Even more difficult problem seems to be one to characterize stationary semi-Markov processes which are $\leq_{\rm sm}$-regular.

# 4 Queueing Applications

## 4.1 Cox/GI/1/$\infty$

We consider Cox/GI/1/$\infty$ queues, that is, customers arrive at a single-server queue according to a stationary Cox process, and the sequence of service times is independent from arrivals and consists of i.i.d. non-negative random variables. The Ross type conjecture on the Cox/GI/1/$\infty$ queues is summarized as, the faster the intensity process changes, the smaller the stationary workload is in some sense of stochastic ordering. These problems were considered by Chang et al. (1991) and by Bäuerle and Rolski (1998) in some general settings. In this section, we reexamine such a statement in terms of $\leq_{\text{idcx}}$-regularity of the arrival intensity process $\{\lambda(t)\}_{t\in\mathbb{R}}$.

For the stability of the system, we assume $\bar{\lambda}\,\mathrm{E}[S_0] < 1$, where $\bar{\lambda} = \mathrm{E}[\lambda(0)]$. Then, it is known that a unique, a.s. finite and stationary workload $W$ exists and is given by

$$W = \sup_{t\leq 0}\{t + M([t,0))\}, \tag{3}$$

where $M(B) = \sum_{n\in\mathbb{Z}} S_n\, 1_B(T_n)$ for $B \in \mathcal{B}(\mathbb{R})$ (see, e.g., Borovkov (1976)). Similarly, the stationary waiting time $W^\circ$ can be expressed as

$$W^\circ = \sup_{t\leq 0}\{t + M^\circ([t,0))\}, \tag{4}$$

where $M^\circ(B) = \sum_{n\in\mathbb{Z}} S_n\, 1_B(T_n^\circ)$, and $T_n^\circ$ are points in the Palm version of the corresponding stationary Cox process. We denote by $W(c)$ the workload in the queue with arrival intensity $\{\lambda_c(t)\}$ (note that $W = W(1)$), $W(\infty)$ for the workload in the associated M/GI/1/$\infty$ queue and $W(0)$ the workload in the system with arrivals according to a Poisson process with fixed but random arrival rate $\lambda^* = \lambda(0)$. In the case of single server queues, we have to assume $\lambda(0)\mathrm{E}S < 1$ a.s. Similarly, by $W^\circ(c)$ we denote the waiting time, $W^\circ(\infty) =_{\mathrm{d}} W(\infty)$ for the waiting time in the associated M/GI/1/$\infty$ queue and $W^\circ(0)$ the waiting time in the system with arrivals according to a Poisson process with fixed but random arrival rate $\lambda' = \lambda^\circ(0)$.

**Theorem 2** *If $\{\lambda(t)\}_{t\in\mathbb{R}}$ is $\leq_{\text{idcx}}$-regular, then the stationary workload $W(c)$ is $\leq_{\text{icx}}$-decreasing in $c$ $(> 0)$, that is, for any $c_1 > c_2$ $(> 0)$,*

$$W(c_1) \leq_{\text{icx}} W(c_2). \tag{5}$$

Similarly, the waiting time $W^\circ(c)$ is $\leq_{\text{icx}}$-decreasing in $c$ $(> 0)$. The proof for $W^\circ(c)$ is similar.

*Proof:* For $f$ increasing and convex, following Rolski (1986) we can express $\mathrm{E}[f(W(c))]$ as a limit of expressions of form $\mathrm{E}[h(\Lambda_c(I_1), \dots, \Lambda_c(I_k))]$, where $h$ is idcx. From Lemma 2, $\{\lambda_c(t)\}_{t\in\mathbb{R}}$ is $\leq_{\text{idcx}}$-decreasing in $c$ when $\{\lambda(t)\}_{t\in\mathbb{R}}$ is $\leq_{\text{idcx}}$-regular, and thus, the result follows from Lemmas 1 and 3. □

Note that $W^\circ =_{\text{st}} W(\infty) \leq_{\text{icx}} W(c)$ is the consequence of Theorem 2 and the monotone convergence theorem (see Rolski (1981,1991)). Similarly, $W(c) \leq_{\text{icx}} W(0)$ was proved by Rolski (1986). Note that we have also $W^\circ(c) \leq_{\text{icx}} W^\circ(0)$.

**Remark 2** It is tempting to conjecture that a converse result to Theorem 2 is true. That is, (5) yields $\{\lambda(t)\}_{t\in\mathbb{R}}$ to be $\leq_{\rm idcx}$-regular.

