Teacher's Manual

for the book

Stochastic Processes for Insurance and Finance

Draft

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2000

Teacher's Manual for *Stochastic Processes in Insurance and Finance* printed by Wiley, Chichester, 1999 was anounced in the Preface of the book. The project has not been completed and the presented version consists of unfinished notes.

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Probability Distributions

2.1 Random Variables and Their Characteristics

By X we denote a *random variable* and by $F(x) = \mathbb{P}(X \le x)$ its *distribution function*. The *tail* of $F(x)$ we denote by $\overline{F}(x) = 1 - F(x)$. We say that X is *discrete* if there exists a denumerable subset $E = \{x_0, x_1, \ldots\}$ of $\mathbb{R}$ such that $\mathbb{P}(X \in E) = 1$. In this case, we define the *probability function* $p : E \to [0,1]$ by $p(x_k) = \mathbb{P}(X = x_k)$. The most important subclass of nonnegative discrete random variables is the lattice case, in which $E \subset h\mathbb{N}$, i.e. $x_k = hk$ for some $h > 0$, where $\mathbb{N} = \{0, 1, \ldots\}$. We then simply write $p(x_k) = p_k$ and say that X is a *lattice* random variable. On the other hand, we say that X is *absolutely continuous* if there exists a measurable function $f : \mathbb{R} \to \mathbb{R}_+$ such that $\int f(x)\, dx = 1$ and $\mathbb{P}(X \in B) = \int_B f(x)\, dx$ for each $B \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ denotes the σ-algebra of Borel sets in $\mathbb{R}$. We call $f(x)$ the *density function* of X.

An important characteristic of the random variable X is its *expectation* $\mu = \mathbb{E}\, X$ which is given by

$$\mathbb{E}\, X = \begin{cases} \sum_k x_k p(x_k) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x f(x)\, \mathrm{d}x & \text{if } X \text{ is absolutely continuous} \end{cases}$$

provided that $\sum_k |x_k| p(x_k) < \infty$ and $\int_{-\infty}^{\infty} |x| f(x)\, \mathrm{d}x < \infty$, respectively. In general, for each measurable function $g : \mathbb{R} \to \mathbb{R}$, the expectation $\mathbb{E}\, g(X)$ is given by the *Lebesgue-Stieltjes integral* [1] $\mathbb{E}\, g(X) = \int_{-\infty}^{\infty} g(x)\, \mathrm{d}F(x)$ provided that $\int_{-\infty}^{\infty} |g(x)|\, \mathrm{d}F(x) < \infty$.

The expectation $\mathbb{E}\,(X^n)$ is denoted by $\mu^{(n)}$ and called the n-th *moment* of X. The *variance* of X is $\operatorname{Var} X = \sigma^2 = \mathbb{E}\,(X - \mu)^2$ and $\sigma = \sqrt{\sigma^2}$ is the *standard deviation*. The *coefficient of variation* is given by $\mathrm{cv}_X = \sigma/\mu$, the *index of dispersion* by $\mathrm{I}_X = \sigma^2/\mu$. For two random variables X, Y the

[1] g or F cannot be discontinuous at the same time, because then Stieltjes integral cannot be defined.

covariance $\mathrm{Cov}(X,Y)$ is defined by $\mathrm{Cov}(X,Y) = \mathbb{E}\big((X-\mathbb{E}\,X)(Y-\mathbb{E}\,Y)\big)$ provided that $\mathbb{E}\,(X^2), \mathbb{E}\,(Y^2) < \infty$. We say that X,Y are *positively correlated* if $\mathrm{Cov}(X,Y) > 0$, *uncorrelated* if $\mathrm{Cov}(X,Y) = 0$, *negatively correlated* if $\mathrm{Cov}(X,Y) < 0$.

A *median* of the random variable X is any number $\zeta_{1/2}$ such that

$$\mathbb{P}(X \le \zeta_{1/2}) \ge \tfrac{1}{2}\,, \qquad \mathbb{P}(X \ge \zeta_{1/2}) \ge \tfrac{1}{2}\,.$$

Let $I = \{s \in \mathbb{R} : \mathbb{E}\,\mathrm{e}^{sX} < \infty\}$. Note that I is an interval which can be the whole real line $\mathbb{R}$, a halfline or even the singleton $\{0\}$. The *moment generating function* $\hat{m} : I \to \mathbb{R}$ of X is defined by $\hat{m}(s) = \mathbb{E}\,\mathrm{e}^{sX}$. We also use the following transforms.

- The *Laplace–Stieltjes transform* $\hat{l}(s) = \mathbb{E}\,\mathrm{e}^{-sX} = \int_{-\infty}^{\infty} \mathrm{e}^{-sx}\,\mathrm{d}F(x)$ of X. Distinguish $\hat{l}(s)$ from the *Laplace transform* $\hat{L}(s) = \int_{-\infty}^{\infty} \mathrm{e}^{-sx} c(x)\,\mathrm{d}x$ of a function $c : \mathbb{R} \to \mathbb{R}_+$.
- For lattice random variables on $\mathbb{N}$ with probability function $\{p_k,\ k \in \mathbb{N}\}$ we define the *probability generating function* $\hat{g} : [-1,1] \to \mathbb{R}$ defined by $\hat{g}(s) = \sum_{k=0}^{\infty} p_k s^k$.
- For X being an arbitrary real-valued random variable with distribution F, the *characteristic function* $\hat{\varphi} : \mathbb{R} \to \mathbb{C}$ of X is given by $\hat{\varphi}(s) = \mathbb{E}\,\mathrm{e}^{\mathrm{i}sX}$.

The n-th *derivative* of a function $h(s)$ will be denoted by $h^{(n)}(s)$, the one-sided n-th derivatives by $h^{(n)}(s-)$ and $h^{(n)}(s+)$ respectively.

Let $X_1, \ldots, X_n$ be an arbitrary sequence of random variables defined on the same probability space. For n fixed and for all $k = 1, 2, \ldots, n$ and $\omega \in \Omega$, let $X_{(k)}(\omega)$ denote the k-th smallest value of $X_1(\omega), \ldots, X_n(\omega)$. The components of the random vector $(X_{(1)}, \ldots, X_{(n)})$ are called the *order statistics* of $(X_1, \ldots, X_n)$.

Exercises

2.1.1 Show that

$$\mathbb{E}\,(X^n) = \int_{-\infty}^{\infty} x^n\,\mathrm{d}F(x) = n\int_0^{\infty} x^{n-1}\overline{F}(x)\;\mathrm{d}x - n\int_{-\infty}^{0} x^{n-1}F(x)\,\mathrm{d}x$$

for all $n = 1, 2, \ldots$, provided $\int_{-\infty}^{\infty} |x|^n\,\mathrm{d}F(x) < \infty$.

2.1.2 Prove that if X is $\mathbb{N}$-valued with probability function $\{p_k\}$, then

$$\mathbb{E}\,(X^n) = \sum_{k=1}^{\infty} k^n p_k = \sum_{k=1}^{\infty} (k^n - (k-1)^n) r_k$$

for all $n = 1, 2, \ldots$, provided $\sum_{k=1}^{\infty} k^n p_k < \infty$, where $r_k = \mathbb{P}(X \ge k)$ is given by

$$r_k = \sum_{i=k}^{\infty} p_i \,, \qquad k \in \mathbb{N} \,. \tag{2.1.1}$$

2.1.3 Show that if $\mathbb{E}\,|X| < \infty$ and $\zeta_{1/2}$ is a median of X then

$$\mathbb{E}\,|X - \zeta_{1/2}| \le \mathbb{E}\,|X - x| \,, \qquad x \in \mathbb{R} \,.$$

Formulate and prove a similar property for the expectation $\mathbb{E}\,X$. [Hint. For the second part consider the function $\mathbb{E}\,(X - x)^2$.]

2.1.4 Show that for the *Cauchy distribution* with density function

$$f(x) = \frac{1}{\pi}\frac{1}{1 + x^2} \,, \qquad x \in \mathbb{R} \,,$$

the moment generating function $\hat{m}(s)$ is finite only at $s = 0$.

2.1.5 Suppose that $F(x)$ is the distribution function of a nonnegative random variable X, i.e. $F(0-) = 0$. Show that

$$\hat{m}(s) = \int_0^{\infty} \mathrm{e}^{sx}\,\mathrm{d}F(x) = 1 + s\int_0^{\infty} \mathrm{e}^{sx}\overline{F}(x)\,\mathrm{d}x$$

for all $s \in \mathbb{R}$ with $\hat{m}(s) < \infty$. Derive a similar formula for the Laplace–Stieltjes transform $\hat{l}(s)$ of X.

2.1.6 Let $F(x)$ be the distribution function of a nonnegative random variable X. Show that $\hat{m}(s_0) < \infty$ for some $s_0 > 0$ if and only if for some $a, b > 0$ the inequality $\overline{F}(x) \le a\,\mathrm{e}^{-bx}$ holds for all $x \ge 0$. Conclude from this that X has all moments finite if $\hat{m}(s_0) < \infty$ for some $s_0 > 0$. Give an example of a distribution F of a nonnegative random variable such that $\hat{m}(s) = \infty$ for all $s > 0$.

2.1.7 Prove that, if $\hat{m}(s)$ and $\hat{l}(s)$ are well-defined only on $(-\infty, 0]$ and $[0, \infty)$, respectively, then

$$\mathbb{E}\,X^n = \hat{m}^{(n)}(0-) = (-1)^n \hat{l}^{(n)}(0+) \,.$$

$\hat{\varphi}$. Show that, if X takes its values in a subset of $\mathbb{N}$ and if $\mathbb{E}\,X^n < \infty$, then

$$\mathbb{E}\,\left(X(X-1)\ldots(X-n+1)\right) = \hat{g}^{(n)}(1-) \,.$$

2.1.8 Let X be an $\mathbb{N}$-valued random variable with probability function $\{p_k\}$ such that $\sum_{k=0}^{\infty} p_k s^k < \infty$ for all $s \in [0, s_0]$, where $s_0 \geq 1$. Show that then

$$(1-s)\sum_{k=0}^{\infty} r_{k+1} s^k = 1 - \hat{g}_X(s)\,, \qquad |s| < s_0\,,$$

where the r_k are given by (2.1.1).

2.1.9 Let $X_1, \ldots, X_n$ be independent and uniformly distributed on $[0, t]$. Show that the density $f(t_1, \ldots, t_n)$ of the order statistics $(X_{(1)}, \ldots, X_{(n)})$ is given by

$$f(t_1, \ldots, t_n) = \begin{cases} n!\,t^{-n} & \text{if } 0 \leq t_1 \leq \ldots \leq t_n \leq t, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.2}$$

Solutions

2.1.1 The first equality can be seen as a definition of $\mathbb{E}(X^n)$. The decomposition

$$\int_{-\infty}^{\infty} x^n \,\mathrm{d}F(x) = \int_{-\infty}^{0} x^n \,\mathrm{d}F(x) + \int_{0}^{\infty} x^n \,\mathrm{d}F(x)$$

and integration by parts give the second equality.

2.1.2 The first equality is just the definition of $\mathbb{E}(X^n)$ for the discrete random variable X. Furthermore, for each $j \in \mathbb{N}$ we have

$$\sum_{k=1}^{j}(k^n - (k-1)^n) r_k = \sum_{k=1}^{j} k^n p_k\,,$$

which gives the second equality.

2.1.3 We have

$$\begin{aligned} \mathbb{E}|X - x| &= \int_{-\infty}^{\infty} |y - x| \,\mathrm{d}F(y) = \int_{-\infty}^{x} (x - y)\,\mathrm{d}F(y) + \int_{x}^{\infty} (y - x)\,\mathrm{d}F(y) \\ &= 2xF(x) - x + \mathbb{E}\,X - 2\int_{-\infty}^{x} y \,\mathrm{d}F(y) \\ &= 2\int_{-\infty}^{x} F(y)\,\mathrm{d}y - x + \mathbb{E}\,X\,, \end{aligned}$$

where we used integration by parts in the last equality. We now see that the function $g(x) = 2\int_{-\infty}^{x} F(y)\,\mathrm{d}y - x$ is nonincreasing for $x \in (-\infty, \zeta_{1/2}]$ and nondecreasing for $x \in [\zeta_{1/2}, \infty)$. In order to prove that

$$\mathbb{E}\left((X - \mathbb{E}\,X)^2\right) \leq \mathbb{E}\left((X - x)^2\right)$$

for all $x \in \mathbb{R}$, we consider the function $\mathbb{E}\left((X-x)^2\right)$ of $x \in \mathbb{R}$, and find the global minimum. The first derivative is $2(x - \mathbb{E}\,X)$, which has the unique root at $x = \mathbb{E}\,X$.

2.1.4 Let $s, t > 0$. Then

$$\int_{-\infty}^{\infty} \mathrm{e}^{sx} f(x)\,\mathrm{d}x \geq \frac{1}{\pi}\int_0^t \frac{\mathrm{e}^{sx}}{1+x^2}\,\mathrm{d}x \longrightarrow_{t\to\infty} \infty\,,$$

since $\lim_{x\to\infty} \mathrm{e}^{sx}/(1+x^2) = \infty$. The case $s < 0$ follows analogously.

2.1.5 The first equality immediately follows from the definition of $\hat{m}(s)$, whereas integration by parts gives the second equality. For the Laplace–Stieltjes transform $\hat{l}(s)$ of F we have

$$\hat{l}(s) = \int_0^{\infty} \mathrm{e}^{-sx}\,\mathrm{d}F(x) = 1 - s\int_0^{\infty} \mathrm{e}^{-sx}\overline{F}(x)\,\mathrm{d}x$$

for all $s \in \mathbb{R}$ with $\hat{l}(s) < \infty$.

2.1.6 Suppose that $\hat{m}(s_0) < \infty$ for some $s_0 > 0$. Then

$$\mathrm{e}^{s_0 x}\overline{F}(x) \leq \int_x^{\infty} \mathrm{e}^{s_0 y}\,\mathrm{d}F(y) \longrightarrow_{x\to\infty} 0\,.$$

Thus there exists a constant $a > 0$ such that $\overline{F}(x) \leq a\,\mathrm{e}^{-s_0 x}$ for all $x \geq 0$. On the other hand, if for some $a, b > 0$ the inequality $\overline{F}(x) \leq a\,\mathrm{e}^{-bx}$ holds for all $x \geq 0$, then by Exercise 2.1.5

$$\begin{aligned}
\int_0^{\infty} \mathrm{e}^{sx}\,\mathrm{d}F(x) &\leq 1 + s\int_0^{\infty} \mathrm{e}^{sx}\overline{F}(x)\,dx \\
&\leq 1 + sa\int_0^{\infty} \mathrm{e}^{sx}\mathrm{e}^{-bx}\,dx \\
&\leq 1 + sa\int_0^{\infty} \mathrm{e}^{-(b-s)x}\,dx < \infty
\end{aligned}$$

for all $n \in \mathbb{N}$. Consider the density

$$f(x) = \begin{cases} \dfrac{2}{\pi}\dfrac{1}{1+x^2} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and use the result of Exercise 2.1.4 to show that then $\hat{m}(s) = \infty$ for all $s > 0$.

2.1.8 Observe that
$$(1-s^k) = (1-s)(1+\ldots+s^{k-1}).$$
Then we have
$$\begin{aligned} 1-\hat{g}_X(s) &= \sum_{k=1}^{\infty} p_k(1-s^k) = (1-s)\sum_{k=1}^{\infty} p_k(1+\ldots+s^{k-1}) \\ &= (1-s)\sum_{k=0}^{\infty} r_{k+1}s^k \, . \end{aligned}$$

2.1.9 Let $x_0, x_1, \ldots, x_n$ be arbitrary real numbers such that $0 \le x_0 \le x_1 \le \ldots \le x_n \le t$. Then, the distribution of the random vector $(X_{(1)}, \ldots, X_{(n)})$ is uniquely determined by the following probabilities:
$$\begin{aligned} &\mathbb{P}(x_0 \le X_{(1)} \le x_1 \le X_{(2)} \le x_2 \le \ldots \le X_{(n)} \le x_n) \\ &= \sum_{(i_1,\ldots,i_n)} \mathbb{P}(x_0 \le X_{i_1} \le x_1 \le X_{i_2} \le x_2 \le \ldots \le X_{i_n} \le x_n) \\ &= n!\,\mathbb{P}(x_0 \le X_1 \le x_1 \le X_2 \le x_2 \le \ldots \le X_n \le x_n) \\ &= n!\,t^{-n}\prod_{k=1}^{n}(x_k - x_{k-1}) \\ &= \int_{x_0}^{x_1} \ldots \int_{x_{n-1}}^{x_n} f(t_1,\ldots,t_n)\,\mathrm{d}t_n \ldots \mathrm{d}t_1 \, , \end{aligned}$$
where the summation $\sum_{(i_1,\ldots,i_n)}$ is taken over all permutations $(i_1,\ldots,i_n)$ of the set $\{1,\ldots,n\}$. Thus, $(X_{(1)},\ldots,X_{(n)})$ is absolutely continuous with density $f(t_1,\ldots,t_n)$ given in (2.1.2).

2.2 Parametrized Families of Distributions

We use the following symbols for parametrized families of distributions:
δ_a – degenerate distribution,
Ber(p) – Bernoulli distribution,
Bin(n,p) – Binomial distribution,
Poi(λ) – Poisson distribution,
Geo(p) – geometric distribution,
NB(α,p) – negative binomial distribution or Pascal distribution,
Del(λ,α,p) – Delaporte distribution,
Log(p) – logarithmic distribution,
UD(n) – (discrete) uniform distribution,
Si(θ,λ,a) – Sichel distribution

$\mathrm{Eng}(\theta, a)$ – Engen distribution,
$\mathrm{N}(\mu, \sigma^2)$ – normal distribution,
$\mathrm{Exp}(\lambda)$ – exponential distribution,
$\mathrm{Erl}(n, \lambda)$ – Erlang distribution distribution,
$\chi^2(n)$ – χ^2-distribution,
$\Gamma(a, \lambda)$ – gamma distribution,
$\mathrm{U}(a, b)$ – uniform distribution,
$\mathrm{Beta}(a, b, \eta)$ – beta distribution,
$\mathrm{IG}(\mu, \lambda)$ – inverse Gaussian distribution,
$\mathrm{EV}(\gamma)$ – extreme value distribution,
$\mathrm{LN}(a, b)$ – logarithmic normal (or lognormal) distribution,
$\mathrm{W}(r, c)$ – Weibull distribution,
$\mathrm{Par}(\alpha, c)$ – Pareto distribution,
$\mathrm{PME}(\alpha)$ – Pareto mixtures of exponentials,
$\mathrm{L\Gamma}(a, \lambda)$ – loggamma distribution,
$\mathrm{BenI}(a, b, c)$ – Benktander type I distribution,
$\mathrm{BenII}(a, b, c)$ – Benktander type II distribution.

The *convolution* $F * G$ of two distribution functions F, G is defined by

$$F * G(x) = \int_{-\infty}^{\infty} F(x-u)\,\mathrm{d}G(u)\,, \qquad x \in \mathbb{R}\,. \tag{2.2.1}$$

If both F and G have densities f and g, respectively, then the density of $F*G$ is given by the (density) convolution $f*g(x) = \int_{-\infty}^{\infty} f(x-u)g(u)\,\mathrm{d}u$ for $x \in \mathbb{R}$. The (discrete) convolution of two probability functions $\{p_k, k \in \mathbb{N}\}$ and $\{p'_k, k \in \mathbb{N}\}$ is given by

$$(p * p')_k = \sum_{i,j \in \mathbb{N}: i+j=k} p_i p'_j\,, \qquad k \in \mathbb{N}\,.$$

The n-fold convolution of F, denoted by F^{*n} is defined iteratively: for $n = 0$, $F^{*0}(x) = \delta_0(x)$ with $\delta_0(x) = 1$ if $x \ge 0$ and $\delta_0(x) = 0$ if $x < 0$ while for $n \ge 1$, $F^{*n} = F^{*(n-1)} * F = F * \ldots * F$ (n times). The n-fold convolution of other functions is similarly defined and denoted. For the tail of F^{*n} we write $\overline{F^{*n}}(x) = 1 - F^{*n}(x)$.

Consider a sequence $F_1, F_2, \ldots$ of distributions on $\mathcal{B}(\mathbb{R})$ and a probability function $\{p_k,\ n = 1, 2, \ldots\}$. Then, the distribution $F = \sum_{k=1}^{\infty} p_k F_k$ is called a *mixture* of $F_1, F_2, \ldots$ with weights $p_1, p_2, \ldots$ We can also have an uncountable family of distributions F_θ parametrized by θ, where θ is chosen from a certain subset Θ of $\mathbb{R}$ according to a distribution G concentrated on Θ. Formally, the mixture $F(x)$ of the family $\{F_\theta,\ \theta \in \Theta\}$ with *mixing distribution* G is given by $F(x) = \int_\Theta F_\theta(x)\,\mathrm{d}G(\theta)$, $x \in \mathbb{R}$.

Let X be a random variable with distribution F and let $C \in \mathcal{B}(\mathbb{R})$ be a certain subset of $\mathbb{R}$. The *truncated distribution* F_C is the conditional distribution of X given that the values of X are restricted to the set $C^c = \mathbb{R} \setminus C$, i.e. $F_C(B) = \mathbb{P}(X \in B \mid X \notin C)$, $B \in \mathcal{B}(\mathbb{R})$. For example, if X is discrete with probability function $\{p_k, k = 0, 1, 2, \ldots\}$, then the *zero truncation* $F_{\{0\}}$ is given by the probability function $\{\mathbb{P}(X = k \mid X \geq 1), k = 1, 2, \ldots\}$. In particular, if X is $\mathrm{Geo}(p)$-distributed, then the zero truncation is given by $\mathbb{P}(X = k \mid X \geq 1) = (1-p)p^{k-1}$ for $k = 1, 2, \ldots$. We refer to this distribution as the *truncated geometric distribution* and use the abbreviation $\mathrm{TG}(p)$.

Exercises

2.2.1 Prove that

(a) $\mathrm{Bin}(n_1, p) * \mathrm{Bin}(n_2, p) = \mathrm{Bin}(n_1 + n_2, p)$,
(b) $\mathrm{Poi}(\lambda_1) * \mathrm{Poi}(\lambda_2) = \mathrm{Poi}(\lambda_1 + \lambda_2)$,
(c) $\mathrm{NB}(\alpha_1, p) * \mathrm{NB}(\alpha_2, p) = \mathrm{NB}(\alpha_1 + \alpha_2, p)$,
(d) $F^{*n} = \mathrm{Erl}(n, \lambda)$ if $F = \mathrm{Exp}(\lambda)$,
(e) $\Gamma(a_1, \lambda) * \Gamma(a_2, \lambda) = \Gamma(a_1 + a_2, \lambda)$.

[Hint. Use a transform.]

2.2.2 Let $F = \mathrm{Exp}(\lambda)$. Prove that

$$\overline{F^{*n}}(x) = \mathrm{e}^{-\lambda x}\left(1 + \frac{\lambda x}{1!} + \ldots + \frac{(\lambda x)^{n-1}}{(n-1)!}\right). \qquad (2.2.2)$$

[Hint. Differentiate both sides of (2.2.2).]

2.2.3 Let the random variable N be Poisson distributed with parameter $\lambda > 0$. Show that $(\lambda - k)\mathbb{P}(N \leq k-1) \leq k\mathbb{P}(N = k)$ for all $k \in \{1, 2, \ldots\}$.

2.2.4 Suppose that $F_\theta = \mathrm{Poi}(\theta)$, $\theta > 0$ and $G = \Gamma(a, \lambda)$. Show that the mixture $F = \int F_\theta \,\mathrm{d}G(\theta)$ is $\mathrm{NB}(a, 1/(1+\lambda))$.

2.2.5 (Continuation) Let the random variables N and Λ have the distributions F and G of Exercise 2.2.4, respectively. Show that, for each $n \in \mathbb{N}$, the conditional (posterior) distribution of Λ given $N = n$ is $\Gamma(a + n, \lambda + 1)$.

2.2.6 Show that, for α tending to 0, the zero truncation of $\mathrm{NB}(\alpha, p)$ converges to $\mathrm{Log}(p)$, i.e. if X is $\mathrm{NB}(\alpha, p)$-distributed, then

$$\lim_{\alpha \to 0+} \mathbb{P}(X = k \mid X \geq 1) = -\frac{1}{\log(1-p)} \frac{p^k}{k}, \qquad k = 1, 2 \ldots .$$

[Hint. Use that $\Gamma(x) = \Gamma(x+1)/x$ for $x > 0$ and that the gamma function is continuous.]

2.2.7 Consider the mixture $F = \int_0^\infty F_\theta \, \mathrm{d}G(\theta)$, where $F_\theta = \mathrm{Poi}(\theta)$ and G is a mixing distribution, which is concentrated on $[0, \infty)$. Prove that $\sigma_F^2 \geq \mu_F$. Furthermore, show that $\mathrm{cv}_F \geq 1$ if $\mu_F \leq 1$.

2.2.8 Show that the tail $\overline{F}(x)$ of the mixture $F = \int_0^\infty F_\theta \, \mathrm{d}G(\theta)$ with $F_\theta = \mathrm{Exp}(\theta)$ and $G = \Gamma(a, \lambda)$ is $\overline{F}(x) = (\lambda^{-1}x + 1)^{-a}$.

2.2.9 Suppose that the duration T of a fire is a random variable with distribution $\mathrm{Exp}(\lambda)$. The loss $w(t)$ associated with a fire of a fixed duration t is assumed to be given by $w(t) = a\mathrm{e}^{bt}$ for some $a, b > 0$. Show that the random variable $w(T)$ has the Pareto distribution $\mathrm{Par}(\lambda/b, a)$.

2.2.10 Let Y be a $\mathrm{L}\Gamma(a, \lambda)$-distributed random variable. Determine the n-th moment of Y for $n < \lambda$. Show that $\mathbb{E}(Y^n) = \infty$ for $n \geq \lambda$. Furthermore, conclude from this that the moment generating function of Y is infinite on $(0, \infty)$. [Comment. The distribution of a random variable with the latter property is called *heavy-tailed*; see Section 2.5.]

2.2.11 Discuss whether or not for the Pareto, lognormal or Weibull $(0 < r < 1)$ distributions, their moment generating functions are finite on $(0, \infty)$.

2.2.12 Show by examples that the following cases are possible for the moment generating function $\hat{m}(s)$:

(a) $\hat{m}(s) < \infty$ for all $s \in \mathbb{R}$
(b) there exists $s_0 > 0$ such that $\hat{m}(s) < \infty$ for $s < s_0$ and $\hat{m}(s) = \infty$ for $s \geq s_0$,
(c) there exists $s_0 > 0$ such that $\hat{m}(s) < \infty$ for $s \leq s_0$ and $\hat{m}(s) = \infty$ for $s > s_0$.

[Hint. Use the inverse Gaussian distribution as an example for (c).]

Solutions

2.2.1 Recall that for arbitrary distributions F, G on $\mathbb{R}_+$ we have

$$\hat{m}_{F*G}(s) = \hat{m}_F(s)\hat{m}_G(s)$$

for all $s \leq 0$. Furthermore, $\hat{m}_{\mathrm{Bin}(n,p)}(s) = (ps + q)^n$ where $q = 1 - p$. Thus,

$$\begin{aligned} \hat{m}_{\mathrm{Bin}(n_1 + n_2, p)}(s) &= (ps + q)^{n_1 + n_2} \\ &= \hat{m}_{\mathrm{Bin}(n_1, p)}(s)\hat{m}_{\mathrm{Bin}(n_2, p)}(s) \,. \end{aligned}$$

Statement (a) now follows from the one-to-one correspondence between distributions and their moment generating functions. Statements (b) to (e) can be proved analogously.

2.2.2 Differentiating both sides of (2.2.2) we get for the density f^{*n} of F^{*n}:

$$f^{*n}(x) = \lambda \mathrm{e}^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}$$

which is the density of the Erlang distribution $\mathrm{Erl}(n, \lambda)$. The statement now follows from the result of Exercise 2.2.1d.

2.2.3 The asserted inequality can be proved by induction with respect to k. For $k = 1$ we obviously have $(\lambda - 1)\mathbb{P}(N \le 0) = (\lambda - 1)\mathrm{e}^{-\lambda} \le \lambda \mathrm{e}^{-\lambda}$. Suppose now that the inequality holds for some $k \ge 1$. Then,

$$\begin{aligned}
(\lambda - (k+1))\mathbb{P}(N \le k) &\le (\lambda - k)\big(\mathbb{P}(N \le k-1) + \mathbb{P}(N = k)\big) \\
&\le k\mathbb{P}(N = k) + (\lambda - k)\mathbb{P}(N = k) = \lambda \mathbb{P}(N = k) \\
&= \lambda \mathbb{P}(N = k) = (k+1)\mathbb{P}(N = k+1)\,.
\end{aligned}$$

2.2.4 Notice that

$$\hat{g}_F(s) = \int_0^\infty \hat{g}_{F_\theta}(s)\,\mathrm{d}G(\theta)\,.$$

Furthermore, for F_θ =Poi(θ), we have $\hat{g}_{F_\theta}(s) = \exp(\theta(s-1))$. Thus,

$$\begin{aligned}
\hat{g}_F(s) &= \frac{1}{\Gamma(a)} \int_0^\infty \exp(\theta(s-1))\lambda^a \theta^{a-1} \mathrm{e}^{-\lambda\theta}\,\mathrm{d}\theta \\
&= \Big(\frac{\lambda}{1+\lambda-s}\Big)^a = \Big(\frac{1-(1+\lambda)^{-1}}{1-s(1+\lambda)^{-1}}\Big)^a,
\end{aligned}$$

which is the generating function of $\mathrm{NB}(a, 1/(1+\lambda))$. The statement now follows from the one-to-one correspondence between distributions on $\mathbb{N}$ and their probability generating functions.

2.2.5 We have

$$\begin{aligned}
\mathbb{P}(\Lambda \le x \mid N = n) &= \frac{\mathbb{P}(\Lambda \le x, N = n)}{\mathbb{P}(N = n)} \\
&= \frac{\lambda^a \int_0^x \mathrm{e}^{-v} \frac{v^n}{n!} v^{a-1} \mathrm{e}^{-\lambda v}\,\mathrm{d}v \Gamma(a) n! (\lambda+1)^{a+n}}{\Gamma(a)\lambda^a \Gamma(a+n)} \\
&= \frac{(\lambda+1)^{a+n}}{\Gamma(a+n)} \int_0^x v^{n+a-1} \mathrm{e}^{-(\lambda+1)v}\,\mathrm{d}v\,.
\end{aligned}$$

2.2.6 For $k = 1, 2, \ldots$ we have

$$\mathbb{P}(X = k \mid X \ge 1) = \frac{\dfrac{\Gamma(\alpha+k)(1-p)^\alpha p^k}{\Gamma(\alpha)\Gamma(k+1)}}{1-(1-p)^\alpha}\,.$$

Since the gamma function is continuous, $\Gamma(1) = 1$ and $\Gamma(\alpha+1)/\Gamma(\alpha) = \alpha$ for $\alpha > 0$, this gives

$$\begin{aligned}
\lim_{\alpha\to 0+} \mathbb{P}(X = k \mid X \geq 1) &= \frac{p^k}{k} \lim_{\alpha\to 0+} \frac{\alpha}{1-(1-p)^\alpha} \\
&= \frac{p^k}{k} \lim_{\alpha\to 0+} \frac{\alpha}{1-\exp(\alpha\log(1-p))} \\
&= -\frac{1}{\log(1-p)}\frac{p^k}{k} .
\end{aligned}$$

2.2.7 Notice that

$$\begin{aligned}
\mu_F &= \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} \int_0^\infty \mathrm{e}^{-\theta} \frac{\theta^k}{(k-1)!}\, \mathrm{d}G(\theta) \\
&= \int_0^\infty \theta \mathrm{e}^{-\theta} \Big(\sum_{k=0}^{\infty} \frac{\theta^k}{k!} \Big)\, \mathrm{d}G(\theta) = \int_0^\infty \theta\, \mathrm{d}G(\theta) = \mu_G .
\end{aligned}$$

In the same way we get that $\mu_F^{(2)} = \mu_G + \mu_G^{(2)}$. Thus,

$$\sigma_F^2 = \mu_F^{(2)} - (\mu_F)^2 = \mu_G + \sigma_G^2 \geq \mu_G = \mu_F .$$

This implies that $\mathrm{cv}_F = \sigma_F/\mu_F \geq 1$ if $\mu_F \leq 1$.

2.2.8 We have

$$\begin{aligned}
\overline{F}(x) &= \frac{\lambda^a}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-\theta x} \theta^{a-1} \mathrm{e}^{-\lambda\theta}\, \mathrm{d}\theta \\
&= \frac{\lambda^a}{\Gamma(a)} \frac{1}{(x+\lambda)^a} \int_0^\infty t^{a-1} \mathrm{e}^{-t}\, \mathrm{d}t = \frac{1}{(\lambda^{-1}+1)^a} ,
\end{aligned}$$

where the substitution $t = (x+\lambda)\theta$ is used in the second equality.

2.2.9 We have

$$\begin{aligned}
\mathbb{P}(w(T) > x) &= \lambda \int_0^\infty \mathbb{P}(w(t) > x) \mathrm{e}^{-\lambda t}\, \mathrm{d}t \\
&= \lambda \int_0^\infty \mathbb{I}(t > b^{-1}\log(x/a)) \mathrm{e}^{-\lambda t}\, \mathrm{d}t \\
&= \begin{cases} \left(\dfrac{a}{\lambda}\right)^{\lambda/b} & \text{if } x \geq a, \\ 1 & \text{if } x < a. \end{cases}
\end{aligned}$$

2.2.10 Let $X = \log Y$. For $n < \lambda$ we have

$$\mathbb{E}(Y^n) = \mathbb{E}\left(\mathrm{e}^{n\log Y}\right) = \mathbb{E}\,\mathrm{e}^{nX} = \Big(\frac{\lambda}{\lambda - n}\Big)^a .$$

Since $\hat{m}_X(s)$ is infinite for all $s \geq \lambda$, we get that $\mathbb{E}(Y^n) = \infty$ for $n \geq \lambda$. Furthermore, using the result of Exercise 2.1.6, this implies that $\hat{m}_Y(s) = \infty$ for all $s > 0$.

2.2.11 Let $s > 0$. If $F = \mathrm{Par}(\alpha, c)$, then for all sufficiently large $c' > 0$ we have

$$\hat{m}_F(s) = \int_c^\infty \exp(sx)\frac{\alpha}{c}\Big(\frac{c}{x}\Big)^{\alpha+1}\,\mathrm{d}x \geq \alpha c^\alpha \int_{c'}^\infty 1\,\mathrm{d}x = \infty .$$

If $F = \mathrm{LN}(a, b)$, then

$$\begin{aligned}\hat{m}_F(s) &= \frac{1}{b\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{sx}\frac{1}{x}\exp\Big(-\frac{(\log x - a)^2}{2b^2}\Big)\,\mathrm{d}x\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \exp\Big(s\mathrm{e}^{tb+a} - \frac{t^2}{2}\Big)\,\mathrm{d}x = \infty ,\end{aligned}$$

where the substitution $t = b^{-1}(\log x - a)$ is used in the second equality. If $F = \mathrm{W}(r, c)$, then

$$\begin{aligned}\hat{m}_F(s) &= \int_0^\infty \mathrm{e}^{sx} rcx^{r-1}\exp(-cx^r)\,\mathrm{d}x\\ &= \int_0^\infty c\exp(st^{1/r} - ct)\,\mathrm{d}x = \infty ,\end{aligned}$$

where the substitution $t = x^r$ is used in the second equality.

2.2.12 (a) Let the random variable X be bounded, i.e. $|X| < c$ for some $c < \infty$. For instance, let X have the uniform distribution $\mathrm{U}(a, b)$. Then, obviously, $\hat{m}(s) < \infty$ for all $s \in \mathbb{R}$. Notice, however, that boundedness is not necessary. For instance, for the generating function of the Poisson distribution $\mathrm{Poi}(\lambda)$ we have $\hat{m}(s) = \exp(\lambda(s-1)) < \infty$ for all $s \in \mathbb{R}$.
(b) Consider the exponential distribution $\mathrm{Exp}(\lambda)$. Then, $\overline{F}(x) = \exp(-\lambda x)$ and

$$\hat{m}(s) = \begin{cases} \dfrac{\lambda}{\lambda - s} & \text{if } s < \lambda,\\ \infty & \text{if } s \geq \lambda.\end{cases}$$

(c) Consider the inverse Gaussian distribution $\mathrm{IG}(\mu, \lambda)$. Then,

$$\hat{m}(s) = \begin{cases} \dfrac{\exp(\lambda/\mu)}{\exp\big(\sqrt{\lambda^2\mu^{-2} - 2\lambda s}\big)} & \text{if } s \leq \dfrac{\lambda}{2\mu^2} ,\\ \infty & \text{if } s > \dfrac{\lambda}{2\mu^2} .\end{cases}$$

2.3 Associated Distributions

Let $s_F^- = \inf\{s \le 0 : \hat{m}_F(s) < \infty\}$ and $s_F^+ = \sup\{s \ge 0 : \hat{m}_F(s) < \infty\}$ be the lower and upper *abscissa of convergence*, respectively, of the moment generating function $\hat{m}_F(s)$. Clearly $s_F^- \le 0 \le s_F^+$. Whenever $s_F^- < s_F^+$, an infinite family of related distributions can be associated with F. For each $t \in (s_F^-, s_F^+)$

$$\tilde{F}_t(x) = \frac{1}{\hat{m}_F(t)} \int_{-\infty}^{x} \mathrm{e}^{ty}\, \mathrm{d}F(y)\,, \qquad x \in \mathbb{R}$$

defines a proper distribution on $\mathbb{R}$ called an *associated distribution* to F. The distribution $\tilde{F}_t$ is also called an *Esscher transform* of F. The whole family $\{\tilde{F}_t; s_F^- < t < s_F^+\}$ is called the *class of distributions associated to F*.

Exercises

2.3.1 Show: if $\hat{m}_F(s_1) < \infty$ and $\hat{m}_F(s_2) < \infty$ for some $s_1 < s_2$, then $\hat{m}_F(s) < \infty$ for all $s \in (s_1, s_2)$.

2.3.2 Let $s_F^- < t < s_F^+$. Show that the moment generating function of $\tilde{F}_t$ is

$$\hat{m}_{\tilde{F}_t}(s) = \frac{\hat{m}_F(s+t)}{\hat{m}_F(t)} \tag{2.3.1}$$

for $s_F^- - t < s < s_F^+ - t$. Moreover, show that all absolute moments of $\tilde{F}_t$ are finite; in particular, the expectation $\mu_{\tilde{F}_t}$ of the associated distribution $\tilde{F}_t$ is given by

$$\mu_{\tilde{F}_t} = \frac{\hat{m}_F^{(1)}(t)}{\hat{m}_F(t)}\,,$$

while the variance $\sigma^2_{\tilde{F}_t}$ of $\tilde{F}_t$ is given by

$$\sigma^2_{\tilde{F}_t} = \frac{\hat{m}_F^{(2)}(t)\hat{m}_F(t) - (\hat{m}_F^{(1)}(t))^2}{(\hat{m}_F(t))^2}\,.$$

2.3.3 Let $\lambda > 0$ and let G be an arbitrary distribution on $\mathbb{R}$ such that $s_G^- < s_G^+$. For the distribution $F(x) = \sum_{k=0}^{\infty}(k!)^{-1}\lambda^k \mathrm{e}^{-\lambda} G^{*k}(x)$, show that $\tilde{F}_t(x) = \sum_{k=0}^{\infty}(k!)^{-1}\tilde{\lambda}^k \mathrm{e}^{-\tilde{\lambda}} \tilde{G}_t^{*k}(x)$ for all $t \in (s_G^-, s_G^+)$, where $\tilde{\lambda} = \lambda \hat{m}_G(t)$. [Hint. Use that the moment generating function of F is given by $\hat{m}_F(s) = \exp(\lambda(\hat{m}_G(s) - 1))$, which will be shown in Exercise 4.2.2.]

2.3.4 For many classical distributions the associated distributions are of the same type. Show that this is true for the binomial, Poisson, negative-binomial, gamma and even normal distribution.

Solutions

2.3.1 Let $s_1 < s_2$ and $s \in (s_1, s_2)$. Then

$$\begin{aligned}\hat{m}_F(s) &= \int_{-\infty}^{0} \mathrm{e}^{sx}\,\mathrm{d}F(x) + \int_{0}^{\infty} \mathrm{e}^{sx}\,\mathrm{d}F(x)\\ &\le \int_{-\infty}^{\infty} \mathrm{e}^{s_1 x}\,\mathrm{d}F(x) + \int_{-\infty}^{\infty} \mathrm{e}^{s_2 x}\,\mathrm{d}F(x)\\ &= \hat{m}_F(s_1) + \hat{m}_F(s_2) < \infty,.\end{aligned}$$

2.3.2 Let $s_F^- < t < s_F^+$ and $s \in (S_F^- - t, s_F^+ - t)$. Then

$$\hat{m}_{\tilde{F}_t}(s) = \frac{1}{\hat{m}_F(t)} \int_{-\infty}^{\infty} \mathrm{e}^{sx}\mathrm{e}^{tx}\,\mathrm{d}F(x) = \frac{\hat{m}_F(s+t)}{\hat{m}_F(t)}\,.$$

In order to prove that $\int_{-\infty}^{\infty} |x|^n\,\mathrm{d}\tilde{F}_t(x) < \infty$ for all $n \in \mathbb{N}$, we can use the result of Exercise 2.1.6. If F is concentrated on $\mathbb{R}_+$ or on $\mathbb{R}_-$, then the finiteness of all absolute moments of $\tilde{F}_t$ follows directly from Exercise 2.1.6. Otherwise, we first represent F as the mixture $F = F(0)F^- + \overline{F}(0)F^+$, where

$$F^-(B) = \frac{F(B \cap \mathbb{R}_-)}{F(0)}\,, \qquad F^+(B) = \frac{F(B \cap \mathbb{R}_+)}{\overline{F}(0)}\,,$$

and use then Exercise 2.1.6 for $\tilde{F}_t^-$ and $\tilde{F}_t^+$ respectively. Furthermore,

$$\mu_{\tilde{F}_t} = \frac{1}{\hat{m}_F(t)} \int_{-\infty}^{\infty} x\mathrm{e}^{tx}\,\mathrm{d}F(x) = \frac{\hat{m}_F^{(1)}(t)}{\hat{m}_F(t)}$$

and

$$\mu_{\tilde{F}_t}^{(2)} = \frac{1}{\hat{m}_F(t)} \int_{-\infty}^{\infty} x^2\mathrm{e}^{tx}\,\mathrm{d}F(x) = \frac{\hat{m}_F^{(2)}(t)}{\hat{m}_F(t)}\,.$$

2.3.3 Using (2.3.1) and the fact that $\hat{m}_F(s) = \exp(\lambda(\hat{m}_G(s) - 1))$, we have

$$\begin{aligned}\hat{m}_{\tilde{F}_t}(s) &= \frac{\exp(\lambda(\hat{m}_G(s+t) - 1))}{\exp(\lambda(\hat{m}_G(t) - 1))}\\ &= \exp\Big(\lambda\hat{m}_G(t)\Big(\frac{\hat{m}_G(s+t)}{\hat{m}_G(t)} - 1\Big)\Big)\,,\end{aligned}$$

where the last expression is the generating function of the compound distribution $\sum_{k=0}^{\infty}(k!)^{-1}\tilde{\lambda}^k\mathrm{e}^{-\tilde{\lambda}}\tilde{G}_t^{*k}(x)$ with $\tilde{\lambda} = \lambda\hat{m}_G(t)$. The statement now follows from the one-to-one correspondence between distributions and their moment generating functions.

2.3.4 We use (2.3.1) and the one-to-one correspondence between distributions and their moment generating functions. If F is the binomial distribution $\mathrm{Bin}(n,p)$, then

$$\hat{m}_{\tilde{F}_t}(s) = \frac{(p\mathrm{e}^{s+t}+q)^n}{p\mathrm{e}^t+q)^n} = (p'\mathrm{e}^s+q')^n\,,$$

where $p' = p\mathrm{e}^t(p\mathrm{e}^t+q)^{-1}$ and $q' = 1-p'$. Thus, $\tilde{F}_t$ is the binomial distribution $\mathrm{Bin}(n,p')$. If F is the Poisson distribution $\mathrm{Poi}(\lambda)$, then

$$\hat{m}_{\tilde{F}_t}(s) = \frac{\exp(\lambda(\mathrm{e}^{s+t}-1))}{\exp(\lambda(\mathrm{e}^t-1))} = \exp(\lambda'(\mathrm{e}^s-1))\,,$$

where $\lambda' = \lambda\mathrm{e}^t$. Thus, $\tilde{F}_t$ $=\mathrm{Poi}(\lambda')$. If F is the negative binomial distribution $\mathrm{NB}(r,p)$, then

$$\hat{m}_{\tilde{F}_t}(s) = \frac{q^r(1-p\mathrm{e}^{s+t})^{-r}}{q^r(1-p\mathrm{e}^t)^{-r}} = (q')^R(1-p'\mathrm{e}^s)^{-r}\,,$$

where $p' = p\mathrm{e}^t$ and $q' = 1-p'$. Thus, $\tilde{F}_t$ $=\mathrm{NB}(r,p')$. If F is the gamma distribution $\Gamma(a,\lambda)$, then

$$\hat{m}_{\tilde{F}_t}(s) = \frac{\left(\dfrac{\lambda}{\lambda-s-t}\right)^a}{\left(\dfrac{\lambda}{\lambda-t}\right)^a} = \left(\frac{\lambda'}{\lambda^\lambda-s}\right)^a,$$

where $\lambda' = \lambda - t$. Thus, $\tilde{F}_t = \Gamma(a,\lambda')$. If F is the normal distribution $\mathrm{N}(\mu,\sigma^2)$, then

$$\hat{m}_{\tilde{F}_t}(s) = \frac{\exp\bigl(\mu(s+t)+\frac{1}{2}\sigma^2(s+t)^2\bigr)}{\exp\bigl(\mu(t)+\frac{1}{2}\sigma^2(t)^2\bigr)} = \exp\Bigl(\mu's+\frac{1}{2}\sigma^2s^2\Bigr)\,,$$

where $\mu' = \mu+\sigma^2t$. Thus, $\tilde{F}_t$ $=\mathrm{N}(\mu',\sigma^2)$.

2.4 Distributions with Monotone Hazard Rates

Let $\{p_k\}$ be the probability function of an $\mathbb{N}$-valued random variable X. Consider the tail probabilities $r_n = \mathbb{P}(X \geq n) = p_n + p_{n+1} + \ldots$ and define, for all $n \in \mathbb{N}$ such that $r_n > 0$,

$$m_n = \mathbb{P}(X = n \mid X \geq n) = \frac{p_n}{r_n}\,. \tag{2.4.1}$$

The quotient m_n in (2.4.1) is called the *hazard rate* of $\{p_k\}$ in the n-th period. The probability function $\{p_k\}$ or, alternatively, the corresponding distribution is $\mathrm{IHR_d}$ if the sequence $\{m_n\}$ is increasing, and $\mathrm{DHR_d}$ if the sequence $\{m_n\}$ is decreasing. If X is a nonnegative random variable with absolutely continuous distribution F and density function f, then we define the *hazard rate function* $m(t)$ by

$$m(t) = \frac{f(t)}{1 - F(t)} \qquad \text{if } F(t) < 1\,. \tag{2.4.2}$$

We say that the distribution F is IHR if $m(t)$ is increasing, and DHR if $m(t)$ is decreasing.

Define the *residual hazard distribution* F_t at t by $F_t(x) = \mathbb{P}(X - t \le x \mid X > t)$ if $F(t) < 1$. Since $\mu_{F_t} = \mathbb{E}\,(X - t \mid X > t)$, the function $\mu_F(t) = \mu_{F_t}$, $t \ge 0$ is called *the mean residual hazard function.*

A distribution F is called *stochastically smaller (larger)* than a distribution G if $\overline{F}(x) \le (\ge)\,\overline{G}(x)$ for all $x \in \mathbb{R}$. In this case, we write $F \le_{\mathrm{st}} G$ and $F \ge_{\mathrm{st}} G$, respectively. Furthermore, we write $X \le_{\mathrm{st}} Y$ if X, Y have distributions F, G respectively such that $F \le_{\mathrm{st}} G$.

Exercises

2.4.1 Show that the geometric distribution has the *lack-of-memory* property (discrete version), which says that a geometrically distributed X satisfies

$$\mathbb{P}(X \ge i + j \mid X \ge j) = \mathbb{P}(X \ge i) \tag{2.4.3}$$

for all $i, j \in \mathbb{N}$. Furthermore, show that the exponential distribution satisfies the following continuous version of the *lack-of-memory* property (2.4.3): if X is exponentially distributed, then

$$\mathbb{P}(X > t + s \mid X > s) = \mathbb{P}(X > t) \tag{2.4.4}$$

for all $s, t \ge 0$. Conclude that for these two types of distributions hazard rates are constant.

2.4.2 Show that the geometric distribution is the only distribution on $\mathbb{N}$ with constant hazard rates and that the exponential distribution is the only absolutely continuous distribution on $\mathbb{R}_+$ with constant hazard rate function.

2.4.3 Show that the sequence $a_k = k!$, $(k = 1, 2, \ldots)$ is logconvex and conclude that the sequence $b_k = \binom{n}{k}$ $(k = 0, \ldots, n)$ is logconvex too.

2.4.4 Show that the probability functions of the binomial distribution $\mathrm{Bin}(n, p)$ and the Poisson distribution $\mathrm{Poi}(\lambda)$ are logconcave. Conclude from this that the distributions $\mathrm{Bin}(n, p)$ and $\mathrm{Poi}(\lambda)$ are $\mathrm{IHR_d}$.

2.4.5 Let $\{p_k\}$ be a probability function and consider the corresponding tail probabilities $r_k = p_k + p_{k+1} + \ldots$. Show that $\{r_k\}$ fulfils

$$r_{k+1}^2 \le r_k r_{k+2} \tag{2.4.5}$$

for all $k \in \mathbb{N}$ if and only if $\{p_k\}$ is $\mathrm{DHR_d}$ and $p_k > 0$ for infinitely many $k \in \mathbb{N}$. State and prove an analogous property for $\mathrm{IHR_d}$. [Hint. The inequality (2.4.5) means that r_{k+1}/r_k is increasing. Now use $r_{k+1}/r_k = 1 - (p_k/r_k)$.]

2.4.6 Show that any mixture of $\mathrm{DHR_d}$-distributions on $\mathbb{N}$ is again a $\mathrm{DHR_d}$-distribution on $\mathbb{N}$.

2.4.7 Let F be an absolutely continuous distribution on $\mathbb{R}_+$ with continuous density f. Show that

$$\overline{F}(x) = \mathrm{e}^{-\int_0^x m(v)\,\mathrm{d}v}, \tag{2.4.6}$$

for $x \ge 0$. The function $M(x) = -\log \overline{F}(x) = \int_0^x m(v)\,\mathrm{d}v$ is called *hazard function*. Show that the hazard function $M(x)$ is differentiable and

$$\mathrm{d}M(x)/\,\mathrm{d}x = -\mathrm{d}\log \overline{F}(x)/\,\mathrm{d}x = m(x) \tag{2.4.7}$$

for all $x \ge 0$ such that $F(x) < 1$, where $m(x) = f(x)/\overline{F}(x)$ is the hazard rate function.

2.4.8 Show that for each Borel-measurable function $m : [0,\infty) \to [0,\infty)$ with $\int_0^x m(t)\,\mathrm{d}t < \infty$ for all $x \in (0,\infty)$ and $\int_0^\infty m(t)\,\mathrm{d}t = \infty$ there exists exactly one absolutely continuous distribution on $\mathbb{R}_+$ such that $m(t)$ is its hazard rate function.

2.4.9 Show that, for the Pareto distribution $\mathrm{Par}(\alpha, c)$, the hazard rate function $m(t)$ is given by $m(t) = \alpha/t$ for all $t \ge c$.

2.4.10 Compute the hazard rate functions for the uniform distribution $\mathrm{U}(a, b)$, the exponential distribution $\mathrm{Exp}(\lambda)$, Erlang distribution $\mathrm{Erl}(n, \lambda)$ and the Weibull distribution $\mathrm{W}(r, c)$. Find out when these functions are increasing or decreasing.

2.4.11 Consider two independent absolutely continuous risks X_1 and X_2 with hazard rate functions $m_1(t)$ and $m_2(t)$, respectively. Show that the hazard rate function of $\min\{X_1, X_2\}$ is $m_1(t) + m_2(t)$. Find an interpretation for the distribution (so-called *Makeham's law*) defined by the hazard rate function $A + Cc^x$. [Comment. In life insurance the distribution with $A = 0$ is called *Gompertz' law*.]

2.4.12 Let X be an absolutely continuous risk with distribution F. Show that $\overline{F}(x)$ is logconvex if and only if F is from DHR. Derive a similar result for IHR.

2.4.13 Let $F = (1-\theta)\delta_0 + \theta G$ for some $\theta \in (0,1)$, where G is an absolute continuous distribution from DHR. Show that $F_{t_1} \le_{\rm st} F_{t_2}$ for $t_1 \le t_2$.

2.4.14 Let F, G be two distributions on $\mathbb{R}$ whose the first order absolute moments are finite. Show that $\mu_F \le \mu_G$ if $F \le_{\rm st} G$.

2.4.15 Let F be an absolutely continuous distribution on $\mathbb{R}_+$. Show that the mean residual hazard function $\mu_F(t)$ is decreasing (increasing) if the hazard rate function $m(t)$ is increasing (decreasing). [Hint. Use Theorem 2.4.2 from RSST and the result of Exercise 2.4.14.]

2.4.16 Let F be the Weibull distribution $\mathrm{W}(r,c)$. Show that the mean residual hazard function $\mu_F(t)$ is decreasing if $r > 1$, increasing if $0 < r < 1$, and constant if $r = 1$.

2.4.17 Let F be a distribution on $\mathbb{R}_+$ with $0 < \mu_F < \infty$. Prove the following relationship between the tail function $\overline{F}(x)$ and the corresponding mean residual hazard function $\mu_F(t)$:

$$\overline{F}(x) = \frac{\mu_F}{\mu_F(x)} \exp\Big(-\int_0^x \frac{1}{\mu_F(t)}\,\mathrm{d}t\Big)\,, \qquad x \ge 0\,. \tag{2.4.8}$$

2.4.18 (Continuation) Show that F is an exponential distribution if and only if the mean residual hazard function $\mu_F(t)$ is constant. [Hint. Use the result of Exercise 2.4.17 for the sufficiency part.]

2.4.19 (Continuation) Assume that $\sup\{\mu_F(x) : x \ge 0\} < \infty$. Show that then $\hat{m}_F(t_0) < \infty$ for some $t_0 > 0$.

2.4.20 Show that for the Benktander distribution $\mathrm{BenI}(a,b,c)$ the mean residual hazard function is given by $\mu_F(t) = t/(a + 2b\log t)$ for $t > 1$ while, for the Benktander distribution $\mathrm{BenII}(a,b,c)$, $\mu_F(t) = t^{1-b}/a$ for $t > 1$.

Solutions

2.4.1 Let X have the geometric distribution $\mathrm{Geo}(p)$, i.e. $\mathbb{P}(X = k) = qp^k$ for $k = 0, 1, \ldots$, where $q = 1-p$. Then,

$$\begin{aligned}
\mathbb{P}(X \ge i+j \mid X \ge j) &= \frac{\sum_{k=i+j}^{\infty} qp^k}{\sum_{k=j}^{\infty} qp^k} = \frac{\sum_{k=i}^{\infty} p^k}{\sum_{k=0}^{\infty} p^k} \\
&= q\sum_{k=i}^{\infty} p^k = \mathbb{P}(X \ge i)\,.
\end{aligned}$$

Furthermore, (2.4.1) gives $m_n = q$. Let now X have the exponential distribution $\mathrm{Exp}(\lambda)$, i.e. $\mathbb{P}(X > x) = \exp(-\lambda x)$ for $x \geq 0$. Then,

$$\mathbb{P}(X > t + s \mid X > s) = \frac{\mathrm{e}^{-\lambda(t+s)}}{\mathrm{e}^{-\lambda s}} = \mathrm{e}^{-\lambda t} = \mathbb{P}(X > t)\,.$$

Furthermore, (2.4.2) gives $m(t) = \lambda$.

2.4.2 Let $\{p_k\}$ be a probability function such that $m_n = c$ for some $c \in (0,1)$ and for all $n \in \mathbb{N}$. Then, from (2.4.1) we immediately get that $p_0 = c$ and $p_1 = c(1-c)$. Suppose now that $p_k = c(1-c)^k$ holds for all $k = 0, 1, \ldots, n$. Then, (2.4.1) gives

$$p_{n+1} = c\Big(1 - \sum_{k=1}^{n} p_k\Big) = c\Big(1 - \sum_{k=1}^{n} c(1-c)^k\Big) = c(1-c)^{n+1}\,.$$

Let now F be an absolutely continuous distribution on $\mathbb{R}_+$ with density f such that $m(t) = \lambda$ for some $\lambda > 0$ and for all $t \geq 0$. Then, (2.4.2) gives $1 - F(t) > 0$ and $f(t) = \lambda(1 - F(t))$ for all $t \geq 0$, which means that the density f is continuous. Hence, $\overline{F}$ is differentiable with $\overline{F}^{(1)}(t) = -f(t)$. Thus, $\overline{F}^{(1)}(t)/\overline{F}(t) = -\lambda$ and consequently

$$\frac{\mathrm{d}}{\mathrm{d}t}\log(\overline{F}(t)) = -\lambda$$

for all $t \geq 0$. Since $\log(\overline{F}(0)) = 0$, this gives $\log(\overline{F}(t)) = -\lambda t$ and therefore $\overline{F}(t) = \exp(-\lambda t)$ for all $t \geq 0$.

2.4.3 The logconvexity of a_k (that is $a_k a_{k+2} \geq a_{k+1}^2$) holds because obviously

$$(k!)^2(k+1)^2 \leq (k!)^2(k+1)(k+2)\,.$$

The logconvexity of b_k can be deduced from the logconvexity of a_k.

2.4.4 Let $\{p_k\}$ be the probability function of $\mathrm{Bin}(n,p)$. Then, for $k = 0, 1, \ldots, n-2$,

$$\begin{aligned} p_{k+1}^2 &= \left(\binom{n}{k+1} p^{k+1}(1-p)^{n-k-1}\right)^2 \\ &\geq \left(\binom{n}{k+2} p^{k+2}(1-p)^{n-k-2}\right)\left(\binom{n}{k} p^{k}(1-p)^{n-k}\right) \\ &= p_{k+2}p_k\,. \end{aligned}$$

By e.g. Theorem 2.4.1 from RSST, since logconcavity implies $\mathrm{IHR_d}$, hence $\mathrm{Bin}(n,p)$ is $\mathrm{IHR_d}$. If $\{p_k\}$ be the probability function of $\mathrm{Poi}(\lambda)$, then

$$p_{k+1}^2 = \left(\mathrm{e}^{-\lambda}\frac{\lambda^{k+1}}{(k+1)!}\right)^2 \geq \left(\mathrm{e}^{-\lambda}\frac{\lambda^{k+2}}{(k+2)!}\right)\left(\mathrm{e}^{-\lambda}\frac{\lambda^{k}}{k!}\right) = p_{k+2}p_k$$

for all $k \in \mathbb{N}$. Thus Poi(λ) is IHR$_\mathrm{d}$.

2.4.5 Suppose that (2.4.5) holds for all $k \in \mathbb{N}$. This means that $r_k > 0$ for all $k \in \mathbb{N}$ and that r_{k+1}/r_k is increasing. Thus, using the identity $r_{k+1}/r_k = 1 - (p_k/r_k)$, we get that $m_k = p_k/r_k$ is decreasing. Suppose now that $\{p_k\}$ is DHR$_\mathrm{d}$ and $p_k > 0$ for infinitely many $k \in \mathbb{N}$. Then, $r_k > 0$ for all $k \in \mathbb{N}$ and the above used arguments can be reversed. This gives (2.4.5). If $\{p_k\}$ is a probability function such that $p_k > 0$ for infinitely many $k \in \mathbb{N}$, then it can be shown by similar arguments that $\{p_k\}$ is IHR$_\mathrm{d}$ if and only if $r_{k+1}^2 \ge r_k r_{k+2}$ for all $k \in \mathbb{N}$.

2.4.6 For each i, let $\{p_{in},\ n = 0, 1, \ldots\}$ be a probability function, and let the mixing distribution be $\{q_i\}$. Each probability function is DHR$_\mathrm{d}$ and we want to show that $\{p_n, n = 0, 1, \ldots\}$ defined by $p_n = \sum_i q_i p_{in}$ is DHR$_\mathrm{d}$ too. For this it suffices to demonstrate that, for $i \neq i'$

$$a_j = \frac{q_i p_{ij} + q_{i'} p_{i'j}}{q_i \sum_{k=j}^{\infty} p_{ik} + q_{i'} \sum_{k=j}^{\infty} p_{i'k}}$$

is decreasing. Let

$$x_\nu = q_\nu p_{\nu j}, \qquad y_\nu = q_\nu p_{\nu j+1}, \qquad z_\nu = q_\nu \sum_{k=j+2} p_{\nu k} \ .$$

for $\nu = i, i'$. Due to our assumption we have

$$\frac{x_\nu}{x_\nu + y_\nu + x_\nu} \ge \frac{y_\nu}{y_\nu + z_\nu} \ ,$$

which is equivalent to

$$\frac{x_\nu}{z_\nu} \ge \left(\frac{y_\nu}{z_\nu}\right)^2 + \frac{y_\nu}{z_\nu} \ . \tag{2.4.9}$$

Notice that $a_j \ge a_{j+1}$ is equivalent to

$$\frac{x_i + x_{i'}}{x_i + y_i + z_i + x_{i'} + y_{i'} + z_{i'}} \ge \frac{y_i + y_{i'}}{y_i + z_i + y_{i'} + z_{i'}} \ .$$

Using (2.4.9) the above will follow from

$$\frac{x_i}{z_i} + \frac{x_{i'}}{z_{i'}} \ge 2\frac{y_i}{z_i}\frac{y_{i'}}{z_{i'}} + \frac{y_i}{z_i} + \frac{y_{i'}}{z_{i'}} \ .$$

However adding inequalities (2.4.9) for $\nu = i, i'$ we obtain

$$\frac{x_i}{z_i} + \frac{x_{i'}}{z_{i'}} \ge (\frac{y_i}{z_i})^2 + (\frac{y_{i'}}{z_{i'}})^2 + \frac{y_i}{z_i} + \frac{y_{i'}}{z_{i'}} \ge 2\frac{y_i}{z_i}\frac{y_{i'}}{z_{i'}} + \frac{y_i}{z_i} + \frac{y_{i'}}{z_{i'}} \ .$$

2.4.7 Since f is continuous, the tail function $\overline{F}(x) = \int_x^\infty f(t)\,\mathrm{d}t$ is differentiable with $\overline{F}^{(1)}(x) = -f(t)$. Thus, $M(x) = -\log \overline{F}(x)$ is differentiable and $M^{(1)}(x) = f(x)/\overline{F}(x)$ for all $x \geq 0$ such that $F(x) < 1$.

2.4.8 Let $\mathcal{M}$ be the equivalence class of all non-negative functions fulfilling conditions of the exercise, and $\mathcal{A}$ be the class of all absolute continuous distributions on $\mathbb{R}_+$. The mapping

$$\mathcal{M} \ni m(x) \to \overline{F}(x) \in \mathcal{A}$$

given by (2.4.6) establishes the one to one correspondence. The validity of (2.4.6) for a general $F \in \mathcal{A}$ now follows from the fact that in this case a differentiation rule similar to (2.4.7) holds; see, for example, Theorem A4.12 in Last and Brandt (1995).

2.4.9 Let $m(t)$ be the hazard rate function of $\mathrm{Par}(\alpha, c)$. Then, for $t \geq c$,

$$m(t) = \frac{(\alpha/c)(c/t)^{\alpha+1}}{(c/t)^\alpha} = \frac{\alpha}{t}\,.$$

2.4.10 If F is the uniform distribution $\mathrm{U}(a,b)$ with $0 \leq a < b$, then the hazard rate function $m(t)$ is increasing and given by

$$m(t) = \begin{cases} 0 & \text{if } 0 \leq t < a, \\ \dfrac{1}{b-t} & \text{if } a \leq t < b. \end{cases}$$

If F is Erlang distribution $\mathrm{Erl}(n, \lambda)$, then

$$m(t) = \frac{\lambda^k x^{k-1} \mathrm{e}^{-\lambda x}}{\mathrm{e}^{-\lambda x}(1 + \lambda x + \ldots + \frac{(\lambda x)^{n-1}}{(n-1)!})},$$

which can be rewritten as

$$\lambda \frac{1}{a^{n-1} + a^{n-2} + \frac{a^{n-3}}{2!} + \ldots + \frac{1}{(n-1)!}},$$

where $a = (\lambda x)^{-1}$. The above function is decreasing in a so increasing in x. If F is the Weibull distribution $\mathrm{W}(r,c)$, then $m(t) = rcx^{r-1}$ for $x > 0$, which is increasing if $r > 1$, constant if $r = 1$, and decreasing if $r < 1$.

2.4.11 Let $F_i(x)$ and $f_i(x)$ denote distribution function and density of X_i respectively. Then, the distribution function $F(x)$ and the density $f(x)$ of $X = \min\{X_1, X_2\}$ are given by $F(x) = 1 - \overline{F}_1(x)\overline{F}_2(x)$ and $f(x) = f_1(x)\overline{F}_2(x) + f_2(x)\overline{F}_1(x)$. Thus, for the hazard rate function $m(t)$ of F we have

$$\begin{aligned} m(t) &= \frac{f_1(x)\overline{F}_2(x) + f_2(x)\overline{F}_1(x)}{\overline{F}_1(x)\overline{F}_2(x)} \\ &= \frac{f_1(t)}{\overline{F}_1(t)} + \frac{f_2(t)}{\overline{F}_2(t)} = m_1(t) + m_2(t) \end{aligned}$$

for all $t \in \mathbb{R}$ with $\overline{F}_1(t)\overline{F}_2(t) > 0$. In view of this result, we can interpret Makeham's law as the distribution of the minimum of an exponentially distributed random variable and another independent IHR-distributed random variable.

2.4.12 ¿¿From the result of Exercise 2.4.8 (see formula (2.4.6)) we have

$$\log \overline{F}(x) = -\int_0^x m(t)\,\mathrm{d}t\,.$$

Thus, the function $\log \overline{F}(x)$ is convex if F is DHR and concave if F is IHR.

2.4.13 For each $t \geq 0$ such that $G(t) < 1$ we have

$$\overline{F}_t(x) = \frac{\overline{G}(t+x)}{\overline{G}(t)} = \overline{G}_t(x)\,.$$

We now use that G is DHR, if and only if for each x, $G_t(x)$ is decreasing function of $t \geq 0$ (see e.g. Theorem 2.4.2 in RSST).

2.4.14 From the result of Exercise 2.1.1 we have

$$\mu_F = \int_0^\infty \overline{F}(x)\,\mathrm{d}x - \int_{-\infty}^0 F(x)\,\mathrm{d}x$$

and analogously

$$\mu_G = \int_0^\infty \overline{G}(x)\,\mathrm{d}x - \int_{-\infty}^0 G(x)\,\mathrm{d}x\,.$$

Thus, $F \leq_{\mathrm{st}} G$ implies that $\mu_F \leq \mu_G$.

2.4.15 Suppose that $m(t)$ is increasing. Then (e.g. by by Theorem 2.4.2 from RSST) we have $F_{t_1} \geq_{\mathrm{st}} F_{t_2}$ if $t_1 \leq t_2$. Thus, using the result of Exercise 2.4.14, we find that the function $\mu_F(t) = \mu_{F_t}$ is decreasing. The case where $m(t)$ is decreasing follows analogously.

2.4.16 The statement immediately follows from the results of Exercises 2.4.10 and 2.4.15.

2.4.17 Notice that (2.4.8) is equivalent to

$$\frac{1}{\mu_F}\int_x^\infty \overline{F}(t)\,\mathrm{d}t = \exp\Big(-\int_0^x \frac{1}{\mu_F(t)}\,\mathrm{d}t\Big)\,, \qquad x \ge 0\,.$$

Furthermore, in order to prove the latter equality, notice that $1/\mu_F(t)$ is the hazard rate function corresponding to the (integrated) tail function $\overline{F^{\mathrm{s}}}(x) = \mu_F^{-1}\int_x^\infty \overline{F}(t)\,\mathrm{d}t$. Then, the statement follows from the result of Exercise 2.4.8.

2.4.18 If $\mu_F(t)$ is constant, then $\mu_F(t) = \mu_F$ for all $t \ge 0$. Thus, the sufficiency part immediately follows from (2.4.8). If F is an exponential distribution, then the lack-of-memory property (2.4.4) implies that $\mu_F(t) = \mu_F$ for all $t \ge 0$.

2.4.19 Using (2.4.8) one can easily show that for some $a, b > 0$ we have $\overline{F}(x) < a\mathrm{e}^{-bx}$ for all $x \ge 0$. Thus, the statement follows from the result of Exercise 2.1.6.

2.5 Heavy-Tailed Distributions

In this section we consider distributions on $\mathbb{R}_+$ fulfilling $F(0-) = 0$. Such a distribution F is called *heavy-tailed* if $\hat{m}_F(s) = \infty$ for all $s > 0$. Let $\alpha_F = \limsup_{x\to\infty} M(x)/x$, where $M(x) = -\log \overline{F}(x)$ is the *hazard function* of F. A distribution F on $\mathbb{R}_+$ is said to be *subexponential* if

$$\lim_{x\to\infty} \frac{1 - F^{*2}(x)}{1 - F(x)} = 2\,.$$

Let $\mathcal{S}$ denote the class of all subexponential distributions. For a distribution F of a nonnegative random variable with finite expectation $\mu > 0$, the *integrated tail distribution* F^{s} is given by

$$F^{\mathrm{s}}(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \mu^{-1}\displaystyle\int_0^x \overline{F}(y)\,\mathrm{d}y & \text{if } x > 0. \end{cases}$$

We say that F belongs to the class $\mathcal{S}^*$ if F has finite expectation μ and

$$\lim_{x\to\infty}\int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)}\overline{F}(y)\,\mathrm{d}y = 2\mu\,.$$

Two distributions F and G on $\mathbb{R}_+$ are said to be *tail-equivalent* if $\lim_{x\to\infty} \overline{G}(x)/\overline{F}(x) = c$ for some $0 < c < \infty$. This will be denoted by $G \sim^{\mathrm{t}} F$. In particular, the asymptotic equivalence $\lim_{x\to\infty} g_1(x)/g_2(x) = 1$ of two functions $g_1(x)$ and $g_2(x)$ we denote by $g_1(x) \sim g_2(x)$. We say that F belongs to $\mathcal{L}$ if for all $y \in \mathbb{R}$ fixed we have $\overline{F}(x-y) \sim \overline{F}(x)$ as $x \to \infty$. Furthermore, we say that a positive function $L : \mathbb{R}_+ \to (0,\infty)$ is a *slowly varying function of x at ∞* if for all $y > 0$, $L(xy) \sim L(x)$ as $x \to \infty$. A distribution F is called *Pareto-type* with exponent $\alpha > 0$ if $\overline{F}(x) \sim L(x)x^{-\alpha}$ as $x \to \infty$ for a slowly varying function $L(x)$. Pareto-type distributions are also called *distributions with regular varying tails.*

Exercises

2.5.1 Show that for F being the distribution function of a bounded random variables, or being Erlang $\mathrm{Erl}(n,c)$ [2]

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = \infty\,.$$

2.5.2 Show that $|\log^p x|$, $\log(\log(1+x))$, functions with positive and finite limits are slowly varying functions. Show that $2+\sin x$ is not a slowly varying function.

2.5.3 Show that the following distribution functions are Pareto-type:

- Pareto $\mathrm{Par}(\alpha,c)$,
- loggamma with density function

$$f(x) = \frac{\lambda^a}{\Gamma(a)}(\log x)^{a-1}x^{-\lambda-1}, \qquad x > 1,$$

- Burr distribution with

$$\overline{F}(x) = \left(\frac{b}{b+x}\right)^{\alpha}, \qquad x \geq 0.$$

2.5.4 Show that F with the first finite moment is heavey tailed if and only if the corresponding integrated tail distribution F^{s} is heavy tailed.

2.5.5 Let F be heavy-tailed. Show that $\limsup_{x\to\infty} \mathrm{e}^{sx}\overline{F}(x) = \infty$ for all $s > 0$.

2.5.6 Let F be an arbitrary distribution of $\mathbb{R}_+$. Show that

$$\liminf_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \geq 2 \quad \text{and} \quad \liminf_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)}\overline{F}(y)\,\mathrm{d}y \geq 2\mu_F\,.$$

[2] I guess this must be true for any light tailed distr., perhaps with lim chanded to lim sup. TR

2.5.7 Let X_1 and X_2 be two independent nonnegative random variables such that $\lim_{x\to\infty} \overline{F_{X_i}}(x)/\overline{F}(x) = c_i$ for some $F \in \mathcal{S}$ and $c_i \in [0,\infty)$, where either $c_1 > 0$ or $c_2 > 0$. Show that $\mathbb{P}(X_1 + X_2 > x)/\overline{F}(x)$ tends to $c_1 + c_2$ as $x \to \infty$.

2.5.8 Suppose that $F \in \mathcal{S}$. Show that if $X_1, \ldots, X_n$ are independent random variables with common distribution F, then

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i > x\Big) \sim \mathbb{P}\Big(\max_{1\le i\le n} X_i > x\Big), \qquad x \to \infty .$$

2.5.9 Consider the nonnegative random variables $X_1, \ldots, X_n$ with distributions $F_1, \ldots, F_n$ respectively. Assume that F_i satisfies $\overline{F}_i(x) \sim c_i \overline{F}(x)$ for some Pareto-type distribution F and $c_i \in [0,\infty)$; $i = 1, \ldots, n$. Show that if $P(X_i > x, X_j > x) = o(\overline{F}(x))$ for all pairs $i \neq j$, then

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i > x\Big) \sim \Big(\sum_{i=1}^{n} c_i\Big)\overline{F}(x) .$$

Notice that we do not need any assumption on the independence nor identical distribution of the random variables $X_1, \ldots, X_n$.

2.5.10 Consider the mean residual hazard function $\mu_F(t)$ of F and assume that $\mu_F(t) \to \infty$ as $t \to \infty$. Show that then F is heavy-tailed.

2.5.11 Let F, G be two distributions on $\mathbb{R}_+$ with finite positive expectations. Show that $F^{\mathrm{s}} \sim^{\mathrm{t}} G^{\mathrm{s}}$ whenever $F \sim^{\mathrm{t}} G$.

2.5.12 (Continuation) Let $F \sim^{\mathrm{t}} G$. Show that $G^{\mathrm{s}} \in \mathcal{S}$ implies $F^{\mathrm{s}} \in \mathcal{S}$.

2.5.13 Let F be an absolutely continuous distribution on $\mathbb{R}_+$ with hazard rate function $m(x)$. Show that $F \in \mathcal{L}$ if $\lim_{x\to\infty} m(x) = 0$. [Hint. Use (2.4.6).]

2.5.14 Show that the lognormal distribution $\mathrm{LN}(a,b)$ belongs to $\mathcal{S}^*$. [Hint. Show first that the hazard rate function $m(x)$ of $\mathrm{LN}(a,b)$ is ultimately decreasing to 0 as $x \to \infty$ and use then Theorem 2.5.7 from RSST.]

2.5.15 Let F be the discrete distribution on $\mathbb{N} \setminus \{0\}$ with atoms of mass $3\cdot 4^{-n}$ at the points 2^n for $n \in \mathbb{N} \setminus \{0\}$. Show that $\mu = 3$ and $F^{\mathrm{s}} \in \mathcal{S}$, but $F \notin \mathcal{S}$.

2.5.16 Let F be the distribution on $\mathbb{R}_+$ defined by

$$\overline{F}(x) = \left(\frac{b}{b + x^r}\right)^{\alpha}, \qquad x \ge 0, \tag{2.5.1}$$

where $\alpha, b, r > 0$. Show that $F \in \mathcal{S}$ and, if $\alpha r > 1$, $F^{\mathrm{s}} \in \mathcal{S}$. [Comment. The distribution F given in (2.5.1) is called a *Burr distribution*. Notice that this distribution is Pareto-type.]

2.5.17 Let $\alpha > 2$ and let F_α be the Pareto mixture of exponentials with density function f_α given by

$$f_\alpha(x) = \int_{(\alpha-1)/\alpha}^{\infty} \theta^{-1} \mathrm{e}^{-\theta^{-1}x} \alpha \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} \theta^{-(\alpha+1)} \, \mathrm{d}\theta \,.$$

Show that $\overline{F}_\alpha(x) = a_\alpha f_{\alpha-1}(a_\alpha x)$ and, for $\alpha > 3$,

$$\overline{F^{\mathrm{s}}_\alpha}(x) = \int_x^\infty \overline{F}_\alpha(y) \, \mathrm{d}y = b_\alpha f_{\alpha-2}(c_\alpha x) \,,$$

where

$$a_\alpha = \frac{\alpha(\alpha-2)}{(\alpha-1)^2} \,, \quad b_\alpha = \frac{(\alpha-1)(\alpha-3)}{(\alpha-2)^2} \,, \quad c_\alpha = \frac{\alpha(\alpha-3)}{(\alpha-1)(\alpha-2)} \,.$$

2.5.18 Let F_α be the Pareto mixture of exponentials PME(α), where $\alpha > 1$, and let F^{s}_α be the corresponding integrated tail distribution. Show that as $x \to \infty$,

$$\overline{F^{\mathrm{s}}_\alpha}(x) \sim \Gamma(\alpha) \left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} x^{-(\alpha-1)} \,.$$

2.5.19 Let $\alpha > 2$. Show that the squared coefficient of variation $\mathrm{cv}^2 = \sigma^2/\mu^2$ of PME(α) is $\mathrm{cv}^2 = 1 + 2/(\alpha(\alpha-2))$.

2.5.20 Show that for the tail function $\overline{F}_2(x)$ and the density function $f_2(x)$ of PME(2),

$$\overline{F}_2(x) = \frac{1}{2x^2} \left(1 - (1+2x)\mathrm{e}^{-2x}\right)$$

and

$$f_2(x) = \frac{1}{x^3} \left(1 - (1 + 2x + 2x^2)\mathrm{e}^{-2x}\right) \,.$$

Solutions

2.5.1 For F being Erlangian use the result of Exercise 2.2.2.

2.5.2 For $x > 1$ and $xy > 1$

$$\frac{(\log xy)^p}{(\log x)^p} = \left(1 + \frac{\log y}{\log x}\right)^p \,.$$

Hence for all y, passing with $x \to \infty$, the RHS tends to 1. To show that $2 + \sin x$ is not a slowly varying function, let $x_n = 2\pi n$ and $y = 1/4$. Then

$$\limsup_{x\to\infty} \frac{2+\sin(x_n y)}{2+\sin(x_n)} = \frac{3}{2} \qquad \liminf_{x\to\infty} \frac{2+\sin(x_n y)}{2+\sin(x_n)} = \frac{1}{2}\,.$$

2.5.3 For the Pareto we write

$$\overline{F}(x) = \left(\frac{x}{c}\right)^{-\alpha} L(x),$$

where

$$L(x) = \begin{cases} \left(\frac{x}{c}\right)^{\alpha} & \text{for } 0 \le x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

The loggamma case is immediately implied by Exercise 2.5.1.

2.5.4 Notice that from Exercise 2.1.5

$$\hat{m}_F(s) = 1 + s\mu_F \hat{m}_{F^s}(s),$$

which shows that F^s is heavy tailed if and only if F is.

2.5.5 Suppose that $\limsup_{x\to\infty} e^{bx}\overline{F}(x) < \infty$ for some $b > 0$. Then, for some $a \in (0,\infty)$, the inequality $\overline{F}(x) \le a\exp(-bx)$ holds for all $x \ge 0$. The result of Exercise 2.1.6 now implies that $\hat{m}(s_0) < \infty$ for some $s_0 > 0$. Thus, F cannot be heavy-tailed.

2.5.6 From definition (2.2.1) of the convolution F^{*2} we immediately get the identity

$$\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 1 + \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)}\, \mathrm{d}F(y)\,.$$

Since $\overline{F}(x-y)/\overline{F}(x) \ge 1$, this identity can be used to see that for each $\varepsilon > 0$ there exists $x_0 > 0$ such that

$$\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \ge 2 - \varepsilon$$

for all $x \ge x_0$. This means that

$$\liminf_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \ge 2\,.$$

In order to prove the second inequality, we can use the following identity

$$\int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)}\overline{F}(y)\,\mathrm{d}y = \int_0^{x/2} \ldots + \int_{x/2}^x \ldots = 2\int_0^{x/2} \frac{\overline{F}(x-y)}{\overline{F}(x)}\overline{F}(y)\,\mathrm{d}y\,,$$

since

$$\int_0^{x/2} \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y)\,\mathrm{d}y \geq \int_0^{x/2} \overline{F}(y)\,\mathrm{d}y \to_{x\to\infty} \mu_F \,.$$

2.5.7 Let F_1 and F_2 denote the distributions of X_1 and X_2 respectively. Suppose without loss of generality that $c_1 > 0$. Then, by Lemma 2.5.4, we have $F_1 \in \mathcal{S}$. Moreover, using Lemma 2.5.2 we can conclude that

$$\frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} = \frac{\overline{F_1 * F_2}(x)}{\overline{F}_1(x)} \frac{\overline{F}_1(x)}{\overline{F}(x)} \to \Big(1 + \frac{c_2}{c_1}\Big) c_1 = c_1 + c_2$$

as $x \to \infty$, since

$$\frac{\overline{F}_2(x)}{\overline{F}_1(x)} = \frac{\overline{F}_2(x)}{\overline{F}(x)} \frac{\overline{F}(x)}{\overline{F}_1(x)} \to_{x\to\infty} c_2 \frac{1}{c_1} \,.$$

2.5.8 We have to show that $\overline{F^{*n}}(x) \sim 1 - (F(x))^n$ as $x \to \infty$. Indeed, using Theorem 2.5.3 from RSST we have

$$\frac{\overline{F^{*n}}(x)}{1 - (F(x)))^n} = \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \frac{\overline{F}(x)}{1 - (F(x)))^n} \to n \frac{1}{n}$$

as $x \to \infty$, since

$$\frac{1 - (F(x)))^n}{1 - F(x)} = 1 + F(x) + \ldots + (F(x))^{n-1} \to_{x\to\infty} n \,.$$

2.5.9 Let $\overline{F}(x) = x^{-\alpha} L(x)$. for $n = 2$ use that for $\epsilon < 1/2$

$$\{X_1 + X_2 > x\} \subset \{X_1 > (1-\epsilon)x\} \cup \{X_2 > (1-\epsilon)x\} \cup \{X_1 > \epsilon x, X_2 > \epsilon x\},$$

from which we have

$$\begin{aligned} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} &\leq \\ &\leq \frac{\overline{F}_1((1-\epsilon)x) + \overline{F}_2((1-\epsilon)x) + \mathbb{P}(X_1 > \epsilon x, X_2 > \epsilon x)}{\overline{F}(x)} \,. \end{aligned}$$

We now have for $i = 1, 2$

$$\frac{\overline{F}_i((1-\epsilon)x)}{\overline{F}(x)} = \frac{\overline{F}_i((1-\epsilon)x)}{\overline{F}((1-\epsilon)x)} (1-\epsilon)^{-\alpha} \frac{L((1-\epsilon)x)}{L(x)} \to c_i (1-\epsilon)^{-\alpha} \,.$$

Furthermore, by our assumption,

$$\frac{\mathbb{P}(X_1 > \epsilon x, X_2 > \epsilon x)}{\overline{F}(x)} \to 0 \,.$$

Hence

$$\limsup_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} \le c_1 + c_2 \; .$$

To prove that

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} \ge c_1 + c_2$$

use that

$$\{X_1 > x\} \cup \{X_2 > x\} \subset \{X_1 + X_2 > x\}$$

from which we have

$$\overline{F}_1(x) + \overline{F}_2(x) - \mathbb{P}(X_1 > x, X_2 > x) \le \mathbb{P}(X_1 + X_2 > x) \; .$$

2.5.10 Let

$$\delta(x) = \frac{\overline{F}(x)}{\int_x^\infty \overline{F}(v)\,\mathrm{d}v}.$$

By the assumption $\delta(x) \to 0$ for $x \to \infty$. Now notice that

$$-\delta(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log\left(\frac{1}{\mu_F}\int_x^\infty \overline{F}(v)\,\mathrm{d}v\right).$$

Thus

$$\frac{1}{\mu_F}\int_x^\infty \overline{F}(v)\,\mathrm{d}v = \mathrm{e}^{-\int_0^x \delta(v)\,\mathrm{d}v}.$$

Clearly

$$\frac{\int_0^x \delta(v)\,\mathrm{d}v}{x} \to 0.$$

Now we can use e.g. Theorem 2.5.1 from RSST to conclude that F is heavy tailed or notice that for all $\epsilon > 0$, there exists x_0 such that

$$M(x) \le \epsilon x, \qquad x \ge x_0$$

or $\overline{F}(x) \ge \exp(-\epsilon x)$. Since ϵ is arbitrary, the function $\exp(sx)\overline{F}(x)$ cannot be integrable for any $s > 0$, and therefore F is heavy-tailed.

2.5.11 Suppose that $\lim_{x\to\infty} \overline{F}(x)/\overline{G}(x) = c$ for some $c \in (0,\infty)$. Then, for each $\varepsilon > 0$ there exists $x_0 > 0$ such that

$$\frac{\int_x^\infty \overline{F}(y)\,\mathrm{d}y}{\int_x^\infty \overline{G}(y)\,\mathrm{d}y} = \frac{\int_x^\infty \overline{G}(y)\frac{\overline{F}(y)}{\overline{G}(y)}\,\mathrm{d}y}{\int_x^\infty \overline{G}(y)\,\mathrm{d}y} \le \frac{(c+\varepsilon)\int_x^\infty \overline{G}(y)\,\mathrm{d}y}{\int_x^\infty \overline{G}(y)\,\mathrm{d}y} = c + \varepsilon$$

for all $x > x_0$. Thus,

$$\limsup_{x\to\infty} \frac{\int_x^\infty \overline{F}(y)\,\mathrm{d}y}{\int_x^\infty \overline{G}(y)\,\mathrm{d}y} \le c\,.$$

In the same way we can show that

$$\liminf_{x\to\infty} \frac{\int_x^\infty \overline{F}(y)\,\mathrm{d}y}{\int_x^\infty \overline{G}(y)\,\mathrm{d}y} \ge c\,.$$

2.5.12 By the result of Exercise 2.5.11 we have $F^{\mathrm{s}} \sim^{\mathrm{t}} G^{\mathrm{s}}$. The statement now follows from Lemma 2.5.4.

2.5.13 Let $y \in \mathbb{R}$ be fixed. Without loss of generality we can assume that $y > 0$. Then, using formula (2.4.6) we have

$$\frac{\overline{F}(x-y)}{\overline{F}(x)} = \frac{\exp\Big(-\int_0^{x-y} m(t)\,\mathrm{d}t\Big)}{\exp\Big(-\int_0^{x} m(t)\,\mathrm{d}t\Big)} = \exp\Big(\int_{x-y}^{x} m(t)\,\mathrm{d}t\Big)\,,$$

where the last expression converges to 1 as $x \to \infty$ provided that $\lim_{t\to\infty} m(t) = 0$.

2.5.14 Let X be $\mathrm{N}(a,b)$-distributed, i.e. the random variable $Y = \mathrm{e}^X$ is $\mathrm{LN}(a,b)$-distributed. Then, $Y = \mathrm{e}^a Z^{\sqrt{b}}$, where $Z = \exp((X-a)/\sqrt{b})$ is $\mathrm{LN}(0,1)$-distributed. Thus,

$$\overline{F}_Y(x) = 1-\Phi(\log(x\mathrm{e}^{-a})^{1/\sqrt{b}}), \qquad m_Y(x) = \frac{\phi\big(\log(x\mathrm{e}^{-a})^{1/\sqrt{b}}\big)}{x\big(1-\Phi\big(\log(x\mathrm{e}^{-a})^{1/\sqrt{b}}\big)\big)}\,,$$

where $\Phi(x)$ is the standard normal distribution function with density $\phi(x)$. Since $\phi(x) \sim x(1-\Phi(x))$ as $x \to \infty$, this shows that the hazard rate function $m_Y(x)$ is ultimately decreasing to 0 as $x \to \infty$. Furthermore, in the same way as in the case of the standard lognormal distribution $\mathrm{LN}(0,1)$ we can show (see Example 2 in Section 2.5.3) that

$$\int_0^\infty \exp(x m_Y(x))\overline{F}_Y(x)\,\mathrm{d}x < \infty\,.$$

Thus, by Theorem 2.5.7 from RSST we have $\mathrm{LN}(a,b) \in \mathcal{S}^*$.

2.5.15 We have

$$\mu_F = \sum_{k=1}^\infty 2^k \frac{3}{4^k} = 3\sum_{k=1}^\infty \Big(\frac{1}{2}\Big)^k = 3\,.$$

Furthermore, $\overline{F}(2^n) = \sum_{k=n+1}^{\infty}(3/4^k) = 4^{-n}$. Thus, $\overline{F}(2^n-1)/\overline{F}(2^n) = 4$ for all $n = 2, 3, \ldots$ and therefore $F \notin \mathcal{S}$ by the result of Lemma 2.5.1. On the other hand, for all $n \in \mathbb{N}$ and $x \in [2^n, 2^{n+1})$ we have for the tail of F^{s}

$$\overline{F^{\mathrm{s}}}(x) = \frac{1}{3}\Big(\frac{2^{n+1}-x}{4^n} + \frac{1}{2^n}\Big)$$

and for the hazard rate function

$$m_{F^{\mathrm{s}}}(x) = \frac{1}{2^{n+1}-x+2^n}\ .$$

This gives

$$\limsup_{x\to\infty} \frac{\overline{F^{\mathrm{s}}}(x/2)}{\overline{F^{\mathrm{s}}}(x)} \le \limsup_{n\to\infty} \frac{\overline{F^{\mathrm{s}}}(2^{n-1})}{\overline{F^{\mathrm{s}}}(2^{n+1})} = 4$$

and $\lim_{x\to\infty} m_{F^{\mathrm{s}}}(x) = 0$. Thus, by the result of Exercise 2.5.13, $F^{\mathrm{s}} \in \mathcal{L}$. Consequently, using the identity

$$\frac{\overline{(F^{\mathrm{s}})^{*2}}(x)}{\overline{F^{\mathrm{s}}}(x)} = 2\int_0^{x/2} \frac{\overline{F^{\mathrm{s}}}(x-y)}{\overline{F^{\mathrm{s}}}(x)}\,\mathrm{d}F^{\mathrm{s}}(y) + \frac{\big(\overline{F^{\mathrm{s}}}(x/2)\big)^2}{\overline{F^{\mathrm{s}}}(x)}\ ,$$

the bounded convergence theorem gives $\lim_{x\to\infty} \overline{(F^{\mathrm{s}})^{*2}}(x)/\overline{F^{\mathrm{s}}}(x) = 2$.

2.5.16 Notice that (2.5.1) can be written in the form $\overline{F}(x) = L(x)x^{-r\alpha}$ for all $x > 0$, where

$$L(x) = \Big(\frac{bx^r}{b+x^r}\Big)^{\alpha}$$

is a slowly varying function. This means that F is a Pareto-type distribution. Hence, by Theorem 2.5.5 from RSTT, $F \in \mathcal{S}$. Assume now that $\alpha r > 1$. Then $\mu_F < \infty$ and, by Karamata's theorem (see e.g. Theorem 2.5.8 in RSST), F^{s} is Pareto-type too. Thus, using Theorem 2.5.5 from RSST anew, we find that $F^{\mathrm{s}} \in \mathcal{S}$. However, there still is another way to show that $F^{\mathrm{s}} \in \mathcal{S}$. Notice that the hazard rate function $m_F(x)$ of the Burr distribution F is given by

$$m_F(x) = \alpha r \frac{x^{r-1}}{b+x^r}\ .$$

Thus, $\limsup_{x\to\infty} x m_F(x) < \infty$ and, by Corollary 2.5.1 from RSST, $F^{\mathrm{s}} \in \mathcal{S}$.

2.5.17

$$\begin{aligned}\bar{F}_\alpha(x) &= \int_x^\infty \left(\int_{\frac{\alpha-1}{\alpha}}^\infty \alpha\left(\frac{\alpha-1}{\alpha}\right)^\alpha \theta^{-(\alpha+2)}\mathrm{e}^{-\theta^{-1}y}\,\mathrm{d}\theta\right)\mathrm{d}y = \\ &= \int_{\frac{\alpha-1}{\alpha}}^\infty \alpha\left(\frac{\alpha-1}{\alpha}\right)^\alpha \theta^{-((\alpha-1)+2)}\mathrm{e}^{-\theta^{-1}x}\,\mathrm{d}\theta\ .\end{aligned}$$

Susbtituting $\theta = (a_\alpha)^{-1}s$ the above equal to

$$\begin{aligned}
(a_\alpha)^{-1} \int_{\frac{\alpha-2}{\alpha-1}} \alpha \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} \theta^{-((\alpha-1)+2)} \mathrm{e}^{-(a_\alpha x)s^{-1}}\, \mathrm{d}s &= \\
= \quad a_\alpha \int_{\frac{\alpha-2}{\alpha-1}}^{\infty} (\alpha-1) \left(\frac{\alpha-2}{\alpha-1}\right)^{\alpha-1} \theta^{-((\alpha-1)+2)} \mathrm{e}^{-(a_\alpha x)s^{-1}}\, \mathrm{d}s.
\end{aligned}$$

Hence

$$\overline{F^{\mathrm{s}}}_\alpha(x) = \int_0^\infty \overline{F}_\alpha(y)\, \mathrm{d}y = \alpha \int_0^\infty f_{\alpha-1}(a_\alpha y)\, \mathrm{d}y = a_{\alpha-1} f_{\alpha-2}(a_\alpha a_{\alpha-1} x).$$

Now $a_{\alpha-1} = b_\alpha$ and $a_\alpha a_{\alpha-1} = c_\alpha$.

2.5.18 In the remark at the end of Section 2.5.4 in RSST it was shown that

$$\overline{F}_\alpha(x) \sim \Gamma(\alpha+1) \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} x^{-\alpha}$$

as $x \to \infty$. Thus, using Theorems 2.5.8 and 2.5.9a from RSST, we get that $\mu_{F_\alpha} = 1$ and

$$\overline{F^{\mathrm{s}}_\alpha}(x) \sim \frac{\Gamma(\alpha+1)\left(\frac{\alpha-1}{\alpha}\right)^{\alpha}}{\alpha-1} x^{-(\alpha-1)} \sim \Gamma(\alpha)\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} x^{-(\alpha-1)},$$

since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. We can also get the result directly from Exercise 2.5.17. Thus

$$\begin{aligned}
f_\alpha(x) &= \alpha \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} \int_{\frac{\alpha-1}{\alpha}}^{\infty} \mathrm{e}^{-x\theta^{-1}} \theta^{-(\alpha+2)}\, \mathrm{d}\theta \\
&= -\alpha \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} x^{-1} \int_{\frac{\alpha-1}{\alpha}}^{\infty} \mathrm{e}^{-x\theta^{-1}} \theta^{-\alpha} \left(-\frac{x}{\theta^2}\right) \mathrm{d}\theta\ .
\end{aligned}$$

Substituting $s = x/\theta$, the above equals to:

$$\begin{aligned}
\alpha \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} x^{-1} \int_0^{x\frac{\alpha}{\alpha-1}} x^{-\alpha} \mathrm{e}^{-s} s^{\alpha}\, \mathrm{d}s &= \\
= \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} \Gamma(\alpha+1) x^{-(\alpha+1)} L(x), \qquad &(2.5.2)
\end{aligned}$$

where $L(x) \nearrow 1$.

2.5.19 Use formula (see e.g. RSST, Theorem 2.5.9a)

$$\mu_\alpha^{(n)} = \frac{n!}{\alpha - n}\alpha\left(\frac{\alpha-1}{\alpha}\right)^n .$$

2.5.20 From (2.5.2)

$$f_2(x) = 2\left(\frac{1}{2}\right)^2 x^3 \int_0^{2x} s^2 \mathrm{e}^{-s}\, \mathrm{d}s .$$

Now from Exercise 2.2.2

$$\int_0^{2x} s^2 \mathrm{e}^{-s}\, \mathrm{d}s = 2(1 - \mathrm{e}^{-2x}(1 + 2x + 2x^2)) .$$

2.6 Quantile Plots

[3] [4] For an increasing and right-continuous function $F(x)$, we define the *generalized inverse function* $F^{-1}(y)$ by $F^{-1}(y) = \inf\{x : \ F(x) \ge y\}$. If F is a distribution function, then function Q_F defined by $Q_F(y) = F^{-1}(y)$ is called the *quantile function* of F. For a sequence of sample variables $\{U_i, 1 \le i \le n\}$, the *empirical distribution* F_n is defined by

$$F_n(x) = n^{-1} \max\left\{i : U_{(i)} \le x\right\}$$

for all $x \in \mathbb{R}$. Consider the quantile function $Q_n(y) = Q_{F_n}(y)$ of the empirical distribution F_n. Then, for the ordered sample $U_{(1)} \le U_{(2)} \le \ldots \le U_{(n)}$ we have

$$\left\{Q_n(y) = U_{(k)}\right\} = \{(k-1)n^{-1} < y \le kn^{-1}\}$$

and a continuity-corrected *quantil plot* will graph the points

$$\left\{\left(Q_F(k/(n+1)), U_{(k)}\right), 1 \le k \le n\right\}.$$

Exercises

2.6.1 Show that the continuity-corrected quantile plot of the exponential distribution $F = \mathrm{Exp}(\lambda)$ is given by

$$\left\{\left(-\lambda^{-1}\log(1 - \frac{k}{n+1}), U_{(k)}\right), 1 \le k \le n\right\}.$$

[3] Should we add some further exercises to this section? (VS) We have no exercises on heavy tailed distributions. TR

[4] Add solutions.

2.6.2 Consider the model of Exercise 2.6.1 and the least-squares statistic $\hat{\lambda}^{-1}$ for λ, which minimizes $\sum_{k=1}^{n} \left(U_{(k)} + \lambda^{-1} \log\left(1 - k/(n+1)\right)\right)^2$. Show that $\hat{\lambda}^{-1}$ is given by

$$\hat{\lambda}^{-1} = \sum_{k=1}^{n} U_{(k)} Q_G(k/(n+1)) \Big/ \sum_{k=1}^{n} \Big(Q_G(k/(n+1))\Big)^2 ,$$

where $G(x) = 1 - \exp(-x)$ is the distribution function of the (standard) exponential distribution.

2.6.3 Show that the normal distribution $\mathrm{N}(\mu, \sigma^2)$ has the quantile function $Q(y) = \mu + \sigma\Phi^{-1}(y)$, where $\Phi^{-1}(y)$ is the quantile function of the standard normal distribution $\mathrm{N}(0,1)$.

2.6.4 Show how the parameters μ and σ of the normal distribution $\mathrm{N}(\mu, \sigma^2)$ can be estimated from the quantil plot

$$\left\{ \left(\Phi^{-1}(k/(n+1)), U_{(k)}\right),\ 1 \le k \le n \right\} .$$

3

Premiums and Ordering of Risks

3.1 Premium Calculation Principles

In actuarial applications, a nonnegative random variable X is frequently called a *risk*. Consider a certain family of risks X. A *premium calculation principle* is a rule that determines the premium as a functional, assigning a value $\Pi(F_X) \in \mathbb{R} \cup \{\pm\infty\}$ to the risk distribution F_X. Following our notational convention we usually write $\Pi(X)$ instead of $\Pi(F_X)$. For example, the simplest premium principle is the (pure) *net premium principle* $\Pi(X) = \mathbb{E}\,X$ provided that $\mathbb{E}\,X < \infty$. In general, however, the difference $\Pi(X) - \mathbb{E}\,X$ is called the *safety loading*.

For some $a > 0$, $p \geq 1$ and $0 < \varepsilon < 1$, further premium calculation principles are the

- *expected value principle*

$$\Pi(X) = (1+a)\mathbb{E}\,X\ , \tag{3.1.1}$$

- *variance principle*

$$\Pi(X) = \mathbb{E}\,X + a\mathrm{Var}\,X\ , \tag{3.1.2}$$

- *standard deviation principle*

$$\Pi(X) = \mathbb{E}\,X + a\sqrt{\mathrm{Var}\,X}\ , \tag{3.1.3}$$

- *modified variance principle*

$$\Pi(X) = \begin{cases} \mathbb{E}\,X + a\mathrm{Var}\,X/\mathbb{E}\,X & \text{if } \mathbb{E}\,X > 0, \\ 0 & \text{if } \mathbb{E}\,X = 0, \end{cases} \tag{3.1.4}$$

- *exponential principle*

$$\Pi(X) = a^{-1}\log \mathbb{E}\,\mathrm{e}^{aX}\ , \tag{3.1.5}$$

- *risk-adjusted principle*

$$\Pi(X) = \int_0^\infty (1 - F_X(x))^{1/p}\,\mathrm{d}x\,, \tag{3.1.6}$$

- *ε-quantile principle*

$$\Pi(X) = F_X^{-1}(1 - \varepsilon)\,, \tag{3.1.7}$$

- *absolute deviation principle*

$$\Pi(X) = \mathbb{E}\,X + a\kappa_X\,, \tag{3.1.8}$$

  where $\kappa_X = \mathbb{E}\,|X - F_X^{-1}(1/2)|$ is the expected absolute deviation from the median $F_X^{-1}(1/2)$ of X.

Let X, Y, Z be arbitrary risks for which the premiums below are well-defined and finite. We have the following list of desirable properties:

- *no unjustified safety loading* if, for all constants $a \ge 0$, $\Pi(a) = a$,
- *proportionality* if, for all constants $a \ge 0$, $\Pi(aX) = a\Pi(X)$,
- *subadditivity* if $\Pi(X + Y) \le \Pi(X) + \Pi(Y)$,
- *additivity* if $\Pi(X + Y) = \Pi(X) + \Pi(Y)$,
- *consistency* if, for all $a \ge 0$, $\Pi(X + a) = \Pi(X) + a$,
- *monotonicity under stochastic order* if $X \le_{\mathrm{st}} Y$ implies $\Pi(X) \le \Pi(Y)$,
- *compatibility under mixing* if, for all $p \in [0,1]$ and for all Z, $\Pi(X) = \Pi(Y)$ implies $\Pi(pF_X + (1-p)F_Z) = \Pi(pF_Y + (1-p)F_Z)$.

A function $v : (b_1, b_2) \to \mathbb{R}$ which is increasing and concave on a certain interval $(b_1, b_2) \subset \mathbb{R}$ is called a *utility function*. A function $w : (b_1, b_2) \to \mathbb{R}$ which is increasing and convex on (b_1, b_2) is called a *loss function*.

Exercises

3.1.1 Show that for the variance principle, $\Pi(X+Y) < \Pi(X) + \Pi(Y)$ holds if X and Y are negatively correlated, where the equality holds if and only if X and Y are uncorrelated.

3.1.2 Show that the standard deviation principle and the modified variance principle are proportional. Moreover, show that these two principles are subadditive provided that $\mathrm{Cov}(X, Y) \le 0$.

3.1.3 Find an example which shows that for the premium calculation principles given by (3.1.2)–(3.1.4), the inequality $\Pi(X) \le \Pi(Y)$ does not always follow from $F_X \le_{\mathrm{st}} F_Y$.

3.1.4 Let the risk X be $\Gamma(b, \lambda)$-distributed. Determine the premium $\Pi(X)$ using the premium calculation principles given by (3.1.1)–(3.1.5) and (3.1.8), respectively.

3.1.5 Show that for the expected absolute deviation from the median, $\kappa_{X+Y} \le \kappa_X + \kappa_Y$ for all nonnegative random variables X, Y such that $\mathbb{E}\,X < \infty$ and $\mathbb{E}\,Y < \infty$. Conclude from this that the absolute deviation principle is subadditive. [Hint. Use the result of Exercise 2.1.3.]

3.1.6 Show that the absolute deviation principle is proportional and consistent.

3.1.7 Show that the risk-adjusted principle has nonnegative safety loading, but no unjustified safety loading. Moreover, show that this principle is proportional, consistent, and monotone with respect to stochastic ordering of risks.

3.1.8 Let $a > 0$ and let X be a nonnegative random variable such that $\mathbb{E}\,\mathrm{e}^{aX} < \infty$. Show that for the exponential utility function $v(x) = (1 - \mathrm{e}^{-ax})/a$, the solution to the equation

$$\mathbb{E}\,(v(\Pi(X) - X)) = v(0) \qquad (3.1.9)$$

is given by $\Pi(X) = a^{-1} \log \mathbb{E}\,\mathrm{e}^{aX}$.

3.1.9 Show that the exponential principle is additive for independent risks.

3.1.10 Show that the solution to (3.1.9), i.e. the *zero utility premium*, does not change if, instead of $v(x)$, the linear transform $av(x) + b$ is considered; $a > 0$ and $b \in \mathbb{R}$.

3.1.11 Consider a premium calculation principle Π which is compatible under mixing. Show that

$$\Pi\Big(\sum_{i=1}^{n} p_i F_{X_i}\Big) = \Pi\Big(\sum_{i=1}^{n} p_i F_{Y_i}\Big)$$

holds provided that $\Pi(F_{X_i}) = \Pi(F_{Y_i})$, $0 \le p_i \le 1$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$.

Solutions

3.1.1 Recall that

$$\mathbb{E}\,(X + Y) = \mathbb{E}\,X + \mathbb{E}\,Y \qquad (3.1.10)$$

and

$$\mathrm{Var}\,(X + Y) = \mathrm{Var}\,X + \mathrm{Var}\,Y + 2\mathrm{Cov}(X, Y)\,. \qquad (3.1.11)$$

Thus, $\mathrm{Var}\,(X + Y) < \mathrm{Var}\,X + \mathrm{Var}\,Y$ if and only if $\mathrm{Cov}(X, Y) < 0$, and $\mathrm{Var}\,(X + Y) = \mathrm{Var}\,X + \mathrm{Var}\,Y$ if and only if $\mathrm{Cov}(X, Y) = 0$.