## 4.2 Infinite server system

In this section, we consider a Cox/GI/$\infty$/$\infty$ system. If $\{\lambda(t)\}_{t\in\mathbb{R}}$ is an arrival intensity process, then the steady state number, $L$, of customers in the system has the following representation. Consider the plane Poisson process on $\mathbb{R}\times\mathbb{R}_+$ with intensity $\lambda(t)\,\mathrm{d}t\,\mathrm{d}B(x)$. Each point of the process represents on its first coordinate an arrival time and on the second one the corresponding service time. Customer $i$, arriving at $T_i<0$ with service time $S_i$, is present at time 0 in the system if $T_i+S_i\geq 0$. Therefore the number in the system $L=L(1)$ at time $t=0$ is the number of points of the planar Poisson point process in

$$\{(t,x)\in\mathbb{R}\times\mathbb{R}_+\,:\,x\geq -t,\ t\leq 0\},$$

that is, it is a Poisson r.v. with mean

$$\int_{-\infty}^{0}\lambda(t)\bar{B}(-t)\,\mathrm{d}t,$$

where $\bar{B}(x)=1-B(x)$. A similar representation is valid for the steady state number, $L^\circ$, of customers just before an arrival, but with arrival intensity process $\{\lambda^\circ(t)\}_{t\in\mathbb{R}}$, where $\{\lambda^\circ(t)\}_{t\in\mathbb{R}}$ is the Palm version of $\{\lambda(t)\}_{t\in\mathbb{R}}$.

**Lemma 9** *In a* Cox/GI/$\infty$/$\infty$ *system, the steady state number $L$ of customers has the probability function*

$$\Pr(L=k)=\mathrm{E}\Big[\frac{(\int_{-\infty}^{0}\lambda(t)\bar{B}(-t)\,\mathrm{d}t)^k}{k!}\exp\Big[-\int_{-\infty}^{0}\lambda(t)\bar{B}(-t)\,\mathrm{d}t\Big]\Big],$$

*for $k=0,1,\ldots$. The steady state number $L^\circ$ of customers just before an arrival has the probability function*

$$\Pr(L^\circ=k)=\mathrm{E}\left[\frac{(\int_{-\infty}^{0}\lambda^\circ(t)\bar{B}(-t)\,\mathrm{d}t)^k}{k!}\exp\left[-\int_{-\infty}^{0}\lambda^\circ(t)\bar{B}(-t)\,\mathrm{d}t\right]\right],$$

*where $\{\lambda^\circ(t)\}$ is a Palm version of $\{\lambda(t)\}$.*

In a similar way, like Lemma 4, we can prove the following modification for $\lambda^\circ(t)$. To avoid unnecessary technical assumptions from now on we assume that $\lambda(t)$ is bounded.

**Lemma 10** *Suppose a stationary intensity process $\{\lambda(t)\}_{t\in\mathbb{R}}$ is $\leq_{\rm idcx}$-regular. Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a non-negative decreasing integrable function. Then*

$$\int_0^\infty f(t)\lambda_c^\circ(t)\,\mathrm{d}t$$

*is $\leq_{\rm icx}$-decreasing.*

*Proof:* Similar to Lemma 4 with the use of Lemma 6. □

Consider now the *associated* M/GI/$\infty$/$\infty$ system, that is, the system with homogeneous Poisson arrivals with rate $\bar{\lambda} = \mathrm{E}\lambda(0)$ and i.i.d. service times with the common distribution $B$. We denote the number of customers in this system by $L(\infty)$. From Lemma 9, we immediately obtain the well known formula:

$$\Pr(L(\infty) = k) = \frac{(\bar{\lambda}\mathrm{E}S)^k}{k!} \exp\left[-\bar{\lambda}\mathrm{E}S\right].$$

In the following theorem, we denote by $L(c)$ the number in the system with arrival intensity process $\{\lambda_c(t)\}_{t\in\mathbb{R}}$.

**Theorem 3** (i) *If $\{\lambda(t)\}_{t\in\mathbb{R}}$ is $\leq_{\mathrm{idcx}}$-regular, then $L(c)$ is $\leq_{\mathrm{icx}}$-decreasing. Furthermore $L^\circ(c)$ is $\leq_{\mathrm{icx}}$-decreasing too.*
(ii)

$$L(\infty) \leq_{\mathrm{icx}} L(c) \leq_{\mathrm{icx}} L(0),$$

*where $L(0)$ denotes the number in the system with the arrivals according to a Poisson process with fixed but random arrival rate $\lambda(0)$ and*

$$L^\circ(\infty) \leq_{\mathrm{icx}} L^\circ(c) \leq_{\mathrm{icx}} L^\circ(0),$$