3.1.2 Proportionality is easily obtained, since $\mathbb{E}\,(cX) = c\mathbb{E}\,X$ and $\mathrm{Var}\,(cX) = c^2 \mathrm{Var}\,X$ for all $c \in \mathbb{R}$. Suppose now that $\mathrm{Cov}(X,Y) \le 0$. Then, using (3.1.10) and (3.1.11), we have for the standard deviation principle

$$\begin{aligned}\Pi(X+Y) &= \mathbb{E}\,(X+Y) + a\sqrt{\mathrm{Var}\,(X+Y)} \\ &\le \mathbb{E}\,X + \mathbb{E}\,Y + a\sqrt{\mathrm{Var}\,X + \mathrm{Var}\,Y} \\ &\le \big(\mathbb{E}\,X + a\sqrt{\mathrm{Var}\,X}\big) + \big(\mathbb{E}\,Y + a\sqrt{\mathrm{Var}\,Y}\big) \\ &= \Pi(X) + \Pi(Y)\,,\end{aligned}$$

and for the modified variance principle

$$\begin{aligned}\Pi(X+Y) &= \mathbb{E}\,(X+Y) + a\frac{\mathrm{Var}\,(X+Y)}{\mathbb{E}\,(X+Y)} \\ &\le \mathbb{E}\,X + \mathbb{E}\,Y + a\frac{\mathrm{Var}\,X + \mathrm{Var}\,Y}{\mathbb{E}\,X + \mathbb{E}\,Y} \\ &\le \Big(\mathbb{E}\,X + a\frac{\mathrm{Var}\,X}{\mathbb{E}\,X}\Big) + \Big(\mathbb{E}\,Y + a\frac{\mathrm{Var}\,Y}{\mathbb{E}\,Y}\Big) \\ &= \Pi(X) + \Pi(Y)\end{aligned}$$

provided that $\mathbb{E}\,X\mathbb{E}\,Y > 0$. If $\mathbb{E}\,X = 0$ or $\mathbb{E}\,Y = 0$, then obviously $\Pi(X+Y) = \Pi(X) + \Pi(Y)$.

3.1.3 Let X have the uniform distribution $\mathrm{U}(0,b)$ for some $b > 0$ and let $Y = c$ for some constant $c > 0$. Then $X \le_{\mathrm{st}} Y$ if $b \le c$. But the inequality

$$\Pi(X) \le \Pi(Y) \tag{3.1.12}$$

does not hold for the premium calculation principles given by (3.1.2)–(3.1.4) provided that the constants $a, b, c > 0$ are appropriately chosen. Indeed, (3.1.12) does not hold for the variance principle if $a > 6(2c-b)b^{-2}$, for the standard deviation principle if $a > \sqrt{12}(2c-b)/(2b)$, and for the modified variance principle if $a > 3(2c-b)b^{-1}$.

3.1.4 We have $\mathbb{E}\,X = b\lambda^{-1}$, $\mathrm{Var}\,X = b\lambda^{-2}$, $\mathbb{E}\,\mathrm{e}^{aX} = \big(\lambda/(\lambda-a)\big)^b$ for $a < \lambda$, and $\kappa_X = \mathbb{E}\,|X - F_X^{-1}(1/2)| = \ldots$ [1] Thus, $\Pi(X) = (1+a)b/\lambda$ for the expected value principle, $\Pi(X) = (\lambda+a)b\lambda^{-2}$ for the variance principle, $\Pi(X) = (b + a\sqrt{b})\lambda^{-1}$ for the standard deviation principle, $\Pi(X) = (b+a)\lambda^{-1}$ for the modified variance principle, $\Pi(X) = a^{-1}\big(\lambda/(\lambda-a)\big)^b$ for the exponential principle, $\Pi(X) = \ldots$ [2] for the absolute deviation principle.

[1] Complete the solution.
[2] Complete the solution.

3.1.5 Suppose that $\mathbb{E}\,X < \infty$ and $\mathbb{E}\,Y < \infty$. Then, by the result of Exercise 2.1.3, we have

$$\begin{aligned}
\kappa_{X+Y} &= \mathbb{E}\,|X+Y-F_{X+Y}^{-1}(1/2)| \\
&\le \mathbb{E}\,|X+Y-F_X^{-1}(1/2)-F_Y^{-1}(1/2)| \\
&\le \mathbb{E}\,|X-F_X^{-1}(1/2)|+\mathbb{E}\,|Y-F_Y^{-1}(1/2)| \\
&= \kappa_X+\kappa_Y\,.
\end{aligned}$$

Using (3.1.10), this shows that the absolute deviation principle is subadditive.

3.1.6 For $a \ge 0$ we have $\mathbb{E}\,(aX) = a\mathbb{E}\,X$ and

$$\begin{aligned}
\kappa_{aX} &= \mathbb{E}\,|aX-F_{aX}^{-1}(1/2)| = \mathbb{E}\,|aX-aF_X^{-1}(1/2)| \\
&= a\mathbb{E}\,|X-F_X^{-1}(1/2)| = a\kappa_X\,.
\end{aligned}$$

Thus, the absolute deviation principle is proportional. The proof of consistency is analogous.

3.1.7 Notice that for $p \ge 1$

$$\int_0^\infty \big(1-F_X(x)\big)^{1/p}\,\mathrm{d}x \ge \int_0^\infty \big(1-F_X(x)\big)\,\mathrm{d}x = \mathbb{E}\,X\,,$$

which means that the risk-adjusted principle has nonnegative safety loading. If $X = c$ for some constant $c \ge 0$, then $\int_0^\infty \big(1-F_X(x)\big)^{1/p}\,\mathrm{d}x = c$. Thus, there is no unjustified safety loading. Moreover, for each $a > 0$, we have

$$\begin{aligned}
\Pi(aX) &= \int_0^\infty \big(1-F_{aX}(x)\big)^{1/p}\,\mathrm{d}x = \int_0^\infty \big(1-F_X(x/a)\big)^{1/p}\,\mathrm{d}x \\
&= a\int_0^\infty \big(1-F_X(y)\big)^{1/p}\,\mathrm{d}y = a\Pi(X)\,,
\end{aligned}$$

which shows proportionality. In order to show consistency, notice that

$$\begin{aligned}
\Pi(X+a) &= \int_0^\infty \big(1-F_{X-a}(x)\big)^{1/p}\,\mathrm{d}x \\
&= a+\int_a^\infty \big(1-F_{X-a}(x)\big)^{1/p}\,\mathrm{d}x = a+\Pi(X)\,.
\end{aligned}$$

Let now X, Y be two nonnegative random variables such that $X \le_{\mathrm{st}} Y$. Then, obviously,

$$\Pi(X) = \int_0^\infty \big(1-F_X(x)\big)^{1/p}\,\mathrm{d}x \le \int_0^\infty \big(1-F_Y(x)\big)^{1/p}\,\mathrm{d}x = \Pi(Y)\,.$$

3.1.8 For $v(x) = (1 - \mathrm{e}^{-ax})/a$, equation (3.1.9) takes the form

$$\mathbb{E}\left(a^{-1}(1 - \exp(-a(\Pi(X) - X)))\right) = 0\,,$$

which is equivalent to $\Pi(X) = a^{-1}\log \mathbb{E}\,\mathrm{e}^{aX}$.

3.1.9 Let the random variables X, Y be nonnegative, independent and such that $\mathbb{E}\,\mathrm{e}^{a(X+Y)} < \infty$ for some $a > 0$. Then,

$$\mathbb{E}\,\mathrm{e}^{a(X+Y)} = \mathbb{E}\left(\mathrm{e}^{aX}\mathrm{e}^{aY}\right) = \mathbb{E}\,\mathrm{e}^{aX}\,\mathbb{E}\,\mathrm{e}^{aY}\,.$$

Thus,

$$\begin{aligned} \Pi(X+Y) &= a^{-1}\log \mathbb{E}\,\mathrm{e}^{a(X+Y)} \\ &= a^{-1}\log \mathbb{E}\,\mathrm{e}^{aX} + a^{-1}\log \mathbb{E}\,\mathrm{e}^{aY} = \Pi(X) + \Pi(Y)\,. \end{aligned}$$

3.1.10 From the linearity properties of the expectation we get that (3.1.9) is equivalent to $\mathbb{E}\left(av(\Pi(X) - X) + b\right) = av(0) + b$.

3.1.11 We have

$$\begin{aligned} \Pi\Big(\sum_{i=1}^{n} p_i F_{X_i}\Big) &= \Pi\Big((1-p_n)\Big(\frac{1}{1-p_n}\sum_{i=1}^{n-1} p_i F_{X_i}\Big) + p_n F_{X_n}\Big) \\ &= \Pi\Big((1-p_n)\Big(\frac{1}{1-p_n}\sum_{i=1}^{n-1} p_i F_{X_i}\Big) + p_n F_{Y_n}\Big) \\ &= \Pi\Big((1-p_{n-1})\frac{1}{1-p_{n-1}}\Big(\sum_{i=1}^{n-2} p_i F_{X_i} + p_n F_{Y_n}\Big) + p_{n-1} F_{X_{n-1}}\Big) \\ &= \Pi\Big((1-p_{n-1})\frac{1}{1-p_{n-1}}\Big(\sum_{i=1}^{n-2} p_i F_{X_i} + p_n F_{Y_n}\Big) + p_{n-1} F_{Y_{n-1}}\Big) \\ &\;\vdots \\ &= \Pi\Big(\sum_{i=1}^{n} p_i F_{Y_i}\Big)\,. \end{aligned}$$

3.2 Ordering of Distributions

The notion of *stochastic order* mentioned in Section 2.4 is equivalent to the following integral version of stochastic ordering. Besides this, we consider two further integral orderings.

Let X, Y be two real-valued random variables. We say that X is *stochastically dominated* by (or *stochastically smaller* than) Y and we write $X \le_{\mathrm{st}} Y$ if for all increasing functions $g : \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}\, g(X) \le \mathbb{E}\, g(Y)\,, \tag{3.2.1}$$

provided the expectations $\mathbb{E}\, g(X), \mathbb{E}\, g(Y)$ exist and are finite.

Assume that $\mathbb{E}\, X_+, \mathbb{E}\, Y_+ < \infty$, where $x_+ = \max\{0, x\}$. We say that X is smaller than Y in *stop-loss order* and we write $X \le_{\mathrm{sl}} Y$ if (3.2.1) holds for all increasing convex functions $g : \mathbb{R} \to \mathbb{R}$ provided the expectations $\mathbb{E}\, g(X), \mathbb{E}\, g(Y)$ exist and are finite.

Assume that $\mathbb{E}\,(-X)_+, \mathbb{E}\,(-Y)_+ < \infty$. We say that X is smaller than Y in *increasing-concave order* and we write $X \le_{\mathrm{icv}} Y$ if (3.2.1) holds for all increasing concave functions $g : \mathbb{R} \to \mathbb{R}$ provided the expectations $\mathbb{E}\, g(X), \mathbb{E}\, g(Y)$ exist and are finite.

Exercises

3.2.1 Show that the absolute deviation principle is monotone with respect to stochastic ordering of risks.

3.2.2 Let the random variables X, Y be stochastically ordered, i.e. $X \le_{\mathrm{st}} Y$. Show that $\overline{F}_X(x) < \overline{F}_Y(x)$ on an interval of positive length provided that $F_X \ne F_Y$.

3.2.3 Let F and F' be two distributions on $\mathbb{N}$ with probability functions $\{p_k, k \in \mathbb{N}\}$ and $\{p'_k, k \in \mathbb{N}\}$ respectively. Show that $F \le_{\mathrm{st}} F'$ if and only if

$$\sum_{j=k}^{\infty} p_j \le \sum_{j=k}^{\infty} p'_j, \qquad k \in \mathbb{N}$$

and also if and only if, for each increasing sequence $g = (g_1, g_2, \ldots)$,

$$\sum_{j=0}^{\infty} p_j g_j \le \sum_{j=0}^{\infty} p'_j g_j\,.$$

3.2.4 Let $b \in (0, \infty)$ be fixed. Show that for each distribution F on $[0, b]$ there exist two sequences of discrete distributions $\{F_n^-\}$ and $\{F_n^+\}$ with finitely many atoms such that $F_n^- \le_{\mathrm{st}} F \le_{\mathrm{st}} F_n^+$ and

$$\lim_{n\to\infty} \Big(\int_0^b c(x)\, \mathrm{d}F_n^+(x) - \int_0^b c(x)\, \mathrm{d}F_n^-(x) \Big) = 0\,,$$

where the function $c : [0, b] \to \mathbb{R}_+$ is continuous and increasing.

3.2.5 Show that the following statements are equivalent:
(a) $X \leq_{\mathrm{sl}} Y$,
(b) $\int_x^\infty \overline{F}_X(y)\,\mathrm{d}y \leq \int_x^\infty \overline{F}_Y(y)\,\mathrm{d}y$ for all $x \in \mathbb{R}$,
(c) $\mathbb{E}\max\{X, x\} \leq \mathbb{E}\max\{Y, x\}$ for all $x \in \mathbb{R}$.

3.2.6 All orderings considered in Section 3.2 belong to the class of *integral orderings*. That is, for some class $\mathcal{H}$ of functions $h : \mathbb{R} \to \mathbb{R}$, we say that $F \prec_{\mathcal{H}} G$ holds for two distributions F and G if

$$\int_{-\infty}^{\infty} h(t)\,\mathrm{d}F(t) \leq \int_{-\infty}^{\infty} h(t)\,\mathrm{d}G(t)$$

for all $h \in \mathcal{H}$ provided the integrals in this inequality exist. Assume that the functions belonging to $\mathcal{H}$ are increasing and have the following invariance property: if $h \in \mathcal{H}$, then the shifted function $h_t(\cdot) = h(\cdot + t)$ also belongs to $\mathcal{H}$ for each $t \in \mathbb{R}$. (Note that the classes of increasing, increasing and convex, increasing and concave functions considered in the definition of the orderings $\leq_{\mathrm{st}}$, $\leq_{\mathrm{sl}}$ and $\leq_{\mathrm{icv}}$ possess these monotonicity and invariance properties.) Show that the following properties (a) and (b) hold for each integral ordering:
(a) $\delta_x \prec_{\mathcal{H}} \delta_y$ if and only if $x \leq y$.
(b) If X_1 is independent of X_2, Y_1 is independent of Y_2, $X_1 \prec_{\mathcal{H}} Y_1$ and $X_2 \prec_{\mathcal{H}} Y_2$ then $X_1 + X_2 \prec_{\mathcal{H}} Y_1 + Y_2$.

3.2.7 Let X, Y have the distributions $\mathrm{Bin}(n, p)$ and $\mathrm{Poi}(np)$ respectively. Show that $X \leq_{\mathrm{sl}} Y$.

3.2.8 Let X, Y have the distributions $\mathrm{Poi}(\lambda)$ and $\mathrm{NB}(\alpha, \lambda/(\alpha + \lambda))$, where $\alpha \in \mathbb{N}$. Show that $X \leq_{\mathrm{sl}} Y$.

3.2.9 For each $t > 0$, let X_t have the ditribution $\Gamma(t, \lambda/t)$. Show that $X_t \leq_{\mathrm{sl}} X_{t+h}$ for all $h \geq 0$.

3.2.10 Let X and Y be absolutely continuous random variables with $\mu_X \leq \mu_Y$. Assume that the difference $f_Y(t) - f_X(t)$ of the densities $f_X(t), f_Y(t)$ changes the sign twice, say at $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, where $f_Y(t) - f_X(t) \geq 0$ for all $t \geq t_2$. Show that then $X \leq_{\mathrm{sl}} Y$. Moreover, show that $X \leq_{\mathrm{st}} Y$ does not hold unless $F_X = F_Y$.

3.2.11 Let X and Y be normally distributed random variables with $\mu_X \leq \mu_Y$ and $\sigma_X^2 \leq \sigma_Y^2$. Show that $X \leq_{\mathrm{sl}} Y$.

3.2.12 Let X and Y have the distributions $\mathrm{LN}(a_X, b_X)$ and $\mathrm{LN}(a_Y, b_Y)$ respectively, where $a_X \leq a_Y$ and $b_X \leq b_Y$. Show that $X \leq_{\mathrm{sl}} Y$.

3.2.13 Let X and Y have the distributions $\mathrm{W}(r_X, c_X)$ and $\mathrm{W}(r_Y, c_Y)$ with $r_X \geq r_Y$, and let $\mathbb{E}X \leq \mathbb{E}Y$. Show that $X \leq_{\mathrm{sl}} Y$.

3.2.14 Let $a > 0$ and consider the utility function $v(x) = 1 - \mathrm{e}^{-ax}$. Show that there exists an increasing and right-continuous function $\eta(t)$ such that

$$v(x) = v(+\infty) - \int_{-\infty}^{\infty} (t-x)_+^2 \, \mathrm{d}\eta(t)\,, \qquad x \geq 0$$

and $\int_{-\infty}^{\infty} t^2 \, \mathrm{d}\eta(t) < \infty$, where $(t-x)_+^2 = ((t-x)_+)^2$. [Comment. An utility function whith this property is called 2-*concave*.]

Solutions

3.2.1 Let $X \leq_{\mathrm{st}} Y$. Then, by Theorem 3.2.1, there exists a probability space and nonnegative random variables X', Z' on it such that $X \stackrel{\mathrm{d}}{=} X'$ and $Y \stackrel{\mathrm{d}}{=} X' + Z'$. This implies that $\mathbb{E}\, X \leq \mathbb{E}\, Y$ and, using the result of Exercise 3.1.5, $\kappa_X \leq \kappa_Y$. Thus, $\Pi(X) \leq \Pi(Y)$ holds.

3.2.2 Recall that, by Theorem 3.2.1, $X \leq_{\mathrm{st}} Y$ means that $\overline{F}_X(x) \leq \overline{F}_Y(x)$ for all $x \in \mathbb{R}$. Suppose now that the set $B = \{x : \overline{F}_X(x) = \overline{F}_Y(x)\}$ is everywhere dense in $\mathbb{R}$. Then, for each $x_0 \in \mathbb{R}$ there exists a decreasing sequence $\{x_n\} \subset B$ such that $x_0 = \lim_{n\to\infty} x_n$. Thus, $\overline{F}_X(x_0) = \lim_{n\to\infty} \overline{F}_X(x_n) = \lim_{n\to\infty} \overline{F}_Y(x_n) = \overline{F}_Y(x_0)$, which is a contradiction to the assumption that $F_X \neq F_Y$.

3.2.3 The assertions immediately follow from Theorem 3.2.1.

3.2.4 Define the distributions $\{F_n^-\}$ and $\{F_n^+\}$ by $F_n^-(x) = F((k+1)/n)$ and $F_n^+(x) = F(k/n)$ if $x \in [k/n, (k+1)/n)$. Then, by Theorem 3.2.1, we have $F_n^- \leq_{\mathrm{st}} F \leq_{\mathrm{st}} F_n^+$. Furthermore,

$$\begin{aligned}
&\int_0^b c(x) \, \mathrm{d}F_n^+(x) - \int_0^b c(x) \, \mathrm{d}F_n^-(x) \\
&= \sum_{k=1}^{\lfloor bn \rfloor + 1} c(k/n)\big(F(k/n) - F((k-1)/n)\big) + c(0)F(0) \\
&\quad - \sum_{k=1}^{\lfloor bn \rfloor} c(k/n)\big(F((k+1)/n) - F(k)/n)\big) + c(0)F(1/n) \\
&= \sum_{k=1}^{\lfloor bn \rfloor} \big(c((k+1)/n) - c(k/n)\big)\big(F((k+1)/n) - F(k)/n)\big) \\
&\leq \max_{0 \leq k \leq \lfloor bn \rfloor} \big(c((k+1)/n) - c(k/n)\big)\,,
\end{aligned}$$

where the last expression converges to zero as $n \to \infty$, since the function $c(x)$ is uniformly continuous on the interval $[0, b]$.

3.2.5 The equivalence of statements (a) and (b) is shown in Theorem 3.2.2. In order to prove that (b) and (c) are equivalent, notice that

$$F_{\max\{X,x\}}(t) = \begin{cases} 0 & \text{if } t \le x \\ F_X(x) & \text{if } t > x \end{cases}$$

and, by the result of Exercise 2.1.1,

$$\mathbb{E}\max\{X,x\} = \int_0^\infty \overline{F}_{\max\{X,x\}}(t)\,\mathrm{d}t - \int_{-\infty}^0 F_{\max\{X,x\}}(t)\,\mathrm{d}t\,.$$

Thus,

$$\mathbb{E}\max\{X,x\} = x + \int_x^\infty \overline{F}_X(t)\,\mathrm{d}t$$

for all $x \in \mathbb{R}$, which shows that (b) and (c) are equivalent.

3.2.6 Let $h \in \mathcal{H}$ and $x \in \mathbb{R}$. Then $h(x) = \int_{-\infty}^{\infty} h(t)\,\mathrm{d}\delta_x(t)$. Since $h(x)$ is an arbitrary increasing function, this shows that $\delta_x \prec_{\mathcal{H}} \delta_y$ if and only if $x \le y$. Assume now that the conditions of statement (b) are fulfilled. Then, for each $h \in \mathcal{H}$,

$$\begin{aligned}
\int h(t)\,\mathrm{d}F_{X_1+X_2}(t) &= \int h(t)\Big(\int \mathrm{d}_t F_{X_1}(t-v)\,\mathrm{d}F_{X_2}(v)\Big) \\
&= \int\int h(t+v)\,\mathrm{d}F_{X_1}(t)\,\mathrm{d}F_{X_2}(v) \\
&\le \int\int h(t+v)\,\mathrm{d}F_{Y_1}(t)\,\mathrm{d}F_{X_2}(v) = \int\int h(t+v)\,\mathrm{d}F_{X_2}(v)\,\mathrm{d}F_{Y_1}(t) \\
&\le \int\int h(t+v)\,\mathrm{d}F_{Y_2}(v)\,\mathrm{d}F_{Y_1}(t) = \int h(t)\,\mathrm{d}F_{Y_1+Y_2}(t)\,.
\end{aligned}$$

3.2.7 By the results of Exercises 2.2.1 and 3.2.6, it suffices to consider the case $n = 1$. In this case we have $1 - p \le \mathrm{e}^{-p}$ and therefore $F_X(t) \le F_Y(t)$ for all $t < 1$ and $F_X(t) \ge F_Y(t)$ for all $t > 1$. Furthermore, $\mathbb{E}X = \mathbb{E}Y$. By the one-cut criterion given in Theorem 3.2.4, this implies that $X \le_{\mathrm{sl}} Y$.

3.2.8 Let us first consider the case $\alpha = 1$. Then, we have to compare the distributions $\mathrm{Poi}(\lambda)$ and $\mathrm{NB}(1, \lambda/(1+\lambda)) = \mathrm{Geo}(\lambda/(1-\lambda))$. Since the function $g : \mathbb{N} \to \mathbb{R}_+$ given by

$$g(k) = \mathrm{e}^{-\lambda}\frac{\lambda^k}{k!} - \Big(\frac{\lambda}{\lambda+1}\Big)^k \frac{1}{1+\lambda}$$

changes the sign twice, where $g(k) \le 0$ for all sufficiently large k, we get that there exists $t_0 > 0$ such that $F_X(t) \le F_Y(t)$ for all $t < t_0$ and $F_X(t) \ge F_Y(t)$ for all $t > t_0$. Thus, by Theorem 3.2.4, we have $X \le_{\mathrm{sl}} Y$. Using the results of Exercises 2.2.1 and 3.2.6, we conclude that $X \le_{\mathrm{sl}} Y$ holds for all $\alpha \in \mathbb{N} \setminus \{0\}$.

3.2.9 Notice that by the result of Exercise 2.2.1 we have

$$\Gamma(t+h,\lambda/(t+h)) = \Gamma(t,\lambda/(t+h)) * \Gamma(h,\lambda/(t+h)) . \qquad (3.2.2)$$

Using the one-cut criterion given in Theorem 3.2.4, it is easily seen that $\Gamma(t,\lambda/(t+h))$ is larger than $\Gamma(t,\lambda/t)$ in stop-loss order. Furthermore, it is obvious that $\Gamma(h,\lambda/(t+h))$ is larger than δ_0. Thus, using (3.2.2) and the result of Exercise 3.2.6, we get that $\Gamma(t+h,\lambda/(t+h))$ is larger than $\Gamma(t,\lambda/t)$ in stop-loss order.

3.2.10 From our assumptions we have $F_X(t) - F_Y(t) \le 0$ for $t < t_1$ and $F_X(t) - F_Y(t) \ge 0$ for $t > t_2$. Since the function $F_X(t) - F_Y(t)$ is continuous, there exists a smallest $t_0 \in [t_1, t_2]$ such that $F_X(t_0) = F_Y(t_0)$. Hence, $F_X(t) \le F_Y(t)$ for all $t \le t_0$ and $F_X(t) \ge F_Y(t)$ for all $t \ge t_0$. By Theorem 3.2.4, this implies that $X \le_{\rm sl} Y$. Furthermore, by the result of Exercise 2.1.1, the inequality $\mu_X \le \mu_Y$ implies that $F_X(t) = F_Y(t)$ for all $t \ge t_0$ if $F_X(t) = F_Y(t)$ for all $t \le t_0$. Suppose now that $F_X(t') < F_Y(t')$ for some $t' < t_0$. Since distribution functions are right-continuous, there exists $\varepsilon > 0$ such that $F_X(t) < F_Y(t)$ for all $t \in [t', t'+\varepsilon)$. By $\mu_X \le \mu_Y$, this implies that $F_X(t) > F_Y(t)$ for some $t \ge t_0$. Thus $X \le_{\rm st} Y$ does not hold.

3.2.11 The inequalities $\mu_X \le \mu_Y$ or $\sigma_X^2 \le \sigma_Y^2$ imply that, for densities $f_X(t)$ and $f_Y(t)$ of X and Y, there exist $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$ and $f_X(t) < f_Y(t)$ if $t \in (-\infty, t_1) \cup (t_2, \infty)$, and $f_X(t) > f_Y(t)$ if $t \in (t_1, t_2)$. Thus, by the result of Exercise 3.2.10, we get that $X \le_{\rm sl} Y$.

3.2.12 Recall that $X = \mathrm{e}^{X'}$ and $Y = \mathrm{e}^{Y'}$, where X' and Y' have the normal distributions $\mathrm{N}(a_X, b_X)$ and $\mathrm{N}(a_Y, b_Y)$ respectively. Furthermore, notice that the function $g(t) = \mathrm{e}^t$ is increasing and convex and that the superposition of increasing and convex functions is an increasing and convex function. Thus, by the result of Exercise 3.2.11, we have $X \le_{\rm sl} Y$.

3.2.13 Since $\overline{F}_X(t) = \exp\bigl(-c_X t^{r_X}\bigr)$ and $\overline{F}_Y(t) = \exp\bigl(-c_Y t^{r_Y}\bigr)$, there exists $t_0 \in \mathbb{R}$ such that $F_X(t) \le F_Y(t)$ for all $t \le t_0$ and $F_X(t) \ge F_Y(t)$ for all $t \ge t_0$. By the one-cut criterion given in Theorem 3.2.4, this implies that $X \le_{\rm sl} Y$.

3.2.14 We show that there exists a differentiable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mathrm{e}^{-ax} = \int_x^\infty (t-x)^2 f(t)\,\mathrm{d}t$$

holds for all $x \ge 0$. Notice that this equation is equivalent to

$$1 = \int_0^\infty \mathrm{e}^{ax} t^2 f(t+x)\,\mathrm{d}t\,, \qquad x \ge 0\,. \qquad (3.2.3)$$

Differentiating both sides of (3.2.3) with respect to x, we arrive at the differential euqation

$$af(t) + f^{(1)}(t) = 0\,, \qquad t \geq 0\,.$$

This means that $f(t) = \exp(-at + c)$ for some $c \in \mathbb{R}$. Inserting this solution into (3.2.3), we see that $c = \log 2 - 3\log a$.

3.3 Some Aspects of Reinsurance

A reinsurance contract specifies the part $X - h(X)$ of the claim amount X which has to be compensated by the reinsurer, after taking off the retained amount $h(X)$. Here $h : \mathbb{R}_+ \to \mathbb{R}_+$, the *retention function*, is assumed to have the following properties:

- $h(x)$ and $x - h(x)$ are increasing,
- $0 \leq h(x) \leq x$ and in particular $h(0) = 0$.

It is reasonable to suppose that both the retention function $h(x)$ and the *compensation function* $k(x) = x - h(x)$ are increasing, i.e. with the growing claim amount, both parts contribute more. Possible choices of retention functions $h(x)$ are

- $h(x) = ax$ for the *proportional contract*, where $0 < a \leq 1$,
- $h(x) = \min\{a, x\}$ for the *stop-loss contract*, where $a > 0$.

For a given risk X, a reinsurance contract with retention function $h(x)$ is said to be *compatible* with respect to a premium calculation principle Π if

$$\Pi(X) = \Pi(h(X)) + \Pi(X - h(X))\,.$$

Consider risks of the form $X = \sum_{i=1}^{N} U_i$, where N is an $\mathbb{N}$-valued random variable and where the nonnegative random variables U_i are interpreted as local risks. We can model *local reinsurance* with local retention functions $h_i(x)$ as follows: for the i-th claim of size U_i the part $U_i - h_i(U_i)$ is carried by the reinsurer. The local retention functions $h_i(x)$ are assumed to have the same properties as their global alternatives $h(x)$. Consider the corresponding local compensation functions $k_i(x) = x - h_i(x)$ and define the function $k(x)$ by

$$k(x) = \mathbb{E}\Big(\sum_{i=1}^{N} k_i(U_i) \mid X = x\Big). \tag{3.3.1}$$

It would be useful if $k(x)$ would be a (global) compensation function; see Theorem 3.3.1. Unfortunately, there exist examples, which show that this is not always the case. In Theorem 3.3.2 we showed that the function $k(x)$ defined in (3.3.1) is a compensation function if the local risks U_i have absolutely continuous distributions with PF_2 densities. Here a function $f : \mathbb{R} \to \mathbb{R}_+$ is said to be a *Pólya frequency function of order* 2, or PF_2 for short, if

$$\det \begin{pmatrix} f(x_1 - y_1) & f(x_1 - y_2) \\ f(x_2 - y_1) & f(x_2 - y_2) \end{pmatrix} \geq 0 \qquad (3.3.2)$$

whenever $x_1 \leq x_2$ and $y_1 \leq y_2$.

Exercises

3.3.1 Assume that the claim amount X is $\Gamma(2, \lambda)$-distributed and partially compensated by an reinsurer via a stop-loss contract with retention level a, where $a = 60000$ and $\lambda = 0.00007$. Compute

(a) the conditional expectation of X given that the claim is not reported to the reinsurer,

(b) the expected claim amount compensated by the reinsurer, given that the claim is reported to the reinsurer.

3.3.2 Let X be a nonnegative random variable. Consider two continuous retention functions $h_1(x)$ and $h_2(x)$, where $h_i(x)$ is strictly increasing for $x \in (0, x_i)$ and constant for $x \in (x_i, \infty)$; $0 \leq x_i \leq \infty$. Assume $\mathbb{E}\, h_1(X) \leq \mathbb{E}\, h_2(X)$ and there exists $x_0 \in [0, \infty)$ such that $h_1(x) \geq h_2(x)$ for $x \in (0, x_0)$ and $h_1(x) \leq h_2(x)$ for $x \in (x_0, \infty)$. Show that $h_1(X) \leq_{\mathrm{sl}} h_2(X)$. [Hint. Use Theorem 3.2.4.]

3.3.3 Let X be an arbitrary nonnegative random variable. For $\beta > 0$, $n \geq 2$ and for fixed numbers $x_1, x_2, \ldots, x_n$ with $0 < x_1 < \ldots < x_n$, consider the family of retention functions $h(x)$ such that $\mathbb{E}\, h(X) = \beta$ and $h(x) = \sum_{i=1}^{n} a_i(x_i - (x_i - x)_+)$ for some $a_1, \ldots, a_n \geq 0$ with $\sum_{i=1}^{n} a_i \leq 1$. Show that, for any loss function $w : \mathbb{R} \to \mathbb{R}$, the retention function

$$h^*(x) = a(x_k - (x_k - x)_+) + (1 - a)(x_{k+1} - (x_{k+1} - x)_+)$$

minimizes the expected loss $\mathbb{E}\, w(h(X))$, where $k = \min\{i : x_{i+1} > b\}$ with $b > 0$ being the solution to $\beta = \mathbb{E}\,(X - b)_+$ and $a \in (0, 1)$ such that $\mathbb{E}\, h^*(X) = \beta$. [Hint. Use the result of Exercise 3.3.2.]

3.3.4 Show that the proportional contract is compatible with respect to the expected value and standard deviation principles.

3.3.5 Assume that the function $f : \mathbb{R} \to \mathbb{R}_+$ is PF_2. Show that $f(x)$ is either everywhere 0 or has the following properties:

(a) $f(x)$ is strictly positive on a (finite or infinite) interval $I \subset \mathbb{R}$ and 0 on I^c.
(b) $f(x+t)/f(x)$ is decreasing in x on the interval I, for each $t > 0$.

Moreover, show that each function $f : \mathbb{R} \to \mathbb{R}_+$, which has the properties (a) and (b), is PF_2.

3.3.6 Show that the densities of the following distributions are PF_2:

(a) the density of the gamma distribution $\Gamma(a, \lambda)$ for $a \geq 1$,
(b) the density of the uniform distribution $\mathrm{U}(a, b)$ for $a < b$,
(c) the density of the Weibull distribution $\mathrm{W}(r, c)$ for $r \geq 1$.

[Hint. Use the result of Exercise 3.3.5.]

3.3.7 Let F be an absolutely continuous distribution on $\mathbb{R}_+$ with density f. Show: if F is PF_2, then F is IHR. [Hint. Show that the reciprocal hazard rate function $(m(x))^{-1} = \overline{F}(x)/f(x)$ is decreasing on the interval $I = \{x : f(x) > 0\}$.]

3.3.8 If the tail function $\overline{F}(x)$ of a nonnegative random variable X is PF_2, then F is IHR. [Hint. Use Theorem 2.4.2 and the result of Exercise 3.3.5.]

Solutions

3.3.1 For the expected claim amount $\mathbb{E}(X \mid X < a)$ given that the claim is not reported to the reinsurer, we have

$$\mathbb{E}(X \mid X < a) = \frac{\displaystyle\int_0^a x^2 \mathrm{e}^{-\lambda x}\, \mathrm{d}x}{\displaystyle\int_0^a x\mathrm{e}^{-\lambda x}\, \mathrm{d}x} .$$

Integration by parts gives

$$\int_0^a x^2 \mathrm{e}^{-\lambda x}\, \mathrm{d}x = 2\lambda^{-3}(1 - \mathrm{e}^{-\lambda a}) - 2\lambda^{-2} a \mathrm{e}^{-\lambda a} - \lambda^{-1} a^2 \mathrm{e}^{-\lambda a}$$

and

$$\int_0^a x\mathrm{e}^{-\lambda x}\, \mathrm{d}x = \lambda^{-2}(1 - \mathrm{e}^{-\lambda a}) - -\lambda^{-1} a \mathrm{e}^{-\lambda a} .$$

Thus, inserting $a = 60000$ and $\lambda = 0.00007$, we find that $\mathbb{E}(X \mid X < a) \approx 24473$. Furthermore,

$$\mathbb{E}(X \mid X > a) = \frac{\mathbb{E}X - \mathbb{E}(X \mathbb{I}(X < a))}{1 - \mathbb{P}(X < a)} = \frac{2\lambda^{-2} - \int_0^a x^2 \mathrm{e}^{-\lambda x}\, \mathrm{d}x}{1 - \int_0^a x\mathrm{e}^{-\lambda x}\, \mathrm{d}x} .$$

This yields the value $\mathbb{E}(X \mid X > a) \approx 17033$.

3.3.2 Let $x_0' \geq 0$ be the smallest number such that $h_1(x) \geq h_2(x)$ for $x \in (0, x_0')$ and $h_1(x) \leq h_2(x)$ for $x \in (x_0', \infty)$. Notice that $x_2 \geq x_0'$. Put $t_0 = h_2(x_0')$. Then, for each $t \in (0, t_0)$ there exists $x \in (0, x_0')$ such that $t = h_1(x)$ and

$$\begin{aligned} \mathbb{P}(h_2(X) \leq t) &= \mathbb{P}(h_2(X) \leq h_1(x)) \\ &\geq \mathbb{P}(h_2(X) \leq h_2(x)) = \mathbb{P}(X \leq x) = \mathbb{P}(h_1(X) \leq h_1(x)) \\ &= \mathbb{P}(h_1(X) \leq t)\,. \end{aligned}$$

Furthermore, put $t_2 = h_2(x_2)$. Then, for each $t \in (t_0, t_2)$ there exists $x \in (x_0', x_2)$ such that $t = h_2(x)$ and

$$\begin{aligned} \mathbb{P}(h_2(X) > t) &= \mathbb{P}(h_2(X) > h_2(x)) \\ &= \mathbb{P}(X > x) = \mathbb{P}(h_1(X) > h_1(x)) \\ &\geq \mathbb{P}(h_1(X) > h_2(x)) = \mathbb{P}(h_1(X) > t)\,. \end{aligned}$$

Notice also that $\mathbb{P}(h_2(X) > t) = \mathbb{P}(h_1(X) > t) = 0$ for $t \in (t_2, \infty)$. Thus, the one-cut criterion of Theorem 3.2.4 is fulfilled.

3.3.3 We have to show that $h^*(x) \leq_{\mathrm{sl}} h(X)$. Notice that the retention functions $h(x)$ and $h^*(x)$ are continuous and piecewise differentiable with

$$h^{(1)}(x) = \begin{cases} \sum_{i=1}^n a_i & \text{if } x < x_1, \\ \sum_{i=j+1}^n a_i & \text{if } x_j < x < x_{j+1}, \\ 0 & \text{if } x > x_n, \end{cases}$$

and

$$(h^*)^{(1)}(x) = \begin{cases} 1 & \text{if } x < x_k, \\ 1-a & \text{if } x_k < x < x_{k+1}, \\ 0 & \text{if } x > x_{k+1}. \end{cases}$$

Thus, it is enough to show that $h_1(x) = h^*(x)$ and $h_2(x) = h(x)$ satisfy the assumptions of Exercise 3.3.2. [3]

3.3.4 Consider the retention function $h(x) = ax$, where $0 < a \leq 1$. Then, for the expected value principle $\Pi(X) = (1+b)\mathbb{E}\,X$ with $b > 0$, we have

$$\begin{aligned} \Pi(h(X)) + \Pi(X - h(X)) &= (1+b)(\mathbb{E}\,(aX) + \mathbb{E}\,(X - aX)) \\ &= (1+b)\mathbb{E}\,X = \Pi(X)\,. \end{aligned}$$

Moreover, for the standard deviation principle $\Pi(X) = \mathbb{E}\,X + b\sqrt{\mathrm{Var}\,X}$ with $b > 0$, we have

$$\begin{aligned} &\Pi(h(X)) + \Pi(X - h(X)) \\ &= \mathbb{E}\,(aX) + b\sqrt{\mathrm{Var}\,(aX)} + \mathbb{E}\,(X - aX) + b\sqrt{\mathrm{Var}\,(X - aX)} \\ &= \mathbb{E}\,X + b\sqrt{\mathrm{Var}\,X} = \Pi(X)\,. \end{aligned}$$

[3] Complete the solution.

3.3.5 Let f be PF_2. Suppose that $f(x) > 0$ for some $x \in \mathbb{R}$ and $f(x') = 0$ for some $x' < x$. Without loss of generality we can assume that $x = 0$. Let $x_2 = y_1$ and $x_1 - y_1 = x' < 0$. Then, (3.3.2) implies that

$$f(x')f(x_2 - y_2) \geq f(x'')f(0)$$

for all $x'' \leq x'$. Thus, $f(x'') = 0$ for all $x'' \leq x'$. In other words, if $f(y')f(y'') > 0$ for some $y' < y''$, then $f(y) > 0$ for all $y \in [y', y'']$. This shows that $f(y)$ is strictly positive on some interval and 0 otherwise. Furthermore, if we put $x_2 - y_2 = x + t$, $x_1 - y_2 = x$, $x_2 - y_1 = x' + t$, $x_1 - y_1 = x'$ for $x \leq x'$, then (3.3.2) implies that

$$f(x')f(x+t) \geq f(x)f(x'+t)\,. \tag{3.3.3}$$

This means that $f(x+t)/f(x)$ is decreasing in x on the interval $I = \{x : f(x) > 0\}$. Suppose now that $f : \mathbb{R} \to \mathbb{R}_+$ is a function with properties (a) and (b). Then, (3.3.3) holds for $x, x' + t \in I$. Notice that, otherwise, (3.3.3) is obviously fulfilled since the right-hand side vanishes in this case. Thus, (3.3.2) holds.

3.3.6 Let $a \geq 1$ and

$$f(x) = \begin{cases} \lambda^a x^{a-1} \mathrm{e}^{-\lambda x}/\Gamma(a) & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $t, x > 0$, we have

$$\frac{f(x+t)}{f(x)} = \left(\frac{x+t}{x}\right)^{a-1} \mathrm{e}^{-\lambda t}$$

which is decreasing in $x > 0$. Thus, by the result of Exercise 3.3.5, the density of $\Gamma(a, \lambda)$ is PF_2. In the same way we get that the densities of $\mathrm{U}(a, b)$ and $\mathrm{W}(r, c)$ are PF_2.

3.3.7 Notice that

$$(m(x))^{-1} = \int_{\{t: f(t) > 0\}} \mathbb{I}(x < t) \frac{f(x+t)}{f(x)}\, \mathrm{d}t\,.$$

For each t fixed, the integrand is decreasing in x by the result of Exercise 3.3.5. Thus, $(m(x))^{-1}$ is decreasing.

3.3.8 Let $\overline{F}(x)$ be PF_2. Consider the residual hazard distribution function $F_t(x) = \mathbb{P}(X - t \leq x \mid X > t)$ for $t \geq 0$ such that $F(t) < 1$. Notice that

$$F_t(x) = 1 - \frac{\overline{F}(t+x)}{\overline{F}(t)}\,.$$

Thus, by the result of Exercise 3.3.5, $F_t(x)$ is decreasing in t for each $x \geq 0$. By Theorem 2.4.2, this implies that F is IHR.

4

Distributions of Aggregate Claim Amount

4.1 Individual and Collective Model

Two models are considered:

- *Individual model.* Consider a portfolio consisting of n policies with individual risks $U_1, \ldots, U_n$. We assume that the nonnegative random variables $U_1, \ldots, U_n$ are independent, but not necessarily identically distributed. Let the distribution F_{U_i} of U_i be the mixture $F_{U_i} = (1-\theta_i)\delta_0 + \theta_i F_{V_i}$, where $0 < \theta_i \le 1$ and where F_{V_i} is the distribution of a (strictly) positive random variable V_i, $i = 1, \ldots, n$. The *aggregate claim amount* in this model, is $X^{\mathrm{ind}} = \sum_{i=1}^n U_i$ with distribution $F_{U_1} * \ldots * F_{U_n}$. A portfolio is called *homogeneous* if $F_{V_1} = \ldots = F_{V_n}$.
- *Collective model.* A portfolio consists of a number of anonymous policies which we do not observe separately. The total number N of claims occurring in a given period is random, where we assume that $\mathbb{P}(N > 0) > 0$. Further, the claim sizes U_i are (strictly) positive and are assumed to form a sequence $U_1, U_2, \ldots$ of independent and identically distributed random variables. We also assume that the sequence $U_1, U_2, \ldots$ of individual claim sizes is independent of the claim number N. The aggregate claim amount is the random variable $X^{\mathrm{col}} = \sum_{i=1}^N U_i$. Here and throughout the whole Teacher's Manual we use the convention that $\sum_{i=1}^0 U_i = 0$.

Exercises

4.1.1 Show that for the individual model

$$\mathbb{E}\, X^{\mathrm{ind}} = \sum_{i=1}^n \theta_i \mathbb{E}\, V_i\,, \qquad \mathrm{Var}\, X^{\mathrm{ind}} = \sum_{i=1}^n \theta_i \mathrm{Var}\, V_i + \sum_{i=1}^n \theta_i (1-\theta_i)(\mathbb{E}\, V_i)^2\,.$$

4.1.2 Show that for the collective model

$$\mathbb{E}\, X^{\mathrm{col}} = \mathbb{E}\, N \mathbb{E}\, U\,, \qquad \mathrm{Var}\, X^{\mathrm{col}} = \mathrm{Var}\, N (\mathbb{E}\, U)^2 + \mathbb{E}\, N \mathrm{Var}\, U\,,$$

$$\mathrm{cv}^2_{X^{\mathrm{col}}} = \mathrm{cv}^2_N + \frac{1}{\mathbb{E}\,N}\mathrm{cv}^2_U\,.$$

4.1.3 Consider the aggregate claim amount $X^{\mathrm{col}} = \sum_{i=1}^N U_i$ in the collective model. Assume that N has the Poisson distribution $\mathrm{Poi}(\lambda)$ and U_i is $\mathrm{Exp}(\delta)$-distributed. Determine the premium $\Pi(X^{\mathrm{col}})$ using the premium calculation principles given by (3.1.1)–(3.1.4).

4.1.4 Assume that the aggregate claim amount per year for a certain class of motorists can be modelled by the random variable $X^{\mathrm{col}} = \sum_{i=1}^N U_i$, where N is $\mathrm{Poi}(\lambda)$-distributed and U_i is $\mathrm{LN}(a_U, b_U)$-distributed with $\lambda = 0.3$, $a_U = 6.8$, $b_U = 1.8$. An insurance company offers to insure the motorists subject to a local stop-loss contract with retention level $a = 1000$. Compute expectation and variance of the claim amount compensated per year by the insurer. [Hint. Use the result of Exercise 4.1.2.]

4.1.5 Let $U_1, U_2, \ldots$ and $U_1', U_2', \ldots$ be two sequences of independent and identically distributed nonnegative random variables. Show that the following comparability properties hold:
(a) if $U \le_{\mathrm{st}} U'$ then $\sum_{j=1}^n U_j \le_{\mathrm{st}} \sum_{j=1}^n U_j'$ for each $n \ge 1$,
(b) If $U \le_{\mathrm{sl}} U'$ then $\sum_{j=1}^n U_j \le_{\mathrm{sl}} \sum_{j=1}^n U_j'$ for each $n \ge 1$.

4.1.6 (Continuation) Consider the function $f(n) = \mathbb{E}\left(\sum_{i=1}^n U_i - x\right)_+$ for some $x \ge 0$ and assume that $\mathbb{E}\,U < \infty$. Show that $f(0), f(1), \ldots$ is a convex sequence, that is $f(n+m) \ge f(n) + f(m)$ and $f(n) \ge nf(1)$ for all $n, m \ge 0$.

4.1.7 Consider the aggregate claim amount $X^{\mathrm{col}} = \sum_{i=1}^N U_i$ in the collective model. Show that $\mathbb{E}\,(\sum_{i=1}^N U_i - x)_+ \ge \mathbb{E}\,N\mathbb{E}\,(U - x)_+$ for all $x \ge 0$.

Solutions

4.1.1 The first formula immediately follows from (3.1.10) and from the definition of the expectation given in Section 2.1. In order to prove the second formula we use (3.1.11) to get $\mathrm{Var}\,X^{\mathrm{ind}} = \sum_{i=1}^n \mathrm{Var}\,U_i$, since $U_1, \ldots, U_n$ are independent. Notice now that

$$\begin{aligned}\mathrm{Var}\,U_i &= \mathbb{E}\,U_i^2 - (\mathbb{E}\,U_i)^2 = \theta_i \mathbb{E}\,V_i^2 - \theta_i^2(\mathbb{E}\,V_i)^2 \\ &= \theta_i \mathrm{Var}\,V_i + \theta_i(1-\theta_i)(\mathbb{E}\,V_i)^2\,.\end{aligned}$$

4.1.2 The first formula easily follows from the law of total probability. Indeed,

$$\mathbb{E}\,X^{\mathrm{col}} = \sum_{n=0}^{\infty} \mathbb{E}\,(X^{\mathrm{col}} \mid N = n)\mathbb{P}(N = n)$$

$$= \sum_{n=0}^{\infty} \mathbb{E}\Big(\sum_{i=1}^{n} U_i\Big)\mathbb{P}(N=n)$$
$$= \mathbb{E}\,U \sum_{n=0}^{\infty} n\mathbb{P}(N=n) = \mathbb{E}\,N\mathbb{E}\,U\,.$$

Similarly,

$$\begin{aligned}\operatorname{Var} X^{\mathrm{col}} &= \sum_{n=0}^{\infty} \mathbb{E}\Big(\sum_{i=1}^{n} U_i\Big)^2 \mathbb{P}(N=n) - (\mathbb{E}\,N\mathbb{E}\,U)^2 \\ &= \sum_{n=0}^{\infty} \big(n\mathbb{E}\,U^2 + n(n-1)(\mathbb{E}\,U)^2\big)\mathbb{P}(N=n) - (\mathbb{E}\,N\mathbb{E}\,U)^2 \\ &= \operatorname{Var} N(\mathbb{E}\,U)^2 + \mathbb{E}\,N\operatorname{Var} U\,.\end{aligned}$$

The formula for the squared coefficient of variation follows immediately.

4.1.3 By the result of Exercise 4.1.2, we have $\mathbb{E}\,X^{\mathrm{col}} = \lambda/\delta$ and $\operatorname{Var} X^{\mathrm{col}} = 2\lambda/\delta^2$. Thus, $\Pi(X^{\mathrm{col}}) = (1+a)\lambda/\delta$ for the expected value principle, $\Pi(X^{\mathrm{col}}) = \lambda/\delta + 2a\lambda/\delta^2$ for the variance principle, $\Pi(X^{\mathrm{col}}) = (\lambda + a\sqrt{2\lambda})/\delta$ for the standard deviation principle, $\Pi(X^{\mathrm{col}}) = (\lambda + 2a)/\delta$ for the modified variance principle.

4.1.4 Notice that the claim amount X compensated per year by the insurer is given by $X = \sum_{i=1}^{N}(U_i - a)_+$. Let $f(t)$ denote the density of U. Then,

$$\mathbb{E}\,(U-a)_+ = \int_a^{\infty} (t-a)f(t)\,\mathrm{d}t \approx 3875.13$$

and

$$\mathbb{E}\,(U-a)_+^2 = \int_a^{\infty} (t-a)^2 f(t)\,\mathrm{d}t \approx 5.17242\cdot 10^8\,.$$

Moreover, $\mathbb{E}\,N = \operatorname{Var} N = \lambda$. Thus, by the result of Exercise 4.1.2

$$\mathbb{E}\,X = \lambda\mathbb{E}\,(U-a)_+ \approx 1162.54$$

and

$$\operatorname{Var} X = \lambda\mathbb{E}\,(U-a)_+^2 \approx 1.55173\cdot 10^8\,.$$

4.1.5 We use induction with respect to n. For $n=1$, statement (a) is obviously true. Let now $U \le_{\mathrm{st}} U'$ and suppose that (a) holds for some $n \ge 1$. Then, for each increasing function $g: \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}\,g\Big(\sum_{i=1}^{n+1} U_i\Big) = \int_0^{\infty} \mathbb{E}\,g\Big(\sum_{i=1}^{n} U_i + t\Big)\,\mathrm{d}F_{U_{n+1}}(t)$$

$$
\begin{aligned}
&\leq \int_0^\infty \mathbb{E}\, g\Big(\sum_{i=1}^n U_i' + t\Big)\, \mathrm{d}F_{U_{n+1}}(t) \\
&\leq \int_0^\infty \mathbb{E}\, g\Big(\sum_{i=1}^n U_i' + t\Big)\, \mathrm{d}F_{U'_{n+1}}(t) \\
&= \mathbb{E}\, g\Big(\sum_{i=1}^{n+1} U_i'\Big),
\end{aligned}
$$

since the functions $g_1(x) = g(x+t)$ and $g_2(x) = \mathbb{E}\, g\big(\sum_{i=1}^n U_i' + x\big)$ are increasing. The proof of (b) is analogous.

4.1.6 Notice that $(a+b-x)_+ \geq (a-x)_+ + (b-x)_+$ for all $a, b, x \geq 0$. Thus, the linearity of expectation implies that

$$
\begin{aligned}
f(n+m) &= \mathbb{E}\Big(\sum_{i=1}^{n+m} U_i - x\Big)_+ \\
&\geq \mathbb{E}\Big(\sum_{i=1}^{n} U_i - x\Big)_+ + \mathbb{E}\Big(\sum_{i=n+1}^{m} U_i - x\Big)_+ \\
&= f(n) + f(m)
\end{aligned}
$$

for arbitrary $n, m \geq 1$. The inequality $f(n) \geq n f(1)$ now follows by induction.

4.1.7 By the result of Exercise 4.1.6, the law of total probability gives

$$
\begin{aligned}
\mathbb{E}\Big(\sum_{i=1}^{N} U_i - x\Big)_+ &= \sum_{n=1}^{\infty} \mathbb{E}\Big(\sum_{i=1}^{n} U_i - x\Big)_+ \mathbb{P}(N=n) \\
&\geq \mathbb{E}\,(U-x)_+ \sum_{n=1}^{\infty} n\mathbb{P}(N=n) \\
&= \mathbb{E}\, N \mathbb{E}\,(U-x)_+ .
\end{aligned}
$$

4.2 Compound Distributions

Let N be a nonnegative integer-valued random variable and $U_1, U_2, \ldots$ a sequence of nonnegative random variables. Then the random variable $X = \sum_{i=1}^N U_i$ is called a *compound*. We assume throughout this chapter that the random variables $N, U_1, U_2, \ldots$ are independent. If not stated otherwise, we also assume that $U_1, U_2, \ldots$ are identically distributed. We say that X has a *compound distribution* determined by the (compounding) probability function

$\{p_k,\ k \in \mathbb{N}\}$ of N and by the distribution F_U of U_i if the distribution of X is given by

$$F_X = \sum_{k=0}^{\infty} p_k F_U^{*k}, \tag{4.2.1}$$

where F_U^{*k} denotes the k-fold convolution of F_U.

In actuarial applications three cases are of special interest:

- *Poisson compounds* where N has a Poisson distribution; in this case the distribution of the compound, determined by $\lambda = \mathbb{E}\,N$ and by the distribution F_U, is called a *compound Poisson distribution* with characteristics (λ, F_U).
- *Pascal* (or *negative binomial*) *compounds* with compounding distribution $\mathrm{NB}(\alpha, p)$.
- *Geometric compounds* where N has a geometric distribution; in this case the distribution of the compound is determined by $p = 1 - \mathbb{P}(N = 0)$ and by the distribution F_U and is called a *compound geometric distribution* with characteristics (p, F_U).

Exercises

4.2.1 Consider the compound $X = \sum_{k=1}^{N} U_k$. Show that the characteristic function $\hat{\varphi}_X(s)$ can be computed from the formula

$$\hat{\varphi}_X(s) = \hat{g}_N(\hat{\varphi}_U(s))\,, \qquad s \in \mathbb{R}\,, \tag{4.2.2}$$

where $\hat{g}_N(z)$ is the generating function of N and $\hat{\varphi}_U(s)$ is the characteristic function of U_i. [Hint. Use the same arguments as in the proof of Theorem 4.2.1.]

4.2.2 Show that mean and variance of a Poisson compound X with characteristics (λ, F) are given by $\mu_X = \lambda\mu_F$, $\sigma_X^2 = \lambda\mu_F^{(2)}$, the Laplace–Stieltjes transform and the moment generating function by

$$\hat{l}_X(s) = \mathrm{e}^{-\lambda(1-\hat{l}_F(s))}\,, \qquad \hat{m}_X(s) = \mathrm{e}^{\lambda(\hat{m}_F(s)-1)}\,.$$

4.2.3 (Continuation) Use the exponential principle given by (3.1.5) to determine the premium $\Pi(X)$ for the Poisson compound X of Exercise 4.2.2.

4.2.4 Show that for a geometric compound X with characteristics (p, F)

$$\mu_X = \frac{p\mu_F}{1-p}\,,\ \sigma_X^2 = \frac{p}{(1-p)^2}(\mu_F^2 + (1-p)\sigma_F^2)\,,\ \hat{l}_X(s) = \frac{1-p}{1-p\hat{l}_F(s)}\,.$$

4.2.5 Show that the compound geometric distribution $\sum_{k=0}^{\infty}(1-p)p^k F^{*k}$, for some $0 < p < 1$, is equal to the pointwise limit $\lim_{n\to\infty} F_n$, where F_n is defined by the recursion

$$F_n = (1-p)\delta_0 + pF * F_{n-1} \tag{4.2.3}$$

for all $n \ge 1$ and F_0 is an arbitrary (initial) distribution on $\mathbb{R}_+$. [Hint. Prove that $F_n = \sum_{k=0}^{n-1}(1-p)p^k F^{*k} + p^n F^{*n} * F_0$.]

4.2.6 Show that the distribution of a zero-truncated geometric compound with characteristics (p, F) is exponential with parameter $\delta(1-p)$ provided that F is the exponential distribution with parameter δ.

4.2.7 Show that if N is $\mathrm{NB}(\alpha, p)$-distributed and U_i has distribution F_U, then the distribution F_X of $X = \sum_{i=1}^{N} U_i$ can be represented as

$$F_X = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathrm{e}^{-\lambda} \tilde{F}^{*k}, \tag{4.2.4}$$

where $\lambda = -\alpha \log(1-p)$ and

$$\tilde{F} = \frac{1}{-\log(1-p)} \sum_{k=1}^{\infty} \frac{p^k}{k} F_U^{*k}.$$

Conclude that the negative binomial distribution can be represented as compound Poisson distribution with characteristics $(\lambda, \tilde{F})$ where $\tilde{F}$ is the logarithmic distribution $\mathrm{Log}(p)$.

4.2.8 Consider the tail $\overline{F}_X(x)$ of the compound $X = \sum_{i=1}^{N} U_i$, where N has the negative binomial distribution $\mathrm{NB}(2, p)$ and U is $\mathrm{Exp}(\delta)$-distributed. Show that for all $x \ge 0$

$$\overline{F}_X(x) = \mathrm{e}^{-(1-p)\delta x}(1 - (1-p)^2 + p^2(1-p)\delta x). \tag{4.2.5}$$

4.2.9 Let $X = \sum_{i=1}^{N} U_i$ be a compound with characteristics $(\{p_k\}, F_U)$. Show that for the tail $\overline{F}_X(x) = \mathbb{P}(X > x)$,

$$\overline{F}_X(x) = \sum_{k=1}^{\infty} p_k \overline{F_U^{*k}}(x), \qquad x \ge 0$$

and

$$\int_0^{\infty} \mathrm{e}^{-sx} \overline{F}_X(x)\, \mathrm{d}x = \frac{1 - \hat{g}_N(\hat{l}_U(s))}{s}, \qquad s > 0. \tag{4.2.6}$$

4.2.10 (Continuation) Assume that $\hat{g}_N(s_0) < \infty$ for some $s_0 > 1$. Show that then

$$\overline{F}_X(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \mathbb{E} \binom{N}{k} (1 - F_U)^{*k}(x) .$$

Solutions

4.2.1 Using the law of total probability and our independence assumptions, the same arguments as in the proof of Theorem 4.2.1 lead to (4.2.2).

4.2.2 Let N be Poi(λ)-distributed. Then $\mathbb{E} N = \operatorname{Var} N = \lambda$. Thus, $\mu_X = \lambda\mu_F$ and $\sigma_X^2 = \lambda\mu_F^{(2)}$ by the result of Exercise 4.1.2; see also Corollary 4.2.1. The Laplace–Stieltjes transform of X is immediately obtained from Theorem 4.2.1, since $\hat{g}_N(s) = \exp(\lambda(s-1))$. The formula for $\hat{m}_X(s)$ follows analogously.

4.2.3 By the result of Exercise 4.2.2, we have $\Pi(X) = a^{-1} \log \hat{m}_X(a) = \lambda a^{-1} \big(\hat{m}_F(a) - 1\big)$.

4.2.4 Recall that $\mathbb{E} N = p/(1-p)$, $\operatorname{Var} N = p/(1-p)^2$ and $\hat{g}_N(s) = (1-p)/(1-ps)$ if N is Geo(p)-distributed. The statements are then obtained in the same way as in Exercise 4.2.2, using the result of Exercise 4.1.2 and Theorem 4.2.1.

4.2.5 Consider the distributions F_n defined recursively by (4.2.3). Notice that $F_1 = (1-p)p^0 F^{*0} + p^0 F^{*0} * F_0$. Suppose now that

$$F_n = \sum_{k=0}^{n-1} (1-p)p^k F^{*k} + p^n F^{*n} * F_0 \tag{4.2.7}$$

for some $n \geq 1$. Then,

$$\begin{aligned} F_{n+1} &= (1-p)\delta_0 + pF * F_n \\ &= \sum_{k=0}^{n} (1-p)p^k F^{*k} + p^{n+1} F^{*(n+1)} * F_0 . \end{aligned}$$

Thus, (4.2.7) holds for all $n \geq 1$. Since $F^{*n} * F_0(t) \leq 1$ for all $t \geq 0$ and $p^n \to 0$ as $n \to \infty$, this shows that

$$\lim_{n\to\infty} F_n(t) = \sum_{k=1}^{\infty} (1-p)p^k F^{*k}(t) , \qquad t \geq 0 .$$

4.2.6 Let $F = \mathrm{Exp}(\delta)$. Then, $\hat{l}_F(s) = \delta/(\delta+s)$ and the Laplace–Stieltjes transform of the compound distribution $\sum_{k=1}^{\infty}(1-p)p^{k-1}F^{*k}$ is given by

$$\begin{aligned}\sum_{k=1}^{\infty}(1-p)p^{k-1}\big(\hat{l}_F(s)\big)^k &= \frac{\delta(1-p)}{\delta+s}\sum_{k=0}^{\infty}\Big(\frac{p\,\delta}{\delta+s}\Big)^k \\ &= \frac{\delta(1-p)}{\delta(1-p)+s}\,,\end{aligned}$$

which is the Laplace–Stieltjes transform of $\mathrm{Exp}(\delta(1-p))$. The statement now follows from the one-to-one correspondence between distributions on $\mathbb{R}_+$ and their Laplace–Stieltjes transforms.

4.2.7 We show that the Laplace–Stieltjes transforms of both sides of (4.2.4) coincide. Let N be $\mathrm{NB}(\alpha,p)$-distributed. Then, $\hat{g}_N(s) = (1-p)^\alpha/(1-ps)^\alpha$ and, by Theorem 4.2.1,

$$\hat{l}_X(s) = \Big(\frac{1-p}{1-p\hat{l}_U(s)}\Big)^\alpha. \tag{4.2.8}$$

On the other hand,

$$\hat{l}_{\tilde{F}}(s) = \frac{\log(1-p\hat{l}_U(s))}{\log(1-p)}$$

since $\hat{g}_{\mathrm{Log}(p)}(s) = \log(1-ps)/\log(1-p)$. Thus, using Theorem 4.2.1 anew, the Laplace–Stieltjes transform of the compound Poisson distribution $\sum_{k=0}^{\infty}(\lambda^k/k!)\mathrm{e}^{-\lambda}\tilde{F}^{*k}$ with $\lambda = -\alpha\log(1-p)$ is given by

$$\begin{aligned}\exp\big(\lambda(\hat{l}_{\tilde{F}}(s)-1)\big) &= \exp\big(-\alpha\log(1-p\hat{l}_U(s))+\alpha\log(1-p)\big) \\ &= \Big(\frac{1-p}{1-p\hat{l}_U(s)}\Big)^\alpha,\end{aligned}$$

which coincides with the right-hand side of (4.2.8). In particular, if $F_U = \delta_1$, then X is $\mathrm{NB}(\alpha,p)$-distributed and $\tilde{F} = \mathrm{Log}(p)$. Thus, (4.2.4) shows that $\mathrm{NB}(\alpha,p)$ can be represented as compound Poisson distribution with characteristics $(\lambda,\tilde{F})$, where $\lambda = -\alpha\log(1-p)$ and $\tilde{F} = \mathrm{Log}(p)$.

4.2.8 We have $\hat{g}_N(s) = (1-p)^2/(1-ps)^2$ and $\hat{l}_U(s) = \delta/(\delta+s)$. Thus, by Theorem 4.2.1,

$$\hat{l}_X(s) = \Big(\frac{1-p}{1-p\delta/(\delta+s)}\Big)^2. \tag{4.2.9}$$

We show that this function coincides with the Laplace–Stieltjes transform of the distribution given in (4.2.5). By the result of Exercise 2.2.2, formula (4.2.5) is equivalent to

$$\overline{F}_X(x) = (1-p)^2\overline{\delta}_0(x) + p(2-p)\Big(\frac{2(1-p)}{2-p}\overline{F}(x) + \frac{p}{2-p}\overline{F^{*2}}(x)\Big)\,,$$

where $F = \mathrm{Exp}((1-p)\delta)$. Moreover, this is equivalent to

$$\begin{aligned}\hat{l}_X(s) &= (1-p)^2 + 2p(1-p)\frac{(1-p)\delta}{(1-p)\delta+s} + p^2\Big(\frac{(1-p)\delta}{(1-p)\delta+s}\Big)^2\\ &= (1-p)^2\Big(1+\frac{p\delta}{(1-p)\delta+s}\Big)^2\\ &= (1-p)^2\Big(\frac{\delta+s}{(1-p)\delta+s}\Big)^2\,,\end{aligned}$$

where the last expression coincides with the right-hand side of (4.2.9).

4.2.9 We have

$$\begin{aligned}\overline{F}_X(x) &= 1-\sum_{k=0}^{\infty} p_k F_U^{*k}(x)\\ &= \sum_{k=0}^{\infty} p_k\big(1-F_U^{*k}(x)\big) = \sum_{k=0}^{\infty} p_k \overline{F_U^{*k}}(x)\,.\end{aligned}$$

Moreover, in Exercise 2.1.5 we showed that $\hat{l}_X(s) = 1 - s\int_0^\infty \mathrm{e}^{-sx}\overline{F}_X(x)\,\mathrm{d}x$. Thus, by Theorem 4.2.1,

$$\int_0^\infty \mathrm{e}^{-sx}\overline{F}_X(x)\,\mathrm{d}x = \frac{1-\hat{g}_N\big(\hat{l}_U(s)\big)}{s}\,.$$

4.2.10 By Taylor series expansion, (4.2.6) takes the form

$$\int_0^\infty \mathrm{e}^{-sx}\overline{F}_X(x)\,\mathrm{d}x = -s^{-1}\sum_{k=1}^{\infty}\frac{\hat{g}^{(k)}(1)}{k!}\big(\hat{l}_U(s)-1\big)^s\,.$$

Thus, using the results Exercises 2.1.5 and 2.1.7, we have

$$\int_0^\infty \mathrm{e}^{-sx}\overline{F}_X(x)\,\mathrm{d}x = s^{-1}\sum_{k=1}^{\infty}(-1)^{k+1}\mathbb{E}\binom{N}{k}\Big(s\int_0^\infty \mathrm{e}^{-sx}\overline{F}_U(x)\,\mathrm{d}x\Big)^k$$

and consequently

$$1-\hat{l}_X(s) = \sum_{k=1}^{\infty}(-1)^{k+1}\mathbb{E}\binom{N}{k}\big(1-\hat{l}_U(s)\big)^k\,.$$

[1]

[1] Complete the solution

4.3 Claim Number Distributions

In this section we study properties of $\mathbb{N}$-valued random variables, denoted by N, which are used to model the number of claims incurred in a given time period. Furthermore, let Λ be a (strictly) positive random variable with distribution F_Λ. We say that N has a *mixed Poisson distribution* with *mixing distribution* F_Λ if

$$\mathbb{P}(N = k) = \int_0^\infty \frac{\lambda^k}{k!}\, \mathrm{e}^{-\lambda}\, \mathrm{d}F_\Lambda(\lambda)$$

for each $k \in \mathbb{N}$. Then we can write

$$\mathbb{P}(N = k) = \mathbb{E}\left(\mathrm{e}^{-\Lambda}\Lambda^k\right)/k!$$

for each $k \in \mathbb{N}$, where Λ is called the *mixing variable.*

Recall that the gamma function is given by $\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\, \mathrm{d}t$, $x \ge 0$. In particular, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N} \setminus \{0\}$.

Exercises

4.3.1 Show that

(a) if $\mathbb{E}\, N^n < \infty$ for some $n \in \mathbb{N}$, then

$$\mathbb{E}\, N^k = \sum_{j=1}^{k} c_k^{(j)} \hat{g}_N^{(j)}(1-)$$

for all $k = 1, \ldots, n$, where $c_k^{(j)}$ are the *Stirling numbers of the second kind* defined by

$$x^k = \sum_{j=1}^{k} c_k^{(j)} x(x-1)\ldots(x-j+1)\,, \qquad x \in \mathbb{R} \tag{4.3.1}$$

[Hint. Use the result of Exercise 2.1.7.],

(b) for $k \ge 1$

$$\mathbb{P}(N \ge k) = \frac{1}{(k-1)!}\int_0^1 v^{k-1}\hat{g}_N^{(k)}(1-v)\, \mathrm{d}v\,,$$

(c) for $k \in \mathbb{N}$ and $k + x > 0$

$$\mathbb{E}\left(\frac{N!}{\Gamma(N+x+1)}\mathbb{I}(N \ge k)\right) = \frac{1}{\Gamma(k+x)}\int_0^1 v^{k+x-1}\hat{g}_N^{(k)}(1-v)\, \mathrm{d}v\,.$$

4.3.2 Assume that $\hat{g}_N(s_0) < \infty$ for some $s_0 > 1$. Show that

$$\hat{g}_N(1+s) = \sum_{k=0}^{\infty} s^k \mathbb{E}\binom{N}{k}, \qquad 0 \le s < s_0 - 1 .$$

[Hint. Use the result of Exercise 2.1.7.]

4.3.3 Show that for the Poisson, negative-binomial, binomial and logarithmic distribution, the probability function $\{p_k, k \in \mathbb{N}\}$ has the following property: There exist constants $a, b \in \mathbb{R}$ such that

$$p_k = \Big(a + \frac{b}{k}\Big) p_{k-1} \tag{4.3.2}$$

for all $k = 1, 2, \ldots$ in case of the Poisson, negative-binomial, binomial distribution, and for all $k = 2, 3, \ldots$ in case of the logarithmic distribution. Find the constants a, b for these distributions.

4.3.4 Let $\{p_k, k \in \mathbb{N}\}$ be a probability function which satisfies (4.3.2) for $k = 1, 2, \ldots$. Show that then

$$p_0 = \begin{cases} \mathrm{e}^{-b} & \text{if } a = 0, \\ (1-a)^{1+b/a} & \text{if } a \ne 0. \end{cases}$$

4.3.5 Let the probability function $\{p_k, k \in \mathbb{N}\}$ of N satisfy (4.3.2) for $k = 1, 2, \ldots$. Show that the generating function $\hat{g}_N(s)$ of N fulfils

$$\hat{g}_N^{(1)}(s) = \frac{a+b}{1-as}\, \hat{g}_N(s)\,, \qquad 0 \le s < 1 . \tag{4.3.3}$$

Conclude from (4.3.3) that the distribution of N is either Poisson, negative-binomial or binomial provided that $a + b > 0$.

4.3.6 Let N have a mixed Poisson distribution with mixing variable Λ. Show that for $n = 1, 2, \ldots$

$$\mathbb{E}\binom{N}{n} = \frac{1}{n!}\, \mathbb{E}\, \Lambda^n .$$

Solutions

4.3.1 Using (4.3.1) and the result of Exercise 2.1.7, we have

$$\begin{aligned} \mathbb{E}\, N^k &= \sum_{j=1}^{k} c_k^{(j)} \mathbb{E}\big(N(N-1)\ldots(N-j+1)\big) \\ &= \sum_{j=1}^{k} c_k^{(j)} \hat{g}_N^{(j)}(1-) . \end{aligned}$$

In order to prove (b), notice that for $k \geq 1$ and $s \in (0,1)$

$$\hat{g}_N^{(k)}(s) = \sum_{j=k}^{\infty} p_j\, j(j-1)\dots(j-k+1)\, s^{j-k}$$

and therefore

$$\begin{aligned}
&\int_0^1 v^{k-1}\hat{g}_N^{(k)}(1-v)\,\mathrm{d}v \\
&= \sum_{j=k}^{\infty} p_j\, j(j-1)\dots(j-k+1)\int_0^1 v^{k-1}(1-v)^{j-k}\,\mathrm{d}v \\
&= \sum_{j=k}^{\infty} p_j\, j(j-1)\dots(j-k+1)B(k, j-k+1) \\
&= (k-1)!\sum_{j=k}^{\infty} p_j\,,
\end{aligned}$$

where $B(k, j-k+1) = (k-1)!(j-k)!/j!$. Statement (c) follows analogously, since

$$\begin{aligned}
&\int_0^1 v^{k+x-1}\hat{g}_N^{(k)}(1-v)\,\mathrm{d}v \\
&= \sum_{j=k}^{\infty} p_j\, j(j-1)\dots(j-k+1)\int_0^1 v^{k+x-1}(1-v)^{j-k}\,\mathrm{d}v \\
&= \sum_{j=k}^{\infty} p_j\, j(j-1)\dots(j-k+1)B(k+x, j-k+1)\,,
\end{aligned}$$

where $B(k+x, j-k+1) = \Gamma(k+x)\Gamma(j-k+1)/\Gamma(j+x+1)$.

4.3.2 By Taylor series expansion we have for $s \in [0, s_0 - 1)$

$$\hat{g}_N(1+s) = \sum_{k=0}^{\infty} \frac{\hat{g}_N^{(k)}(1)}{k!}\, s^k = \sum_{k=0}^{\infty} s^k\, \mathbb{E}\binom{N}{k}\,,$$

where we used the result of Exercise 2.1.7 in the last equality.

4.3.3 If $p_k = \mathrm{e}^{-\lambda}\lambda^k/k!$ for all $k \in \mathbb{N}$ and some $\lambda > 0$, then (4.3.2) holds for all $k \geq 1$ with $a = 0$ and $b = \lambda$. If

$$p_k = \frac{\Gamma(\alpha+k)(1-p)^{\alpha}p^k}{\Gamma(\alpha)k!}$$

for all $k \in \mathbb{N}$ and some $\alpha > 0$, $p \in (0,1)$, then (4.3.2) holds for all $k \geq 1$ with $a = p$ and $b = p(\alpha - 1)$. If $p_k = \binom{n}{k} p^k (1-p)^{n-k}$ for all $k = 0, 1, \ldots, n$ and some $n \in \mathbb{N}$, $p \in (0,1)$, then (4.3.2) holds for all $k \geq 1$ with $a = p/(p-1)$ and $b = (n+1)p/(1-p)$. If

$$p_k = \frac{-p^k}{k \log(1-p)}$$

for all $k \geq 1$ and some $p \in (0,1)$, then (4.3.2) holds for all $k \geq 2$ with $a = p$ and $b = -p$.

4.3.4 For a solution, see the proof of Theorem 4.3.1.

4.3.5 Let $s \in (0,1)$. Then

$$\begin{aligned} (1-as)\hat{g}_N^{(1)}(s) &= \sum_{k=1}^{\infty} k p_k s^{k-1} - \sum_{k=1}^{\infty} a k p_k s^k \\ &= \sum_{k=0}^{\infty} \big((k+1)p_{k+1} - akp_k\big) s^k \\ &= \sum_{k=0}^{\infty} (a+b) p_k s^k = (a+b)\hat{g}_N(s)\,, \end{aligned}$$

where we used (4.3.2) in the last but one equality. Thus, (4.3.3) holds. Let now $a + b > 0$. If $a = 0$, then (4.3.3) gives $\log \hat{g}_N(s) = b(s-1)$. Thus, $\hat{g}_N(s) = \exp(b(s-1))$, which means that N is Poi(b)-distributed. If $0 < a < 1$, then (4.3.3) gives

$$\begin{aligned} \log \hat{g}_N(s) &= -\frac{a+b}{a} \int_0^s \frac{-a}{1-av}\,\mathrm{d}v + \log c \\ &= -\frac{a+b}{a} \log(1-as) + \log c \end{aligned}$$

for some constant $c > 0$. Since $\hat{g}_N(1) = 0$, we have $c = (1-a)^{(a+b)/a}$. Thus,

$$\hat{g}_N(s) = \Big(\frac{1-a}{1-as}\Big)^{(a+b)/a},$$

which means that N is NB($(a+b)/a, a$)-distributed. If $a < 0$, then

$$\begin{aligned} \hat{g}_N(s) &= \Big(\frac{1-as}{1-a}\Big)^{-(a+b)/a} \\ &= \Big(\frac{a}{a-1}s + \frac{1}{1-a}\Big)^{-(a+b)/a}, \end{aligned}$$

where $-(a+b)/a > 0$. Suppose now that $-(a+b)/a \notin \mathbb{N}$. Then, $\hat{g}_N^{(k)}(0) < 0$ for some $k \in \mathbb{N}$, which is a contradiction to $\hat{g}_N^{(k)}(0) = k!p_k \geq 0$. Thus, $-(a+b)/a \in \mathbb{N}$ and therefore N is $\mathrm{Bin}(-(a+b)/a, a/(a-1))$-distributed.

4.3.6 By the result of Exercise 2.1.7 we have

$$\begin{aligned}\mathbb{E}\binom{N}{n} &= \frac{1}{n!}\hat{g}_N^{(n)}(1-) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\int_0^\infty \exp(\lambda(s-1))\,\mathrm{d}F_\Lambda(\lambda)\Big|_{s=1-} \\ &= \frac{1}{n!}\int_0^\infty \lambda^n\,\mathrm{d}F_\Lambda(\lambda) = \frac{1}{n!}\mathbb{E}\,\Lambda^n\,.\end{aligned}$$

4.4 Recursive Computation Methods

[2]

The following individual model describes a portfolio of n independent (not necessarily identically distributed) insurance policies. Suppose that each policy is related with a risk. The claim amount generated by this risk is a random integer multiple of some monetary unit. The portfolio can be divided into a number of classes by gathering all policies with the same claim probability and the same claim amount distribution. Let n_{ij} be the number of policies with claim probability $\theta_j < 1$ and with claim amount probabilities $p_1^{(i)},\ldots,p_{m_i}^{(i)}$. This means that the individual claim amount distribution F_{ij} generated by each policy of this class is given by the mixture $F_{ij} = (1-\theta_j)\delta_0+\theta_j\sum_{k=1}^{m_i}p_k^{(i)}\delta_k$. The generating function $\hat{g}(s)$ of the aggregate claim amount X^{ind} in this model is

$$\hat{g}(s) = \prod_{i=1}^a\prod_{j=1}^b\Bigl(1-\theta_j+\theta_j\sum_{k=1}^{m_i}p_k^{(i)}s^k\Bigr)^{n_{ij}} \tag{4.4.1}$$

where a is the number of possible conditional claim amount distributions and b is the number of different claim probabilities. In Theorem 4.4.1 we showed that the probability function $\{p_k\}$ of X^{ind} can be computed recursively by

$$p_0 = \prod_{i=1}^a\prod_{j=1}^b(1-\theta_j)^{n_{ij}}\,, \qquad p_k = \frac{1}{k}\sum_{i=1}^a\sum_{j=1}^b n_{ij}v_{ij}(k) \tag{4.4.2}$$

for $k = 1,\ldots,m$, where $m = \sum_{i=1}^a\sum_{j=1}^b n_{ij}m_i$ is the maximal aggregate claim amount,

$$v_{ij}(k) = \frac{\theta_j}{1-\theta_j}\sum_{l=1}^{m_i}p_l^{(i)}(lp_{k-l}-v_{ij}(k-l)) \tag{4.4.3}$$

[2] Should we add some further exercises to this section?

for $k = 1, \ldots, m$ and $v_{ij}(k) = 0$ otherwise.

In the special case of an individual model which describes a portfolio of independent life insurance policies, i.e. $p_i^{(i)} = 1$ for all $i = 1, \ldots, a$, then

$$v_{ij}(k) = \frac{\theta_j}{1-\theta_j}(ip_{k-i} - v_{ij}(k-i)) . \tag{4.4.4}$$

Formulae (4.4.2) and (4.4.4) yield an efficient reformulation of another recursive scheme. Let $p_i^{(i)} = 1$ for all $i = 1, \ldots, a$. Then we showed in Corollary 4.4.2 that, besides (4.4.2) and (4.4.4), the probability function $\{p_k\}$ of X^{ind} satisfies the following recursion formula, which is called *De Pril's algorithm*:

$$p_k = \frac{1}{k} \sum_{i=1}^{\min\{a,k\}} \sum_{l=1}^{\lfloor k/i \rfloor} c_{il} p_{k-li} , \tag{4.4.5}$$

where $\lfloor x \rfloor = \max\{n \in \mathbb{N} : n \le x\}$ and

$$c_{il} = (-1)^{l+1} i \sum_{j=1}^{b} n_{ij} \left(\frac{\theta_j}{1-\theta_j} \right)^l . \tag{4.4.6}$$

If the coefficients c_{il} can be neglected for $l > r$, then the following *r-th order approximation* $p_{k,r}$ to p_k is obtained:

$$p_{0,r} = \prod_{i=1}^{a} \prod_{j=1}^{b} (1-\theta_j)^{n_{ij}} , \qquad p_{k,r} = \frac{1}{k} \sum_{i=1}^{\min\{a,k\}} \sum_{l=1}^{\min\{r,\lfloor k/i \rfloor\}} c_{il} p_{k-li,r} . \tag{4.4.7}$$

The following recursive method, called *Panjer's algorithm*, can be used to calculate the probabilities $p_k^X = \mathbb{P}(X = k)$ of the compound $X = \sum_{i=1}^N U_i$ provided that (4.3.2) holds for $k = 1, 2, \ldots$ and $U_1, U_2, \ldots$ are (discrete) independent and identically distributed random variables taking their values in $\mathbb{N}$. Thus if we denote the probability function of U_i by $\{q_k, k \in \mathbb{N}\}$, then

$$p_j^X = \begin{cases} \hat{g}_N(q_0), & \text{for } j = 0, \\ (1-aq_0)^{-1} \sum_{k=1}^{j} \left(a + bkj^{-1}\right) q_k p_{j-k}^X, & \text{for } j = 1, 2, \ldots. \end{cases} \tag{4.4.8}$$

If the distribution F_U is not discrete but, for instance, absolutely continuous with density $f(x)$, then we cannot directly apply Panjer's algorithm (4.4.8). However, for each $h > 0$, we can consider the discrete approximation $U_{1,h}, U_{2,h}, \ldots$ of the random variables $U_1, U_2, \ldots$, where

$$\mathbb{P}(U_{i,h} = kh) = \int_{kh}^{(k+1)h} f(x)\, dx . \tag{4.4.9}$$

Then, we can apply (4.4.8) to calculate the probabilities $p^X_{k,h} = \mathbb{P}(X_h = k)$ of the discretized compound $X_h = \sum_{i=1}^N U_{i,h}$.

In Theorems 4.4.3 and 4.4.4 we derived continuous versions of (4.4.8) for the case of an arbitrary (not necessarily discrete) distribution F_U. If the compounding probability function $\{p_k\}$ is governed by (4.3.2) with parameters a and b and $F_U(0) = 0$, then the compound distribution $F_X = \sum_{k=0}^\infty p_k F_U^{*k}$ satisfies the integral equation

$$F_X(x) = p_0 + aF_U * F_X(x) + b\int_0^x v \int_0^{x-v} \frac{\mathrm{d}F_X(y)}{v+y}\,\mathrm{d}F_U(v)\,, \qquad x > 0\,. \tag{4.4.10}$$

If $F_U(0) = \alpha > 0$, then $F_X(0) = ((1-a)/(1-a\alpha))^{(a+b)a^{-1}}$, where this expression is interpreted as $\mathrm{e}^{(\alpha-1)b}$ if $a = 0$. If F_U is absolutely continuous with bounded density $f(x)$, then the density $\tilde{f}_X(x)$ of the absolutely continuous part of the compound distribution $F_X = \sum_{k=0}^\infty p_k F_U^{*k}$ satisfies

$$\tilde{f}_X(x) - \frac{1}{x}\int_0^x (ax+by) f_U(y) \tilde{f}_X(x-y)\,\mathrm{d}y = p_1 f_U(x)\,, \qquad x > 0\,. \tag{4.4.11}$$

Exercises

4.4.1 Let $h > 0$. Consider the compounds $\sum_{k=1}^N U_k$ and $\sum_{k=1}^N U_{k,h}$, where the distribution of the random variables $U_{k,h}$ is given by (4.4.9). Show that for each $x \in \mathbb{R}$

$$\mathbb{E}\Big(\sum_{k=1}^N U_{k,h} - x\Big)_+ \le \mathbb{E}\Big(\sum_{k=1}^N U_k - x\Big)_+ \le \mathbb{E}\Big(\sum_{k=1}^N U_{k,h} - x\Big)_+ + h\mathbb{E}\,N\,.$$

4.4.2 Consider the compound $X = \sum_{i=1}^N U_i$. Let the compounding probability function $\{p_k\}$ satisfy (4.3.2) for some parameters a and b and let $F_U(0) = 0$. Show that the *stop-loss transform* $\Pi(x) = \int_x^\infty (1 - F_X(v))\,\mathrm{d}v$ satisfies the integral equation

$$\begin{aligned}\Pi(x) &= a\Pi(0) + (a+b)\int_x^\infty (1 - F_U(v))\,\mathrm{d}v + a\,F_U * \Pi(x)\\ &\quad - bx\int_0^x \int_{x-v}^\infty \frac{v}{(v+w)^2} - bx\int_x^\infty \int_0^\infty \frac{v}{(v+w)^2}\,\mathrm{d}\Pi(w)\,\mathrm{d}F_U(v)\,.\end{aligned}$$

[Hint. Prove the following auxiliary results:

(a) Show first that $F_X(\infty) = (1 - p_0 - a)/b$.
(b) Show then that $\Pi(x) = a\,A(x) + b\,B(x)$ where $A(x) = \int_x^\infty (1 - F_U * F_X(v))\,\mathrm{d}v$ and $B(x) = \int_x^\infty (F_X^{\mathrm{s}}(\infty) - F_X^{\mathrm{s}}(v))\,\mathrm{d}v$.

(c) Show that

$$A(x) = \Pi(0) + \int_x^\infty (1 - F_U(v))\,\mathrm{d}v + F_U * \Pi(x)\,.$$

(d) Find a similar expression for $B(x)$.
(e) Check that $\Pi(0) = (a+b)\mu_U/(1-a)$.]

4.4.3 Let the functions $g : (0,\infty) \to \mathbb{R}$ and $f : (0,\infty) \to \mathbb{R}_+$ be integrable and bounded respectively. For arbitrary $a, b \in \mathbb{R}$ let the mapping $g \to \boldsymbol{A}g$ be defined by

$$(\boldsymbol{A}g)(x) = \frac{1}{x}\int_0^x (ax + by) f(y) g(x-y)\,\mathrm{d}y\,, \qquad x > 0\,.$$

Complete the proof of Theorem 4.4.4, i.e. show that for all $x > 0$ and $n = 1, 2, \ldots$

$$|(\boldsymbol{A}^n g)(x)| \le c_1 c_2^n (|a| + |b|)^n \frac{x^{n-1}}{(n-1)!} \tag{4.4.12}$$

where $c_1 = \int_0^\infty |g(x)|\,\mathrm{d}x$ and $c_2 = \sup_{x>0} f(x)$. [Hint. Use induction with respect to n.]

Solutions

4.4.1 Notice that $U_{k,h} \le_{\rm st} U_k$. Thus, by the result of Exercise 4.1.5b, we have $\sum_{k=1}^N U_{k,h} \le_{\rm st} \sum_{k=1}^N U_k$. This gives the first inequality since the function $g(t) = (t - x)_+$ is increasing, for each $x \in \mathbb{R}$. The second inequality is obtained in the same way, noticing that $U_k \le_{\rm st} U_{k,h} + h$.

4.4.2 [3]

4.4.3 Notice that

$$\left|\frac{ax + by}{x}\right| \le |a| + |b|$$

for all $y \in [0, x]$. Thus, (4.4.12) is obviously satisfied for $n = 1$. Suppose now that (4.4.12) holds for some $n \ge 1$. Then,

$$\begin{aligned}
\big|(\boldsymbol{A}^{n+1}g)(x)\big| &\le \big(|a| + |b|\big) c_2 \int_0^x \big|(\boldsymbol{A}^n g)\big|(x-y)\,\mathrm{d}y \\
&\le \big(|a| + |b|\big)^{n+1} c_1 c_2^{n+1} \frac{1}{(n-1)!}\int_0^x (x-y)^{n-1}\,\mathrm{d}y \\
&= \big(|a| + |b|\big)^{n+1} c_1 c_2^{n+1} \frac{x^n}{n!}\,.
\end{aligned}$$

Thus, for each $x > 0$, we have $\lim_{n\to\infty} (\boldsymbol{A}^n g)(x) = 0$.

[3] Add a solution.

4.5 Lundberg Bounds

A *two-sided Lundberg-type bound* for the tail $\overline{F}_X(x) = \mathbb{P}(X > x)$ of the compound $X = \sum_{i=1}^{N} U_i$ is given if we can show that there exist constants $\gamma > 0$ and $0 \le a_- \le a_+ < \infty$ such that

$$a_- \mathrm{e}^{-\gamma x} \le \overline{F}_X(x) \le a_+ \mathrm{e}^{-\gamma x}, \qquad x \ge 0. \tag{4.5.1}$$

It is clear that the case when $a_- > 0$ is of particular interest. In Theorem 4.5.1 we showed that the following is true. If X is a geometric compound with characteristics (p, F_U) such that the equation

$$\hat{m}_U(\gamma) = p^{-1} \tag{4.5.2}$$

admits a positive solution γ, then (4.5.1) holds, where

$$a_- = \inf_{x \in [0,x_0)} \frac{\mathrm{e}^{\gamma x}\overline{F}_U(x)}{\int_x^\infty \mathrm{e}^{\gamma y}\,\mathrm{d}F_U(y)}, \qquad a_+ = \sup_{x \in [0,x_0)} \frac{\mathrm{e}^{\gamma x}\overline{F}_U(x)}{\int_x^\infty \mathrm{e}^{\gamma y}\,\mathrm{d}F_U(y)} \tag{4.5.3}$$

and $x_0 = \sup\{x : F_U(x) < 1\}$. Notice that $a_+ \le 1$. The solution $\gamma > 0$ to (4.5.2) is usually called the *adjustment coefficient.*

Similar Lundberg bounds can be derived for more general compounding distributions. For instance, let the probability function $\{p_k\}$ be *logconcave*, i.e. $p_{k+1}^2 \ge p_{k+2}p_k$ for all $k \in \mathbb{N}$. If $p_0 + p_1 < 1$, then we showed in Corollary 4.5.2 that

$$\overline{F}_X(x) \le \frac{(1-p_0)^2}{1-p_0-p_1}\, a_+ \mathrm{e}^{-\gamma x}, \qquad x \ge 0\,, \tag{4.5.4}$$

where a_+ is defined as in (4.5.3) and $\gamma > 0$ is the solution to the equation

$$\hat{m}_U(\gamma) = \frac{1-p_0}{1-p_0-p_1}\,. \tag{4.5.5}$$

Suppose that we have a sample $U_1, U_2, \ldots, U_n$ of n independent claim sizes with common distribution F_U. How do we get an estimate for the unknown solution $\gamma > 0$ to

$$\hat{m}_U(s) = c > 1 \tag{4.5.6}$$

based on this sample? A rather natural procedure is to replace the moment generating function $\hat{m}_U(s)$ by its empirical analogue, the *empirical moment generating function* $\hat{m}_{F_n}(s)$, which is obtained as the moment generating function of the empirical distribution F_n, where $F_n(x) = n^{-1}\max\{i : U_{(i)} \le x\}$.

Exercises

4.5.1 Consider the tail $\overline{F}_X(x)$ of the compound $X = \sum_{i=1}^N U_i$, where N has the negative binomial distribution $\mathrm{NB}(2,p)$ and U is $\mathrm{Exp}(\delta)$-distributed. Show that if $\gamma > 0$ is computed from (4.5.5), then

$$\gamma = \frac{2(1-p)^2\delta}{2-p} < (1-p)\delta\,.$$

Conclude that in this case the Lundberg bound (4.5.4) is not optimal. [Hint. Use the result of Exercise 4.2.8.]

4.5.2 Consider the stop-loss transform $\Pi(x) = \mathbb{E}\,(X-x)_+$ of the compound $X = \sum_{i=1}^N U_i$ and assume that the distribution of the compounding variable N is $\mathrm{IHR_d}$. Show that

$$\mathbb{E}\,(X-x)_+ \le \frac{a_+}{\gamma}\mathrm{e}^{-\gamma x}, \qquad x \ge 0\,,$$

where a_+ is given in (4.5.3) and $\gamma > 0$ is the root of

$$1 + \frac{1}{\mathbb{E}\,N} = \hat{m}_U(\gamma)\,. \tag{4.5.7}$$

4.5.3 (Continuation) Assume that N is $\mathrm{NB}(2,p)$-distributed and U is $\mathrm{Exp}(\delta)$-distributed as in Exercise 4.5.1. Compute the solution γ to (4.5.7) and compare it with the solution to (4.5.5). [Hint. Recall that the probability function of N is logconcave; see the proof of Corollary 2.4.2. Furthermore, note that $N \le_{\mathrm{st}} N'$ if N' has the distribution $\mathrm{Geo}(\mathbb{E}\,N/(1+\mathbb{E}\,N))$ and use the result of the Exercise 4.5.1.]

4.5.4 Consider the tail $\overline{F}_X(x)$ of the compound $X = \sum_{i=1}^N U_i$ and let N have the binomial distribution $\mathrm{Bin}(n,p)$. Show that if $\gamma > 0$ is the unique solution to

$$\hat{m}_{F_U}(\gamma) = \frac{1-(1-p)^n}{1-(1-p)^n - np(1-p)^{n-1}}\,,$$

then

$$\overline{F}_X(x) \le \frac{(1-(1-p)^n)^2}{1-(1-p)^n - np(1-p)^{n-1}}a_+\mathrm{e}^{-\gamma x}, \qquad x \ge 0\,,$$

where a_+ is defined as in (4.5.3).

4.5.5 Let N be Poisson-distributed with parameter λ and let $\gamma > 0$ be the solution to

$$\hat{m}_{F_U}(\gamma) = \frac{\mathrm{e}^\lambda - 1}{\mathrm{e}^\lambda - 1 - \lambda}\,.$$

Show that, for the tail $\overline{F}_X(x)$ of the Poisson compound $X = \sum_{i=1}^N U_i$,

$$\overline{F}_X(x) \le \Bigl(1 + \frac{e^{-\lambda} - 1 + \lambda}{e^{\lambda} - 1 - \lambda}\Bigr) a_+ e^{-\gamma x}, \qquad x \ge 0\,,$$

where a_+ is defined as in (4.5.3).

4.5.6 Let $\alpha_U > 0$ and $\hat{m}_U(\alpha_U) = \infty$, where $\alpha_U = \limsup_{x\to\infty} - \log \overline{F}_U(x)/x$. Show that for any closed interval $I \subset (-\infty, \alpha_U)$ and any $k \in \mathbb{N}$

$$\lim_{n\to\infty} \sup_{s\in I} |\hat{m}^{(k)}_{F_n}(s) - \hat{m}^{(k)}_U(s)| = 0\,. \tag{4.5.8}$$

4.5.7 Define the estimator $\hat{\gamma}_n$ for the adjustment coefficient $\gamma > 0$ by the equation

$$\hat{m}_{F_n}(\hat{\gamma}_n) = \frac{1}{n}\sum_{i=1}^n e^{\hat{\gamma}_n U_i} = c\,, \tag{4.5.9}$$

where $\gamma > 0$ is the solution to (4.5.6). Show that the estimator $\hat{\gamma}_n$ is *strongly consistent*, i.e. $\hat{\gamma}_n \to \gamma$ with probability 1 as $n \to \infty$. [Hint. Use the result of Exercise 4.5.6.]

Solutions

4.5.1 Recall that for the probability function $\{p_k\}$ of the distribution NB$(2,p)$ we have $p_k = (k+1)(1-p)^2 p^k$ for all $k \in \mathbb{N}$. Furthermore, in the proof of Corollary 2.4.2 we showed that $\{p_k\}$ is logconcave. Thus, (4.5.4) holds and (4.5.5) takes the form

$$\frac{\delta}{\delta - s} = \frac{2-p}{2-p-2(1-p)^2}\,,$$

which has the solution

$$\gamma = \frac{2(1-p)^2\delta}{2-p}\,.$$

However, by the result of Exercise 4.2.8, the tail of X can be exponentially bounded with $\gamma' = (1-p)\delta > \gamma$.

4.5.2 [4]

4.5.3 [5]

4.5.4 By the result of Exercise 2.4.4, the probability function of Bin(n,p) is logconcave. Thus, (4.5.4) holds and the statement follows noticing that $p_0 = (1-p)^n$ and $p_1 = np(1-p)^{n-1}$.

[4] Add a solution.

[5] Add a solution.

4.5.5 By the result of Exercise 2.4.4, the probability function of $\mathrm{Poi}(n,p)$ is logconcave. Thus, (4.5.4) holds and the statement follows noticing that $p_0 = \mathrm{e}^{-\lambda}$ and $p_1 = \lambda \mathrm{e}^{-\lambda}$.

4.5.6 [6]

4.5.7 [7]

4.6 Approximation by Compound Distributions

Consider the aggregate claim amount $X^{\mathrm{ind}} = \sum_{i=1}^{n} U_i$ in the individual model, where the risks $U_1, \ldots, U_n$ are independent, but not necessarily identically distributed. As in Section 4.1, let the distribution F_{U_i} of U_i be the mixture $F_{U_i} = (1-\theta_i)\delta_0 + \theta_i F_{V_i}$, where $0 < \theta_i \le 1$ and F_{V_i} is the distribution of some (strictly) positive random variable V_i. We will approximate X^{ind} by a collective model with the aggregate claim amount $X^{\mathrm{col}} = \sum_{i=1}^{N} U_i'$, where the probability function of N satisfies (4.3.2) and the $U_1', U_2', \ldots$ are independent and identically distributed. Thus, by the results of Section 4.4, we can consider the following three cases:

- *the compound Poisson approximation*, which relies on the approximation of each random variable U_i by a Poisson compound Y_i with characteristics (θ_i, F_{V_i}). This yields $\mathbb{E}\, U_i = \mathbb{E}\, Y_i$. Furthermore, taking $Y_1, \ldots, Y_n$ independent, Theorem 4.2.2 implies that $Y = Y_1 + \ldots + Y_n$ is a Poisson compound with characteristics (λ, F) given by

$$\lambda = \sum_{i=1}^{n} \theta_i \,, \qquad F = \sum_{i=1}^{n} \frac{\theta_i}{\lambda} F_{V_i} \,, \tag{4.6.1}$$

- *the compound binomial approximation*, where the compounding variable N is $\mathrm{Bin}(n,p)$-distributed with $p = \lambda/n$, U_i' has the distribution F, and λ, F are as in the compound Poisson approximation.
- *the compound negative binomial approximation*, where the compounding variable N is $\mathrm{NB}(n, p/(1+p))$-distributed with $p = \lambda/n$, U_i' has the distribution F, and λ, F are as in the compound Poisson approximation.

Let F and G be two distributions on $\mathbb{R}$. The *total variation distance* between F and G is defined by

$$d_{\mathrm{TV}}(F,G) = \sup_{B \in \mathcal{B}(\mathbb{R})} |F(B) - G(B)| \,. \tag{4.6.2}$$

[6] Add a solution.
[7] Add a solution.

Analogously, the total variation distance $d_{\mathrm{TV}}(X,Y)$ between two random variables X and Y on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is defined by

$$d_{\mathrm{TV}}(X,Y) = \sup_{B \in \mathcal{B}(\mathbb{R})} |\mathbb{P}(X \in B) - \mathbb{P}(Y \in B)| . \tag{4.6.3}$$

Let F and G be the distributions of X and Y respectively. Then, obviously, $d_{\mathrm{TV}}(X,Y) = d_{\mathrm{TV}}(F,G)$.

A (finite) *signed measure* M is a σ-additive set function $M : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ for which $|M(B)| < \infty$ for all Borel sets $B \in \mathcal{B}(\mathbb{R})$. Let $M_1, M_2 : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ be two signed measures. The *convolution* $M_1 * M_2$ of M_1 and M_2 is the signed measure given by

$$M_1 * M_2(B) = \iint \mathbb{I}_B(x+y) \mathrm{d}M_1(x) \mathrm{d}M_2(y) , \qquad B \in \mathcal{B}(\mathbb{R}) .$$

The n-fold convolution M^{*n} of a signed measure M is defined recursively by $M^{*0} = \delta_0$, and $M^{*n} = M^{*(n-1)} * M, n \geq 1$. It is well known that each signed measure M can be represented as the difference $M = M_+ - M_-$ of two (nonnegative) measures M_+, M_-, called the *Hahn–Jordan decomposition* of M. The *total variation* of M is given by $\|M\| = M_+(\mathbb{R}) + M_-(\mathbb{R})$. The *characteristic function* $\hat{\varphi}_M(s)$ of the signed measure M is given by $\hat{\varphi}_M(s) = \int \mathrm{e}^{\mathrm{i}st} \, dM(t)$.

Exercises

4.6.1 Let $X^{\mathrm{ind}} = \sum_{i=1}^n U_i$ have the distribution $F_{U_1} * \ldots * F_{U_n}$ with $F_{U_i} = (1-\theta_i)\delta_0 + \theta_i F_{V_i}$, where $0 < \theta_i \leq 1$ and F_{V_i} is the distribution of some (strictly) positive random variable V_i; $i = 1, \ldots, n$. Show that for the approximation by the collective model with Poisson-distributed compounding variable N specified by (4.6.1), we have $\mathbb{E}\, X^{\mathrm{ind}} = \mathbb{E}\, X^{\mathrm{col}}$ and $\operatorname{Var} X^{\mathrm{ind}} < \operatorname{Var} X^{\mathrm{col}}$.

4.6.2 (Continuation) Show that the result of Exercise 4.6.1 remains true if, alternatively, the (above specified) compound binomial approximation with $n \geq 2$ or the compound negative binomial approximation are considered.

4.6.3 (Continuation) Consider three approximations to a certain individual model by the following collective models. In all these collective models let $U_1', U_2', \ldots$ be an arbitrary sequence of independent and identically distributed random variables. Let the (above specified) compounding variable N be (a) binomial, (b) Poisson, (c) negative binomial distributed and denote the corresponding aggregate claim amounts by X_1^{col}, X_2^{col} and X_3^{col}, respectively. Show that $X_1^{\mathrm{col}} \leq_{\mathrm{sl}} X_2^{\mathrm{col}} \leq_{\mathrm{sl}} X_3^{\mathrm{col}}$ and hence $\operatorname{Var} X_1^{\mathrm{col}} \leq \operatorname{Var} X_2^{\mathrm{col}} \leq \operatorname{Var} X_3^{\mathrm{col}}$.

4.6.4 Let F and G be two arbitrary distributions on $\mathbb{R}$. Show that the mapping $(F,G) \to d_{\mathrm{TV}}(F,G)$ is a *metric*, i.e. $d_{\mathrm{TV}}(F,G) = 0$ if and only if $F = G$, $d_{\mathrm{TV}}(F,G) = d_{\mathrm{TV}}(G,F)$ and $d_{\mathrm{TV}}(F_1,F_2) \le d_{\mathrm{TV}}(F_1,F_3) + d_{\mathrm{TV}}(F_3,F_2)$ for arbitrary distributions F_1, F_2, F_3 on $\mathbb{R}$.

4.6.5 Show that the inequality $d_{\mathrm{TV}}(X,Y) \le \mathbb{P}(X \neq Y)$ holds for arbitrary random variables X, Y defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

4.6.6 Let the random variables X_1 and X_2 have Poisson distributions with parameters λ_1 and λ_2 respectively. Show that

$$d_{\mathrm{TV}}(X_1, X_2) \le |\lambda_1 - \lambda_2| \,.$$

[Hint. Use the result of Exercise 4.6.5. Assume that $\lambda_1 < \lambda_2$ and put $X_2 = X_1 + Z$, where Z is $\mathrm{Poi}(\lambda_2 - \lambda_1)$-distributed and independent of X_1.]

4.6.7 Let $I_1, \ldots, I_n$ be independent Bernoulli distributed random variables (assuming the values 0 and 1) with $\mathbb{P}(I_i = 1) = \theta_i$ and, for some $\lambda > 0$, let N be a $\mathrm{Poi}(\lambda)$-distributed random variable, which is independent of $I_1, \ldots, I_n$. Show that

$$d_{\mathrm{TV}}(N, N') \le \min\Big\{\sum_{i=1}^n \theta_i^2, \Big(\sum_{i=1}^n \theta_i^2 \Big/ \sum_{i=1}^n \theta_i\Big)\Big\} + \Big|\sum_{i=1}^n \theta_i - \lambda\Big|,$$

where $N' = \sum_{i=1}^n I_i$. [Hint/Comment. Let N'' be $\mathrm{Poi}(\sum_{i=1}^n \theta_i)$-distributed. Use the inequality $d_{\mathrm{TV}}(N,N') \le d_{\mathrm{TV}}(N,N'') + d_{\mathrm{TV}}(N'',N')$ and the result of Exercise 4.6.6. Notice that, by the result of Exercise 4.6.7, the bound derived in Theorem 4.6.2 can be slightly improved.]

4.6.8 Show that the exponential set function $\exp(M) = \sum_{k=0}^\infty M^{*k}/k!$ of a signed measure M is a well-defined signed measure.

4.6.9 Let M, M_1, M_2 be arbitrary signed measures on $\mathbb{R}$. Then
(a) $\exp(\hat{\varphi}_M(s))$ is the characteristic function of $\exp(M)$,
(b) $\exp(M_1 + M_2) = \exp(M_1) * \exp(M_2)$,
(c) $\|M_1 * M_2\| \le \|M_1\|\,\|M_2\|$,
(d) $\|\exp(M) - \delta_0\| \le \mathrm{e}^{\|M\|} - 1$.