*where $L^\circ(0)$ denotes the number in the system with the arrivals according to a Poisson process with fixed but random arrival rate $\lambda^\circ(0)$.*

*Proof:* (i) Let $N(x)$ be a Poisson random variable with mean $x$. Then, for an icx function $g$: $\mathbb{Z}_+ \to \mathbb{R}$, the function $\phi_g$: $\mathbb{R}_+ \to \mathbb{R}$ defined by $\phi_g(x) = \mathrm{E}g(N(x))$ is also icx by Lemma 1(ii). Therefore,

$$\begin{aligned}\mathrm{E}[g(L(c))] &= \mathrm{E}\Big[\mathrm{E}\Big[g\big(N\big(\int_{-\infty}^{0} \lambda_c(t)\,\bar{B}(-t)\,\mathrm{d}t\big)\big)\;\Big|\;\int_{-\infty}^{0} \lambda_c(t)\,\bar{B}(-t)\,\mathrm{d}t\Big]\Big]\\ &= \mathrm{E}\Big[\phi_g\big(\int_{-\infty}^{0} \lambda_c(t)\,\bar{B}(-t)\,\mathrm{d}t\big)\Big],\end{aligned}$$

and the proof of the monotonicity of $L(c)$ follows by Lemma 4. To prove the monotonicity of $L^\circ(c)$ we write

$$\mathrm{E}g(L(c)) = \mathrm{E}\phi_g\Big(\int_{-\infty}^{0} \lambda_c^\circ(t)\bar{B}(-t)\,\mathrm{d}t\Big) = \frac{1}{\bar{\lambda}}\mathrm{E}\Big[\lambda(0)\phi_g\Big(\Big(\int_{-\infty}^{0} \lambda_c(t)\bar{B}(-t)\,\mathrm{d}t\Big)\Big].$$

We now have to use Lemma 10.
(ii) The proof follows from part (i) and the monotone convergence theorem. □

## 4.3 Other systems

In his paper from 1978, Ross also posed a question on the monotonicity of the loss probability in Coxian loss systems. Unfortunately, this problem turns out to be very hard and only some very special cases are solved till now; see e.g. Kokotushkin (1974), Rolski (1983), Daley and Servi (2002A, 2002B). The only case when a general model is tractable are single server loss systems, but of course such systems are the least interesting. The point is that for Cox/M/1/0, the system at zero is full if and only if the last customer arriving and served before 0 is still present at time 0. Let $\mu$ be the service rate. Thus, we have that the customer stationary probability of loss equals

$$\begin{aligned} p^\circ_{\text{loss}} &= \mathrm{E}\int_{-\infty}^{0} \lambda^\circ(s)\, e^{-\int_s^0 \lambda^\circ(u)\mathrm{d}u+\mu s}\,\mathrm{d}s = \mathrm{E}\int_{-\infty}^{0} e^{-s+\mu\Lambda^{\circ -1}(-s)}\,\mathrm{d}s \\ &= \int_{-\infty}^{0} \mathrm{E}e^{-s+\mu\Lambda^{\circ -1}(-s)}\,\mathrm{d}s \geq \int_{-\infty}^{0} e^{-s+\mu\mathrm{E}\Lambda^{\circ -1}(-s)}\mathrm{d}s \\ &= \int_{0}^{\infty} e^{-s-\frac{\mu}{\bar\lambda}s}\,\mathrm{d}s = \frac{\bar\lambda}{\bar\lambda+\mu}, \end{aligned}$$

where $\{\lambda^\circ(t)\}$ is a Palm version of $\{\lambda(t)\}$. The inequality above is justified by Jensen's inequality because the function $\exp(-s+\mu x)$ of $x$ is convex for all $s$. Using some convexity property of two server loss systems, Rolski (1984) proved Ross's conjecture for Cox/M/2/0. Unfortunately, the convexity property for more servers is unknown and seems to be difficult to verify and therefore other techniques are required.

Many server queues lack any representation in the form of idcx functions and therefore the techniques used in this note cannot be directly applied. In consequence, it is not clear whether such strong results like obtained in this note are possible, however following the paper by Asmussen and O'Cinneide (2002), it is tempting to conjecture that in Cox/GI/$k$/$\infty$ if for the workload $W(c)$ we have $\Pr(W(c)>x) \sim e^{-\gamma(c)x}$ for some $\gamma(c)>0$ and if the arrival intensity process $\{\lambda(t)\}_{t\in\mathbb{R}}$ is $\leq_{\text{icdx}}$-regular, then $\gamma(c)$ is monotonic with respect to $c>0$.

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