Solutions

4.6.1 Recall that $\mathbb{E}\, X^{\mathrm{ind}} = \sum_{i=1}^n \theta_i \mathbb{E}\, V_i$ by the result of Exercise 4.1.1. On the other hand, using (4.6.1) and the result of Exercise 4.1.2, we have

$$\mathbb{E}\, X^{\mathrm{col}} = \mathbb{E}\, N \mathbb{E}\, U = \lambda \sum_{i=1}^n \frac{\theta_i}{\lambda} \mathbb{E}\, V_i = \mathbb{E}\, X^{\mathrm{ind}} \,.$$

In the same way we find that

$$\begin{aligned}
\operatorname{Var} X^{\mathrm{col}} &= \operatorname{Var} N (\mathbb{E}\, U)^2 + \mathbb{E}\, N \operatorname{Var} U \\
&= \lambda\Big(\sum_{i=1}^{n} \frac{\theta_i}{\lambda} \mathbb{E}\, V_i\Big)^2 + \lambda\Big(\sum_{i=1}^{n} \frac{\theta_i}{\lambda} \mathbb{E}\, V_i^2 - \Big(\sum_{i=1}^{n} \frac{\theta_i}{\lambda} \mathbb{E}\, V_i\Big)^2\Big) \\
&= \sum_{i=1}^{n} \theta_i \mathbb{E}\, V_i^2 \\
&> \sum_{i=1}^{n} \theta_i \mathbb{E}\, V_i^2 - \sum_{i=1}^{n} \theta_i^2 (\mathbb{E}\, V_i)^2 \\
&= \operatorname{Var} X^{\mathrm{ind}}.
\end{aligned}$$

4.6.2 Let N be $\mathrm{Bin}(n,p)$-distributed with $p = \lambda/n$. Then $\mathbb{E}\, N = np = \lambda$ and $\operatorname{Var} N = np(1-p) = \lambda(1-\lambda/n)$. Thus, in the same way as in Exercise 4.6.1, we find that $\mathbb{E}\, X^{\mathrm{ind}} = \mathbb{E}\, X^{\mathrm{col}}$ and

$$\begin{aligned}
\operatorname{Var} X^{\mathrm{col}} &= \sum_{i=1}^{n} \theta_i \mathbb{E}\, V_i^2 - \frac{\lambda^2}{n}\Big(\sum_{i=1}^{n} \frac{\theta_i}{\lambda} \mathbb{E}\, V_i\Big)^2 \\
&> \sum_{i=1}^{n} \theta_i \mathbb{E}\, V_i^2 - \sum_{i=1}^{n} \theta_i^2 (\mathbb{E}\, V_i)^2 \\
&= \operatorname{Var} X^{\mathrm{ind}},
\end{aligned}$$

since $n(a_1^2 + \ldots + a_n^2) > a_1^2 + \ldots + a_n^2$ for all $n \geq 2$ and $a_1, \ldots, a_n > 0$. Let now N be $\mathrm{NB}(n, p/(1+p))$-distributed with $p = \lambda/n$. Then, $\mathbb{E}\, N = np = \lambda$ and $\operatorname{Var} N = np(1+p) = \lambda(1+\lambda/n)$. Thus, $\mathbb{E}\, X^{\mathrm{ind}} = \mathbb{E}\, X^{\mathrm{col}}$ and

$$\begin{aligned}
\operatorname{Var} X^{\mathrm{col}} &= \sum_{i=1}^{n} \theta_i \mathbb{E}\, V_i^2 + \frac{\lambda^2}{n}\Big(\sum_{i=1}^{n} \frac{\theta_i}{\lambda} \mathbb{E}\, V_i\Big)^2 \\
&> \sum_{i=1}^{n} \theta_i \mathbb{E}\, V_i^2 - \sum_{i=1}^{n} \theta_i^2 (\mathbb{E}\, V_i)^2 \\
&= \operatorname{Var} X^{\mathrm{ind}}.
\end{aligned}$$

4.6.3 Let N_1, N_2, N_3 denote the binomial, Poisson and negative binomial distributed compounding variables respectively. By the results of Exercises 3.2.7 and 3.2.8 we have $N_1 \leq_{\mathrm{sl}} N_2 \leq_{\mathrm{sl}} N_3$. By Theorem 4.2.3b, this implies that $X_1^{\mathrm{col}} \leq_{\mathrm{sl}} X_2^{\mathrm{col}} \leq_{\mathrm{sl}} X_3^{\mathrm{col}}$ and in particular $\mathbb{E}\,(X_1^{\mathrm{col}})^2 \leq \mathbb{E}\,(X_2^{\mathrm{col}})^2 \leq \mathbb{E}\,(X_3^{\mathrm{col}})^2$. Furthermore, notice that $\mathbb{E}\, X_1^{\mathrm{col}} = \mathbb{E}\, X_2^{\mathrm{col}} = \mathbb{E}\, X_3^{\mathrm{col}}$. Hence, $\operatorname{Var} X_1^{\mathrm{col}} \leq \operatorname{Var} X_2^{\mathrm{col}} \leq \operatorname{Var} X_3^{\mathrm{col}}$.

4.6.4 Definition 4.6.2 immediately gives that $d_{\rm TV}(F,G) = 0$ if and only if $F = G$. The fact that $d_{\rm TV}(F,G) = d_{\rm TV}(G,F)$ is also obvious. Furthermore,

$$\begin{aligned} d_{\rm TV}(F_1,F_2) &= \sup_{B\in\mathcal{B}(\mathbb{R})} |F_1(B)-F_2(B)| \\ &\le \sup_{B\in\mathcal{B}(\mathbb{R})} \big(|F_1(B)-F_3(B)|+|F_3(B)-F_2(B)|\big) \\ &\le \sup_{B\in\mathcal{B}(\mathbb{R})} |F_1(B)-F_3(B)| + \sup_{B\in\mathcal{B}(\mathbb{R})} |F_3(B)-F_2(B)| \\ &= d_{\rm TV}(F_1,F_3)+d_{\rm TV}(F_3,F_2)\,. \end{aligned}$$

4.6.5 [8]

4.6.6 Without loss of generality we can assume that $\lambda_1 < \lambda_2$. Let now X_1 and Z be two independent random variables with distributions $\mathrm{Poi}(\lambda_1)$ and $\mathrm{Poi}(\lambda_2)$ respectively, which are given on the same probability space. The construction of such a (product) space is always possible since X_1 and Z are independent. Then, $X_2 = X_1 + Z$ is also defined on the same probability space and, by the result of Exercise 2.2.1b, the random variable X_2 is $\mathrm{Poi}(\lambda_2)$-distributed. Thus, by the result of Exercise 4.6.5, we have

$$\begin{aligned} d_{\rm TV}(X_1,X_2) &\le \mathbb{P}(X_1 \ne X_2) \\ &= \mathbb{P}(X_1 \ne X_1 + Z) = \mathbb{P}(Z>0) \\ &= 1-\mathrm{e}^{-(\lambda_2-\lambda_1)} \le \lambda_2-\lambda_1\,. \end{aligned}$$

4.6.7 By the inequality $d_{\rm TV}(N,N') \le d_{\rm TV}(N,N'') + d_{\rm TV}(N'',N')$ and the result of Exercise 4.6.6, it suffices to show that for $\lambda = \sum_{i=1}^n \theta_i$

$$d_{\rm TV}(N,N') \le \min\Big\{\sum_{i=1}^n \theta_i^2, \Big(\sum_{i=1}^n \theta_i^2 \Big/ \sum_{i=1}^n \theta_i\Big)\Big\}\,.$$

However, this inequality is an immediate consequence of the arguments given in the proof of Theorem 4.6.2.

4.6.8 Since $|M^{*k}(B)| \le \|M\|^k$ for each $B \in (\mathbb{R})$ and $\sum_{k=0}^\infty \|M\|^k/k! < \infty$, the bounded convergence theorem implies that $\exp(M)$ is a well-defined (finite) signed measure.

4.6.9 For each $s \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}st}\,\mathrm{d}\exp(M)(t) \quad = \quad \sum_{k=0}^\infty \frac{1}{k!}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}st}\,\mathrm{d}M^{*k}(t)$$

[8] Add a solution.

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}st}\,\mathrm{d}M(t)\Big)^k \\
&= \exp\big(\hat{\varphi}_M(s)\big)\,.
\end{aligned}
$$

This proves statement (a). Thus, for the characteristic function of $\exp(M_1 + M_2)$ we have

$$
\begin{aligned}
\exp\big(\hat{\varphi}_{M_1+M_2}(s)\big) &= \exp\big(\hat{\varphi}_{M_1}(s) + \hat{\varphi}_{M_2}(s)\big) \\
&= \exp\big(\hat{\varphi}_{M_1}(s)\big)\exp\big(\hat{\varphi}_{M_2}(s)\big) \\
&= \hat{\varphi}_{\exp(M_1)}(s) \cdot \hat{\varphi}_{\exp(M_2)}(s) \\
&= \hat{\varphi}_{\exp(M_1)*\exp(M_2)}(s)\,.
\end{aligned}
$$

Statement (b) now follows from the one-to-one correspondence between signed measures and their characteristic functions. In order to prove (c), notice that

$$(M_1 * M_2)_+(\mathbb{R}) = (M_1)_+(\mathbb{R})(M_2)_+(\mathbb{R})$$

and

$$(M_1 * M_2)_-(\mathbb{R}) = (M_1)_-(\mathbb{R})(M_2)_-(\mathbb{R})\,.$$

Thus,

$$
\begin{aligned}
\|M_1 * M_2\| &= (M_1)_+(\mathbb{R})(M_2)_+(\mathbb{R}) + (M_1)_-(\mathbb{R})(M_2)_-(\mathbb{R}) \\
&\le \|M_1\|\|M_2\|\,.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\|\exp(M) - \delta_0\| &\le \sum_{k=1}^{\infty} \frac{1}{k!}\|M^{*k}\| \\
&\le \sum_{k=1}^{\infty} \frac{1}{k!}\|M\|^k = \mathrm{e}^{\|M\|} - 1\,.
\end{aligned}
$$

4.7 Inverting the Fourier Transform

Recall that the characteristic function of an $\mathbb{N}$-valued random variable with probability function $\{p_k\}$ is given by the Fourier transform

$$\hat{\varphi}(s) = \sum_{k=0}^{\infty} \mathrm{e}^{\mathrm{i}sk} p_k\,, \qquad s \in \mathbb{R}\,. \tag{4.7.1}$$

Suppose now that we know the Fourier transform $\hat{\varphi}(s) = \sum_{k=0}^{\infty} \mathrm{e}^{\mathrm{i}sk} p_k$ of the probability function $\{p_k\}$. Then it is possible to obtain an approximation to the first n terms $p_0, \ldots, p_{n-1}$ of the sequence $p_0, p_1, \ldots$ by sampling the Fourier transform $\hat{\varphi}(s)$ at the points $s_j = 2\pi j/n$, $j = 0, 1, \ldots, n-1$. Since for each $k = 1, 2, \ldots$ the function $\{\mathrm{e}^{\mathrm{i}sk},\ s \geq 0\}$ has the period 2π, we obtain

$$\hat{\varphi}(s_j) = \sum_{k=0}^{\infty} p_k \mathrm{e}^{\mathrm{i}s_j k} = \sum_{\ell=0}^{\infty} \sum_{k=0}^{n-1} p_{k+n\ell} \mathrm{e}^{\mathrm{i}s_j(k+n\ell)} = \sum_{k=0}^{n-1} \tilde{p}_k \mathrm{e}^{\mathrm{i}s_j k},$$

where $\tilde{p}_k = \sum_{\ell=0}^{\infty} p_{k+n\ell}$ for $k = 0, 1, \ldots, n-1$. The values $\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_{n-1}$ can be calculated from $\{\hat{\varphi}(s_j), j = 0, 1, \ldots, n-1\}$ by the following formula:

$$\tilde{p}_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{\varphi}(s_j) \mathrm{e}^{-\mathrm{i}s_j k}. \tag{4.7.2}$$

Note that $\tilde{p}_k$ approximates p_k for each $k = 0, \ldots, n-1$, since the error $\tilde{p}_k - p_k = \sum_{\ell=1}^{\infty} p_{k+n\ell}$ becomes arbitrarily small if n tends to infinity. When bounds on the tail of $\{p_k\}$ are available, we can even estimate the error $\tilde{p}_k - p_k$.

Consider the compound $X = \sum_{k=1}^{N} U_k$, where $U_1, U_2, \ldots$ are independent and identically distributed $\mathbb{N}$-valued random variables. Assume that the generating function $\hat{g}_N(z)$ of N and the values $\{\hat{\varphi}_U(s_j), j = 0, 1, \ldots, n-1\}$ of the characteristic function $\hat{\varphi}_U(s)$ are known. Then the values $\{\hat{\varphi}_X(s_j),\ j = 0, 1, \ldots, n-1\}$ of the characteristic function $\hat{\varphi}_X(s)$ can be computed from formula (4.2.2) in Exercise 4.2.1. Thus, the probability function $\{p_k\}$ of X can be calculated or, at least, approximated in the way given above.

Exercises

4.7.1 Consider Gerber's portfolio given in Table 4.7.1. Compute λ and F defined in (4.6.1) for the compound Poisson approximation. Compute also p needed for the compound binomial and negative binomial approximations to the aggregate claim amount in this portfolio.

4.7.2 Write a program which computes the p_k for the portfolio given in Table 4.7.1 (using the Fourier transform method) and the compound Poisson, binomial and negative binomial approximations as shown in Figure 4.7.1 (using Panjer's algorithm).

4.7.3 Write a program (using the Fourier transform method) which computes the stop-loss transform $\Pi(n) = \mathbb{E}\,(X^{\mathrm{ind}} - n)_+$ for Gerber's portfolio given in Table 4.7.1. Compute also the stop-loss transforms for the compound Poisson, binomial and negative binomial approximations.

4.7.4 Consider a Poisson compound $X = \sum_{k=1}^{N} U_i$ specified by (λ, F), where F is the geometric distribution Geo(1/2). Suppose that we want to compute the probability function $\{p_k\}$ of X for $k = 0, \ldots, 10$. In how many points one has to sample the characteristic function $\hat{\varphi}_X(s)$ to have an error less than 10^{-2}. Show the calculations for $\lambda = 1$ and $\lambda = 10$. [Hint. Use inequality (4.5.4).]

5

Risk Processes

5.1 Time-Dependent Risk Models

Consecutive claim sizes $X_1, X_2, \ldots$, are assumed to be independent and identically distributed, taking values in $\mathbb{N}$. The common probability function is $\{p_k\} = \{\mathbb{P}(X_n = k)\}$. The random variables $\{R_n,\, n \in \mathbb{N}\}$, where

$$R_n = u + n - \sum_{i=1}^{n} X_i\,,$$

describe the evolution of a risk reserve at the end of a sequence of time periods. For simplicity we take the premium per period equal to 1 and suppose that the initial reserve is equal to $u \in \mathbb{N}$. We will call the sequence $\{R_n,\, n \in \mathbb{N}\}$ a *discrete-time risk reserve process.*

If $p_0 + p_1 < 1$, then the risk reserve R_n can be negative for some $n \in \mathbb{N}$. The event $\{R_1 < 0\} \cup \{R_2 < 0\} \cup \ldots$ is called the (technical) *ruin* of the portfolio. Formally, the probability of ruin is defined in the following way. The epoch $\tau_{\mathrm{d}}(u) = \min\{n \geq 1 : \; R_n < 0\}$ when the risk reserve process becomes negative for the first time is called the *time of ruin* (or *ruin time*) (we set $\tau_{\mathrm{d}}(u) = \infty$ if $R_n \geq 0$ for all $n \in \mathbb{N}$). Furthermore,

$$\psi(u) = \mathbb{P}(\{R_1 < 0\} \cup \{R_2 < 0\} \cup \ldots) = \mathbb{P}(\tau_{\mathrm{d}}(u) < \infty)$$

is called the (infinite-horizon) *ruin probability* for the initial reserve $u \in \mathbb{N}$. If $\psi(u)$ is seen as a function of the variable u, then $\psi(u)$ is called the *ruin function.*

Instead of the risk reserve process, it is sometimes preferable to consider the *claim surplus process* $\{S_n\}$ defined by

$$S_n = \sum_{i=1}^{n} X_i - n\,, \qquad n \in \mathbb{N}\,.$$

A more general risk reserve model in continuous time is defined as follows. We are given

- random epochs $\sigma_1, \sigma_2, \ldots$ with $0 < \sigma_1 < \sigma_2 < \ldots$ at which the claims occur, where the random variables σ_n can be discrete or continuous; $T_n = \sigma_n - \sigma_{n-1}$,
- the corresponding positive (individual or aggregate) claim sizes $U_1, U_2, \ldots$, which can also be arbitrary nonnegative random variables,
- the initial risk reserve $u \geq 0$, and
- the premiums which are collected at a constant rate $\beta > 0$, so that the premium income is a linear function of time.

The random sequence $\{\sigma_n\}$ is called a *point process* and $\{(\sigma_n, U_n)\}$ a *marked point process*. Another approach is based on the *cumulative arrival process* $\{X(t), t \geq 0\}$, which is given by

$$X(t) = \sum_{k=1}^{\infty} U_k \mathbb{I}(\sigma_k \leq t) = \sum_{k=1}^{N(t)} U_k \,, \tag{5.1.1}$$

where $X(t)$ is the aggregate amount of all claims arriving in the interval $(0, t]$ and the *counting process* $\{N(t), t \geq 0\}$ is given by $N(t) = \sum_{k=1}^{\infty} \mathbb{I}(\sigma_k \leq t)$. The *risk reserve process* $\{R(t),\ t \geq 0\}$ is then given by

$$R(t) = u + \beta t - \sum_{i=1}^{N(t)} U_i \,,$$

while the *claim surplus process* $\{S(t),\ t \geq 0\}$ is $S(t) = \sum_{i=1}^{N(t)} U_i - \beta t$.

The *time of ruin* $\tau(u) = \min\{t : R(t) < 0\} = \min\{t : S(t) > u\}$ is the first epoch when the risk reserve process becomes negative or, equivalently, when the claim surplus process crosses the level u. We are interested in the (infinite-horizon) *ruin probabilities* $\psi(u) = \mathbb{P}(\tau(u) < \infty)$. We further need the notion of the *survival probability* $\overline{\psi}(u) = 1 - \psi(u)$.

The largest value $M = \max_{t \geq 0} S(t)$ of the claim surplus process can be given by

$$M = \max_{n \geq 0} \sum_{k=1}^{n} (U_k - \beta T_k) \tag{5.1.2}$$

and consequently $\psi(u) = \mathbb{P}(M > u)$.

Exercises

5.1.1 Consider the risk model in discrete time. Show that the set $\{\omega : \tau_{\mathrm{d}}(u, \omega) = k\}$ is measurable, i.e. $\{\omega : \tau_{\mathrm{d}}(u, \omega) = k\} \in \mathcal{F}$ for all $k \in \mathbb{N}$.

5.1.2 Consider the risk model in discrete time. Show that if $\mathbb{E}\, X < 1$ then $\psi(u) < 1$ for all $u \in \mathbb{N}$. [Hint. Note that $M = \max\{0, S_1, S_2, \ldots\}$ is finite with probability 1. Thus, it suffices to show that $\mathbb{P}(M = 0) > 0$.]

5.1.3 Consider the risk model in discrete time. Show that the solution to the recurrence equation

$$L_n = (L_{n-1} + X_n - 1)_+$$

is given by

$$L_n = \max\{0, Y_n, Y_{n-1} + Y_n, \ldots, L_0 + Y_1 + \ldots + Y_n\},$$

where $Y_i = X_i - 1$ and L_0 is an arbitrary (initial) random variable.

5.1.4 Consider the risk model in discrete time. Assume that X_n only can assume two values: 0 with probability q and 2 with probability $p = 1-q$, where $p < q$. Show that $\psi(u) = (p/q)^{u+1}$. [Comment: In this case also $Y_n = X_n - 1$ only assumes two values: -1 with probability q and 1 with probability p as in the classical *gambler's ruin problem*, i.e. in each game the gambler can win one currency unit with probability p and lose one currency unit (that is to win -1 currency unit) with probability q, $p > q$. Assume that initially the gambler has u currency units and that the game is against an infinitely rich player. If by being ruined we mean that the gambler's capital reaches the level -1, then the probability of ruin is $(q/p)^{u+1}$.]

5.1.5 Consider a discrete-time stochastic process $\{S_n, n \in \mathbb{N}\}$ defined by the sums $S_n = \sum_{i=1}^n Y_i$ of arbitrary independent and identically distributed (not necessarily integer-valued) random variables $Y_1, Y_2, \ldots$, which is called a *random walk*. Show that $S_n \to \infty$ if $\mathbb{E}\, Y > 0$ and that $S_n \to -\infty$ if $\mathbb{E}\, Y < 0$, i.e. the maximum $\max\{S_1, S_2, \ldots\}$ is finite with probability 1 in the latter case. [Hint. Use the strong law of large numbers.]

5.1.6 Let $\ldots, Y_{-1}, Y_0, Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables with $\mathbb{E}\, Y < 0$. Let L_0 be independent of $\{Y_n\}$ and define recursively

$$L_n = (L_{n-1} + Y_n)_+ ,$$

for $n = 0, 1, \ldots$. Show that

$$\begin{aligned} L_n &= \max\{0, Y_n, Y_{n-1} + Y_n, \ldots, L_0 + Y_1 + \ldots + Y_n\} \\ &\stackrel{\mathrm{d}}{=} \max\{0, Y_{-1}, Y_{-1} + Y_{-2}, \ldots, L_0 + Y_{-1} + \ldots + Y_{-n}\}. \end{aligned}$$

[Hint. Use a similar argument as in the proof of Lemma 5.1.1.]

5.1.7 Consider the model of Exercise 5.1.6. Show that $F(x) = \lim_{x\to\infty} \mathbb{P}(L_n \le x)$ is a proper distribution function and

$$\lim_{x\to\infty} \mathbb{P}(L_n \le x) = \mathbb{P}(M \le x)$$

where $M = \max\{0, Y_1, Y_1 + Y_2, \ldots\}$. [Hint. Use the result of Exercise 5.1.6.]

5.1.8 Consider the risk model in continuous time. Let the inter-occurrence times T_n be independent and identically distributed with distribution $\mathrm{TG}(p)$ and let the claim sizes U_n be positive $\mathbb{N}$-valued random variables which are independent and identically distributed with probability function $\{p_k\}$, independent of the inter-occurrence times. We also assume that the premium rate β is equal to 1 and that $\sum_{n=1}^{\infty} np_n < 1/(1-p)$. Compute the generating function $\hat{g}_M(s)$ of the random variables M given in (5.1.2). [Hint. Represent this model as risk model in discrete time and use Theorem 5.1.1a.]

5.2 Poisson Arrival Processes

We now consider the special case of the continuous-time risk model where the claim sizes $\{U_n\}$ are independent and identically distributed with distribution F_U and independent of the sequence $\{\sigma_n\}$ of claim occurrence epochs. Furthermore, we assume that the sequence $\{\sigma_n\}$ forms a *Poisson point process.* By this we mean that the inter-occurrence times $T_n = \sigma_n - \sigma_{n-1}$ are independent and (identically) exponentially distributed. These assumptions lead to the classical *compound Poisson model* of risk theory.

If $\{T_n\}$ is a sequence of independent random variables with exponential distribution $\mathrm{Exp}(\lambda)$, then the counting process $\{N(t)\}$ is called a (homogeneous) *Poisson process* with *intensity* λ. The cumulative arrival process $\{X(t),\ t \ge 0\}$ defined in (5.1.1) is called a *compound Poisson process* with characteristics (λ, F_U).

A real-valued stochastic process $\{X(t), t \ge 0\}$ is said to have

- *independent increments* if for all $n = 1, 2, \ldots$ and $0 \le t_0 < t_1 < \ldots < t_n$, the random variables $X(0), X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent,
- *stationary increments* if for all $n = 1, 2, \ldots$, $0 \le t_0 < t_1 < \ldots < t_n$ and $h \ge 0$, the distribution of $(X(t_1+h) - X(t_0+h), \ldots, X(t_n+h) - X(t_{n-1}+h))$ does not depend on h.

Exercises

5.2.1 Assume that $\{X(t), t \ge 0\}$ is a process with independent increments and for each $t \ge 0$ the distribution of $X(t+h) - X(h)$ does not depend on $h \ge 0$. Show that then the process $\{X(t)\}$ has stationary increments.

5.2.2 Let $\{N(t), t \geq 0\}$ be a stochastic process which has stationary and independent increments and satisfies as $h \downarrow 0$,

$$\mathbb{P}(N(h) = 0) = 1 - \lambda h + o(h)\,, \qquad \mathbb{P}(N(h) = 1) = \lambda h + o(h)$$

for some $\lambda > 0$. Show that then the probability $p_n(t) = \mathbb{P}(N(t) = n)$ is given by

$$p_n(t) = \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t}$$

for all $t \geq 0$ and $n \in \mathbb{N}$.

5.2.3 Let $\{N(t), t \geq 0\}$ be a Poisson process with intensity λ and let $c > 0$ be some constant. Show that $\{N(ct), t \geq 0\}$ is a Poisson process and determine its intensity.

5.2.4 Let $\{X(t)\}$ be a compound Poisson process with characteristics (λ, F_U). Show that for all $t \geq 0$

$$\hat{m}_{X(t)}(s) = \mathrm{e}^{\lambda t(\hat{m}_U(s) - 1)}$$

and consequently $\mathbb{E}\, X(t) = \lambda t \mu_U$,

$$\operatorname{Var} X(t) = \lambda t \mu_U^{(2)}\,, \qquad \mathbb{E}\left((X(t) - \mathbb{E}\, X(t))^3\right) = -\lambda t \mu_U^{(3)}\,.$$

[Hint. Notice that the random variable $X(t)$ has a compound Poisson distribution and use the result of Exercise 4.2.2.]

5.2.5 Let $\{X_1(t)\}$ and $\{X_2(t)\}$ be two independent compound Poisson processes with characteristics (λ_1, F_1), (λ_2, F_2). Show that $\{X(t)\} = \{X_1(t) + X_2(t)\}$ is a compound Poisson process and determine its characteristics.

5.2.6 Consider a compound Poisson process with claim occurrence times σ_n and claim sizes U_n. Define, for some fixed $u \geq 0$,

$$X_1(t) = \sum_{k=1}^{\infty} U_k \mathbb{I}(U_k \leq u) \mathbb{I}(\sigma_k \leq t)$$

and

$$X_2(t) = \sum_{k=1}^{\infty} U_k \mathbb{I}(U_k > u) \mathbb{I}(\sigma_k \leq t)\,.$$

Show that $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ are two independent compound Poisson processes. Determine their characteristics.

5.2.7 Assume that there is a *delay in claim settlement* modelled by the sequence $D_1, D_2, \ldots$ of nonnegative independent and identically distributed random variables which are independent of $\{(\sigma_n, U_n)\}$. Show that the process $\{X(t), t \geq 0\}$ defined by $X(t) = \sum_{k=1}^{\infty} U_k \mathbb{I}(\sigma_k + D_k < t)$ has independent increments. Determine the distribution of $X(t+h) - X(h)$ for $t, h \geq 0$.

5.2.8 Let $\{(\sigma_n, U_n)\}$ and $\{D_n\}$ be the same as in Exercise 5.2.7 and consider the following model of *gradual claim settlement*. Let $g : \mathbb{R} \times \mathbb{R}_+^2 \to \mathbb{R}_+$ be a measurable function. Then $\{X(t)\}$ with $X(t) = \sum_{k=1}^{\infty} g(\sigma_k - t, D_k, U_k)$ is called a *shot-noise process* with *response function* g. Determine the Laplace–Stieltjes transform of $X(t)$.

5.3 Ruin Probabilities: The Compound Poisson Model

In the sequel of this chapter we consider the continuous-time risk model introduced in Section 5.1, where we assume that the claim arrival process is modelled by a compound Poisson process with characteristics (λ, F_U). We also assume that $\beta > \lambda\mu_F$, which is called the *net proft condition*.

We study the ruin function $\psi(u) = \mathbb{P}(M > u)$ and the survival function $\overline{\psi}(u) = 1 - \psi(u)$, which fulfils the integro-differential equation (see Theorem 5.3.1)

$$\beta\overline{\psi}_+^{(1)}(u) = \lambda\Big(\overline{\psi}(u) - \int_0^u \overline{\psi}(u-y)\,\mathrm{d}F_U(y)\Big)\,, \tag{5.3.1}$$

where M is given by (5.1.2). [1]

If the claim sizes U_n are $\mathrm{Exp}(\delta)$-distributed, then (5.3.1) can be solved analytically with the solution

$$\psi(u) = \frac{\lambda}{\beta\delta}\mathrm{e}^{-(\delta-\lambda/\beta)u}\,. \tag{5.3.2}$$

Notice that integrating (5.3.1) yields the integral equation (see Theorem 5.3.2)

$$\beta\psi(u) = \lambda\Big(\int_u^{\infty} \overline{F}_U(x)\,\mathrm{d}x + \int_0^u \psi(u-x)\overline{F}_U(x)\,\mathrm{d}x\Big)\,. \tag{5.3.3}$$

The Laplace transforms $\hat{L}_{\overline{\psi}}(s)$ and $\hat{L}_{\psi}(s)$ are given by (see Theorem 5.3.3)

$$\hat{L}_{\overline{\psi}}(s) = \frac{\beta - \lambda\mu}{\beta s - \lambda(1 - \hat{l}_U(s))}\,, \qquad s > 0\,,$$

[1] Did we introduc notations for derivatives, righ hand derivatives, etc.

and

$$\hat{L}_\psi(s) = \frac{1}{s} - \frac{\beta - \lambda\mu}{\beta s - \lambda(1 - \hat{l}_U(s))}, \qquad s > 0.$$

Taking Laplace transforms on both the sides of (5.3.3) gives the *Pollaczek–Khinchin formula*

$$\psi(u) = \Big(1 - \frac{\lambda\mu}{\beta}\Big) \sum_{n=1}^{\infty} \Big(\frac{\lambda\mu}{\beta}\Big)^n \overline{(F_U^s)^{*n}}(u) \tag{5.3.4}$$

for each $u \geq 0$; see Theorem 5.3.4.

We also consider the *multivariate ruin function* $\psi(u, x, y)$ given by

$$\psi(u, x, y) = \mathbb{P}(\tau(u) < \infty, X^+(u) \leq x, Y^+(u) > y)$$

for $u, x, y \geq 0$, and its dual

$$\varphi(u, x, y) = \mathbb{P}(\tau(u) < \infty, X^+(u) > x, Y^+(u) > y),$$

where $X_+(u) = R(\tau(u)-)$ and $Y_+(u) = -R(\tau(u))$ is the surplus just before and at the ruin time $\tau(u)$ respectively.

Exercises

5.3.1 Let the claim sizes U_n be $\text{Erl}(2, \delta)$-distributed. Show, that

$$\psi(u) = a\,\mathrm{e}^{-r_1 u} - b\,\mathrm{e}^{-r_2 u}, \tag{5.3.5}$$

where $r_1 < r_2$ are the solutions to the equation

$$\beta r^2 - (2\delta\beta - \lambda)r + \delta(\delta\beta - 2\lambda) = 0$$

and

$$a = \frac{\lambda(2\lambda - \beta\delta + 2\beta r_2)}{\beta^2\delta(r_2 - r_1)}, \qquad b = \frac{\lambda(\beta\delta - 2\lambda - 2\beta r_1)}{\beta^2\delta(r_2 - r_1)}.$$

[Hint. Differentiate (5.3.1) twice.]

5.3.2 Let the claim sizes U_n be $\text{Erl}(2, \delta)$-distributed. Determine the Laplace transforms of $\overline{\psi}(u)$ and $\psi(u)$. Invert the Laplace transforms in order to verify (5.3.5).

5.3.3 Let $\beta = \lambda = 1$ and $F_U(x) = 1 - \frac{1}{3}(\mathrm{e}^{-x} + \mathrm{e}^{-2x} + \mathrm{e}^{-3x})$. Show that

$$\psi(u) = 0.550790\mathrm{e}^{-0.485131u} + 0.0436979\mathrm{e}^{-1.72235u} + 0.0166231\mathrm{e}^{-2.79252u}.$$

[Hint. Calculate the Laplace transform $\hat{L}_\psi(s)$ and invert it.]

5.3.4 Put $\rho = \lambda\mu\beta^{-1}$. Let $\rho = 0.75$, $\beta = 1$ and let the claim sizes have the distribution $F_U = p\mathrm{Exp}(a_1) + (1-p)\mathrm{Exp}(a_2)$, where $p = 2/3$, $a_1 = 2$ and $a_2 = 1/2$. Show that the ruin function $\psi(u)$ is given by

$$\psi(u) = 0.75\left(0.935194\mathrm{e}^{-0.15693u} + 0.0648059\mathrm{e}^{-1.59307u}\right).$$

5.3.5 Let the distribution F_U of claims sizes U_n be the exponential distribution $\mathrm{Exp}(\delta)$ with parameter $\delta > 0$. Determine the integrated tail distribution F_U^{s}. Use this result to recover formula (5.3.2) for $\psi(u)$ using the Pollaczek-Khinchin formula (5.3.4).

5.3.6 Let the claims sizes U_n be exponentially distributed with parameter $\delta > 0$. Find the multivariate ruin function $\varphi(u, 0, y)$.

5.4 Bounds, Asymptotics and Approximations

Let $\theta(s) = \lambda(\hat{m}_U(s) - 1) - \beta s$ and assume that the equation

$$\theta(s) = 0 \tag{5.4.1}$$

has a positive solution which is called the *adjustment coefficient* or the *Lundberg exponent* and denoted by γ. Then, we have the two-sided *Lundberg bound*

$$a_-\mathrm{e}^{-\gamma u} \le \psi(u) \le a_+\mathrm{e}^{-\gamma u} \tag{5.4.2}$$

for all $u \ge 0$, see Theorem 5.4.1, where

$$a_- = \inf_{x\in[0,x_0)} \frac{\mathrm{e}^{\gamma x}\int_x^\infty \overline{F}_U(y)\,\mathrm{d}y}{\int_x^\infty \mathrm{e}^{\gamma y}\overline{F}_U(y)\,\mathrm{d}y}, \qquad a_+ = \sup_{x\in[0,x_0)} \frac{\mathrm{e}^{\gamma x}\int_x^\infty \overline{F}_U(y)\,\mathrm{d}y}{\int_x^\infty \mathrm{e}^{\gamma y}\overline{F}_U(y)\,\mathrm{d}y}.$$

Furthermore, in Theorem 5.4.2 we showed that

$$\lim_{u\to\infty} \psi(u)\mathrm{e}^{\gamma u} = \frac{\beta - \lambda\mu}{\lambda\hat{m}_U^{(1)}(\gamma) - \beta}. \tag{5.4.3}$$

This asymptotic result gives rise to the so-called *Cramér–Lundberg approximation*

$$\psi_{\mathrm{app}}(u) = \frac{\beta - \lambda\mu}{\lambda\hat{m}_U^{(1)}(\gamma) - \beta}\mathrm{e}^{-\gamma u} \tag{5.4.4}$$

for the ruin probability $\psi(u)$. In the case of heavy-tailed claim sizes, the asymptotic behaviour of $\psi(u)$ is very different from that in (5.4.3). Indeed, if the integrated tail distribution F_U^{s} is subexponential, then

$$\lim_{u\to\infty} \frac{\psi(u)}{1 - F_U^s(u)} = \frac{\rho}{1-\rho}, \tag{5.4.5}$$

where $\rho = \lambda\mu\beta^{-1}$; see Theorem 5.4.3.

Exercises

5.4.1 Show that an easy application of integral equation (5.3.3) leads to (5.4.2).

5.4.2 Show that it follows from the definition of the adjustement coefficient γ that

$$\int_0^\infty \lambda\beta^{-1}\overline{F}_U(x)\,\mathrm{e}^{\gamma x}\,\mathrm{d}x = 1$$

and that the mean value of the integrating distribution $\mathrm{d}F(x) = \lambda\beta^{-1}\overline{F}_U(x)\mathrm{e}^{\gamma x}\,\mathrm{d}x$ is

$$\int_0^\infty x\lambda\beta^{-1}\overline{F}_U(x)\mathrm{e}^{\gamma x}\,\mathrm{d}x = \lambda\hat{m}_U^{(1)}(\gamma) - \beta/(\beta\gamma)\,.$$

5.4.3 Show that the function

$$z(u) = \frac{\lambda}{\beta}\,\mathrm{e}^{\gamma u}\int_u^\infty \overline{F}_U(x)\,\mathrm{d}x$$

can be factored in the following way: $z(x) = z_1(x)z_2(x)$, where the function $z_1 : \mathbb{R}_+ \to (0,\infty)$ is increasing and $z_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing, such that

$$\int_0^\infty z_1(x)z_2(x)\,\mathrm{d}x < \infty$$

and $\lim_{h\to 0}\sup\{z_1(x+y)/z_1(x) : x \ge 0, 0 \le y \le h\} = 1$.

5.4.4 Show that

$$\int_0^\infty \frac{\lambda}{\beta}\mathrm{e}^{\gamma u}\int_u^\infty \overline{F}_U(x)\,\mathrm{d}x\,\mathrm{d}u = \frac{\beta - \lambda\mu}{\beta\gamma}\,.$$

5.4.5 Let $\beta = \lambda = 1$ and $F_U(x) = 1 - \frac{1}{3}(\mathrm{e}^{-x} + \mathrm{e}^{-2x} + \mathrm{e}^{-3x})$. Show that the mean value of claim sizes is $\mu = 0.611111$ and that the Cramér–Lundberg approximation (5.4.4) to $\psi(u)$ is $\psi_{\mathrm{app}}(u) = 0.550790\mathrm{e}^{-0.485131u}$. Compare this approximation to the exact formula obtained in Exercise 5.3.3.

5.4.6 Let the claim sizes U_n have distribution $\mathrm{Erl}(2,\delta)$. Determine the adjustment coefficient γ and the Cramér-Lundberg approximation (5.4.4). Compare this result to the exact formula for the ruin probability $\psi(u)$ obtained in Exercise 5.3.1.

5.4.7 Find the asymptotic behaviour of the ruin function $\psi(u)$ as $u \to \infty$ if the claim sizes U_n have a Pareto-type distribution. Show in particular that the ruin function $\psi(u)$ has the asymptotics
(a) $\psi(u) \sim \frac{3}{2}u^{-1}$ as $u \to \infty$ if the claim sizes are PME(2)-distributed and $\rho = 0.75$,
(b) $\psi(u) \sim 9(1+u)^{-10}$ as $u \to \infty$ if $\beta = 1$, $\lambda = 9$ and $F_U(x) = 1-(1+x)^{-11}$,
(c) $\psi(u) \sim \rho(\alpha(1-\rho))^{-1}(u/c)^{-(\alpha-1)}$ as $u \to \infty$ if the claim sizes are $\mathrm{Par}(\alpha, c)$-distributed with $\alpha > 1$.

5.4.8 Consider the distribution function $F(x) = 1 - \beta\psi(x)/(\lambda\mu)$, which is used in the Beekman–Bowers approximation of $\psi(u)$. Show that

$$\hat{m}_F(s) = 1 - \frac{\beta}{\lambda\mu} + \frac{\beta(\beta-\lambda\mu)}{\lambda\mu}\,\frac{s}{\beta s - \lambda(\hat{m}_U(s)-1)}\,.$$

[Hint. Use integration by parts.]

5.4.9 Consider the ruin functions $\psi(u)$ and $\psi'(u)$ of two compound Poisson models with arrival rates λ and λ', premium rates β and β', and claim size distributions F_U and $F_{U'}$, respectively. Assume that

$$\lambda \le \lambda', \quad \mu_U \le \mu_{U'}, \quad \beta \ge \beta'$$

and

$$F_U^{\mathrm{s}} \le_{\mathrm{st}} F_{U'}^{\mathrm{s}}\,.$$

Show that $\psi(u) \le \psi'(u)$ for all $u \ge 0$. [Hint. Recall that the right-hand side of (5.3.4) is the tail function of a geometric compound with characteristics (ρ, F_U^{s}) and use Theorem 4.2.3a.]

5.5 Numerical Evaluation of Ruin Functions

[2] The Laplace transform of a function $c(u)$ is

$$\hat{L}(z) = \int_0^\infty \mathrm{e}^{-zu} c(u)\,\mathrm{d}u\,. \tag{5.5.1}$$

In this section we assume that $c(u)$ is a bounded continuous function on $\mathbb{R}_+$ with locally bounded variation such that $0 \le c(u) \le 1$. Let $a > 0$ be an arbitrary but fixed real number. The function $c(u)$ can then be expressed in

[2] Should we add a few further exercises to this section?

terms of its Laplace transform $\hat{L}(z)$ by the formula

$$\begin{aligned} c(u) &= \frac{e^{a/2}}{2u}\Big(\Re\hat{L}\big(\frac{a}{2u}\big) - 2\sum_{k=1}^{\infty}(-1)^{k+1}\Re\hat{L}\big(\frac{a+i2k\pi}{2u}\big)\Big) \\ &\quad - \sum_{k=1}^{\infty} e^{-ak} c((2k+1)u) \end{aligned} \tag{5.5.2}$$

for all $u \geq 0$; see Theorem 5.5.2. Using Euler transformation, the right-hand side of (5.5.2) can be approximated by

$$C(u,m,n) \stackrel{\text{def}}{=} \sum_{k=0}^{m} \binom{m}{k} 2^{-m} s_{n+k}(u)\,, \tag{5.5.3}$$

where $m, n \in \mathbb{N}$ and

$$s_n(u) = \frac{e^{a/2}}{2u}\Re\hat{L}\Big(\frac{a}{2u}\Big) + \frac{e^{a/2}}{u}\sum_{k=1}^{n}(-1)^k\Re\hat{L}\Big(\frac{a+i2\pi k}{2u}\Big)\,.$$

Exercises

5.5.1 Consider the compound Poisson model with $\lambda = 1$, $\beta = 3$ and the claim size distribution $F_U = \mathrm{Erl}(2,1)$. Use the approximation formula (5.5.3) with $m = 11$, $n = 15$ and $a = 18,5$ in order to compute the ruin function $\psi(u)$ numerically for $u = 0.1, 0.3, 0.5, 1.0, 2.0, 3.0$. Compare the obtained values to those obtained by the exact formula for $\psi(u)$ given in Exercise 5.3.1 [Hint. Use the result of Exercise 5.3.2.]

5.5.2 Let $\{a_k\}$ be an arbitrary sequence of real numbers such that $s_n = \sum_{k=0}^{n}(-1)^k a_k$ converges to $s \in \mathbb{R}$ as $n \to \infty$. Complete the proof of (5.5.3), i.e. show that

$$s_n + (-1)^{n+1}\sum_{k=0}^{m-1} 2^{-(k+1)}\Delta^k a_{n+1} = \sum_{k=0}^{m}\binom{m}{k} 2^{-m} s_{n+k}$$

for all $m, n \in \mathbb{N}$, where

$$\Delta a_n = a_n - a_{n+1}\,, \qquad \Delta^k a_n = \Delta(\Delta^{k-1}a_n)\,, \qquad \Delta^0 a_n = a_n\,.$$

[Hint. Use induction.]

5.6 Finite-Horizon Ruin Probabilities

In this section we consider the *finite-horizon ruin probabilities* $\psi(u;x) = \mathbb{P}(\tau(u) \le x)$ and the *finite-horizon survival probabilities* $\overline{\psi}(u;x) = 1 - \psi(u;x)$.

Formulae (5.6.1) and (5.6.2) below are known in actuarial mathematics as *Seal's formulae*. Assume that $\mathbb{P}(U > 0) = 1$. Then, see Theorem 5.6.2,

- for initial risk reserve $u = 0$,

$$\overline{\psi}(0;x) = \frac{1}{\beta x}\mathbb{E}\,(R(x)_+) = \frac{1}{\beta x}\int_0^{\beta x} F_{X(x)}(y)\,\mathrm{d}y\,, \tag{5.6.1}$$

- if $u > 0$ and U has density $f_U(y)$,

$$\overline{\psi}(u;x) = F_{X(x)}(u + \beta x) - \beta\int_0^u \overline{\psi}(0, x-y)\tilde{f}_{X(y)}(u + \beta y)\,\mathrm{d}y\,. \tag{5.6.2}$$

We introduce the auxiliary function

$$J(x) = \sum_{n=0}^{\infty} \frac{z^n}{n!n!} = I_0(2\sqrt{x})\,.$$

where

$$I_0(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!\Gamma(k+1)}$$

is the *modified Bessel function.*

Assume that $F_U(x) = 1 - \mathrm{e}^{-\delta x}$ for all $x \ge 0$. Then, see Theorem 5.6.3,

$$\psi(u;x) = 1 - \mathrm{e}^{-\delta u-(1+c)\lambda x} g(\delta u + c\lambda x, \lambda x)\,, \tag{5.6.3}$$

where $c = \delta\beta/\lambda$ and

$$g(z,\theta) = J(\theta z) + \theta J^{(1)}(\theta z) + \int_0^z \mathrm{e}^{z-v} J(\theta v)\,\mathrm{d}v - \frac{1}{c}\int_0^{c\theta} \mathrm{e}^{c\theta - v} J(zc^{-1}v)\,\mathrm{d}v\,. \tag{5.6.4}$$

Exercises

5.6.1 Show that $\psi(u;x)$ and $\overline{\psi}(u;x)$ satisfy the integro-differential equations

$$\begin{aligned}\beta\frac{\partial\psi(u;x)}{\partial u} &- \frac{\partial\psi(u;x)}{\partial x} - \lambda\psi(u;x)\\ &+ \lambda\int_0^u \psi(u-y;x)\,\mathrm{d}F_U(y) + \lambda\overline{F}_U(u) = 0\end{aligned}$$

and

$$\beta\frac{\partial}{\partial u}\overline{\psi}(u;x) - \frac{\partial}{\partial x}\overline{\psi}(u;x) + \lambda\int_0^u \overline{\psi}(u-y;x)\,\mathrm{d}F_U(y) - \lambda\overline{\psi}(u;x) = 0\,.$$

5.6.2 Let $\{X(t)\} = \{\sum_{i=1}^{N(t)} U_i\}$ be a compound Poisson process such that $\mathbb{P}(U > 0) = 1$ and let $y \in \mathbb{R}_+$. Complete the proof of Seal's formula (5.6.1), i.e. show that for each $x > 0$,

$$\mathbb{P}\Big(\bigcap_{t\le x}\{X(t) \le \beta t\} \;\Big|\; X(x) = y\Big) = \Big(1 - \frac{y}{\beta t}\Big)_+ .$$

[Hint. Assume first that there exists a natural number $m \in \{1, 2, \ldots\}$ such that $\mathbb{P}\left(U 2^m (\beta t)^{-1} \in \mathbb{N}\right) = 1$. Let $j > m$ and consider the random variables

$$Y_i = \frac{2^j}{\beta x}\left(X(2^{-j} i x) - X(2^{-j}(i-1)x)\right).$$

Use now the result of Exercise 5.6.3 below. Do not forget to verify that the limit $j \to \infty$ yields the desired result. For an arbitrary distribution of U use an appropriate discretization.]

5.6.3 Let $n \ge 1$ be an arbitrary integer. Let $\{Y_i : 1 \le i \le n\}$ be a sequence of independent random variables such that $\mathbb{P}(Y_i \in \{1, 2, \ldots\}) = 1$ for all $i = 1, \ldots, n$ and let $X_m = \sum_{i=1}^{m} Y_i$ for $m = 1, \ldots, n$. Prove that for all $k, n \in \{1, 2, \ldots\}$

$$\mathbb{P}\Big(\bigcap_{i=1}^{n}\{X_i < i\} \;\Big|\; X_n = k\Big) = \Big(1 - \frac{k}{n}\Big)_+ .$$

[Hint. The result is trivial for $n = 1$ and $k \ge n$. Use induction with respect to n.]

5.6.4 Prove the following identity for binomial coefficients which is used in the proof of Lemma 5.6.2:

$$\sum_{s=0}^{k} (-1)^s \binom{k}{s}\binom{t+s}{n} = (-1)^k \binom{t}{n-k}$$

for $0 \le k \le n$ and $t \in \mathbb{R}$. [Hint. First prove the formula for $t = 0$. Then generalize by employing the classical identity

$$\sum_{m=0}^{k} \binom{\alpha}{m}\binom{\beta}{k-m} = \binom{\alpha+\beta}{k} .]$$

5.6.5 Consider the function $g(z, \theta)$ defined in (5.6.4) and introduce the notation $g_c(z) = c g(\xi, z/c)$. [3] Show that $g_c(z)$ can be written in the

[3] I do not understand the meaning of ξ. I guess that this is a typo in formula (5.6.19) of our book. Any idea how to correct it? (VS)

following power series with respect to z:

$$g_c(z) = \sum_{r=0}^{\infty} C_r(c)\, z^r \,, \tag{5.6.5}$$

where

$$C_r(c) = \begin{cases} c & \text{if } r = 0, \\ \dfrac{c^{1-m}}{m!m!} + \displaystyle\sum_{n=0}^{m-1} \dfrac{(c-1)c^{-n}}{n!(2m-n)!} & \text{if } r = 2m > 0, \\ \dfrac{c^{-m}}{m!(m+1)!} + \displaystyle\sum_{n=0}^{m} \dfrac{(c-1)c^{-n}}{n!(2m-n+1)!} & \text{if } r = 2m+1. \end{cases}$$

5.6.6 Show that for the coefficients $C_r(c)$ in the power series (5.6.5), the following identity holds:

$$\sum_{r=0}^{\ell} C_{2r+1}(\lambda) \frac{(-1)^{\ell-r}(r+\ell+1)!}{(\ell-r)!} \lambda^r = \lambda^{\ell+1}$$

for $\ell \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. Conclude from this that

$$\sum_{r=0}^{\infty} C_{2r+1}(\lambda) \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2r+n+1}(2r+n+1)!}{n!(2r+2n+2)!} w^{2r+2n+2} = \sum_{\ell=0}^{\infty} \frac{(w\lambda)^{2\ell+2}}{(2\ell+2)!} \,.$$

5.6.7 Let $s > \lambda$. Show that

$$\int_0^{\infty} \mathrm{e}^{-(s+\lambda s^{-1})v} \mathrm{e}^{(y-\lambda v)/s} g_\lambda(\lambda v)\, \mathrm{d}v \;=\; \frac{\lambda}{s-\lambda} \,.$$

[Hint. Use the result of Lemma 5.6.2b by taking Laplace transforms.]

5.6.8 Put $\overline{\psi}(u;x) = \mathrm{e}^{-\delta u + \lambda x(1+c)}\, \omega(x, u+\beta x)$. Prove that $\omega(0,y) = \mathrm{e}^{\delta y}$ and that $\omega(x,y)$ satisfies the partial differential equation

$$\frac{\partial}{\partial x}\omega(x,y) = \lambda\delta \int_{\beta x}^{y} \omega(x,z)\, \mathrm{d}z \,.$$

5.6.9 Let $Z(t)$ be the amount of work at time t in a single-server queueing system having a compound Poisson arrival process with characteristics (λ, F_U) such that $Z(0) = 0$. Furthermore, let $\psi(u;x)$ be the finite-horizon ruin probability in the compound Poisson risk model with the same characteristics. Show that $\psi(u;x) = \mathbb{P}(Z(x) > u)$. [Hint. Proceed as in the proof of Theorem 5.1.2, see also Exercise 5.1.6.]

6

Renewal Processes and Random Walks

6.1 Renewal Processes

Exercises

6.1.1 Show that if F is defective, then $\lim_{t\to\infty} N(t) < \infty$ with probability 1 and $\sigma_{N(\infty)}$ is a geometric compound specified by $p = F(\infty)$ and $\tilde{F}(x) = F(x)/F(\infty)$.

6.1.2 Show that if $\mathbb{P}(T = a) = p$ and $\mathbb{P}(T = 0) = 1 - p$ for some $a > 0$, $0 < p < 1$, then for $k \geq n$

$$\mathbb{P}(N(an) = k) = \binom{k-1}{k-n} p^n (1-p)^{k-n} .$$

6.1.3 Assume F is the Erlang distribution with density $f(x) = x\,\mathrm{e}^{-x}$ for $x \geq 0$. Show that $4H(t) = 2t + \mathrm{e}^{-2t} - 1$ for $t \geq 0$.

6.1.4 (Continuation) Let F be an arbitrary Erlang distribution, i.e. a gamma distribution with $\hat{l}_F(s) = (1 + \lambda s)^{-n}$ where $\lambda > 0$ and $n \in \mathbb{N}$. Show that the renewal function $H(t)$ is given by

$$H(t) = (n\lambda)^{-1} t - \sum_{m=1}^{n-1} a_{n,m}(1 - \exp(-(b^m - 1)t/(n\lambda)))$$

where $b = \mathrm{e}^{2\pi \mathrm{i}/n}$ and

$$a_{n,m} = (1 - b^m)^{-2} \prod_{\substack{1 \leq r < n \\ r \neq m}} \frac{b^r - 1}{b^r - b^m} .$$

6.1.5 Show that if $F(x) = 1 - p\,\mathrm{e}^{-\lambda x} - q\,\mathrm{e}^{-\lambda' x}$ is a mixture of exponential distributions, where $0 < p = 1 - q < 1$, $0 < \lambda \le \lambda'$, then

$$H(t) = \frac{\lambda\lambda'}{p\lambda' + q\lambda}\, t + \frac{pq(\lambda - \lambda')^2}{(p\lambda' + q\lambda)^2}\left(1 - \mathrm{e}^{-(p\lambda' + q\lambda)t}\right).$$

6.1.6 Show that if $\hat{l}_F(s) = (1 + s/\lambda)^{-1/2}$, which means that F is a gamma distribution, then $\mu = 1/(2\lambda)$ and

$$H(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} - \frac{\sqrt{\lambda}\,\mathrm{e}^{-\lambda t}}{\pi}\int_0^\infty \frac{\sqrt{x}\,\mathrm{e}^{-xt}}{(\lambda + x)^2}\,\mathrm{d}x$$

where σ^2 denotes the variance of F.

6.1.7 Show that if $F(x) = 1 - \mathrm{e}^{-x^b}$ for some $b > 0$, then

$$H(t) = \sum_{k=1}^{\infty} \frac{(-1)^k a_k}{\Gamma(bk + 1)}\, t^{bk}$$

where $\{a_k, k \ge 1\}$ satisfies $a_{n+1} = c_{n+1} - \sum_{k=1}^{n} c_k\, a_{n+1-k}$ with $c_n = \Gamma(bn + 1)/n!$.

6.1.8 Show that if T is an $\mathbb{N}$-valued random variable with probability function $\{p_k\}$, then for the renewal sequence $\{h_n\}$

$$\sum_{n=0}^{\infty} h_n s^n = \frac{\sum_{k=0}^{\infty} p_k s^k}{1 - \sum_{k=0}^{\infty} p_k s^k}.$$

6.1.9 Let $\{N(t)\}$ be a renewal process with inter-occurrence time distribution F and let $x \ge 0$ be fixed.
(a) Show that the function $g(t) = \overline{F}_{T(t)}(x)$ where $T(t)$ denotes the excess $T(t) = \sigma_{N(t)+1} - t$ at time t, satisfies the following renewal equation:

$$\overline{F}_{T(t)}(x) = \overline{F}(t + x) + \int_0^t \overline{F}_{T(t-y)}(x)\,\mathrm{d}F(y)\,, \qquad t, x \ge 0\,.$$

(b) Assume that F is nonlattice and $\mu < \infty$. Show that

$$\lim_{t\to\infty} \mathbb{P}(T(t) > x) = \mu^{-1}\int_x^\infty \overline{F}(y)\,\mathrm{d}y\,.$$

[Hint. Use the result of part (a) and Theorem 6.1.11.]

6.1.10 Let F be a distribution on $\mathbb{R}_+$. Show that for $H(t) = \sum_{n=1}^{\infty} F^{*n}(t)$ and for each $h > 0$ the following renewal equation holds:

$$H(t) - H(t-h) = (F(t) - F(t-h)) + \int_0^t (H(t-y) - H(t-h-y))\,\mathrm{d}F(y)\,.$$

6.1.11 (a) Show that for a distribution F on $\mathbb{R}_+$ with density $f(t)$ the renewal function $H(t)$ has density $h(t)$ (i.e. $H(t) = \int_0^t h(v)\,\mathrm{d}v$) (called the *renewal density function*) which fulfils the renewal equation $h(t) = f(t) + \int_0^t h(t-v)f(v)\,\mathrm{d}v$. Deduce that $\lim_{t\to\infty} h(t) \to 1/\mu$ if $f(t)$ is directly Riemann integrable.
(b) (Difficult) Show that $\lim_{t\to\infty} h(t) - f(t) = 1/\mu_F$ if the density $f(t)$ is bounded. [Hint. See Feller (1971), Theorem XI.3.2.]

6.1.12 Assume that F is nonlattice and $\mu^{(2)} < \infty$. Show that for $h > 0$

$$\lim_{t\to\infty} (\operatorname{Var} N(t+h) - \operatorname{Var} N(t)) = ch$$

for some constant $c > 0$ if and only if

$$\lim_{t\to\infty} t(H(t+h) - H(t) - \mu^{-1}h) = 0\,.$$

6.1.13 Consider the *renewal triad* $(T(t), U(t), Z(t))$ where $T(t) = T_{N(t)+1} - t$, $U(t) = t - T_{N(t)}$ and $Z(t) = T_{N(t)+1}$. Show that
(a) for $B \in \mathcal{B}(\mathbb{R}_+^3)$ and $t \geq 0$

$$\begin{aligned}&\mathbb{P}((T(t), U(t), Z(t)) \in B)\\ &\quad= \int_0^t \mathbb{P}((T + v - t, t - v, T) \in B, T > t - v)\,\mathrm{d}H(v)\,,\end{aligned}$$

(b) if $g : \mathbb{R}_+^3 \to \mathbb{R}_+$ is Borel measurable, then

$$\mathbb{E}\, g(T(t), U(t), Z(t)) = \int_0^\infty \int_{(t-x)_+}^t g(v + x - t, t - v, x)\,\mathrm{d}H(v)\,\mathrm{d}F(x)\,.$$

(c) Find $\lim_{t\to\infty} \mathbb{E}\, g(T(t), U(t), Z(t))$ if F is nonlattice and $\mu < \infty$.

6.1.14 Let $U(t)$ and $Z(t)$ be defined as in Exercise 6.1.13.
(a) Show: if $y < t$, then

$$\mathbb{P}(Z(t) \leq y) = \mathbb{P}(U(t) \leq y) - (1 - F(y))(H(t) - H(t-y))\,.$$

(b) Using (a), prove that $Z(t) \geq_{\mathrm{st}} T$ for each $t \geq 0$.
(c) The result of part (b) is sometimes called the *renewal paradox*. Find a plausible explanation of this inequality.
(d) Show: if $F(x) = 1 - \mathrm{e}^{-\lambda x}$, then $\mathbb{E}\, Z(t) = \lambda^{-1}\left(2 - (1 + \lambda t)\,\mathrm{e}^{-\lambda t}\right)$.

6.1.15 The *Stirling numbers of the second kind* $\left\{ {n \atop k} \right\}$ are defined by the relation $j^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} j_{(k)}$ where $j_{(k)} = j(j-1)\dots(j-k+1)$. Show that for $k \in \mathbb{N}$

$$\mathbb{E}\, N^k(t) = \sum_{m=1}^{k} m! \left\{ {m \atop k} \right\} H^{*m}(t) \, .$$

6.1.16 Show that $k!\,\mathbb{E}\binom{N(t)+k}{k} = H_0^{*k}(t)$ for $k \in \mathbb{N}$.

6.1.17 Assume that F is NBU or NWU respectively. Prove that

$$\mathbb{P}(T(t) \geq x) \leq (\geq)\; 1 - F(x)\,, \qquad H(t+s) \geq (\leq)\; H(t) + H(s)\,.$$

6.1.18 Assume that the distribution F has density $f(t)$ and hazard rate function $m(t)$. Show: if $a \leq m(t) \leq b$ for some constants $a, b > 0$, then $(\mu b)^{-1} \leq H_0(t) - \mu^{-1} t \leq (\mu a)^{-1}$ where for the lower bound one needs that for all t sufficiently small, $f(t)$ is bounded away from 0.

6.1.19 Let $\{X(t)\}$ denote the renewal reward process considered in Section 6.1.5 with generic inter-occurrence time T and generic claim size U. Assume that $0 < \mu_F = \mathbb{E}\, T < \infty$ and $\mathbb{E}\, |U| < \infty$. Show that $\lim_{t\to\infty} t^{-1} \mathbb{E}\, X(t) = \mu_F^{-1} \mathbb{E}\, U$.

6.1.20 Use the same notation as in Exercise 6.1.19. Prove that the *current claim* $U_{N(t)+1}$ satisfies the renewal equation

$$A_r(t) = P(U_{N(t)+1} \leq r) = a_r(t) + A_r * F(t)\ .$$

Show that if $\mathbb{E}\,(|U|T) < \infty$ then $\lim_{t\to\infty} \mathbb{E}\,(U_{N(t)+1}) = \mathbb{E}\, U + (\mathbb{E}\, T)^{-1} \mathrm{Cov}\,(U, T)$.

Bibliographical Notes. For Exercise 6.1.20, see Wilson (1983).

6.2 Extensions and Actuarial Applications

Exercises

6.2.1 Take $\mu_G < \infty$ and $0 < d < 1$ and assume that $a_n \sim cn^{-d}$ for some constant $c > 0$. Show that for $x \to \infty$

$$\sum_{n=1}^{\infty} a_n G^{*n}(x) \sim c \int_1^{x/\mu} y^{-b}\, \mathrm{d}y \sim c \frac{\mu^{d-1}}{1-b}\, x^{1-d}\, .$$

6.2.2 Let $a_k = (1-p)^\alpha \binom{\alpha+k-1}{k}$ and $b_k = (k+1)a_{k+1} - ka_k$ where $\alpha > 0$ and $0 < p < 1$. Show that $b_k = \alpha a_k$.

6.2.3 Let $a_k = c\Gamma(k-\theta)/k! \sim c\, k^{-\theta}$ where $c = \theta((1-(1-a)^\theta)\Gamma(1-\theta))^{-1}$, $0 < \theta < 1$ and $0 < a < 1$. Show that $b_k = (k+1)a_{k+1} - ka_k = (1-\theta)a_k$.

6.3 Random Walks

Exercises

6.3.1 Assume that the random walk $\{S_n\}$ has a negative drift and $\nu^+ < \infty$. Show that the new random walk $S_{\nu^++1} - S_{\nu^+}, S_{\nu^++2} - S_{\nu^+}, \ldots$ is an identically distributed copy of the original random walk $\{S_n\}$ which is independent of the sequence $S_1, S_2, \ldots, S_{\nu^+}$.

6.3.2 Show that, for the random walk without drift ($\mathbb{E}\, Y = 0$), the ascending ladder heights $\{S_{\nu^+_{n+1}} - S_{\nu^+_n},\ n \in \mathbb{N}\}$ and the descending ladder heights $\{S_{\nu^-_{n+1}} - S_{\nu^-_n},\ n \in \mathbb{N}\}$ form two sequences of independent and identically distributed random variables. Formulate and prove similar statements in the presence of a drift.

6.3.3 Show that M is finite with probability 1 if and only if the ladder height distribution G^+ is defective.

6.3.4 Assume $\mathbb{E}\, Y < 0$. Show that $\mathbb{E}\,(\nu^-)\, \mathbb{E}\, Y = \mathbb{E}\,(S_{\nu^-})$. [Hint. Verify first that

$$\Big(\frac{1}{n}\sum_{m=1}^{n}(\nu^-_m - \nu^-_{m-1})\Big)\Big(\frac{1}{\nu^-_n}\sum_{k=1}^{\nu^-_n} Y_k\Big) = \frac{1}{n}\sum_{l=1}^{n} Y^-_n$$

and use the law of large numbers.]

6.4 The Wiener-Hopf Factorization

Exercises

6.4.1 Let $H_0^+ = \sum_{k=0}^\infty (G^+)^{*k}$ and define the pre-occupation measure γ^+ by $\gamma^+(B) = \mathbb{E} \sum_{n=0}^{\nu^- -1} \mathbb{I}(S_n \in B)$ for $B \in \mathcal{B}(\mathbb{R})$. Show that $\gamma^+ = H_0^+$.

6.4.2 Assume that $\hat{m}_F(s) < \infty$ for some $s > 0$. Show that then $\hat{m}_{G^+}(s) < \infty$. [Hint. Use Corollary 6.4.1 and (6.4.3) showing first that

$$\int_{-\infty}^{\infty} \mathrm{e}^{sy} H_0^-(\mathrm{d}y) = \sum_{k=0}^{\infty}\Big(\int_{-\infty}^{0} \mathrm{e}^{sy} G^-(\mathrm{d}y)\Big)^k < \infty\,. \quad]$$

6.4.3 Suppose that for some $\varepsilon > 0$ the function $1 - \hat{g}_F(z)$ is well-defined in $1-\varepsilon \le |z| \le 1+\varepsilon$. Prove that $1 - \hat{g}_F(z) = d^+(z)d^-(z)$ for $1-\varepsilon \le |z| \le 1+\varepsilon$ where $d^+(z), d^-(z)$ are defined in (6.4.27) and (6.4.28).

6.4.4 Consider a random walk with the integer-valued generic increment Y. Assume that the probability function $\{p_k\}$ of Y fulfils $p_k = 0$ for $k \ne -1, 0, 1$. Compute G^+ and $\psi(u)$ for $p_{-1} = 2/3$ and $p_1 = 1/3$.

6.4.5 Consider the Sparre Andersen model with premium rate $\beta = 2, 3, \ldots$. Suppose that the inter-occurrence times T_n are geometrically distributed with parameter $p = 2/3$ and that all claim sizes U_n are equal to 2. Compute the ruin function $\psi(u)$.

6.4.6 Suppose Y is $\mathbb{Z}$-valued and such that (6.4.16) holds. Assume the roots are different and arranged so that $1 < |z_1| < \ldots < |z_b|$. Show that z_1 is real and the only solution to $1 = \hat{g}_F(s)$ in $s > 1$. Conclude

$$\mathbb{P}(M > n) \sim \prod_{k=1}^{b}(1 - z_k^{-1})\frac{c_1}{1 - z_1^{-1}} z_1^{-(n+1)}, n \to \infty.$$

[Hint. Show first that there is $s > 1$ such that $\hat{g}_F(s) = 1$ (check the sign of the derivative $\hat{g}_F^{(1)}(s)$ at $s = 1$) or equivalently $\hat{g}_{G^+}(s) = 1$. Assume next that there is z such that $1 < |z| < s$ and $\hat{g}_{G^+}(z) = 1$ and show that this is impossible because $|\hat{g}_{G^+}(z)| < 1$.]

6.4.7 Show that if $F_T = \mathrm{Exp}(\lambda)$ and $F_U = \Gamma(n, \delta)$ for some $n = 2, 3, \ldots$, then $G_0 = n^{-1}\sum_{k=0}^{n-1}\Gamma(n-k, \delta)$.

6.4.8 Assume that $F_T = \mathrm{Exp}(\lambda)$ and $F_U = \sum_{k=1}^{n} p_k \mathrm{Exp}(\delta_k)$ for some $n = 2, 3, \ldots$ and some probability function $\{p_1, \ldots, p_n\}$. Show that

$$G_0 = q_n \sum_{k=1}^{n} \frac{p_k}{\delta_k} \mathrm{Exp}(\delta_k),$$

where $q_n = \prod_{i=1}^{n} \delta_i (\sum_{j=1}^{n} \delta_j)^{-1}$.

6.4.9 Let $F_T = \Gamma(2, 2\lambda)$ and $F_U = \mathrm{Exp}(\delta)$, where $\rho = \lambda(\delta\beta)^{-1} < 1$. Show that then $p = \frac{1}{2}(1 + \rho - \sqrt{1 + 2\rho})$. [Hint. Show first that (6.4.37) leads to a cubic equation with root $s = 1$.]

6.4.10 Assume that $F_T = \sum_{k=1}^{2} p_k \mathrm{Exp}(\lambda_k)$ and $F_U = \mathrm{Exp}(\delta)$ for some probability function $\{p_1, p_2\}$ such that $\delta\beta(p_1\lambda_1^{-1} + p_2\lambda_2^{-1}) > 1$. Show that $\delta\beta(1-p)$ is a solution to the quadratic equation

$$x^2 + (\lambda_1 + \lambda_2 - \delta\beta)x + \lambda_1\lambda_2 - \delta\beta(\lambda_1(1-p_1) + \lambda_2(1-p_2)) = 0.$$

[Hint. Proceed in the same way as in Exercise 6.4.9.]

6.4.11 Show that, for the compound Poisson model,

$$\int_0^\infty \mathrm{e}^{-su}\psi(u)\,\mathrm{d}u = \frac{\lambda - \lambda\hat{l}_U(-s) + \lambda\mu_U s}{s(\lambda - \beta s - \beta s\hat{l}_U(-s))}, \qquad s \geq 0.$$

[Hint. Show first that the Laplace-Stieltjes transform $\hat{l}_{F^s}(s)$ of the integrated tail distribution F^s corresponding to the distribution F of a nonnegative random variable can be given by $\hat{l}_{F^s}(s) = (1 - \hat{l}_F(s))(s\mu_F)^{-1}$ for all $s \geq 0$; see also Exercise 2.1.8.]

6.5 Ruin Probabilities: Sparre Andersen Model

Exercises

6.5.1 Show that the ruin function $\psi(u)$ in the Sparre Andersen model satisfies the defective renewal equation

$$\psi(u) = G^+(\infty) - G^+(u) + \int_0^u \psi(u-v)\,\mathrm{d}G^+(v).$$

6.5.2 Show that the multivariate ruin function $\psi(u,x,y,z)$ satisfies the defective renewal equation (6.5.19).

6.5.3 Show how (6.5.20) specifies in the case of the compound Poisson model. [Hint. Use (6.5.15) and the fact that a compound Poisson process has independent increments.]

6.5.4 Show that in the compound Poisson model the distribution of the maximum deficit $\tilde{Z}^+(u)$ during the first time in the red is given by

$$\mathbb{P}(\tau(u) < \infty, \tilde{Z}^+(u) < z) = \frac{\psi(u) - \psi(u+z)}{1 - \psi(z)}, \qquad u, z \geq 0.$$

[Hint. Use the fact that a compound Poisson process has independent increments and, consequently,

$$\psi(u) - \psi(u,\infty,0,z) = (1-\psi(u)) + \mathbb{P}(\tau(u) < \infty, \tilde{Z}^+(u) \leq z)(1-\psi(z)). \quad]$$

6.5.5 Conclude from the result of Exercise 6.5.4 that, for the compound Poisson model,

$$\mathbb{E}\,(\tilde{Z}^+(u) \mid \tau(u) < \infty) = \frac{1}{\psi(u)} \int_0^\infty \frac{\psi(u+z) - \psi(u)\psi(z)}{1 - \psi(z)}\,\mathrm{d}z.$$

6.5.6 Consider the multivariate ruin functions $\psi(u,\infty,y)$ and $\psi'(u,\infty,y)$ of two compound Poisson models with arrival rates λ and λ', premium rates β and β', and claim size distributions F_U and $F_{U'}$, respectively.
(a) Show that $\psi(u,\infty,y) \leq \psi'(u,\infty,y)$ holds for all $u,y \geq 0$ provided that the conditions (5.4.30) and (5.4.31) are fulfilled. [Hint. Use (6.5.17).]
(b) Find conditions on $F_U, F_{U'}$, in addition to (5.4.30) and (5.4.31) such that $\psi(u,x,y) \leq \psi'(u,x,y)$ holds for all $u,x,y \geq 0$.

6.5.7 Consider two Sparre Andersen models with premium rates β, β', generic inter-occurrence times T, T' and claim sizes U, U', respectively, and let $Y = U - \beta T$, $Y' = U' - \beta' T'$. Assume that, for both models, equation (6.5.21) has the positive solution γ_Y and $\gamma_{Y'}$, respectively. Show that then $F_Y \leq_{\rm sl} F_{Y'}$ implies $\gamma_Y \geq \gamma_{Y'}$.

6.5.8 Assume that the distribution F_Y of the increments of a random walk is nonlattice. Show that then the ladder height distribution G^+ corresponding to F_Y is nonlattice, too.

6.5.9 Prove a version of the Cramér-Lundberg approximation to the multivariate ruin functions $\psi(u,x,y)$ and $\psi(u,x,y,z)$, respectively.

6.5.10 Show that, for the constant $c(h)$ in (6.5.38) $\lim_{h\to 0} c(h) = c$, where c is the constant appearing in the original Cramér-Lundberg approximation (6.5.29) for the corresponding compound Poisson model with permanent inspection of risk reserve.

6.5.11 Consider the risk reserve process in a compound Poisson model inspected at the instants $\sigma'_1, \sigma'_2, \ldots$ of an independent renewal process $\{\sigma'_n\}$. In this model ruin can occur at the inspection times σ'_n only (a special case is when $\sigma_n = nh$). Show that the adjustment coefficient in the corresponding versions of the Cramér-Lundberg approximation and Lundberg's inequality is the same as in the original compound Poisson model.

6.5.12 Let $F_U \in \mathcal{S}$ be the Pareto distribution with density

$$f_U(x) = \begin{cases} \alpha c^\alpha x^{-(\alpha+1)} & \text{if } x \geq c, \\ 0 & \text{if } x < c, \end{cases}$$

with $\alpha > 1, c > 0$. Show that $\mu = \alpha c/(\alpha - 1)$, $F_U^{\rm s} \in \mathcal{S}$ and

$$\psi(u) \sim \frac{c}{\beta \mathbb{E}\, T(\alpha-1) - \alpha c} \left(\frac{c}{u}\right)^{\alpha-1}$$

as $u \to \infty$. [Hint. Show that the condition of Corollary 2.5.1 is fulfilled and use Theorem 6.5.11, see also Exercise 2.4.9.]

6.5.13 Let $F_U \in \mathcal{S}$ be the lognormal distribution $\mathrm{LN}(a, b)$ with $-\infty < a < \infty$, $b > 0$. Show that $F_U^{\mathrm{s}} \in \mathcal{S}$ and

$$\psi(u) \sim c \frac{u}{(\log u - a)^2} \exp\left(-\frac{(\log u - a)^2}{2b^2}\right), \qquad u \to \infty,$$

where $c = b^3(\sqrt{2\pi}(\beta \mathbb{E}\, t - \exp(a + b^2/2)))^{-1}$. [Hint. Show first that

$$\overline{F_U^{\mathrm{s}}}(x) \sim \frac{b^3 \exp(-b^2/2)}{\mathrm{e}^a \sqrt{2\pi}} \frac{x}{(\log x - a)^2} \exp\left(-\frac{(\log x - a)^2}{2b^2}\right).$$

and that the right-hand side belongs to $\mathcal{S}$. Conclude then $F_U^{\mathrm{s}} \in \mathcal{S}$.]

Bibliographical Notes. Exercises 6.5.4 and 6.5.5 are taken from Picard (1994). Exercise 6.5.8 is from Bergmann and Stoyan (1976).

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7

Markov Chains

7.1 Definition and Basic Properties

Exercises

7.1.1 Let $\{X_n\}$ be a Markov chain on E. Show that for any integers $n, m \geq 1$ and any states $i, i_0, \ldots, i_{n-1}, j_1, \ldots, j_m \in E$

$$\begin{aligned}
&\mathbb{P}(X_{n+1} = j_1, \ldots, X_{n+m} = j_m \mid X_0 = i_0, \ldots, X_n = i) \\
&= \mathbb{P}(X_{n+1} = j_1, \ldots, X_{n+m} = j_m \mid X_n = i) \\
&= \mathbb{P}(X_1 = j_1, \ldots, X_m = j_m \mid X_0 = i)
\end{aligned}$$

whenever $\mathbb{P}(X_0 = i_0, \ldots, X_n = i) > 0$ and $\mathbb{P}(X_0 = i) > 0$.

7.1.2 Let $E = \{1, 2\}$ and

$$\boldsymbol{P} = \begin{pmatrix} 1-p & p \\ p' & 1-p' \end{pmatrix}$$

with $0 < p, p' \leq 1$. Show that for $n \geq 1$

$$\boldsymbol{P}^n = \frac{1}{p+p'} \begin{pmatrix} p' & p \\ p' & p \end{pmatrix} + \frac{(1-p-p')^n}{p+p'} \begin{pmatrix} p & -p \\ -p' & p' \end{pmatrix}.$$

7.1.3 Let $\{Y_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with values in $\mathbb{N}$; $p_k = \mathbb{P}(Y_n = k)$. For some fixed $\ell, \ell' \geq 1$ with $\ell' < \ell$, let

$$X_n = \begin{cases} (X_{n-1} + 1 - Y_n)_+ & \text{if } 0 \leq X_{n-1} \leq \ell', \\ (X_{n-1} - Y_n)_+ & \text{if } \ell' < X_{n-1} \leq \ell, \end{cases}$$

where X_0 is independent of $\{Y_n\}$ and takes values in $E = \{1, \ldots, \ell\}$. Show that $\{X_n\}$ is a Markov chain and determine its transition matrix. (Note that this definition of the sequence $\{X_n\}$ has an interpretation as a discrete-time risk process with state-dependent increments: the premiums are added to the portfolio only when the risk reserve process is below the critical level ℓ'. Besides this, any downcrossing below the zero level is compensated instantaneously.)

7.1.4 Let $Y_1, Y_2, \ldots$ be independent random variables with $\mathbb{P}(Y_n = 1) = \mathbb{P}(Y_n = -1) = \frac{1}{2}$ for all $n \geq 1$, and $S_n = Y_1 + \ldots + Y_n, S_0 = 0$. Is $\{X_n\}$ with $X_n = \max\{S_0, S_1, \ldots, S_n\}$ a Markov chain?

7.1.5 Consider the Finnish bonus-malus system. There are 13 bonus classes labeled from 1 to 13; new policies are placed in class 3. The bonus rules and the premium scale are given in Table 1. Model this system by a Markov chain and determine the corresponding transition matrix provided that the number of yearly reported claims is Poisson distributed with parameter λ.

Class	Premium scale	Class after one year (per no. of claims)					
		0	1	2	3	4	$5, 6, \ldots$
13	140	11	13	13	13	13	13
12	120	11	13	13	13	13	13
11	100	10	12	13	13	13	13
10	100	9	11	12	13	13	13
9	80	8	11	12	13	13	13
8	70	7	10	11	12	13	13
7	60	6	9	11	12	13	13
6	50	5	8	10	11	12	13
5	50	4	8	10	11	12	13
4	50	3	8	10	11	12	13
3	50	2	8	10	11	12	13
2	50	1	8	10	11	12	13
1	40	1	7	9	11	12	13

Table 1 Finnish bonus-malus system

7.1.6 The Danish bonus-malus system has four classes labeled from 1 to 4, where 1 means superbonus and 4 supermalus. New policies are placed in class 3. For every claim-free year the policy holder goes up one class. After one claim the policy holder will be moved from classes 1 and 2 to class 3. If a policy holder being in class 3 reports one claim in a year, but no claim in the year before, then he remains in class 3. After two or more claims during one year or after one claim during both of two consecutive years the new class will be 4. Model the Danish bonus-malus system by a Markov chain. [Hint. Define the states as (tariff group, indicator), where the indicator is 0 or 1 depending on whether there was a claim reported in the year before or not; thus possible states are (1,0), (2,0), (3,0), (3,1), (4,1).]

7.2 Stationary Markov Chains

Exercises

7.2.1 Show that a Markov chain is irreducible and aperiodic if its transition matrix $\boldsymbol{P}$ is regular.

7.2.2 Give an example of a (nonergodic) Markov chain which has more than one stationary initial distribution [Hint. Consider the 4×4 matrix $\boldsymbol{P}$ having the block structure

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_2 \end{pmatrix}$$

where $\boldsymbol{P}_1, \boldsymbol{P}_2$ are as in the example given in Section 7.2.3, and let $\boldsymbol{\eta}_1 = (\frac{1}{2}, \frac{1}{2}), \boldsymbol{\eta}_2 = (\frac{1}{3}, \frac{2}{3})$. Show by inspection that, for each $0 \le \theta \le 1$, the probability function $\boldsymbol{\pi} = (\theta\boldsymbol{\eta}_1, (1-\theta)\boldsymbol{\eta}_2)$ fulfils (7.2.15).]

7.2.3 Show that a Markov chain with stationary initial distribution is stationary in the sense of Definition 7.2.2.

7.2.4 Let θ be an eigenvalue of the matrix $\boldsymbol{A}$. Show that then $\sum_{k=0}^{n} c_k \theta^k$ is an eigenvalue of the matrix $\sum_{k=0}^{n} c_k \boldsymbol{A}^k$.

7.2.5 Prove the following statements. If in a nonnegative matrix $\boldsymbol{A}$ the sums over all rows are strictly less than 1, then $\boldsymbol{I} - \boldsymbol{A}$ is nonsingular. In particular, if $\boldsymbol{P}$ is a stochastic matrix, then for $0 < v < 1$ the matrix $\boldsymbol{I} - v\boldsymbol{P}$ is nonsingular. Using Exercise 7.2.4 show that all eigenvectors of $\boldsymbol{I} - v\boldsymbol{P}$ are in the complex circle $\{z : |z-1| \le v\}$.

7.2.6 Let $\boldsymbol{P}$ be a regular stochastic matrix. Show that $\boldsymbol{\Pi}^2 = \boldsymbol{\Pi}$ and $\boldsymbol{\Pi}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{\Pi} = \boldsymbol{\Pi}$. [Hint. Use that $\boldsymbol{P}^n \to \boldsymbol{\Pi}$ as $n \to \infty$.]

7.2.7 Let $\boldsymbol{P}(\lambda) = (p_{ij}(\lambda))_{i,j=1,\ldots,\ell}$ be the transition matrix of a bonus-malus system given in (7.1.27), i.e. $p_{ij}(\lambda) = \sum_{k=0}^{\infty} (k!)^{-1} \lambda^k \mathrm{e}^{-\lambda} t_{ij}(k)$. Show that $\boldsymbol{P}(\lambda)$ is regular if there is a natural number n such that, for each $i, j = 1, \ldots, \ell$, there is a sequence $(i_1, k_1), \ldots, (i_{n-1}, k_{n-1})$ with

$$t_{ii_1}(k_1) t_{i_1 i_2}(k_2) \ldots t_{i_{n-1} j}(k_{n-1}) = 1.$$

7.2.8 Assume that the matrix $\boldsymbol{P}(\lambda) = (p_{ij}(\lambda))$ given in (7.1.27) is regular for some $\lambda > 0$. Show that then $\boldsymbol{P}(\lambda)$ is regular for all $\lambda > 0$.

7.2.9 Consider the following bonus-malus system. There are $n + 1$ classes labeled by $0, 1, \ldots, n$. Assume now that 0 is the superbonus and n the supermalus. For every claim-free year a policy holder advances from i one class down to $\max(i-1, 0)$. If he reports one claim, the class is unchanged and for k claims the policy is moved from class i to $\max(i+k-1, n)$. Let $\{p_k\}$ be the probability function of the yearly reported number of claims and assume that $0 < p_0 < 1$. Model the process of consecutive policy classes by a Markov chain. Show that this Markov chain is ergodic and its stationary initial distribution $\boldsymbol{\pi} = (\pi_0, \ldots, \pi_n)$ can be determined from $\pi_i = a_i/a_n$ for $i = 0, 1, \ldots, n-1$, where $a_0, a_1, \ldots, a_n$ satisfy the recursive equations $p_0 a_{j+1} = a_j - \sum_{i=0}^{j} a_{j-i} p_{i+1}$.

Bibliographical Notes. Exercise 7.2.8 is from Dufresne (1984).

7.3 Markov Chains with Rewards

Exercises

7.3.1 Let $\{X_n\}$ be a Markov chain with regular transition matrix $\boldsymbol{P}$. Furthermore, let $\boldsymbol{\pi}$ denote its stationary distribution and $R_n^{\mathrm{u}}(\boldsymbol{\alpha})$ the undiscounted reward for visits at times $0, \ldots, n-1$ when $\{X_n\}$ has the (arbitrary) initial distribution $\boldsymbol{\alpha}$. Show that $\lim_{n\to\infty} n^{-1} R_n^{\mathrm{u}}(\boldsymbol{\alpha}) = \bar{\beta}$ with probability 1, where $\bar{\beta} = \boldsymbol{\pi}\boldsymbol{\beta}^{\top}$. [Hint. The proof of this statement is based on the law of large numbers for sums of independent random variables, see e.g. Section 2.3 in Tijms (1994).]

7.3.2 Show that under the assumptions of Exercise 7.3.1 for any initial distribution $\boldsymbol{\alpha}$ $\lim_{n\to\infty} n^{-1}(\mathrm{Var}\,(R_n^{\mathrm{u}}(\boldsymbol{\alpha})) - \mathrm{Var}\,(R_n^{\mathrm{u}}(\boldsymbol{\pi}))) = 0$. [Hint. Represent R_n^{u} as a sum of independent random variables and use Theorem 7.2.1.]

7.3.3 Consider the following modified reward model. Assume that each time the Markov chain $\{X_n\}$ with regular transition matrix $\boldsymbol{P}$ changes from state i to state j, a reward $g(i, j)$ is obtained. Determine the long-run reward rate

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} \mathbb{E}\, g(X_k, X_{k+1}). \tag{7.3.1}$$

Show that, as $n \to \infty$, the random variables $n^{-1} \sum_{k=0}^{n-1} g(X_k, X_{k+1})$ converge to the long-run reward rate considered in (7.3.1).

7.3.4 Define by $\boldsymbol{\eta}^{\mathrm{d}}(\lambda) = (\eta_1^{\mathrm{d}}(\lambda), \ldots, \eta_\ell^{\mathrm{d}}(\lambda))$ the efficiency in the case of discounting, where $\eta_i^{\mathrm{d}}(\lambda) = (d\bar{\beta}_i^{\mathrm{d}}(\lambda)/d\lambda) / (\bar{\beta}_i^{\mathrm{d}}(\lambda)/\lambda)$ for $i = 1, \ldots, \ell$, and $\bar{\beta}_i^{\mathrm{d}}(\lambda) = \boldsymbol{e}_i(\boldsymbol{I} - v\boldsymbol{P}(\lambda))^{-1}\boldsymbol{\beta}^\top$ is the expected total discounted premium, see (7.3.3). Show that

$$\frac{\mathrm{d}\boldsymbol{\eta}^{\mathrm{d}}(\lambda)}{\mathrm{d}\lambda} = v(\boldsymbol{I} - v\boldsymbol{P}(\lambda))^{-1}\frac{\mathrm{d}\boldsymbol{P}(\lambda)}{\mathrm{d}\lambda}(\boldsymbol{\eta}^{\mathrm{d}}(\lambda))^\top.$$

7.4 Monotonicity and Stochastic Ordering

Exercises

7.4.1 Let $X_0, X_1, \ldots$ be a stationary and ergodic Markov chain with state space $E = \mathbb{Z}$ and with a stochastically monotone transition matrix $\boldsymbol{P}$. Show that for each increasing function $f : E \to \mathbb{R}$ the covariance $\gamma_n = \mathrm{Cov}\,(f(X_0), f(X_n))$ is monotonously decreasing to 0 as $n \to \infty$.

7.4.2 Show that the following $\ell \times \ell$ matrix

$$\boldsymbol{P} = \begin{pmatrix} p & \frac{1-p}{\ell-1} & \cdots & \frac{1-p}{\ell-1} \\ \frac{1-p}{\ell-1} & p & \cdots & \frac{1-p}{\ell-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1-p}{\ell-1} & \frac{1-p}{\ell-1} & \cdots & p \end{pmatrix}$$

is stochastically monotone if $p \geq 1/\ell$. Find the stationary initial distribution and the covariance function of the stationary Markov chain with transition matrix $\boldsymbol{P}$. [Hint. Find all eigenvalues and eigenvectors of $\boldsymbol{P}$ and show that $\boldsymbol{P}^k = \boldsymbol{e}^\top\boldsymbol{\pi} + \xi^k(\boldsymbol{I} - \boldsymbol{e}^\top\boldsymbol{\pi})$, where $\xi = \ell p - 1/\ell - 1$].

7.4.3 Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables and let $\phi(i, y) : \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ be nondecreasing with respect to i. Define the sequence $X_0, X_1, \ldots$ recursively by $X_{n+1} = \phi(X_n, Y_n)$. Show that, if $X_0 \leq_{\mathrm{st}} X_1$ then $X_n \leq_{\mathrm{st}} X_{n+1}$ for all $n \in \mathbb{N}$.

7.4.4 Let $\{Y_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables with values in $\mathbb{Z}$. For some fixed $k, \ell' \in \mathbb{Z}$ let

$$X_n = \begin{cases} (X_{n-1} + k - Y_n)_+ & \text{if } X_{n-1} \leq \ell', \\ (X_{n-1} - Y_n)_+ & \text{if } \ell' < X_{n-1}, \end{cases}$$

where X_0 is independent of $\{Y_n\}$ and takes values in $\mathbb{Z}$. Show that the transition matrix of the Markov chain $\{X_n\}$ is stochastically monotone if and only if $k \leq 1$.

7.4.5 Let $\boldsymbol{P}, \boldsymbol{P}'$ be two regular transition matrices such that $\boldsymbol{P} \leq_{\rm st} \boldsymbol{P}'$. Show that then $\boldsymbol{\pi} \leq_{\rm st} \boldsymbol{\pi}'$ for the stationary probability functions $\boldsymbol{\pi}, \boldsymbol{\pi}'$ corresponding to $\boldsymbol{P}, \boldsymbol{P}'$, respectively.

7.4.6 For $\lambda \geq 0$, let $Y(\lambda)$ be Poisson distributed with mean λ. Show that $Y(\lambda) \leq_{\rm st} Y(\lambda')$ for $\lambda \leq \lambda'$. [Hint. Use the results of Exercise 3.2.11.]

7.4.7 Show that the Finnish bonus-malus system fulfils conditions (A), (B) and (C) of Section 7.4.3.

7.4.8 Consider a regular bonus-malus system with transition matrix $\boldsymbol{P}(\lambda)$ given in (7.1.27) for $\lambda > 0$ and such that conditions (A), (B), and (C) of Section 7.4.3 are fulfilled. Putting $\boldsymbol{P}(0) = \boldsymbol{T}(0)$, show that the mapping $\lambda \mapsto \boldsymbol{P}(\lambda)$ is continuous for $\lambda \geq 0$. Moreover, for $\lambda > 0$, let $\boldsymbol{\pi}(\lambda)$ denote the stationary probability function corresponding to $\boldsymbol{P}(\lambda)$. Putting $\boldsymbol{\pi}(0) = \boldsymbol{e}_1$, show that the mapping $\lambda \mapsto \boldsymbol{\pi}(\lambda)$ is continuous for $\lambda \geq 0$.

Bibliographical Notes. The result of Exercise 7.4.1 is from Daley (1968), and that of Exercise 7.4.2 from Szekli, Disney and Hur (1994).

7.5 An Actuarial Application of Branching Processes

Exercises

7.5.1 Show that the branching process $\{X_n\}$ defined in (7.5.1) is a Markov chain with state space $\mathbb{N}$ and transition matrix $\boldsymbol{P} = (p_{ij})$ where $p_{ij} = (\{p_k\}^{*i})_j$.

7.5.2 Prove Corollary 7.5.1.

7.5.3 Show $\mathbb{P}(X_n = 0) = \mathbb{E}\,(p_0^{X_{n-1}})$ for the branching process (7.5.1).

7.5.4 Find $\lim_{n\to\infty} \mathbb{P}(X_n = 0)$. [Hint. Show first that $\mathbb{P}(X_n = 0) = \hat{g}_R(\mathbb{P}(X_{n-1} = 0))$. Verify that the solution fulfils $s = \hat{g}_R(s)$. Show, that in the case $\mu_R \leq 1$ the only solution is 1 and that there exists a non-trivial solution in the case $\mu_R > 1$.]

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8

Continuous-Time Markov Models

8.1 Homogeneous Markov Processes

Exercises

8.1.1 Show in the case of a finite state space that the uniform continuity of $\boldsymbol{P}(h)$ for $h \geq 0$ follows from (8.1.1) and (8.1.2).

8.1.2 Show that, for $a \geq 0$,

$$\begin{array}{rclcrcl} \|\boldsymbol{x}+\boldsymbol{y}\| & \leq & \|\boldsymbol{x}\|+\|\boldsymbol{y}\|, & & \|\boldsymbol{A}+\boldsymbol{B}\| & \leq & \|\boldsymbol{A}\|+\|\boldsymbol{B}\|, \\ \|\boldsymbol{A}\boldsymbol{x}^\top\| & \leq & \|\boldsymbol{A}\|\|\boldsymbol{x}\|, & & \|\boldsymbol{A}\boldsymbol{B}\| & \leq & \|\boldsymbol{A}\|\|\boldsymbol{B}\|, \\ \|a\boldsymbol{x}\| & = & a\|\boldsymbol{x}\|, & & \|a\boldsymbol{A}\| & = & a\|\boldsymbol{A}\|. \end{array}$$

8.1.3 Let $\boldsymbol{A}, \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3 \ldots$ be a sequence of $\ell \times \ell$ matrices such that $\lim_{n\to\infty} \|\boldsymbol{A}_n - \boldsymbol{A}\| = 0$. Show that $\lim_{n\to\infty} \|\mathrm{e}^{\boldsymbol{A}_n} - \mathrm{e}^{\boldsymbol{A}}\| = 0$.

8.1.4 Prove that $\exp(\boldsymbol{A}+\boldsymbol{A}') = \exp(\boldsymbol{A})\exp(\boldsymbol{A}')$, provided that $\boldsymbol{A}\boldsymbol{A}' = \boldsymbol{A}'\boldsymbol{A}$. Show that in particular $\exp(h(\boldsymbol{I}+\boldsymbol{A})) = \exp(h)\exp(h\boldsymbol{A})$, where $\boldsymbol{I}$ is the identity matrix.

8.1.5 Let $\boldsymbol{A}$ be an arbitrary matrix such that $a_{ij} \geq 0$ for $i \neq j$, and $(\boldsymbol{A}\boldsymbol{e}^\top)_i \leq 0$ for each $i = 1, \ldots, \ell$. Show that all entries of $\exp(\boldsymbol{A})$ are nonnegative. [Hint. Use a similar argument as in the proof of Lemma 8.1.3.]

8.1.6 Suppose that the eigenvalues $\theta_1, \ldots, \theta_\ell$ of $\boldsymbol{A}$ are distinct. Show that

$$\mathrm{e}^{t\boldsymbol{A}} = \boldsymbol{\Phi}\mathrm{diag}(\mathrm{e}^{\theta_1 t}, \ldots, \mathrm{e}^{\theta_\ell t})\boldsymbol{\Phi}^{-1}$$

where $\boldsymbol{\Phi}$ is the matrix consisting of the right (column) eigenvectors.

8.1.7 Show that for each fixed pair $i, j \in E$ the function $h \mapsto p_{ij}(h)$ vanishes either everywhere or nowhere in $(0, \infty)$.

8.1.8 Show that each $\mathbb{N}$-valued stochastic process with independent and stationary increments is a homogeneous Markov process.

8.1.9 Let $\{\boldsymbol{P}(h), h \ge 0\}$ be a matrix transition function on $E = \{1, 2\}$ with the matrix of transition intensities

$$\boldsymbol{Q} = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}, \qquad 0 < a, b < \infty.$$

Show that

$$\boldsymbol{P}(h) = \frac{1}{a+b} \begin{pmatrix} b + a\mathrm{e}^{-(a+b)h} & a - a\mathrm{e}^{-(a+b)h} \\ b - b\mathrm{e}^{-(a+b)h} & a + b\mathrm{e}^{-(a+b)h} \end{pmatrix}.$$

[Hint. By differentiation check that $\boldsymbol{Q}$ is the matrix of transition intensities corresponding to $\{P(h),\ h \ge 0\}$.]

8.1.10 Assume that E has no absorbing state. Show that, with probability 1, the sequence $\sigma_1, \sigma_2, \ldots$ of jump epochs increases unboundedly to infinity.

8.1.11 Let $\boldsymbol{Q}$ be a stochastically monotone intensity matrix and consider the transition matrices $\boldsymbol{P}(h) = \exp(h\boldsymbol{Q})$; $h \ge 0$. Show that $\boldsymbol{P}(h)$ is stochastically monotone for each $h \ge 0$.

8.1.12 Let $\boldsymbol{Q} = (q_{ij})$ be an intensity matrix and $\boldsymbol{\alpha}$ an initial distribution on $E = \{1, \ldots, \ell\}$. Let $\{N(t)\}$ be a Poisson process with intensity a and let $\{X_n\}$ be a Markov chain with transition matrix $\tilde{\boldsymbol{P}}$ and initial distribution $\boldsymbol{\alpha}$. Assume that $\{N(t)\}$ and $\{X_n\}$ are independent. Show that the stochastic process $\{X(t)\}$ with $X(t) = X_{N(t)}$ is a Markov process with intensity matrix $\boldsymbol{Q}$ and initial distribution $\boldsymbol{\alpha}$.

8.1.13 Show that each birth-and-death process is stochastically monotone.

8.1.14 Let $a_1, \ldots a_\ell \in \mathbb{R}_+$ and $a > 0$ where $a = \sum_{i=1}^{\ell} a_i$. Show that the intensity matrix $\boldsymbol{Q} = (q_{ij})$ given by

$$q_{ij} = \begin{cases} a_j & \text{if } i \ne j, \\ -\sum_{k \ne j} a_k & \text{if } i = j, \end{cases}$$

is stochastically monotone and that moreover $\boldsymbol{\pi} = (a_1/a, \ldots a_\ell/a)$ is the stationary initial distribution corresponding to $\boldsymbol{Q}$.

8.1.15 Show that a Markov process with intensity matrix $\boldsymbol{Q}$ is irreducible if and only if for each pair $i, j \in E$ with $i \ne j$ there exists a sequence $i_1, \ldots, i_n \in E$ $(i_k \ne i_l)$ such that $q_{ii_1} q_{i_1 i_2} \ldots q_{i_{n-1} j} > 0$. [Hint. Use (8.1.37).]

8.1.16 Show that the stationary initial distribution of an irreducible homogeneous Markov process with finite state space is uniquely determined and satisfies $\boldsymbol{\pi}\boldsymbol{Q} = \boldsymbol{0}$.

8.2 Phase-Type Distributions

Exercises

8.2.1 Let $\boldsymbol{A}' = a\boldsymbol{A} + b\boldsymbol{I}$ for some constants $a, b \in \mathbb{R}$. Show that

$$\theta_i(\boldsymbol{A}') = a\theta_i(\boldsymbol{A}) + b, \qquad i = 1, \ldots, \ell.$$

8.2.2 Let $\boldsymbol{A}(t)$ and $\boldsymbol{A}'(t)$ be two differentiable matrix functions. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}(t)\boldsymbol{A}'(t)\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}(t)\right)\boldsymbol{A}'(t) + \boldsymbol{A}(t)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}'(t)\,.$$

8.2.3 Show that, for all $i \in E$ and $h > 0$, the sum

$$c_i = \sum_{n=0}^{\infty} \sum_{j_1,\ldots,j_n \in E} p_{ij_1}(h)p_{j_1j_2}(h)\ldots p_{j_n0}(h)$$

converges, is independent of h and $0 \le c_i \le 1$.

8.2.4 Let $\boldsymbol{Q}$ be an intensity matrix of the form

$$\boldsymbol{Q} = \begin{pmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{b}^\top & \boldsymbol{B} \end{pmatrix}$$

where $\boldsymbol{b}^\top = (b_1, \ldots, b_\ell)^\top = -\boldsymbol{B}\boldsymbol{e}^\top$ is an ℓ-dimensional vector with nonnegative components, and $\boldsymbol{B}$ a subintensity matrix. Show that

$$\exp(t\boldsymbol{Q}) = \begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{e}^\top - \exp(t\boldsymbol{B})\boldsymbol{e}^\top & \exp(t\boldsymbol{B}) \end{pmatrix}, \qquad t \ge 0.$$

[Hint. Show first that

$$\boldsymbol{Q}^n = \begin{pmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{x}_n & \boldsymbol{B}^n \end{pmatrix}$$

for each $n = 1, 2, \ldots$, where the $\boldsymbol{x}_n$ are suitable vectors.]

8.2.5 Prove (8.2.28).

8.2.6 Let $\alpha_0 = 0$, and $\boldsymbol{B}$ a nonsingular subintensity matrix. Show that the moment generating function $\hat{m}(s)$ of the distribution $\mathrm{PH}(\boldsymbol{\alpha}, \boldsymbol{B})$ is well-defined for all $s < \min\{-b_{ii} : i \in E\}$ where $\hat{m}(s) = -\boldsymbol{\alpha}(s\boldsymbol{I} + \boldsymbol{B})^{-1}\boldsymbol{b}^\top$.

8.2.7 Show that $\{\boldsymbol{P}(t), t \geq 0\}$ given by

$$(\boldsymbol{P}(t))_{ik} = \begin{cases} (\boldsymbol{P}_1(t))_{ik} & \text{if } i, k \in E_1, \\ \displaystyle\sum_{k' \in E_1, i' \in E_2} \int_0^t (\boldsymbol{P}_1(t-x))_{ik'} (\boldsymbol{b}_1)_{k'} (\boldsymbol{\alpha}_2)_{i'} (\boldsymbol{P}_2(x))_{i'k} \, \mathrm{d}x & \\ & \text{if } i \in E_1, k \in E_2 \cup \{0\}, \\ (\boldsymbol{P}_2(t))_{ik} & \text{if } i, k \in E_2 \cup \{0\}, \\ 0 & \text{if } i \in E_2 \cup \{0\}, k \in E_1. \end{cases}$$

is a matrix transition function on the state space $E' = E_1 \cup E_2 \cup \{0\}$ such that its intensity matrix $\boldsymbol{A}$ has the form (8.2.14) where $\boldsymbol{B}$ is given by (8.2.30).

8.2.8 Prove Theorem 8.2.7.

8.2.9 Show that the family of Erlang distributions $\{\mathrm{Erl}(n, n), n \geq 1\}$ possesses the following optimality property: among all phase-type distributions such that their state space E has n elements, $\mathrm{Erl}(n, n)$ is the best approximation to δ_1 (in the sense of the L_2-norm). Furthermore, show that $\mathrm{Erl}(n, n)$ converges to δ_1 as $n \to \infty$.

8.2.10 Let η_1 and η_2 be independent random variables with distribution $\mathrm{PH}(\boldsymbol{\alpha}_1, \boldsymbol{B}_1, E_1)$ and $\mathrm{PH}(\boldsymbol{\alpha}_2, \boldsymbol{B}_2, E_2)$, respectively. Show that the random variables $\min\{\eta_1, \eta_2\}$ and $\max\{\eta_1, \eta_2\}$ are phase-type distributed. Find their characteristics.

8.3 Risk Processes with Phase-Type Distributions

Exercises

8.3.1 Redo Exercise 5.3.1 using the phase-type representation of the claim size distribution function.

8.3.2 Redo Exercise 5.3.3 using the phase-type representation of the claim size distribution function.

8.4 Nonhomogeneous Markov Processes

Exercises

8.4.1 Consider the stochastic process $\{X(t), t \geq 0\}$ defined in (8.4.1) where $\mathbb{P}(T > x) = \exp\left(-\int_0^x m(y)\,\mathrm{d}y\right)$ and $m(t)$ is an arbitrary hazard rate function. Show that $\{X(t)\}$ fulfils the Markov property (8.4.2).

8.4.2 Show that either $\sigma_n = \infty$ for some $n \in \mathbb{N}$, or the sequence $\sigma_1, \sigma_2, \dots$ of jump epochs increases unboundedly to infinity.

8.4.3 Let $\{\boldsymbol{P}(t,t'),\ t \le t'\}$ be a matrix transition function with the matrix intensity function $\{\boldsymbol{Q}(t), t \ge 0\}$. Show that the entries $p_{ij}(t,t')$ of $\boldsymbol{P}(t,t')$ satisfy

$$\frac{\partial}{\partial t} p_{ij}(t,t')\Big|_{t'=0} = q_{ij}(t) = \begin{cases} -q_i(t) & \text{if } i = j, \\ q_i(t)p^{\circ}_{ij}(t) & \text{if } i \ne j. \end{cases}$$

for all $t > 0$ with the exception of a set of Lebesgue measure zero. [Hint. Use similar arguments like in the proof of Corollary 8.1.3.]

8.4.4 Consider the following insurance model with three states: 1-active, 2-disabled and 3-dead with possible transitions depicted on Figure 1, where the matrix intensity function $\{\boldsymbol{Q}(t)\}$ is given by $q_{12}(t) = a(t)$ and

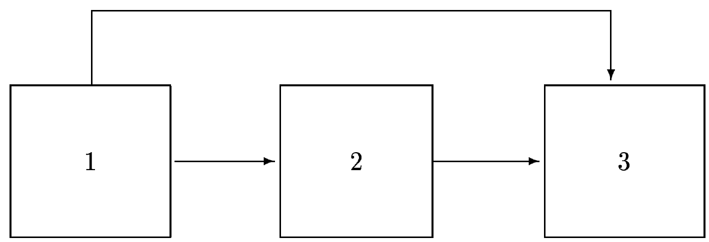

Figure 1 Transition graph

$q_{13}(t) = q_{23}(t) = b(t)$ for some nonnegative and continuous functions $a(t)$, $b(t)$. Show that the corresponding matrix transition function $\{\boldsymbol{P}(t,t')\}$ has the following entries:

$$\begin{aligned} p_{11}(t) &= \exp\Big(-\int_t^{t'} (a(y)+b(y))\,\mathrm{d}y\Big) \\ p_{12}(t) &= \exp\Big(-\int_t^{t'} b(y)\,\mathrm{d}y\Big)\Big(1-\exp\Big(-\int_t^{t'} a(y)\,\mathrm{d}y\Big)\Big) \\ p_{22}(t) &= \exp\Big(-\int_t^{t'} b(y)\,\mathrm{d}y\Big). \end{aligned}$$

8.4.5 Show that if $\delta(x)$ is a Riemann integrable function on $[0,1]$ and if $j,n \to \infty$ so that $0 \le j/n \to t \le 1$, then

$$\lim_{n\to\infty} \prod_{j=1}^{n} \Big(1 + \frac{\delta(j/n)}{n}\Big)^n = \exp\Big(\int_0^t \delta(x)\,\mathrm{d}x\Big).$$

8.4.6 Show that the net prospective premium reserve defined in (8.4.23) satisfies Thiele's differential equation (8.4.24).

8.4.7 A married couple buys a combined life insurance and widow's pension policy specifying that premiums are to be paid at rate β as long as both husband and wife are alive, pensions are to be paid at rate b as long as the wife is widowed and a life assurance of amount c is paid immediately upon the death of the husband if the wife is already dead (as a benefit to their dependant). Assuming that the interest rate is constant equal to δ, write down the system of Thiele's differential equations for the net prospective premium reserves $\mu_i(t)$.

8.5 Mixed Poisson Processes

Exercises

8.5.1 Let $\{X(t)\}$ be a Pólya process with parameters $a, b \geq 0$. Show that the probabilities $\alpha_i(t) = \mathbb{P}(X(t) = i \mid X(0) = 0)$ are given by

$$\alpha_i(t) = \binom{a+i-1}{i} \left(\frac{b}{t+b}\right)^a \left(\frac{t}{t+b}\right)^i, \qquad i \in \mathbb{N}.$$

8.5.2 Let $\{X(t)\}$ be a Pólya process with parameters $a, b \geq 0$. Show that

$$\mathbb{E}\, X(t) = \frac{at}{b}, \qquad \operatorname{Var} X(t) = \frac{at(t+b)}{b^2}, \qquad t \geq 0.$$

8.5.3 Let $\{X(t)\}$ be a Pólya process with parameters $a, b > 0$. Show: if $a/b \to \lambda$ as $a, b \to \infty$, then

$$\lim_{a,b\to\infty} \mathbb{P}(X(t) = i \mid X(0) = 0) = \frac{(\lambda t)^i}{i!}\, \mathrm{e}^{-\lambda t}, \qquad i \in \mathbb{N},\ t \geq 0.$$

8.5.4 Let $\{N(t)\}$ be a mixed Poisson process with mixing random variable Λ. Show that for all $n \geq 1$, $k_1, \ldots, k_n \in \mathbb{N}$ and $0 = x_0 \leq x_1 \leq \cdots \leq x_n$

$$\begin{aligned} &\mathbb{P}(N(x_1) = k_1, \ldots, N(x_n) = k_1 + \cdots + k_n) \\ &= \prod_{r=1}^{n} \frac{(x_r - x_{r-1})^{k_r}}{k_r!} (-1)^{k_1+\cdots+k_r} \alpha_0^{(k_1+\cdots+k_r)}(x_n), \end{aligned}$$

where $\alpha_0(t) = \hat{l}_\Lambda(t)$.

8.5.5 Show that the transition probabilities $p_{ij}(t, t')$ of a mixed Poisson process given by (8.5.3) and (8.5.22) satisfy Kolmogorov's forward equations (8.4.10).

8.5.6 Prove Theorem 8.5.5.

8.5.7 Derive (8.5.34) and (8.5.35) in the setting of mixed Poisson processes.

8.5.8 Let $\{N_1(t)\}$ and $\{N_2(t)\}$ be independent counting processes where $\{N_1(t)\}$ is a Poisson process and $\{N_2(t)\}$ is a Pólya process. Show that the process $\{N(t)\}$ with $N(t) = N_1(t) + N_2(t)$ is a Delaporte process. [Hint. Recall that a mixed Poisson process is uniquely determined by the function $\alpha_0(t)$.]

8.5.9 Let $\{N(t),\ t \geq 0\}$ be a Poisson process with intensity 1 and $\Lambda \geq 0$ a random variable. Show that the process $\{N'(t)\}$, where $N'(t) = N(\Lambda t)$, is a mixed Poisson process and that all mixed Poisson processes can be represented in this way. Conclude that if $\{N'(t)\}$ is a mixed Poisson process with mixing random variable Λ, then $\lim_{t\to\infty} t^{-1}N'(t)t = \Lambda$.

8.5.10 Let the random variable Λ have the distribution $\mathrm{Beta}(a, b, \eta)$. Then the corresponding mixed Poisson process is called a *beta-Poisson process*. Show that for the beta-Poisson process:

$$\alpha_n(t) = \frac{(\eta t)^n B(n+a,b)}{n! B(a,b)} M(n+a, n+a+b; -\eta t)\ ,$$

where $M(y, z; x)$ is the confluent hypergeometric function defined in (2.2.3).

8.5.11 Consider a *uniform-Poisson process*, that is a mixed Poisson process with the uniform mixing distribution $\mathrm{U}(0, \eta)$. Show that the transition intensities $q_j(t)$ are given by

$$q_j(t) = \frac{\int_0^\eta x^{j+1} \mathrm{e}^{-\eta x}\, \mathrm{d}x}{\int_0^\eta x^j \mathrm{e}^{-\eta x}\, \mathrm{d}x}$$

and then conclude that

$$q_0(t) = \frac{1-(t\eta+1)\mathrm{e}^{-t\eta}}{t(1-\mathrm{e}^{-t\eta})}\ , \qquad q_1(t) = \frac{2-(t^2\eta^2+2t\eta+2)\mathrm{e}^{-t\eta}}{t(1-(t\eta+1)\mathrm{e}^{-t\eta})}\ .$$

8.5.12 Consider a *double-Poisson process* which is a discrete mixture of Poisson processes with a two-point mixing distribution with atoms at λ_0 and λ_1 whose weights are p and $q = 1 - p$ respectively. Show that

$$q_j(t) = \frac{p\lambda_0^{j+1}\mathrm{e}^{-\lambda_0 t} + q\lambda_1^{j+1}\mathrm{e}^{-\lambda_1 t}}{p\lambda_0^{j}\mathrm{e}^{-\lambda_0 t} + q\lambda_1^{j}\mathrm{e}^{-\lambda_1 t}}\ .$$

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9

Martingale Techniques I

9.1 Discrete-Time Martingales

Exercises

9.1.1 Let $\{X_n\}$ be adapted to a filtration $\{\mathcal{F}_n\}$ and τ be an $\{\mathcal{F}_n\}$-stopping time. Show that

$$\mathcal{F}_\tau = \{A : \{\tau = n\} \cap A \in \mathcal{F}_n, \text{ for all } n \in \mathbb{N}\},$$

the family of all *events prior to stopping time* τ, is a σ-algebra. Moreover, show that τ and X_τ are $\mathcal{F}_\tau$-measurable.

9.1.2 Show that the last exit time τ of a process $\{X_n\}$ from $B \in \mathcal{B}(\mathbb{R})$, where

$$\tau = \begin{cases} \sup\{n : \ X_n \in B\} & \text{if } X_n \in B \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise,} \end{cases}$$

is not an $\{\mathcal{F}_n^X\}$-stopping time.

9.1.3 Let τ_1 and τ_2 be stopping times with respect to a filtration $\{\mathcal{F}_n\}$. Show that $\min\{\tau_1, \tau_2\}$, $\max\{\tau_1, \tau_2\}$, $\tau_1 + \tau_2$ are stopping times, too.

9.1.4 Show that the ascending ladder epochs $\nu_1^+, \nu_2^+, \ldots$ and the descending ladder epochs $\nu_1^-, \nu_2^-, \ldots$ of a random walk $\{S_n\}$ are $\{\mathcal{F}_n^S\}$-stopping times.

9.1.5 Show that every increasing sequence $\{X_n\}$ adapted to a filtration $\{\mathcal{F}_n\}$ is a submartingale.

9.1.6 Show that if the random variable Z is measurable with respect to $\mathcal{F}_0$ for some filtration $\{\mathcal{F}_n\}$, and if $\mathbb{E}\,|Z| < \infty$, then the sequence $\{X_n\}$ defined by $X_n \equiv Z$ is an $\{\mathcal{F}_n\}$-martingale. Show also that if $\{X_n\}$ is another $\{\mathcal{F}_n\}$-martingale which is independent of Z, then $\{ZX_n, \ n \in \mathbb{N}\}$ is an $\{\mathcal{F}_n\}$-martingale, too.

9.1.7 Let $\{X_n\}$ be a martingale with increments $Y_n = X_n - X_{n-1}$ having finite second moments $\mathbb{E}\, X_n^2 < \infty$. Show that

$$\mathbb{E}\, Y_n = 0, \qquad \mathrm{Cov}(Y_n, Y_{n+k}) = 0$$

and consequently,

$$\mathrm{Var}\,(X_n) = \sum_{i=0}^{n} \mathrm{Var}\,(Y_i).$$

9.1.8 Suppose that $\{X_n\}$ is a submartingale, $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and nondecreasing, and $\mathbb{E}\,|\varphi(X_n)| < \infty$ for all $n \in \mathbb{N}$. Show that then $\{X_n'\}$ with $X_n' = \varphi(X_n)$ is an submartingale. [Hint. Use Jensen's inequality for conditional expectation.]

9.1.9 Let $\{X_n\}$ be a martingale and let τ be a stopping time. Show that then $\{X_n'\}$ with $X_n' = X_{n\wedge\tau}$ is a martingale.

9.1.10 Show that the stopping time τ considered in the example of Section 9.1.6 fulfils the conditions of Theorem 9.1.5. [Hint: Use the inequality $\mathrm{e}^{\gamma S_{k+1}} \mathbb{I}(\tau = k) \le \mathrm{e}^{\gamma(x+Y_{k+1})} \mathbb{I}(\tau = k)$ and the fact that $\mathbb{E}\, N < \infty$.]

9.1.11 Prove part (b) of Theorem 9.1.8.

9.1.12 Show: if $\{X_n\}$ is a supermartingale with respect to a filtration $\{\mathcal{F}_n\}$, then there exists an $\{\mathcal{F}_n\}$-martingale $\{M_n\}$ and an $\{\mathcal{F}_n\}$-predictable sequence $\{V_n\}$ which is decreasing from zero such that $X_n = X_0 + M_n + V_n$ (for all $n \in \mathbb{N}$), where this decomposition is unique modulo indistinguishability. [Hint: Use the fact that $\{-X_n\}$ is a submartingale when $\{X_n\}$ is a supermartingale.]

9.2 Change of the Probability Measure

Exercises

9.2.1 Show that the sequence $\{X_n, n \in \mathbb{N}\}$ defined in (9.2.14) with

$$X_n = \exp\Big(-\gamma \sum_{i=1}^{n} Y_i\Big), \qquad n \in \mathbb{N}$$

is an $\{\mathcal{F}_n^Y\}$-martingale on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \tilde{\mathbb{P}})$.

9.2.2 Let $\{X_n\}$ be a martingale and τ a stopping time. Show that then $\mathbb{E}\,(X_n \mid \mathcal{F}_\tau) = X_{\tau\wedge n}$ for each $n \in \mathbb{N}$, and equivalently

$$\mathbb{E}\,[X_n; A \cap \{\tau \le n\}] = \mathbb{E}\,[X_\tau; A \cap \{\tau \le n\}], \quad A \in \mathcal{F}_\tau.$$

9.2.3 Let $s \in \mathbb{R}$ such that $\hat{m}_F(s) < \infty$.
(a) Show that $\{X_n\}$ with

$$X_n = \exp\Big(-s\sum_{i=1}^{n} Y_i + n\log\hat{m}_F(s)\Big)$$

is an $\{\mathcal{F}_n^Y\}$-martingale on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mathbb{P}^{(s)})$.
(b) Let $\tau_{\mathrm{d}}(u)$ be the ruin time defined in (9.2.16). Show that, for each $A \in \mathcal{F}^Y_{\tau_{\mathrm{d}}(u)}$ such that $A \subset \{\tau_{\mathrm{d}}(u) < \infty\}$,

$$\mathbb{P}(A) = \mathbb{E}^{(s)}\Big[\exp\Big(-s\sum_{i=1}^{\tau_{\mathrm{d}}(u)} Y_i + \tau_{\mathrm{d}}(u)\log\hat{m}_F(s)\Big); A\Big]$$

provided that $\mathbb{P}^{(s)}(\tau_{\mathrm{d}}(u) < \infty) = 1$.

9.2.4 Assume that there exists a positive solution γ to $\hat{m}_F(s) = 1$. Show that then there exist $s' \in \mathbb{R}$ such that $0 < s' < \gamma$ and $\hat{m}_F^{(1)}(s') = 0$. Furthermore, prove that $\mathbb{P}^{(s)}(\tau_{\mathrm{d}}(u) < \infty) = 1$ for all $s > s'$. [Hint. Show first that $\mathbb{E}^{(s)}Y = \hat{m}_F^{(1)}(s)(\hat{m}_F(s))^{-1}$.]

9.2.5 Let $\chi(s) = \log\hat{m}_F(s)$. Show that $\chi(s)$ is convex on $\mathbb{R}$.

9.2.6 Consider the random walk $\{S_n\}$, $S_n = \sum_{i=1}^n Y_i$, defined by a sequence $\{Y_n\}$ of independent and identically distributed random variables.
(a) Suppose there exists a positive solution γ to $\hat{m}_F(s) = 1$. Using (9.2.22) prove the Lundberg inequality

$$\mathbb{P}(\max\{0, S_1, S_2, \ldots\} > u) \le \mathrm{e}^{-\gamma u}, \qquad u \ge 0.$$

(b) Prove a version of Lundberg's inequality with $\gamma = \max\{s > 0 : \hat{m}_F(s) \le 1\}$ using Doob's inequality (9.1.41). [Hint. For (b) use the martingale $\{X_n\}$ with

$$X_n = \frac{\mathrm{e}^{sS_n}}{(\hat{m}_F(s))^n}$$

and notice that

$$\begin{aligned}\mathbb{P}(\max\{0, S_1, \ldots, S_n\} > u) &= \mathbb{P}\left(\max\left\{1, \mathrm{e}^{sS_1}, \ldots, \mathrm{e}^{sS_n}\right\} > \mathrm{e}^{su}\right)\\ &\le \mathbb{P}\Big(\max_{0\le k\le n}\frac{\mathrm{e}^{sS_k}}{[\hat{m}_F(s)]^k} > \mathrm{e}^{su}\Big)\end{aligned}$$

for $s \le \gamma$.]

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10

Martingale Techniques II

10.1 Continuous-Time Martingales

Exercises

10.1.1 Let $\mathcal{T}$ is an arbitrary set of parameters. Show that the intersection $\bigcap_{t\in\mathcal{T}} \mathcal{F}_t$ of any family of σ-algebras $\{\mathcal{F}_t, t \in \mathcal{T}\}$ is a σ-algebra.

10.1.2 Consider the claim surplus process $\{S(t)\}$ on its canonical probability space, i.e. restrict $\Omega = D(\mathbb{R}_+)$ to the set $\Omega_0 \subset \Omega$ of those functions from Ω which have only finitely many jumps in each bounded interval and which decrease linearly between the jumps. Show that on this probability space, the ruin time $\tau(u) = \min\{t \ge 0 : S(t) > u\}$ is an $\{\mathcal{F}_t^S\}$-stopping time.

10.1.3 Let $\{S(t)\}$ be the claim surplus process in the compound Poisson model. Show that $\mathbb{P}(\tau^*(u) \le x) = \mathbb{P}(\tau(u) \le x)$ and consequently $\mathbb{P}(\tau^*(u) < \infty) = \mathbb{P}(\tau(u) < \infty)$ for all $u, x \ge 0$.

10.1.4 Let $\{Y(t)\}$ be a process with stationary and independent increments such that $\mathbb{E}\,|Y(1)| < \infty$. Show that the process $\{X(t)\}$ with

$$X(t) = Y(t) - \mathbb{E}\,Y(1), \qquad t \ge 0$$

is a martingale with respect to the filtration $\{\mathcal{F}_t^Y\}$. [Hint. It suffices to show that $\mathbb{E}\,(X(t) \mid X(s_1), \ldots, X(s_n), X(s)) = X(s)$ whenever $0 \le s_1 < s_2 < \ldots < s_n < s < t$.]

10.1.5 (a) Let $\{\mathcal{F}_t\}$ be an arbitrary filtration and $\{\tau_n\}$ a sequence of $\{\mathcal{F}_t\}$-stopping times such that $\tau_n \uparrow \tau$. Show that τ is an $\{\mathcal{F}_t\}$-stopping time. (b) Assume that the filtration $\{\mathcal{F}_t\}$ is right-continuous and let $\{\tau_n\}$ be a sequence of $\{\mathcal{F}_t\}$-stopping times such that $\tau_n \downarrow \tau$. Show that τ is an $\{\mathcal{F}_t\}$-stopping time. [Hint. If $\tau_n \uparrow \tau$, then $\{\tau \le t\} = \bigcap_{n=1}^{\infty}\{\tau_n \le t\}$. Similarly, if $\tau_n \downarrow \tau$, then $\{\tau < t\} = \bigcup_{n=1}^{\infty}\{\tau_n < t\}$.]

10.1.6 Let $f : \mathbb{R}_+ \to \mathbb{R}$ be bounded and $\{X(t),\ t \geq 0\}$ a càdlàg process such that, with probability 1, the trajectories of $\{X(t)\}$ have locally bounded variation. For fixed $t \geq s \geq 0$, show that the sequence

$$\sum_{\lfloor ns+1 \rfloor \leq i \leq \lfloor nt \rfloor} f\Big(\frac{i-1}{n}\Big)\left(X\Big(\frac{i}{n}\Big) - X\Big(\frac{i-1}{n}\Big)\right), \qquad n = 1, 2, \ldots$$

is uniformly integrable.

10.1.7 For two $\{\mathcal{F}_t\}$-martingales $\{X(t)\}$ and $\{Y(t)\}$, show that the process $\{X(t) + Y(t)\}$ is also an $\{\mathcal{F}_t\}$-martingale.

10.1.8 Show that if the random variable Z is measurable with respect to $\mathcal{F}_0$ for some filtration $\{\mathcal{F}_t, t \geq 0\}$, and if $\mathbb{E}\,|Z| < \infty$, then the process $\{X(t)\}$ defined by $X(t) \equiv Z$ is an $\{\mathcal{F}_t\}$-martingale. Show also that if $\{X(t)\}$ is another $\{\mathcal{F}_t\}$-martingale which is independent of Z, then the process $\{ZX(t),\ t \geq 0\}$ is an $\{\mathcal{F}_t\}$-martingale provided $\mathbb{E}\,(X(0)Z) < \infty$.

10.1.9 Let $\{R(t)\}$ be a classical risk model such that $\mu_U^{(2)} < \infty$. Show that

$$\left(R(t) - \lambda\mu_U^{(1)}t\right)^2 - \lambda t \mu_U^{(2)}$$

is a martingale.

10.1.10 Let $\{W(t)\}$ be a standard Brownian motion. Show that the distribution of the random vector $(W(t_1), \ldots, W(t_n))$ is normal with mean vector zero and covariance matrix $(\mathbb{E}\,(W(t_i)W(t_j))) = (t_i \wedge t_j)$, for all $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, and $n = 1, 2, \ldots$.

10.1.11 Let $\{W(t)\}$ be a stochastic process fulfilling:

- $W(0) = 0$,
- the distribution of the random vector $(W(t_1), \ldots, W(t_n))$ is normal with $\mathbb{E}\,W(t_i) = 0$ and $\mathbb{E}\,(W(t_i)W(t_j)) = t_i \wedge t_j$, for all $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, and $n = 1, 2, \ldots$,
- the sample paths of $\{W(t)\}$ are continuous.

Show that $\{W(t)\}$ is a standard Brownian motion.

10.2 Some Fundamental Results

Exercises

10.2.1 Suppose that $\{X(t)\}$ is a submartingale, $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and nondecreasing, and $\mathbb{E}\,|\varphi(X(t))| < \infty$ for all $t \geq 0$. Show that then $\{X'(t)\}$ with $X'(t) = \varphi(X(t))$ is a submartingale. [Hint. Use Jensen's inequality for conditional expectation.]

10.2.2 Show: if τ is a stopping time, then the random variable $t \wedge \tau = \min\{t, \tau\}$ is a stopping time, too, for each $t \geq 0$.

10.2.3 Let τ be a stopping time. Show that the family of events $\mathcal{F}_\tau$ consisting of those $A \in \mathcal{F}$ for which $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$ is a σ-algebra.

10.2.4 Let $\tau^{(n)}$ be the stopping time defined in (10.2.9). Show that $\mathcal{F}_\tau \subset \mathcal{F}_{\tau^{(n)}}$.

10.2.5 Let τ be a stopping time and assume that there exist nonnegative numbers $\{t_1, t_2, \ldots\}$ such that $\mathbb{P}(\tau \in \{t_1, t_2, \ldots\}) = 1$. Show that

$$\mathbb{E}\,(X(t) \mid \mathcal{F}_\tau) = X(\tau \wedge t)\,.$$

[Hint. See also Exercise 9.2.2.]

10.2.6 Let $\{X(t), t \geq 0\}$ be a stochastic process with left-continuous trajectories. Assume that $\{X(t)\}$ has stationary and independent increments such that $\mathbb{E}\,|X(1)| < \infty$. Show that

$$\lim_{n \to \infty} \mathbb{E}\,(X(s_n) \mid X(t)) = \mathbb{E}\,(X(s) \mid X(t)) \tag{10.2.1}$$

for all $s, t \geq 0$ with $s \leq t$ and for each sequence $s_1, s_2, \ldots$ of nonnegative real numbers such that $s_n \leq s$ and $s_n \uparrow s$. [Hint. Use the fact that $\{X'(t), t \geq 0\}$ with $X'(t) = X(t) - t\mathbb{E}\,X(1)$ is a martingale. Then, proceeding similarly as in the proof of Lemma 10.2.1, show that the sequence $X(s_1), X(s_2), \ldots$ is uniformly integrable. Now, (10.2.1) follows from a well-known convergence theorem for conditional expectations, see e.g. the corollary on p.16 in Liptser and Shiryayev (1977).]

10.2.7 Let $\{X(t)\}$ be a right-continuous submartingale. Show: if $\{X(t)\}$ is a martingale or if $\{X(t)\}$ is bounded from below, then $\{X(t)\}$ belongs to the class DL.

10.3 Ruin Probabilities and Martingales

Exercises

10.3.1 Let $\{X(t)\}$ be an additive process. Show that the function $h_r(t) = \mathbb{E}\,\mathrm{e}^{rX(t)}$ is continuous for all $t \geq 0$.

10.3.2 Show that (10.3.3) and (10.3.4) hold if if $\{X(t)\}$ is the (μ, σ^2)-Brownian motion.

10.3.3 Let $\{X(t)\}$ be the $(-\mu, \sigma^2)$-Brownian motion with negative drift and $\tau(u) = \inf\{t \ge 0,\ X(t) > u\}$. Show that

$$\mathbb{P}(\tau(u) > x) = \Phi\Big(\frac{u - \mu x}{\sigma x^{1/2}}\Big) - \exp\Big(\frac{2\mu}{\sigma^2}u\Big)\Phi\Big(\frac{-u - \mu x}{\sigma x^{1/2}}\Big),\ x \ge 0$$

and that

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(\tau(u) \le x) = \frac{u}{\sigma\sqrt{2\pi x^3}}\exp\Big(-\frac{(u-\mu x)^2}{2\sigma^2 x}\Big),\ x \ge 0$$

where $\Phi(x)$ denotes the distribution function of the standard normal distribution. [Hint. Use the fact that the Laplace transform of

$$h(x) = \frac{k}{2\sqrt{\pi x^3}}\exp\Big(-\frac{k^2}{4x}\Big)$$

is given by

$$\int_0^\infty \mathrm{e}^{-sx} h(x)\,\mathrm{d}x = \exp(-k\sqrt{s})$$

for all $s > 0$; see e.g. the table of Laplace transforms in Korn and Korn (1968).]

10.3.4 Show that for each additive process $\{X(t)\}$, the following equality is true:

$$\mathbb{E}\,[\mathrm{e}^{rX(\tau(u))-\tau(u)g(r)}; \tau(u) < \infty] = 1,$$

where $\tau(u) = \inf\{t \ge 0 : X(t) > u\}$ and $g(r)$ is given by (10.3.1).

10.3.5 Consider the compound Poisson model with arrival rate λ, premium rate β and exponentially distributed claim sizes with parameter δ. Show that

(a) $$\gamma = \delta(1 - \lambda/(\delta\beta)),$$

(b) $$\psi(u) = \frac{\lambda}{\delta\beta}\mathrm{e}^{-\gamma u}, \qquad u \ge 0,$$

(c) $$\mathbb{E}\,[\mathrm{e}^{-s\tau(u)}; \tau(u) < \infty] = \Big(1 - \frac{g^{-1}(s)}{\delta}\Big)\mathrm{e}^{-ug^{-1}(s)}, \qquad s \ge 0,$$

where

$$g^{-1}(s) = \frac{-(s + \lambda - \beta\delta) + \sqrt{(s + \lambda - \beta\delta)^2 + 4\beta\delta}}{2\beta}.$$

[Hint. Use the result of Exercise 10.3.4 substituting $X(\tau(u)) = V(u) + u$, where $V(u) = X(\tau(u)) - u$ is the exponentially distributed overshoot which is independent of τ.]

10.3.6 Prove (10.3.29), i.e. show that for the Poisson compound model with arrival rate λ, premium rate β and claim size distribution F_U,

$$\psi(0;x) = 1 - \mathbb{E}\left(1 - \frac{Y(x)}{\beta x}\right)_+,$$

where $\{Y(t)\}$ is the left-continuous compound Poisson process defined by (10.3.27).

10.3.7 Consider the setting of Exercise 10.3.6 and assume that the claim size distribution is exponential, i.e. $F_U = \mathrm{Exp}(\delta)$ for some $\delta > 0$. Show that

$$\psi(u;x) \le 1 - \frac{1}{u+\beta x}\sum_{k=0}^{\infty}\frac{(\lambda x)^k}{k!}\mathrm{e}^{-\lambda x}\int_0^{u+\beta x}(1 - \mathrm{e}^{-\delta v}\sum_{i=0}^{k-1}\frac{(\delta v)^i}{i!})\,\mathrm{d}v.$$

10.3.8 Consider the setting of Exercise 10.3.6 and assume that $\lambda \mathbb{E}\, U\beta^{-1} < 1$. Show that

$$\psi(u) = \frac{\lambda \mathbb{E}\, U}{\beta} - \mathbb{E}\int_{\tau_\infty^0}^{\infty}\frac{Y(v)}{v}\frac{u}{(u+\beta v)^2}\,\mathrm{d}v,$$

where τ_∞^0 is the last time the risk reserve process $\{S(t)\}$ is above u; $\tau_\infty^0 = 0$ if $S(t) \le u$ for all $t \ge 0$.

11

Piecewise Deterministic Markov Processes

In this chapter E is a normed space with Borel σ-algebra $\mathcal{B}(E)$ and norm $\|\cdot\|$. By $M_{\mathrm{b}}(E)$ we denote the space of all bounded measurable real-valued functions on E. This space is endowed with the supremum norm. The collection of all probability measures on $\mathcal{B}(E)$ is denoted by $\mathcal{P}(E)$. $\{X(t)\}$ denotes a càdlàg stochastic process. If nothing else is mentioned we assume that E is the state space of $\{X(t)\}$.

11.1 Markov Processes with Continuous State Space

A function $P : \mathbb{R}_+ \times E \times \mathcal{B}(E) \to [0,1]$ is called a *transition kernel* if for all $h, h_1, h_2 \geq 0$, $x \in E$ and $B \in \mathcal{B}(E)$

$$\begin{aligned} P(h,x,\cdot) &\in \mathcal{P}(E)\,, \\ P(0,x,\{x\}) &= 1\,, \\ P(\cdot,\cdot,B) &\in M(\mathbb{R}_+ \times E)\,, \\ P(h_1+h_2,x,B) &= \int_E P(h_2,y,B)P(h_1,x,dy)\,. \end{aligned}$$

A Markov process is called *homogeneous* if there exists a transition kernel P such that $\mathbb{P}[X(t+h) \in B \mid \mathcal{F}_t^X] = P(h, X(t), B)$ for all $t, h \geq 0$ and $B \in \mathcal{B}(E)$.

Let $M \subset M_{\mathrm{b}}(E)$ be a subspace. A *semigroup* $\{\boldsymbol{T}(h)\}$ on M is a family of linear mappings from M to M such that $\boldsymbol{T}(0) = \boldsymbol{I}$ and $\boldsymbol{T}(h_1 + h_2) = \boldsymbol{T}(h_1)\boldsymbol{T}(h_2)$ for $h_1, h_1 \geq 0$. The semigroup is called *contraction semigroup* if $\|\boldsymbol{T}(h)g\| \leq \|g\|$ for all $h \geq 0$ and $g \in M$. A semigroup is called *stronglycontinuous* if $\lim_{h\downarrow 0} \boldsymbol{T}(h)g = g$ for all $g \in M$. There is a contraction semigroup $\{\boldsymbol{T}(h)\}$ connected to a homogeneous Markov process $\{X(t)\}$ with transition kernel P,

$$\boldsymbol{T}(h)g(x) = \int_E g(y)P(h,x,\mathrm{d}y) = \mathbb{E}\left(g(X(h)) \mid X(0) = x\right).$$

The *infinitesimal generator* $\boldsymbol{A}$ connected to a semigroup is the linear operator $\boldsymbol{A}g = \lim_{h\downarrow 0} h^{-1}(bfT(h) - \boldsymbol{I})g$. It is defined for all $g \in M$ for which the limit exists. The collection of functions g for which $\boldsymbol{A}g$ exists is denoted by $\mathcal{D}(\boldsymbol{A})$ and called the *domain* of the generator.

Exercises

11.1.1 Let $\{\mathcal{F}_t\}$ be a filtration such that $\{X(t)\}$ is adapted and $\{P(h,x,B)\}$ be a transition kernel. Show that the following are equivalent.
(a) For all $t, h \geq 0$, $B \in \mathcal{B}(E)$,

$$\mathbb{P}(X(t+h) \in B \mid \mathcal{F}_t) = P(h, X(t), B)\,.$$

(b) For all $t, h \geq 0$ and $g \in M_{\mathrm{b}}(E)$,

$$\mathbb{E}\left(g(X(t+h)) \mid \mathcal{F}_t\right) = \int_E g(y)P(h, X(t), \mathrm{d}y)\,.$$

11.1.2 Show that each E-valued process with independent and stationary increments is a homogeneous Markov process.

11.1.3 Let $\{\boldsymbol{T}(h), h \geq 0\}$ be a strongly continuous contraction semigroup. Show that the mapping $t \mapsto T(t)g$ is continuous in t.

11.1.4 Let $\{\boldsymbol{T}(h), h \geq 0\}$ be a strongly continuous contraction semigroup. Show that the Riemann integral $\int_0^t \boldsymbol{T}(v+h)g\,\mathrm{d}v$ exists for all $t, h \geq 0$.

11.1.5 Let $\{\boldsymbol{T}(t)\}$ be a strongly continuous contraction semigroup with generator $\boldsymbol{A}$. Let $g \in \mathcal{D}(\boldsymbol{A})$ and $t > 0$. Show that

$$\frac{\mathrm{d}^-}{\mathrm{d}t}\boldsymbol{T}(t)g = \boldsymbol{T}(t)\boldsymbol{A}g$$

where $\mathrm{d}^-/\mathrm{d}t$ denotes the derivative from the left.

11.1.6 Let $a < b$. Assume that $g : [a,b) \to M$ is a continuous function with a continuous derivative from the right $\mathrm{d}^+/\mathrm{d}tg(t)$.
(a) Show that if $\mathrm{d}^+/\mathrm{d}tg(t) = 0$ then $g(t) = g(a)$.
(b) Show that $\int_a^t \mathrm{d}^+/\mathrm{d}vg(v)\,\mathrm{d}v = g(t) - g(a)$.

11.1.7 Let $\{X(t)\}$ be a homogeneous continuous time Markov chain on $\{1, 2, \ldots, \ell\}$. Find its generator and the domain of the generator.

11.1.8 Suppose $\{W(t)\}$ is a standard Brownian motion and $X(t) = X(0) + W(t)$. Let g be a bounded, twice continuously differentiable function such that $g^{(2)} \in M_{\mathrm{b}}(\mathbb{R})$. Show that $g \in \mathcal{D}(\boldsymbol{A})$ and $\boldsymbol{A}g = \frac{1}{2}g^{(2)}$ where $\boldsymbol{A}$ is the infinitesimal generator of $\{X(t)\}$. [Hint. Use that

$$g(x+y) = g(x) + yg^{(1)}(x) + \tfrac{1}{2}y^2 g^{(2)}(x) + y^2\varepsilon(y)$$

where $\varepsilon(y)$ is a continuous bounded function converging to 0 as $y \to 0$.]

11.1.9 Let $\{\boldsymbol{T}(t)\}$ be a strongly continuous contraction semigroup with generator $\boldsymbol{A}$. Show that the set

$$\{(g, \boldsymbol{A}g) : g \in \mathcal{D}(\boldsymbol{A})\}$$

is closed.

11.1.10 Let $\{\boldsymbol{T}(t)\}$ be a strongly continuous contraction semigroup on M with generator $\boldsymbol{A}$. Show that $\mathcal{D}(\boldsymbol{A})$ is dense in M.

11.1.11 Let $\{\boldsymbol{T}(t)\}$ be a strongly continuous contraction semigroup on M with generator $\boldsymbol{A}$. Let $\lambda > 0$, $g \in M$ and $g' = \int_0^\infty \mathrm{e}^{-\lambda t} \boldsymbol{T}(t) g \, \mathrm{d}t$.
(a) Show that g' is well-defined.
(b) Show that $g' \in \mathcal{D}(\boldsymbol{A})$ and $\boldsymbol{A}g' = \lambda g' - g$.

Solutions

11.1.1 Assume (a) and let $\mathcal{M} \subset M_{\mathrm{b}}(E)$ be the class of functions satisfying the condition in (b). Clearly $\mathcal{M}$ is linear. By (a) all indicator functions $g(x) = \mathbb{I}(x \in B)$ belong to $\mathcal{M}$. Let $g_1 \le g_2 \le \ldots$ be an increasing sequence of functions with $g_n \in \mathcal{M}$. Assume $\{g_n\}$ converges pointwise to a function $g \in M_{\mathrm{b}}(E)$, i.e. $\lim_{n\to\infty} g_n(x) = g(x)$ for all $x \in E$. By bounded convergence $g \in \mathcal{M}$. By the monotone class theorem we have $M_{\mathrm{b}}(E) \subset \mathcal{M}$. That (a) follows from (b) is trivial.

11.1.2 This follows with the transition kernel

$$P(h, x, B) = \mathbb{P}(X_h - x \in B \mid X_0 = x)\,.$$

11.1.3 Let $h, t \ge 0$. Then

$$\boldsymbol{T}(h+t)g - \boldsymbol{T}(t)g = \boldsymbol{T}(t)\boldsymbol{T}(h)g - \boldsymbol{T}(t)g = \boldsymbol{T}(t)(\boldsymbol{T}(h)g - g)\,.$$

Thus because $\{\boldsymbol{T}(h)\}$ is a contraction semigroup

$$\|\boldsymbol{T}(h+t)g - \boldsymbol{T}(t)g\| \le \|\boldsymbol{T}(h)g - g\|\,.$$

The latter tends to zero because $\{\boldsymbol{T}(h)\}$ is strongly continuous. Thus $\{\boldsymbol{T}(h)\}$ is right continuous. Similarly, for $0 \le h \le t$

$$\|\boldsymbol{T}(t-h)g - \boldsymbol{T}(t)g\| \le \|\boldsymbol{T}(h)g - g\|$$

and also left continuity follows.

11.1.4 Recall that by Exercise 11.1.3 the mapping $v \mapsto \boldsymbol{T}(v+h)g$ is continuous and therefore uniformly continuous on $[0, t]$. Start with a special partition of the interval $[0, t]$, $t_{n,i} = 2^{-n} it$. Let $\varepsilon > 0$. Then there is n_0 such that $\|\boldsymbol{T}(v+h)g-\boldsymbol{T}(v'+h)g\|<\varepsilon$ for $|v-v'|\le 2^{-n_0}t$. Hence for $m\ge n\ge n_0$

$$\left\|t2^{-m}\sum_{i=1}^{2^m}\boldsymbol{T}(t_{m,i}+h)g-t2^{-n}\sum_{i=1}^{2^n}\boldsymbol{T}(t_{n,i}+h)g\right\|<t\varepsilon\,.$$

This shows that $\{t2^{-n}\sum_{i=1}^{2^n}\boldsymbol{T}(t_{n,i}+h)g\}$ is a Cauchy sequence and therefore convergent. Let now $t'_0=0<t'_1<\ldots<t'_n=t$ and $v'_i\in[t'_{i-1},t'_i]$. Assume that $\sup\{t'_i-t'_{i-1}\}<2^{-n_1}$ where $n_1=n_0+1$. Let $t''_0=0<t''_1<\ldots<t''_m=t$ such that $\{t''_0,t''_1,\ldots,t''_m\}$ contains $\{t'_i\}$ and $\{t_{i,n_1}\}$. Choose $v_i=t_{n_1,k}$ for $t''_{i-1}<t_{n_1,k}\le t''_i$ and $v''_i=v'_k$ for $t''_{i-1}<t'_k\le t''_i$. Then

$$t2^{-n_1}\sum_{i=1}^{2^{n_1}}\boldsymbol{T}(t_{n_1,i}+h)g=\sum_{i=1}^{m}(t''_i-t''_{i-1})\boldsymbol{T}(v_i+h)g$$

and

$$\sum_{i=1}^{n}(t'_i-t'_{i-1})\boldsymbol{T}(v'_i+h)g=\sum_{i=1}^{m}(t''_i-t''_{i-1})\boldsymbol{T}(v''_i+h)g\,.$$

From $|v''_i-v_i|\le t2^{-n_0}$ we claim that

$$\left\|t2^{-n_1}\sum_{i=1}^{2^{n_1}}\boldsymbol{T}(t_{n_1,i}+h)g-\sum_{i=1}^{n}\boldsymbol{T}(v'_i+h)g(t'_i-t'_{i-1})\right\|<t\varepsilon\,.$$

This show that the Riemann integral exists.

11.1.6 (a) This follows from $\|g(t+h)-g(t)\|\ge|\|g(t+h)\|-\|g(t)\||$ and therefore $\mathrm{d}^+/\mathrm{d}t\|g(t)\|=0$ for all t.
(b) It follows readily that $\mathrm{d}^+/\mathrm{d}t\int_a^t\mathrm{d}^+/\mathrm{d}vg(v)\,\mathrm{d}v=\mathrm{d}^+/\mathrm{d}tg(t)$. Thus the right derivative of $g(t)-\int_a^t\mathrm{d}^+/\mathrm{d}vg(v)\,\mathrm{d}v$ is zero. The result follows now from (a).

11.1.7 Let $\boldsymbol{Q}=(q_{ij})_{i,j=1,\ldots,\ell}$ be the intensity matrix of the Markov chain. Let $\mathbb{P}_i$ be the conditional measure if $X(0)=i$. Then for $g\in M_{\mathrm{b}}(\{1,\ldots,\ell\})$

$$\frac{1}{h}(\mathbb{E}_i g(X(h))-g(i))=-\frac{1-\mathbb{P}_i(X(h)=i)}{h}g(i)+\sum_{j\ne i}\frac{\mathbb{P}_i(X(h)=j)}{h}g(j)$$

and it follows that $g\in\mathcal{D}(\boldsymbol{A})$ with

$$\boldsymbol{A}g(i)=\sum_{j=1}^{\ell}q_{ij}g(j)\,.$$

Thus $\mathcal{D}(\boldsymbol{A})=M_{\mathrm{b}}(\{1,\ldots,\ell\})$.

11.1.8 It follows that

$$\mathbb{E}_x g(X(h)) - g(x) = \mathbb{E}\left(W(h)g^{(1)}(x) + \tfrac{1}{2}W^2(h)g^{(2)}(x) + W^2(h)\varepsilon(W(h))\right)$$

where $\mathbb{P}_x$ be the measure conditioned on $X(0) = x$. Because $\mathbb{E}\,W(h) = 0$ and $\mathbb{E}\,W^2(h) = h$ it remains to show that $h^{-1}\mathbb{E}\,(W^2(h)\varepsilon(W(h)))$ tends to zero. But this follows by bounded convergence.

11.1.9 Let $\{g_n\}$ be a sequence of functions from $\mathcal{D}(\boldsymbol{A})$ such that $\{g_n\}$ converges to a function g and $\{\boldsymbol{A}g_n\}$ converges to a function g'. Then $\boldsymbol{T}(t)g_n - g_n = \int_0^t \boldsymbol{T}(v)\boldsymbol{A}g_n(v)\,\mathrm{d}v$. Letting $n \to \infty$ gives $\boldsymbol{T}(t)g - g = \int_0^t \boldsymbol{T}(v)g'(v)\,\mathrm{d}v$. Dividing by t and letting $t \to 0$ shows that $g \in \mathcal{D}(\boldsymbol{A})$ and $\boldsymbol{A}g = g'$.

11.1.10 Let $g \in M$. We have $\int_0^t \boldsymbol{T}(v)g\,\mathrm{d}v \in \mathcal{D}(\boldsymbol{A})$ and therefore also $t^{-1}\int_0^t \boldsymbol{T}(v)g\,\mathrm{d}v \in \mathcal{D}(\boldsymbol{A})$. Letting $t \to 0$ the latter converges to to $\boldsymbol{T}(0)g$, i.e. $g \in \overline{\mathcal{D}(\boldsymbol{A})}$.

11.1.11 (a) We have

$$\left\| \int_t^\infty \mathrm{e}^{-\lambda v}\boldsymbol{T}(v)g\,\mathrm{d}v \right\| \le \|g\|\lambda^{-1}\mathrm{e}^{-\lambda t}\,.$$

Thus $\lim_{t\to\infty}\int_0^t \mathrm{e}^{-\lambda v}\boldsymbol{T}(v)g\,\mathrm{d}v$ exists.
(b) For $h > 0$ we have

$$\begin{aligned}
\frac{\boldsymbol{T}(h)g' - g'}{h} &= \frac{\int_0^\infty \mathrm{e}^{-\lambda t}\boldsymbol{T}(t+h)g\,\mathrm{d}t - \int_0^\infty \mathrm{e}^{-\lambda t}\boldsymbol{T}(t)g\,\mathrm{d}t}{h} \\
&= \frac{\mathrm{e}^{\lambda h}\int_0^\infty \mathrm{e}^{-\lambda(t+h)}\boldsymbol{T}(t+h)g\,\mathrm{d}t - \int_0^\infty \mathrm{e}^{-\lambda t}\boldsymbol{T}(t)g\,\mathrm{d}t}{h} \\
&= \frac{\mathrm{e}^{\lambda h} - 1}{h}g' - \mathrm{e}^{\lambda h}\frac{1}{h}\int_0^h \mathrm{e}^{-\lambda t}\boldsymbol{T}(t)g\,\mathrm{d}t\,.
\end{aligned}$$

Letting $t \to 0$ gives the result.

11.2 Construction and Properties of PDMP

Let I be a finite set and C_ν be an open subset of $\mathbb{R}^{d_\nu}$ for $\nu \in I$ and some number $d_\nu \in \mathbb{N}$. The state space is $E = \{(\nu, z) : \nu \in I, z \in C_\nu\}$ endowed with the usual topology. A *piecewise deterministic Markov process (PDMP)* $\{X(t)\}$ on E is of the form $X(t) = (J(t), Z(t))$. Between jumps its paths are determined by a *vector field* $\boldsymbol{X}$. Let $t \mapsto \varphi_\nu(t, z)$ be the corresponding *integral* *curves*. We normalize the integral curves such that $\varphi_\nu(0,z) = z$. We assume that through every point (ν, z) there is exactly one integral curve. By

$$t^*(\nu,z) = \sup\{t > 0 : \varphi_\nu(t,z) \text{ exists and } \varphi_\nu(t,z) \in C_\nu\}$$

we denote the maximal time for which the integral curve is defined. The *active boundary* Γ is defined from

$$\begin{aligned} \partial^* C_\nu &= \{\tilde{z} \in \partial C_\nu : \tilde{z} = \varphi_\nu(t,z) \text{ for some } (t,z) \in \mathbb{R}^+ \times C_\nu\}, \\ \Gamma &= \{(\nu,z) \in \partial E : \nu \in I, z \in \partial^* C_\nu\}. \end{aligned}$$

We assume that $\varphi_\nu(t^*(\nu,z),z) \in \Gamma$ if $t^*(\nu,z) < \infty$. We use the natural filtration of $\{X(t)\}$, i.e. $\mathcal{F}_t = \mathcal{F}_t^X$.

Let $\sigma_0 = 0$. The path of $\{X(t)\}$ is constructed recursively. If for some $n \in \mathbb{N}$ the path is constructed until σ_n and $X(\sigma_n) = (\nu, z)$ then

$$\mathbb{P}(\sigma_{n+1} > \sigma_n + t \mid \mathcal{F}_{\sigma_n}) = \exp\Big(-\int_0^t \lambda(\nu, \varphi_\nu(v,z))\, \mathrm{d}v\Big)\, \mathbb{I}(t < t^*(\nu,z))$$

for some measurable integrable function $\lambda : E \to \mathbb{R}_+$ For $\sigma_n \le t < \sigma_{n+1}$ we have $X(t) = \varphi_\nu(t - \sigma_n, z)$. Finally

$$\mathbb{P}(X(\sigma_{n+1}) \in \cdot \mid \mathcal{F}_{\sigma_n}, \sigma_{n+1}) = Q(\varphi_\nu(\sigma_{n+1} - \sigma_n, z), \cdot)$$

for some transition measure $Q : (E \cup \gamma) \times \mathcal{B}(E) \to [0,1]$. Let $N(t) = \sum_{i=1}^\infty \mathbb{I}(\sigma_i \le t)$ denote the number of jumps in $(0,t]$. We assume that $\mathbb{E}\, N(t) < \infty$ for all $t \ge 0$. The PDMP $\{X(t)\}$ is a strong Markov process. Let $\boldsymbol{A}$ be the corresponding (full) generator.

Exercises

11.2.1 Show that $\mathbb{E}\, N(t) < \infty$ holds if the jump intensity $\lambda(x)$ is bounded and if one of the following conditions is fulfilled: (b_1) $t^*(x) = \infty$ for each $x \in E$, or (b_2) for some $\varepsilon > 0$ we have $Q(x, B_\varepsilon) = 1$ for all $x \in \Gamma$, where $B_\varepsilon = \{x \in E : t^*(x) \ge \varepsilon\}$.

11.2.2 Show from the construction of the PDMP $\{X(t)\}$ that $\{X(t)\}$ is a Markov process.

11.2.3 Assume $\mathbb{E}\, N(t) < \infty$. Let $g^* : E \cup \Gamma \to \mathbb{R}$ be a bounded function. Show that

$$\mathbb{E}\Big(\sum_{i:\sigma_i \le t} |g^*(X(\sigma_i)) - g^*(X(\sigma_i-))|\Big) < \infty$$

for all $t \ge 0$.

11.2.4 Let $\{X(t)\}$ be a PDMP with full generator $\boldsymbol{A}$. Let $g \in \mathcal{D}(\boldsymbol{A})$. Show that g is absolutely continuous along integral paths.

11.2.5 Let $\{X(t)\}$ be a PDMP with full generator $\boldsymbol{A}$. Suppose $\Gamma \neq \emptyset$ and let $z' \in \Gamma$. Let ν and z such that $z' = \varphi_\nu(t^*(\nu, z), z)$. Let $g \in \mathcal{D}(\boldsymbol{A})$. Show that

$$\lim_{t \uparrow t^*(\nu,z)} g(\varphi_\nu(t,z)) = \int_E g(y) Q(z', \mathrm{d}y)\,.$$

11.2.6 Show that the risk process $\{R(t)\}$ and the claim surplus process $\{S(t)\}$ in the compound Poisson model are PDMP. Determine the state space E, the vector fields $\{\boldsymbol{c}_\nu, \nu \in I\}$, the jump intensity $\lambda(x)$ and the transition kernel Q of these PDMP.

11.2.7 Let $\{X(t)\}$ be a nonhomogeneous Markov chain with matrix intensity function $\{\boldsymbol{Q}(t) = (q_{ij}(t))_{i,j \in E}\}$. Assume that $\|\boldsymbol{Q}(t)\| = \max_{i \in E} |q_{ii}(t)|$ is integrable on finite intervals. Show that $\mathbb{E}\, N(t) < \infty$, where $N(t)$ denotes the number of jumps in the interval $(0, t]$.

11.2.8 Let $\{X(t)\}$ be a nonhomogeneous Markov chain with matrix intensity function $\{\boldsymbol{Q}(t) = (q_{ij}(t))_{i,j \in E}\}$. The process $\{X(t)\}$ is then a PDMP. Assume that $\|\boldsymbol{Q}(t)\| = \max_{i \in E} |q_{ii}(t)|$ is integrable on finite intervals. Therefore $\mathbb{E}\, N(t) < \infty$ by Exercise 11.2.7. Let $N_{ij}(t)$ be the number of transitions from i to j in the interval $(0,t]$. We denote by $p_{ij}(t,t') = \mathbb{P}(X(t') = j \mid X(t) = i)$. The process $\{M_{ij}(t)\}$ defined by $M_{ij}(t) = N_{ij}(t) - \int_0^t \mathbb{I}(X(v) = i) q_{ij}(v)\,\mathrm{d}v$ is then a martingale. Show that

$$\mathbb{E}\Big(\int_t^{t'} a(v)\,\mathrm{d}N_{ij}(v) \;\Big|\; X(t) = k\Big) = \int_t^{t'} a(v) p_{ki}(t,v) q_{ij}(v)\,\mathrm{d}v \quad (11.2.1)$$

for any locally integrable real function a.

11.2.9 Let $\{X(t)\}$ be a PDMP with $I = \{1, 2, \ldots, \ell\}$. Let $g_\nu : C_\nu \to \mathbb{R}$ be a real measurable function for each $\nu \in I$. Let $\tau = \inf\{t \geq 0 : g_{J(t)}(Z(t)) \leq 0\}$. Consider the process $\{X'(t)\}$ with $X'(t) = (J(t)\mathbb{I}(\tau > t), Z(t))$. Show that $\{X'(t)\}$ is a PDMP and find its characteristics.

11.2.10 Let $\{X(t)\}$ be a nonhomogeneous Markov chain on $\{1,2,3\}$, where 3 is an absorbing state. Let $\{\sigma_i\}$ be the jump times, $\sigma_0 = 0$ and $\{Y(t)\}$ be defined as $Y(t) = t - \sigma_{N(t)}$ if $N(t) \neq 0$ where $N(t)$ is the number of jumps in $(0,t]$. If $N(t) = 0$ we allow more generally $Y(t) = Y(0) + t$. Let $t_0 > 0$ be fixed. Define the functions $g_1^0(t) = g_2^0(t) = 0$ and recursively,

$$\begin{aligned} g_1^{n+1}(t) &= \mathbb{E}\,(g_2^n(\sigma \wedge t_0, 0)\,\mathbb{I}(J(\sigma) = 2) \mid J(t) = 1)\,, && (11.2.2)\\ g_2^{n+1}(t,y) &= \mathbb{E}\,((\sigma \vee y_0 - y \vee y_0) + g_1^n(\sigma \wedge t_0) \\ &\qquad \times \mathbb{I}(J(\sigma) = 1) \mid J(t) = 2, Y(t) = y) && (11.2.3) \end{aligned}$$

where $\sigma = \sigma_{N(t)+1}$ is the time of the next jump. Show that

$$g_1^n(t) = \mathbb{E}\left(\int_t^{\sigma_n \wedge t_0} \mathbb{I}(J(v) = 2, Y(v) \geq y_0)\,\mathrm{d}v \;\middle|\; J(t) = 1, N(t) = 0\right)$$

and

$$\begin{aligned} g_2^n(t,y) \quad &= \quad \mathbb{E}\left(\int_t^{\sigma_n \wedge t_0} \mathbb{I}(J(v) = 2, Y(v) \geq y_0)\,\mathrm{d}v \;\middle|\right. \\ &\qquad\qquad \left. J(t) = 2, Y(t) = y, N(t) = 0\right). \end{aligned}$$

11.2.11 Let $g_1^n(t)$ and $g_2^n(t)$ be the functions defined in (11.2.2) and (11.2.3) of Exercise 11.2.10. Show that for $n \in \mathbb{N}$ these functions fulfil

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} g_1^{n+1}(t) + \lambda_{12}(t) g_2^n(t) - (\lambda_{12}(t) + \lambda_{13}(t)) g_1^{n+1}(t) = 0\,, \\ &\frac{\partial g_2^{n+1}}{\partial t}(t,y) + \frac{\partial g_2^{n+1}}{\partial y}(t,y) + \lambda_{21}(t,y) g_1^n(t) \\ &\qquad - (\lambda_{21}(t,y) + \lambda_{23}(t,y)) g_2^{n+1}(t,y) + \mathbb{I}(y \geq y_0) = 0\,, \end{aligned}$$

where $\lambda_{ij}(t)$ are the transition intensities.

Solutions

11.2.1 Let $\overline{\lambda} = \sup_{x\in E} \lambda(x)$.
(b_1) Let $\{N'(t)\}$ be a Poisson process with rate $\overline{\lambda}$ and $\{\sigma_n'\}$ be the corresponding jump times. Then by construction of the PDMP $\mathbb{P}(\sigma_n' > t) \leq \mathbb{P}(\sigma_n > t)$ and therefore $\mathbb{P}(N'(t) \leq n) \geq \mathbb{P}(N(t) \leq n)$. Thus $\mathbb{E}\,N(t) \leq \mathbb{E}\,N'(t) < \infty$.
(b_2) Let $T_n = \sigma_n - \sigma_{n-1}$. Let $\{T_n'\}$ be independent random variables with $\mathbb{P}(T_n' > t) = \mathbb{I}(t < \varepsilon)\mathrm{e}^{-\overline{\lambda}t}$. If $X(\sigma_{n-1}) \in B_\varepsilon$ then

$$\mathbb{P}(T_n' > t) \leq \mathbb{P}(T_n > t \mid X(\sigma_{n-1}) \in B_\varepsilon)\,.$$

Let $\{N'(t)\}$ be a renewal process with interarrival times $\{T_n'\}$. Because $Q(x, B_\varepsilon) = 1$ for $x \in \Gamma$ we find

$$\mathbb{E}\,N(t) \leq 2\mathbb{E}\,N'(t) < \infty\,.$$

11.2.2 First we note that $\varphi_\nu(h, \varphi_\nu(t,z)) = \varphi(t+h, z)$ and therefore we have $t^*(\nu, \varphi_\nu(t,z)) = t^*(\nu,z) - t$. Let $t \geq 0$. Define $\sigma' = \sigma_{N(t)}$ and $\sigma'' = \sigma N(t) + 1$. Then for $(\nu, z) = X(t)$ and $(\nu, z') = X(\sigma')$ we find

$$\mathbb{P}(\sigma'' > t + v \mid \mathcal{F}_t) = \mathbb{P}(\sigma'' - \sigma' > t - \sigma' + v \mid \mathcal{F}_{\sigma'}, \sigma'' - \sigma' > t - \sigma')$$

$$\begin{aligned}
&= \frac{\exp(-\int_0^{t-\sigma'+v} \lambda(\nu, \varphi(w, z'))\, \mathrm{d}w)\, \mathbb{I}(t - \sigma' + v < t^*(\nu, z'))}{\exp(-\int_0^{t-\sigma'} \lambda(\nu, \varphi(w, z'))\, \mathrm{d}w)} \\
&= \exp\Bigl(-\int_{t-\sigma'}^{t-\sigma'+v} \lambda(\nu, \varphi(w, z'))\, \mathrm{d}w\Bigr)\, \mathbb{I}(t - \sigma' + v < t^*(\nu, z')) \\
&= \exp\Bigl(-\int_0^{v} \lambda(\nu, \varphi(w, z))\, \mathrm{d}w\Bigr)\, \mathbb{I}(v < t^*(\nu, z))\,.
\end{aligned}$$

By this fact and the construction of the PDMP we have

$$\begin{aligned}
&\mathbb{P}(X(t+v) \in B \mid \mathcal{F}_t) = \mathbb{E}\,(\mathbb{P}(X(t+v) \in B \mid \sigma'', X(\sigma''), \mathcal{F}_t) \mid \mathcal{F}_t) \\
&\quad = \mathbb{E}\,(\mathbb{P}(X(t+v) \in B \mid \sigma'', X(\sigma''), X(\sigma')) \mid \mathcal{F}_t) \\
&\quad = \mathbb{E}\,(\mathbb{P}(X(t+v) \in B \mid \sigma'', X(\sigma')) \mid \mathcal{F}_t) \\
&\quad = \mathbb{E}\,(\mathbb{P}(X(t+v) \in B \mid X(t)) \mid \mathcal{F}_t) \\
&\quad = \mathbb{P}(X(t+v) \in B \mid X(t))\,.
\end{aligned}$$

Therefore $\{X(t)\}$ is a Markov process.

11.2.4 Let $(\nu, z) \in E$ and $X(0) = (\nu, z)$. Choose $t < t^*(\nu, z)$. For simplicity write $\lambda(v) = \lambda(\nu, \varphi_\nu(v, z))$ and $x(v) = (\nu, \varphi_\nu(v, z))$. Then by the martingale property

$$g(\nu, z) = \mathbb{E}\,\Bigl(g(X(\sigma_1 \wedge t)) - \int_0^{\sigma_1 \wedge t} \boldsymbol{A} g(x(v))\, \mathrm{d}v\Bigr)\,.$$

The first term on the right hand side can be written as

$$\begin{aligned}
&g(x(t)) \exp\Bigl(-\int_0^t \lambda(v)\, \mathrm{d}v\Bigr) \\
&\qquad + \int_0^t \lambda(v) \exp\Bigl(-\int_0^v \lambda(w)\, \mathrm{d}w\Bigr) \int_E g(y) Q(x(v), \mathrm{d}y)\, \mathrm{d}v\,.
\end{aligned}$$

The second term is

$$\begin{aligned}
&-\int_0^t \boldsymbol{A} g(x(v))\, \mathrm{d}v \exp\Bigl(-\int_0^t \lambda(v)\, \mathrm{d}v\Bigr) \\
&\qquad - \int_0^t \lambda(v) \exp\Bigl(-\int_0^v \lambda(w)\, \mathrm{d}w\Bigr) \int_0^v \boldsymbol{A} g(x(w))\, \mathrm{d}w\, \mathrm{d}v\,.
\end{aligned}$$

Putting these integrals together shows that the function

$$t \mapsto g(x(t)) \exp\Bigl(-\int_0^t \lambda(v)\, \mathrm{d}v\Bigr)$$

is absolutely continuous. Thus $t \mapsto g(x(t))$ is absolutely continuous.

11.2.5 For simplicity write $\lambda(t) = \lambda(\nu, \varphi_\nu(t,z))$, $x(t) = (\nu, \varphi_\nu(t,z))$ and $t^* = t^*(x(0))$. Let $0 \le t < t^*$. Then by the martingale property and noting that $\sigma_1 \le t^*$

$$g(x(t)) = \mathbb{E}\left(g(X(\sigma_1)) - \int_t^{\sigma_1} \boldsymbol{A}g(x(v))\,\mathrm{d}v \;\middle|\; \sigma_1 > t\right).$$

The right hand side can be expressed similarly as in Exercise 11.2.4. Letting $t \uparrow t^*$ gives the result.

11.2.6 The state space is $E = \{0\} \times \mathbb{R}$ and the jump intensity is $\lambda(z) = \lambda$ where λ is the claim arrival intensity. For $\{R(t)\}$ the vector field is $\boldsymbol{X} = \beta\,\mathrm{d}/\mathrm{d}z$ where β is the premium rate. And the transition kernel is given by $Q(x,B) = \mathbb{P}(x - U \in B)$ where U is a generic variable for the claims sizes. For $\{S(t)\}$ the vector field is $\boldsymbol{X} = -\beta\,\mathrm{d}/\mathrm{d}z$ and the transition kernel is given by $Q(x,B) = \mathbb{P}(x + U \in B)$.

11.2.7 Denote the jump times of $\{X(t)\}$ by $\{\sigma_k\}$. Let $\{N'(t)\}$ be a non-homogeneous Poisson process with intensity function $\|\boldsymbol{Q}(t)\|$ and jump times $\{\sigma'_k\}$. Assume $X(\sigma_k) = i$. Then

$$\begin{aligned}\mathbb{P}(\sigma_{k+1} - \sigma_k > t \mid \mathcal{F}_{\sigma_k}) &= \exp\Bigl(-\int_0^t q_{ii}(\sigma_k + v)\,\mathrm{d}v\Bigr)\\ &\ge \exp\Bigl(-\int_0^t \|\boldsymbol{Q}(\sigma_k + v)\|\,\mathrm{d}v\Bigr).\end{aligned}$$

In particular, $\mathbb{P}(\sigma_1 \le t) \le \mathbb{P}(\sigma'_1 \le t)$. By induction it follows that $\mathbb{P}(\sigma_k \le t) \le \mathbb{P}(\sigma'_k \le t)$. This means $\mathbb{P}(N(t) > n) \le \mathbb{P}(N'(t) > n)$ and

$$\mathbb{E}\,N(t) \le \mathbb{E}\,N'(t) = \int_0^t \|\boldsymbol{Q}(v)\|\,\mathrm{d}v < \infty.$$

11.2.8 Assume first that a is an indicator function. It is no loss of generality to assume $a(v) = \mathbb{I}(t'' < a \le t')$ for some $t'' \in [t, t')$. The case $t'' = t$ follows readily from the martingale property of $\{M_{ij}(v)\}$. For $t'' > t$ one gets

$$\begin{aligned}&\mathbb{E}\,(N_{ij}(t') - N_{ij}(t'') \mid X(t) = k)\\ &\quad= \sum_{\ell \in E} \mathbb{E}\,(N_{ij}(t') - N_{ij}(t'') \mid X(t'') = \ell)\mathbb{P}(X(t'') = \ell \mid X(t) = k)\\ &\quad= \sum_{\ell \in E} \int_{t''}^{t'} p_{\ell i}(t'', v) q_{ij}(v)\,\mathrm{d}v\, p_{k\ell}(t, t'')\\ &\quad= \int_{t''}^{t'} p_{ki}(t, v) q_{ij}(v)\,\mathrm{d}v.\end{aligned}$$

It is clear that the space of all functions fulfilling (11.2.1) is linear. Let $\{a_n(v)\}$ be an increasing sequence of bounded measurable functions such that the pointwise limit $a(v) = \lim_{n\to\infty} a_n(v)$ is a bounded function. By bounded convergence (11.2.1) holds for $a(v)$. Thus by the monotone class theorem, (11.2.1) holds for all bounded measurable functions. If a is a locally integrable positive function and $a_n(v) = a(v) \wedge n$ then it follows by monotone convergence that (11.2.1) holds. For an arbitrary locally integrable function the result follows from $a(v) = (a(v))_+ - (a(v))_-$.

11.2.10 The result is trivial for $n = 0$. Let now $n \in \mathbb{N}$ and assume the result holds for n. Then

$$\begin{aligned} g_1^{n+1}(t) &= \mathbb{E}\,(g_2^n(\sigma_1 \wedge t_0, 0)\,\mathbb{I}(J(\sigma_1) = 2) \mid J(t) = 1, N(t) = 0) \\ &= \mathbb{E}\Big(\mathbb{E}\Big(\int_{\sigma_1\wedge t_0}^{\sigma_{n+1}\wedge t_0} \mathbb{I}(J(v) = 2, Y(v) \geq y_0)\,\mathrm{d}v \;\Big| \\ &\qquad J(\sigma_1) = 2, N(t) = 0\Big) \;\Big|\; J(t) = 1, N(t) = 0\Big) \\ &= \mathbb{E}\Big(\int_{\sigma_1\wedge t_0}^{\sigma_{n+1}\wedge t_0} \mathbb{I}(J(v) = 2, Y(v) \geq y_0)\,\mathrm{d}v \;\Big|\; J(t) = 1, N(t) = 0\Big) \\ &= \mathbb{E}\Big(\int_{t}^{\sigma_{n+1}\wedge t_0} \mathbb{I}(J(v) = 2, Y(v) \geq y_0)\,\mathrm{d}v \;\Big|\; J(t) = 1, N(t) = 0\Big) \end{aligned}$$

and the result for $g_2^{n+1}(t, y)$ follows analogously.

11.2.11 Consider the PDMP with $I = \{1, 2, 3\} \times \{0, 1, \ldots, n + 1\}$, $C_{j,j'} = \{0\}$ if $j \neq 2$, $C_{2,j'} = \mathbb{R}_+$. $J'(0) = n+1$ and $J'(\sigma_k) = n+1-k$, $J(t) = X(t)$, where $\{X(t)\}$ is the Markov chain of Exercise 11.2.10. If $J(t) = 2$ then $Z(t) = Y(t)$, as defined in Exercise 11.2.10, and $Z(t) = 0$ otherwise. The quantity to find is

$$\mathbb{E}\Big(\int_0^{\sigma_{n+1}\wedge t_0} \mathbb{I}(J(v) = 2, Y(v) \geq y_0)\,\mathrm{d}v \;\Big|\; N(t) = 0\Big)\,.$$

The result is now a direct application of Theorem 11.2.3.

11.3 The Compound Poisson Model Revisited

In this section we consider the compound Poisson model $\{R(t)\}$ with initial capital u, premium rate β, claim intensity λ and claim size distribution F_U with mean $\mu = \mu_U$. By $\tau(u)$ we denote the ruin time and by $\psi(u)$ the ruin probability. Letting $\theta(s) = \lambda(\hat{m}_U(s) - 1) - \beta s$ we get the martingales $\{M(t)\}$ with

$$M(t) = \exp(-s(R(t) - u) - \theta(s)t)\,,$$

provided $\hat{m}_U(s) < \infty$. We define the new measures

$$\mathbb{P}^{(s)}(A) = \mathbb{E}\,[M(t); A]\,, \qquad A \in \mathcal{F}_t\,.$$

The measure $\mathbb{P}^{(s)}$ can be extended to a measure on $\mathcal{F}$ and $\{R(t)\}$ is a compound Poisson model under $\mathbb{P}^{(s)}$. The claim intensity is $\lambda^{(s)} = \lambda \hat{m}_U(s)$ and the claim size distribution is $F_U^{(s)}(x) = \int_0^x \mathrm{e}^{sy}\,\mathrm{d}F_U(y)/\hat{m}_U(s)$. A possible solution $\gamma > 0$ to $\theta(s) = 0$ is called the adjustment coefficient. If nothing else is said we assume the net profit condition $\beta > \lambda\mu$.

Exercises

11.3.1 Suppose $g(u)$ is absolutely continuous, $g(0) = 0$ and fulfils

$$\beta g^{(1)}(u) + \lambda\Big(\int_0^y g(u-y)\,\mathrm{d}F(y) - g(u)\Big) = 0 \tag{11.3.1}$$

where F is a distribution function with $F(0) = 0$ and $\beta, \lambda > 0$. We do not assume a net profit condition or finite mean. Show that $g(y) \equiv 0$. [Hint. Integrate (11.3.1) first.]

11.3.2 Let $\{L(t)\}$ be a martingale with $\mathbb{E}\,L(0) = 1$. Let $\tilde{\mathbb{P}}$ be the measure with $\tilde{\mathbb{P}}(A) = \mathbb{E}\,[L(t); A]$ for $A \in \mathcal{F}_t$ and assume that $\tilde{\mathbb{P}}$ can be extended to a measure on $\mathcal{F}$. Show that $\{L^{-1}(t)\}$ is a martingale under $\tilde{\mathbb{P}}$ with $\tilde{\mathbb{E}}\,L^{-1}(t) = 1$.

11.3.3 Show that for each $s \in \mathbb{R}$ such that $\hat{m}_U(s) < \infty$ we have

$$\psi(u) = \mathbb{E}^{(s)}[\mathrm{e}^{sR(\tau(u))+\theta(s)\tau(u)}; \tau(u) < \infty]\mathrm{e}^{-su}$$

and

$$\psi(u;x) = \mathbb{E}^{(s)}[\mathrm{e}^{sR(\tau(u))+\theta(s)\tau(u)}; \tau(u) \le x]\mathrm{e}^{-su}\,.$$

11.3.4 Assume $c < \lambda\mu$ and $u = 0$. Let $X_+ = R(\tau-)$ and $Y_+ = -R(\tau)$. Consider the solution $\gamma_0 < 0$ to $\theta(s) = 0$. Show that γ_0 exists and

$$\mathbb{P}(X_+ > x, Y_+ > y) = \frac{\lambda}{c}\int_x^\infty \overline{F}_U(z+y)\mathrm{e}^{\gamma_0 z}\,\mathrm{d}z\,.$$

11.3.5 Let $\hat{l}_\tau(s)$ be the Laplace transform of τ and $\eta_w = \mathbb{E}\int_0^\tau \mathrm{e}^{-wv}\,\mathrm{d}v$. Show that for $u \ge 0$ and $w > 0$ we have $\eta_w = w^{-1}(1 - \hat{l}_\tau(w))$.

11.3.6 Let F be a distribution function with $F(0) = 0$ and $w > 0$. Consider the integro-differential equation

$$\beta g^{(1)}(u) + \lambda\Big(\int_0^u g(u-v)\,\mathrm{d}F(v) - g(u)\Big) - wg(u) + 1 = 0\,, \tag{11.3.2}$$

where $g : \mathbb{R}_+ \to \mathbb{R}$, and the derivative is in the sense of an absolutely continuous function. Show that there is at most one solution to (11.3.2) with $g(0) = g_0$ for each $g_0 \in \mathbb{R}$.

11.3.7 Let F be a distribution function with $F(0) = 0$ and $w > 0$. Show that there is at most one bounded solution to (11.3.2).

11.3.8 Assume $\mu_U^{(2)} = \infty$. Show that $\mathbb{E}[\tau(u); \tau(u) < \infty] = \infty$.

11.3.9 Consider the compound Poisson model with exponential claim size distribution $\mathrm{Exp}(\delta)$. Show that

$$\mathbb{E}(\tau(u) \mid \tau(u) < \infty) = \frac{\beta + \lambda u}{\beta(\beta\delta - \lambda)}.$$

Solutions

11.3.1 Integration of (11.3.1) yields

$$\beta g(u) = \lambda \int_0^u g(u-y)\overline{F}(y)\,\mathrm{d}y\,,$$

see also the proof of Theorem 5.3.2. Choose $u_0 > 0$ such that $\beta > \lambda \int_0^{u_0} \overline{F}(y)\,\mathrm{d}y$. Assume $g(v) = 0$ for $v \in [0, ku_0]$ for some $k \in \mathbb{N}$. Then

$$\beta g(u) = \lambda \int_0^{u-ku_0} g(u-y)\overline{F}(y)\,\mathrm{d}y\,.$$

Let $u \in [ku_0, (k+1)u_0]$ such that $|g(u)| = \sup_{v\in[0,(k+1)u_0]} |g(v)|$. Then

$$\beta|g(u)| \le \lambda \int_0^{u-ku_0} |g(u-y)|\overline{F}(y)\,\mathrm{d}y \le |g(u)|\lambda \int_0^{u_0} \overline{F}(y)\,\mathrm{d}y\,,$$

yielding $|g(u)| = 0$. This gives $g(v) = 0$ for $v \in [0, (k+1)u_0]$. By induction, $g(v) = 0$ for all $v \ge 0$.

11.3.2 Let $t > v \ge 0$. Then

$$\tilde{\mathbb{E}}(L^{-1}(t) \mid \mathcal{F}_v) = \frac{\mathbb{E}(L^{-1}(t)L(t)\mathcal{F}_v)}{L(v)} = L^{-1}(v)\,.$$

That $\tilde{\mathbb{E}}\,L^{-1}(t) = 1$ follows analogously.

11.3.4 Because $\hat{m}_U(s) < \infty$ for all $s \le 0$, $\theta^{(1)}(0) > 0$ and $\lim_{s\to-\infty} \theta(s) = \infty$ it follows that γ_0 exists. Under the measure $\mathbb{P}^{(\gamma_0)}$ the process $\{R(t)\}$ has a positive drift. Indeed, $\beta - \lambda^{(\gamma_0)}\mu_{U^{(\gamma_0)}} = -\theta^{(1)}(\gamma_0) > 0$. The probability we are looking for can be expressed as

$$\mathbb{P}(X_+ > x, Y_+ > y) = \mathbb{E}^{(\gamma_0)}[\mathrm{e}^{\gamma_0 R(\tau)}; \tau < \infty, X_+ > x, Y_+ > y]\,.$$

We know that

$$\mathbb{P}^{(\gamma_0)}(\tau < \infty, X_+ > x, Y_+ > y) = \frac{\lambda^{(\gamma_0)}}{\beta}\int_{x+y}^{\infty} \overline{F_U^{(\gamma_0)}}(z)\,\mathrm{d}z\,.$$

This yields

$$\begin{aligned}\mathbb{P}(X_+ > x, Y_+ > y) &= \frac{\lambda^{(\gamma_0)}}{\beta}\int_x^\infty\int_y^\infty \mathrm{e}^{-\gamma_0 y'} F_U^{(\gamma_0)}(x' + \mathrm{d}y')\,\mathrm{d}x' \\ &= \frac{\lambda^{(\gamma_0)}}{\beta}\int_x^\infty\int_{y+x'}^\infty \mathrm{e}^{-\gamma_0(y'-x')}\,\mathrm{d}F_U^{(\gamma_0)}(y')\,\mathrm{d}x' \\ &= \frac{\lambda}{\beta}\int_x^\infty\int_{y+x'}^\infty \mathrm{e}^{-\gamma_0(y'-x')}\mathrm{e}^{\gamma_0 y'}\,\mathrm{d}F_U(y')\,\mathrm{d}x'\end{aligned}$$

which is the desired result.

11.3.6 Let $g'(u)$ and $g''(u)$ be two solutions. Then $g(u) = g'(u) - g''(u)$ solves

$$\beta g^{(1)}(u) + \lambda\Big(\int_0^u g(u-v)\,\mathrm{d}F(v) - g(u)\Big) - wg(u) = 0 \qquad (11.3.3)$$

with $g(0) = 0$. Integrating the above equation yields

$$\beta g(u) = \lambda\int_0^u g(u-v)\overline{F}(v)\,\mathrm{d}v + w\int_0^u g(v)\,\mathrm{d}v\,.$$

Choose $u_0 > 0$ such that $\beta > \lambda\int_0^{u_0}\overline{F}(v)\,\mathrm{d}v + wu_0$. The problem can now be solved similarly to Exercise 11.3.1.

11.3.7 Let $g'(u)$ and $g''(u)$ be two bounded solutions. Then $g(u) = g'(u) - g''(u)$ solves (11.3.3). We can assume $g(0) \geq 0$. The case $g(0)$ is considered in Exercise 11.3.6. We show first that $g^{(1)}(u) \geq wg(0)u/\beta$. Let $u_0 = \inf\{u : g^{(1)}(u) \leq 0\}$ and assume $u_0 < \infty$. Because $g^{(1)}(u) - w/\beta g(0) > 0$ it follows that $u_0 > 0$. Then $g(u)$ is increasing on $[0, u_0]$, in particular

$$g(u_0) - \int_0^{u_0} g(u_0 - v)\,\mathrm{d}F(v) \geq g(u_0) - g(u_0)F(u_0) \geq 0\,.$$

Thus $\beta g(u_0) \geq w\int_0^{u_0} g(v)\,\mathrm{d}v \geq wu_0 g(0) > 0$, which is a contradiction. Thus g is increasing. The above calculation shows that $g^{(1)}(u) \geq wg(0)u/\beta$. Thus $g(u)$ cannot be bounded.

11.3.8 This follows directly from the proof of Theorem 11.3.5.

11.3.9 For exponentially distributed claims we know $\psi(u) = \lambda/(\beta\delta)\mathrm{e}^{-Ru}$ where $R = \delta - \lambda/c$. It follows that

$$\int_0^u \psi(u-y)\overline{\psi}(y)\,\mathrm{d}y = \frac{\lambda}{\beta\delta R}\Big(1 - \mathrm{e}^{-Ru} - \frac{\lambda}{\beta\delta}Ru\mathrm{e}^{-Ru}\Big)\,.$$

From Theorem 11.3.5 we obtain after a straightforward calculation

$$\mathbb{E}\,[\tau(u);\tau(u)<\infty] = \frac{\lambda(\lambda u+\beta)}{\beta^2\delta(\beta\delta-\lambda)}\mathrm{e}^{-Ru}\,.$$

The desired formula follows now readily.

11.4 Compound Poisson Model in an Economic Environment

In an economic environment the effect of interest and inflation can be expressed by an *economic factor* $\{e(t)\}$, that we model to be a continuous deterministic function with $e(0) = 1$. The surplus process $\{R(t)\}$ in an economic environment is modelled as

$$R(t) = u + \beta\int_0^t e(v)\,\mathrm{d}v - \sum_{i=1}^{N(t)} U_i e(\sigma_i)$$

where $\{\sigma_i\}$ are the claim occurrence times, and $\{N(t)\}$ is a Poisson process with rate λ. We denote by $\psi(u)$ the ruin probability. As in the classical case we let $\theta(s) = \lambda(\hat{m}_U(s)-1) - \beta s$. Recall that $\theta(s)$ is a strictly convex function. If nothing else is said we assume that there exists $s > 0$ such that $\theta(s) < \infty$. If $s \in \mathbb{R}$ is chosen such that $\theta(se(t)) < \infty$ for all $t \geq 0$ then the process $\{M(t)\}$ given by

$$M(t) = \exp\Big(-s(R(t)-u) - \int_0^t \theta(se(v))\,\mathrm{d}v\Big)$$

is a martingale. We define the measures $\mathbb{P}^{(s)}$ on $\mathcal{F}_t$ via the Radon-Nikodym derivative $\mathrm{d}\mathbb{P}^{(s)}/\mathrm{d}\mathbb{P} = M(t)$. We define

$$\gamma = \sup\Big\{s \geq 0 : \sup_{t\geq 0}\int_0^t \theta(se(v))\,\mathrm{d}v < \infty\Big\}\,.$$

For any $\varepsilon > 0$ one has $\lim_{u\to\infty}\psi(u)\mathrm{e}^{(\gamma-\varepsilon)u} = 0$. In many cases γ is the *adjustment coefficient*. By s^+ we denote the upper abscissa of convergence of $\hat{m}_U(s)$.

Exercises

11.4.1 Let $g(u) \geq 0$ be a real function and $\gamma \in \mathbb{R}$. Show that the following are equivalent.
(a) For all $\varepsilon > 0$ one has

$$\lim_{u\to\infty} g(u)\mathrm{e}^{(\gamma-\varepsilon)u} = 0\,, \qquad \lim_{u\to\infty} g(u)\mathrm{e}^{(\gamma+\varepsilon)u} = \infty\,.$$

(b) γ is the Lyapunov exponent, i.e.

$$\lim_{u\to\infty} -\frac{\log g(u)}{u} = \gamma\,.$$

11.4.2 Consider the compound Poisson model (without economic environment, i.e. $e(t) = 1$). Assume $\hat{m}_U(s) = \infty$ for all $s > 0$. Show that

$$\lim_{u\to\infty} -\frac{\log \psi(u)}{u} = 0\,.$$

11.4.3 Consider the compound Poisson model (without economic environment, i.e. $e(t) = 1$). Suppose that there exists $\gamma > 0$ satisfying $\theta(\gamma) < 0$ and $\theta(s) = \infty$ for $s > \gamma$. Suppose $\theta^{(1)}(\gamma) < 0$. Show that $\lim_{u\to\infty} \psi(u)\exp(\gamma u) = 0$.

11.4.4 Consider the compound Poisson model in an economic environment with $e(t) = \mathrm{e}^{-\delta t}$; $\delta > 0$. Show that $\overline{F}(u + \beta/\delta) \leq \psi(u) \leq \overline{F}(u)$, where $F(u) = \mathbb{P}(\sum_{i=1}^{\infty} U_i \mathrm{e}^{-\delta\sigma_i} \leq u)$.

11.4.5 Consider a compound Poisson model in an economic environment. Assume that $\{e(t)\}$ is decreasing. Show that at least one of (a) $\gamma = s^+$, and (b) $\theta(\gamma e(\infty)) = 0$, holds.

11.4.6 Consider a compound Poisson model with decreasing economic factor $\{e(t)\}$ such that $e(\infty) > 0$. Assume that $\gamma = s^+$. Show that $\lim_{u\to\infty} \psi(u)\mathrm{e}^{su} = \infty$ for all $s > \gamma$.

11.4.7 Consider a compound Poisson model with decreasing economic factor $\{e(t)\}$ such that $e(\infty) > 0$. Assume that $\gamma < s^+$. Let $s \in (\gamma, s^+)$. Show that $\lambda(\hat{m}_U(se(\infty)) - 1) - \beta e(t)s > 0$ for t large enough. Conclude that $\lim_{u\to\infty} \psi(u)\mathrm{e}^{su} = \infty$ for all $s > \gamma$.

11.4.8 Consider a compound Poisson model in an economic environment. Let $s \in \mathbb{R}$ be chosen such that $\theta(se(t)) < \infty$ for all $t \geq 0$. Show that under the measure $\mathbb{P}^{(s)}$ the process $\{R(t)\}$ has the following law. The claim occurrence times $\sigma_1, \sigma_2, \ldots$ are the occurrence times of a nonhomogeneous Poisson process (see the definition in Chapter 12) with intensity function $\{\hat{m}_U(se(t))\}$. Given $\sigma_n = t$ the random variable U_i has distribution

$$F_U^t(x) = \frac{\int_0^x \exp(se(t)y)\,\mathrm{d}F_U(y)}{\hat{m}_U(se(t))}\,.$$

That is the jump size at σ_i is $e(\sigma_i)U_i$, where U_i has the conditional distribution function $F_U^{\sigma_i}(y)$ given σ_i.

11.4.9 Consider a compound Poisson model in an economic environment. Let $s \in \mathbb{R}$ be chosen such that $\theta(se(t)) < \infty$ for all $t \geq 0$. Show that under the measure $\mathbb{P}^{(s)}$ the formula

$$\mathbb{E}^{(s)}R(t) - u = -\int_0^t e(v)\theta^{(1)}(se(v))\,\mathrm{d}v$$

holds. Conclude that for $v \geq t$

$$\mathbb{E}^{(s)}(R(v) - R(t) \mid \mathcal{F}_t) = -\int_t^v e(w)\theta^{(1)}(se(w))\,\mathrm{d}w\,,$$

that means the drift at t is $-e(t)\theta^{(1)}(se(t))$.

11.4.10 Consider a compound Poisson model with decreasing economic factor $\{e(t)\}$ such that $e(\infty) > 0$. Assume that $\gamma < s^+$. Show that $\lim_{u\to\infty} \mathbb{P}(\tau(u) \leq t \mid \tau(u) < \infty) = 0$ for all $t \geq 0$. Conclude that $\lim_{u\to\infty} \mathbb{E}\,(\tau(u) \mid \tau(u) < \infty) = \infty$.

Solutions

11.4.1 Assume (a). Choose $\varepsilon > 0$. Then there exists a constant $C > 0$ and u_0 such that for $u \geq u_0$ the inequality $g(u) \leq C\mathrm{e}^{-(\gamma-\varepsilon)u}$ holds. Thus

$$-\frac{\log g(u)}{u} \geq -\frac{\log C}{u} + \gamma - \varepsilon\,.$$

Thus

$$\liminf_{u\to\infty} -\frac{\log g(u)}{u} \geq \gamma - \varepsilon\,.$$

Analogously it follows that

$$\limsup_{u\to\infty} -\frac{\log g(u)}{u} \leq \gamma + \varepsilon\,.$$

Because ε is arbitrary, (b) follows.
Assume (b). Let $\varepsilon > 0$. There exists u_0 such that for all $u \geq u_0$

$$\gamma - \varepsilon/2 \leq -\frac{\log g(u)}{u} \leq \gamma + \varepsilon/2\,.$$

This is equivalent to

$$e^{-(\gamma+\varepsilon/2)u} \le g(u) \le e^{-(\gamma-\varepsilon/2)u}.$$

Thus

$$g(u)e^{(\gamma-\varepsilon)u} = g(u)e^{(\gamma-\varepsilon/2)u}e^{-\varepsilon u/2} \le e^{-\varepsilon u/2}$$

which tends to zero as $u \to \infty$. The other limit follows similarly.

11.4.2 It is clear that $\lim_{u\to\infty} \psi(u)e^{su} = 0$ for all $s \le 0$. From $\hat{m}_U(s) = \infty$ it follows that $\lim_{u\to\infty} \overline{G}(u)e^{su} = \infty$ for all $s > 0$. Using

$$\psi(u) = \frac{\lambda}{c}\int_0^\infty \psi(u-y)\overline{G}(y)\,dy > \frac{\lambda}{c}\int_u^\infty \overline{G}(y)\,dy > \frac{\lambda}{c}\overline{G}(u+1)$$

it follows that $\lim_{u\to\infty} \psi(u)e^{su} = \infty$ for all $s > 0$. The result follows now from Exercise 11.4.1

11.4.3 By change of measure

$$\psi(u)e^{\gamma u} = \mathbb{E}^{(\gamma)}[e^{sR(\tau)+\theta(s)\tau}; \tau < \infty] < \mathbb{P}^{(\gamma)}(\tau < \infty).$$

Under the measure $\mathbb{P}^{(\gamma)}$ the net profit condition is fulfilled, and therefore $\mathbb{P}^{(\gamma)}(\tau < \infty)$ tends to zero as $u \to \infty$.

11.4.4 Consider the claim surplus process

$$S(t) = \sum_{i=1}^{N(t)} U_i e^{-\delta\sigma_i} - \beta\int_0^t e^{-\delta v}\,dv.$$

Then

$$\sum_{i=1}^{N(t)} U_i e^{-\delta\sigma_i} - \frac{\beta}{\delta} < S(t) < \sum_{i=1}^{N(t)} U_i e^{-\delta\sigma_i}.$$

This shows

$$\overline{F}(u+\beta/\delta) \le \mathbb{P}(\sup_{t\ge 0} S(t) > u) = \psi(u) \le \overline{F}(u).$$

11.4.5 Suppose $\gamma < s^+$. Assume first $\theta(\gamma e(\infty)) > 0$. Then there is $s_0 < \gamma$ such that for any $\theta(s_0 e(\infty)) > 0$. For any $s \in [s_0, \gamma]$ we have $\theta(se(t)) \ge \theta(se(\infty)) \ge \theta(s_0 e(\infty))$. Thus $\int_0^\infty \theta(se(t))\,dt = \infty$ contradicting the definition of γ. Thus $\theta(\gamma e(\infty)) \le 0$. Assume now $\theta(\gamma e(\infty)) < 0$. Then there is $s > \gamma$ such that $\theta(se(\infty)) < 0$. Therefore there exists v_0 such that $\theta(se(v_0)) < 0$. But then $\int_0^\infty \theta(se(t))\,dt = -\infty < \infty$ contradicting the definition of γ. Therefore $\theta(\gamma e(\infty)) = 0$.

11.4.6 Consider the compound Poisson model $\{R'(t)\}$ with claim sizes $e(\infty)U_i$, claim arrival process $\{N(t)\}$ and premium rate β. Then $R'(t) \geq R(t)$ and therefore $\psi'(u) \leq \psi(u)$. Because the moment generating function of the claim sizes of $\{R'(u)\}$ does not exist at s, we have $\psi'(u)\mathrm{e}^{su}$ tends to infinity as $u \to \infty$. Hence the same holds for $\psi(u)$.

11.4.7 Clearly $\theta(se(\infty)) > \theta(\gamma e(\infty)) = 0$ by Exercise 11.4.5. Because $e(t) \to e(\infty)$ the existence of t_0 such that $\lambda(\hat{m}_U(se(\infty)) - 1) - \beta e(t_0)s > 0$ follows. Consider the compound Poisson model $\{R'(t)\}$ with claim sizes $e(\infty)U_i$, claim arrival process $\{N(t)\}$ and premium rate $\beta e(t_0)$. Then $R'(t_0+t) - R'(t_0) \geq R(t_0+t) - R(t_0)$. Thus

$$\psi'\Bigl(u + \beta\int_0^{t_0} e(v)\,\mathrm{d}v\Bigr) \leq \psi'(R(t_0 \wedge \tau)) \leq \psi(u)\,.$$

It follows from the Cramér-Lundberg approximation to $\psi'(u)$ that $\psi(u)\mathrm{e}^{su}$ tends to infinity.

11.4.8 Recall that given $N(t) = n$ the occurrence times have under $\mathbb{P}$ the same distribution as the order statistics of n independent uniformly on $(0,t)$ distributed points. This gives for $n \in \mathbb{N}$, $0 \leq t_1 \leq \ldots \leq t_n \leq t$, and Borel sets $A_1, A_2, \ldots, A_n$

$$\begin{aligned}
&\mathbb{P}^{(s)}(N(t) = n, \sigma_i \leq t_i, U_i \in A_i, 1 \leq i \leq n)\\
&= \mathbb{E}\,[\mathrm{e}^{-sR(t) - \int_0^t \theta(se(v))\,\mathrm{d}v}; N(t) = n, \sigma_i \leq t_i, U_i \in A_i, 1 \leq i \leq n]\\
&= \mathrm{e}^{-\lambda\int_0^t(\hat{m}_U(se(v))-1)\,\mathrm{d}v}\frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t}\frac{n!}{t^n}\int_0^{t_1}\int_{v_1}^{t_2}\cdots\int_{v_{n-1}}^{t_n}\\
&\qquad \int_{B_1}\mathrm{e}^{se(v_1)y}\,\mathrm{d}F_U(y)\ldots\int_{B_n}\mathrm{e}^{se(v_n)y}\,\mathrm{d}F_U(y)\,\mathrm{d}v_n\ldots\mathrm{d}v_1\\
&= \frac{(\lambda\int_0^t \hat{m}_U(se(v))\,\mathrm{d}v)^n}{n!}e^{-\lambda\int_0^t \hat{m}_U(se(v))\,\mathrm{d}v}\\
&\qquad \frac{n!}{(\int_0^t \hat{m}_U(se(v))\,\mathrm{d}v)^n}\int_0^{t_1}\hat{m}_U(se(v_1))\ldots\int_{v_{n-1}}^{t_n}\hat{m}_U(se(v_n))\\
&\qquad \int_{B_1}\frac{\mathrm{e}^{se(v_1)y}}{\hat{m}_U(se(v_1))}\,\mathrm{d}F_U(y)\ldots\int_{B_n}\frac{\mathrm{e}^{se(v_n)y}}{\hat{m}_U(se(v_n))}\,\mathrm{d}F_U(y)\,\mathrm{d}v_n\ldots\mathrm{d}v_1\,.
\end{aligned}$$

This is the proposed distribution.

11.4.9 It follows from the martingale property that

$$\mathbb{E}\,\mathrm{e}^{-s(R(t)-u)} = \exp\Bigl(\int_0^t \theta(se(v))\,\mathrm{d}v\Bigr)\,.$$

Hence

$$\begin{aligned}
\mathbb{E}^{(s)}R(t)-u &= \mathbb{E}\left((R(t)-u)\mathrm{e}^{-s(R(t)-u)}\right)\mathrm{e}^{-\int_0^t \theta(se(v))\,\mathrm{d}v} \\
&= -\mathrm{e}^{-\int_0^t \theta(se(v))\,\mathrm{d}v}\frac{\partial}{\partial s}\mathbb{E}\,\mathrm{e}^{-s(R(t)-u)} \\
&= -\mathrm{e}^{-\int_0^t \theta(se(v))\,\mathrm{d}v}\int_0^t e(v)\theta^{(1)}(se(v))\,\mathrm{d}v\,\mathrm{e}^{\int_0^t \theta(se(v))\,\mathrm{d}v} \\
&= -\int_0^t e(v)\theta^{(1)}(se(v))\,\mathrm{d}v\,.
\end{aligned}$$

The rest of the assertion follows from the independent increments property, see Exercise 11.4.8

11.4.10 Let $s \in (\gamma, s^+)$. Note that since $\theta(\gamma e(\infty)) = 0$ by Exercise 11.4.5 we have

$$\theta(se(t)) \geq \theta(se(\infty)) > \theta(\gamma e(\infty)) = 0\,.$$

This implies $\theta^{(1)}(se(t)) \geq \theta^{(1)}(se(\infty)) > 0$. Expressing $\mathbb{P}(\tau \leq t \mid \tau < \infty)$ under the measure $\mathbb{P}^{(s)}$ one finds

$$\mathbb{P}(\tau \leq t \mid \tau < \infty) = \frac{\mathbb{E}^{(s)}\left[e^{sR(\tau)+\int_0^\tau \theta(se(v))\,\mathrm{d}v}; \tau \leq t\right]}{\mathbb{E}^{(s)}\left(e^{sR(\tau)+\int_0^\tau \theta(se(v))\,\mathrm{d}v}\right)}$$

The numerator is bounded and, because $\mathbb{P}^{(s)}(\tau < \infty) = 1$ (see Exercise 11.4.9), it tends to zero. The denominator is $\psi(u)\mathrm{e}^{su}$ and tends to infinity by Exercise 11.4.7. Finally

$$\mathbb{E}(\tau \mid \tau < \infty) = \int_0^\infty \mathbb{P}(\tau > t \mid \tau < \infty)\,\mathrm{d}t$$

tends to infinity as $u \to \infty$ by Fatou's lemma.

11.5 Exponential Martingales: the Sparre Andersen Model

In this section we consider the *renewal risk model*

$$R(t) = u + \beta t - \sum_{i=1}^{N(t)} U_i$$

where $\{N(t)\}$ is a renewal process. The claim sizes $\{U_i\}$ are positive, independent and identically distributed, independent of $\{N(t)\}$. The occurrence times are denoted by $\{\sigma_i\}$ ($\sigma_0 = 0$) and the inter-occurrence times by $\{T_i\}$. For simplicity we consider the ordinary case only, i.e. all $\{T_i\}$ have the same distribution. We assume that not both the claim sizes and the inter-occurrence times are deterministic. The function $\theta(s)$ is defined as the solution to $\hat{m}_U(s)\hat{l}_T(\beta s + \theta) = 1$, provided such a solution exists. Throughout this section we assume the net profit condition $\beta > \lambda\mu_U$ where $\lambda = (\mu_T)^{-1}$, interpreted as zero if $\mu_T = \infty$. The function $\theta(s)$ is then strictly convex with derivative $\theta^{(1)}(0) = -(\beta - \lambda\mu_U)$ in zero. A possible positive solution γ to $\theta(\gamma) = 0$ is called the *adjustment coefficient*.

We define the time $T'(t) = t - \sigma_{N(t)}$ since the last occurrence and the time $T(t) = \sigma_{N(t)+1}$ till the next occurrence. We consider the processes $\{(R(t), T'(t), t)\}$ and $\{(R(t), T(t), t)\}$. We use the natural filtrations $\{\mathcal{F}_t\}$ and $\{\mathcal{F}'_t\}$, respectively, of these two processes according to which process we are considering. If we consider $\{(R(t), T'(t), t)\}$ we assume that T_i has an absolutely continuous distribution with density $f_T(y)$. We denote the generator of $\{(R(t), T'(t), t)\}$ by $\boldsymbol{A}'$ and the generator of $\{(R(t), T(t), t)\}$ by $\boldsymbol{A}$.

If $\theta(s)$ exists then the process $\{M'(t)\}$ defined as

$$M'(t) = \hat{m}_U(s)\frac{\mathrm{e}^{(\theta(s)+\beta s)T'(t)-s(R(t)-u)-\theta(s)t}}{\overline{F}_T(T'(t))}\int_{T'(t)}^{\infty} \mathrm{e}^{-(\theta(s)+\beta s)v} f_T(v)\,\mathrm{d}v$$

is a $\{\mathcal{F}'_t\}$-martingale, and the process $\{M(t)\}$ defined as

$$M(t) = \hat{m}_U(s)\,\mathrm{e}^{-(\theta(s)+\beta s)T(t)-s(R(t)-u)-\theta(s)t}$$

is a $\{\mathcal{F}_t\}$-martingale.

Exercises

11.5.1 Show that $\mathbb{E}\, a^{N(t)} < \infty$ for all $a > 0$ and all $t \geq 0$.

11.5.2 Show that $\theta(s) + \beta s \geq 0$ is equivalent to $s \geq 0$. Show, moreover, that equality only holds if $s = 0$.

11.5.3 Find the characteristics of the PDMP $(R(t), T'(t), t)$.

11.5.4 Let

$$g(y, w, t) = \frac{\mathrm{e}^{(\theta(s)+\beta s)w-sy-\theta(s)t}}{\overline{F}_T(w)}\int_w^{\infty} \mathrm{e}^{-(\theta(s)+\beta s)v} f_T(v)\,\mathrm{d}v\,.$$

Show that $g(y, w, t)$ is in the domain $\mathcal{D}(\boldsymbol{A}')$ of the generator $\boldsymbol{A}'$.

11.5.5 Show that $\mathbb{E}\, M'(t) = 1$. For $s \in \mathbb{R}$ such that $\theta(s)$ exists define the measure $\mathbb{P}'^{(s)}$ such that for $A \in \mathcal{F}'_t$ we have $\mathbb{P}'^{(s)}(A) = \mathbb{E}\,[M'(t); A]$.

Show that under the measure $\mathbb{P}'^{(s)}$ the process $\{(R(t), T'(t), t)$ is a Sparre Andersen model with premium rate β, claim size distribution

$$\tilde{F}_U(x) = \hat{l}_T(\beta s + \theta(s)) \int_0^x \mathrm{e}^{sy} \, \mathrm{d}F_U(y) \tag{11.5.1}$$

and inter-occurrence time distribution

$$\tilde{F}_T(t) = \hat{m}_U(s) \int_0^t \mathrm{e}^{-(\beta s + \theta(s))v} \, \mathrm{d}F_T(v) \,. \tag{11.5.2}$$

Verify the the net profit condition is fulfilled if and only if $\theta^{(1)}(s) < 0$.

11.5.6 Assume the adjustment coefficient γ exists. Prove Lundberg's inequality $\psi(u) < \mathrm{e}^{-\gamma u}$ and the Cramér-Lundberg approximation for the Sparre Andersen model using the change of measure technique. Use the measure $\mathbb{P}'^{(\gamma)}$ defined in Exercise 11.5.5. In order to prove the Cramér-Lundberg approximation assume $\hat{m}_U^{(1)}(\gamma) < \infty$.

11.5.7 Assume the adjustment coefficient γ exists. Assume $\hat{m}_U^{(1)}(\gamma) = \infty$. Show that $\psi(u)\mathrm{e}^{Ru}$ tends to zero as $u \to \infty$.

11.5.8 Find the characteristics of the PDMP $(R(t), T(t), t)$.

11.5.9 Let $g(y, w, t) = \exp(-(\theta(s) + \beta s)w - sy - \theta(s)t)$. Show that $g \in \mathcal{D}(\boldsymbol{A})$.

11.5.10 Show that $\mathbb{E}\, M(t) = 1$. For $s \in \mathbb{R}$ such that $\theta(s)$ exists define the measure $\mathbb{P}^{(s)}$ such that for $A \in \mathcal{F}_t$ we have $\mathbb{P}^{(s)}(A) = \mathbb{E}\,[M(t); A]$. Show that under the measure $\mathbb{P}^{(s)}$ the process $\{(R(t), T(t), t)$ is a Sparre Andersen model with the distributions given by (11.5.1) and (11.5.2)

11.5.11 Assume the adjustment coefficient γ exists. Prove Lundberg's inequality $\psi(u) < \mathrm{e}^{-\gamma u}$ and the Cramér-Lundberg approximation for the Sparre Andersen model using the change of measure technique. Use the measure $\mathbb{P}^{(\gamma)}$ defined in Exercise 11.5.10. In order to prove the Cramér-Lundberg approximation assume $\hat{m}_U^{(1)}(\gamma) < \infty$.

Solutions

11.5.1 We can assume $a > 1$. The result is trivial if $F_T(t) = 0$ for some $t > 0$. We therefore assume $F_T(t) > 0$ for all $t > 0$. Using $F_T^{*n}(t) \le (F_T(t))^n$ we find

$$\sum_{n=0}^{\infty} a^n \mathbb{P}(N(t) = n) \le \sum_{n=0}^{\infty} a^n \mathbb{P}(N(t) \le n) \le \sum_{n=0}^{\infty} (aF_T(t))^n$$

and the result is proved for $t \leq t_0$ where $t_0 > 0$ such that $F_T(t_0) < a^{-1}$. Assume the result is proved for $t \leq kt_0$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned}\mathbb{E}\, a^{N((k+1)t_0)} &= \mathbb{E}\, a^{N(kt_0)+1+N((k+1)t_0)-N(\sigma_{N(kt_0)+1})}\\ &\leq a\,\mathbb{E}\, a^{N(kt_0)}\,\mathbb{E}\, a^{N(t_0)} < \infty\,.\end{aligned}$$

The assertion follows now by induction.

11.5.2 The following are equivalent

$$s \geq 0 \iff \hat{m}_U(s) \geq 1 \iff \hat{l}_T(\theta(s)+\beta s) \leq 1 \iff \theta(s)+\beta s \geq 0\,.$$

Analogously,

$$s = 0 \iff \hat{m}_U(s) = 1 \iff \hat{l}_T(\theta(s)+\beta s) = 1 \iff \theta(s)+\beta s = 0\,.$$

11.5.3 The state space is $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$. The vector field at the point (y, w, t) is given by

$$\boldsymbol{X} = \beta\frac{\partial}{\partial y} + \frac{\partial}{\partial w} + \frac{\partial}{\partial t}\,.$$

This implies in particular that the active boundary $\Gamma = \emptyset$. The jump intensity is $\lambda(y, w, t) = \lambda(w) = f_T(w)/\overline{F}_T(w)$. Finally, the transition kernel is $Q((y, w, t), B_1 \times B_2 \times B_3) = F_U(y - B_1)\mathbb{I}(0 \in B_2, t \in B_3)$.

11.5.4 We verify the conditions of Theorem 11.2.2. Because $g(y, w, t)$ is absolutely continuous it is absolutely continuous along integral paths. The boundary condition is trivial because $\Gamma = \emptyset$. It remains to show the integrability condition

$$\mathbb{E}\Big(\sum_{i=1}^{N(t)} |g(R(\sigma_i), 0, \sigma_0) - g(R(\sigma_i-), T_i, \sigma_0)|\Big) < \infty\,.$$

Note that σ_i and T_i are bounded by t. If $s \leq 0$ then because $R(v) \leq u+\beta t$ the above expected value is obviously finite. Assume therefore $s > 0$. Then the expected value is bounded by

$$c\mathbb{E}\Big(\sum_{i=1}^{N(t)} \exp\Big(s\sum_{j=1}^{i} U_i\Big)\Big)$$

for some constant c. The result follows now analogously to the compound Poisson case, using that $\mathbb{E}\, a^{N(t)} < \infty$ for all $t \geq 0$.

11.5.5 Fix $n \in \mathbb{N}$ and Borel sets B_i and B'_i for $1 \le i \le n$. Note that $T'(\sigma_n) = 0$. Then

$$\begin{aligned}
&\mathbb{P}'^{(s)}(U_i \in B_i, T_i \in B'_i, 1 \le i \le n) \\
&= \mathbb{E}\,[e^{-s(R(\sigma_n)-u)-\theta(s)\sigma_n}; U_i \in B_i, T_i \in B'_i, 1 \le i \le n] \\
&= \mathbb{E}\left[\exp\Bigl(\sum_{i=1}^{n}(sU_i - (\beta s + \theta(s))T_i)\Bigr); U_i \in B_i, T_i \in B'_i, 1 \le i \le n\right] \\
&= \prod_{i=1}^{n} \hat{l}_T(\beta s + \theta(s))\mathbb{E}\,[\mathrm{e}^{sU_i}; U_i \in B_i]\,\hat{m}_U(s)\mathbb{E}\,[\mathrm{e}^{-(\beta s+\theta s)T_i}; T_i \in B'_i]
\end{aligned}$$

from which the result follows. The net profit condition is

$$\beta > -\frac{\hat{l}_T(\beta s + \theta(s))\hat{m}_U^{(1)}(s)}{\hat{l}_T^{(1)}(\beta s + \theta(s))\hat{m}_U(s)}$$

which is equivalent to $\theta^{(1)}(s) < 0$.

11.5.6 Note that $T'(\tau) = 0$. Thus

$$\psi(u) = \mathbb{E}\,'^{(\gamma)}[\mathrm{e}^{sR(\tau)}; \tau < \infty]\,\mathrm{e}^{-Ru} = \mathbb{E}\,'^{(\gamma)}\mathrm{e}^{sR(\tau)}\mathrm{e}^{-Ru}\,.$$

The last equality follows from $\mathbb{P}'^{(\gamma)}(\tau < \infty) = 1$. Lundberg's inequality follows immediately. In order to proof the Cramér-Lundberg approximation let $g(u) = \mathbb{E}^{(\gamma)}(\mathrm{e}^{sR(\tau)} \mid R(0) = 0)$ and $G^-(x) = \mathbb{P}^{(\gamma)}(X_\tau \ge -x \mid R(0) = 0)$. Then $g(u)$ fulfils the renewal equation

$$g(u) = \int_0^u g(u-y)\,\mathrm{d}G^-(y) + \int_u^\infty \mathrm{e}^{-R(y-u)}\,\mathrm{d}G^-(y)\,.$$

By the key renewal theorem $g(u)$ converges to a constant c. Since U_i has finite mean $\hat{m}_U^{(1)}(\gamma)/\hat{m}_U(\gamma)$, it follows that $G^-(y)$ is a distribution function of a random variable with finite mean, see Feller (1971). Therefore, $c > 0$.

11.5.7 The result follows as in Exercise 11.5.6 noting that $G^-(y)$ is a distribution function with an infinite mean value.

11.5.8 The state space is $\mathbb{R} \times (0,\infty) \times \mathbb{R}_+$. The vector field at the point (y, w, t) is

$$\boldsymbol{X} = \beta\frac{\partial}{\partial y} - \frac{\partial}{\partial w} + \frac{\partial}{\partial t}\,.$$

This gives that $\Gamma = \{(y, 0, t) : y \in \mathbb{R}, t \ge 0\}$. The jump intensity is $\lambda(y, w, t) = 0$. Finally, the jump measure needs only to be given on Γ, and is $Q((y, 0, t), B_1 \times B_2 \times B_3) = F_U(y - B_1)F_T(B_2)\mathbb{I}(t \in B_3)$.

11.5.9 We verify the conditions of Theorem 11.2.2. Clearly, $g(y,w,t)$ is absolutely continuous along integral curves. Moreover,

$$\begin{aligned}\int_0^\infty\int_0^\infty g(y-z,w,t)\,\mathrm{d}F_T(w)\,\mathrm{d}F_U(z) &= \mathrm{e}^{-sy-\theta(s)t}\hat{m}_U(s)\hat{l}_U(\beta s+\theta(s))\\ &= g(y,0,t)\end{aligned}$$

and therefore the boundary condition is fulfilled. It remains to verify the integrability condition. Consider first the case $s \ge 0$. Then $\beta s+\theta(s)\ge 0$ and therefore $\mathrm{e}^{-(\beta s+\theta(s))T(v)-\theta(s)v}$ is bounded for $v\in[0,t]$. It remains to show that

$$\mathbb{E}\sum_{i=1}^{N(t)}\mathrm{e}^{-sR(\sigma_i)}<\infty\,.$$

This follows in the same way as in Exercise 11.5.4. Assume now $s<0$. Then $\beta s+\theta(s)<0$. Now $\mathrm{e}^{-sR(v)}$ is bounded. Moreover, $T(v)$ is bounded by t for all $v<\sigma_{N(t)}$. Because $\mathbb{E}\,N(t)<\infty$ it remains to prove that $\mathbb{E}\,\mathrm{e}^{-(\beta s+\theta(s))T_{N(t)+1}}<\infty$. We can assume that $F_T(v)<1$ for all $v>0$. Otherwise, the statement is trivial. Thus we have to show that

$$\sup_{0\le v\le t}\frac{\int_v^\infty \mathrm{e}^{-(\beta s+\theta(s))w}\,\mathrm{d}F_T(w)}{\overline{F}_T(v)}<\infty\,.$$

But now the assertion follows from $\hat{l}_T(\beta s+\theta(s))<\infty$.

11.5.10 That $\mathbb{E}\,M(t)=1$ follows from $T(0)=T_1$. Fix $n\in\mathbb{N}$ and Borel sets B_i and B_i' for $1\le i\le n$. Note that $T(\sigma_n)=T_{n+1}$. Then

$$\begin{aligned}&\mathbb{P}^{(s)}(U_i\in B_i,T_i\in B_i',1\le i\le n)\\ &\quad= \hat{m}_U(s)\mathbb{E}\,[\mathrm{e}^{-(\beta s+\theta(s))T(\sigma_n)-s(R(\sigma_n)-u)-\theta(s)\sigma_n};\\ &\qquad\qquad U_i\in B_i,T_i\in B_i',1\le i\le n]\\ &\quad= \mathbb{E}\,[\mathrm{e}^{-s(R(\sigma_n)-u)-\theta(s)\sigma_n};U_i\in B_i,T_i\in B_i',1\le i\le n]\\ &\quad= \prod_{i=1}^n\hat{l}_T(\beta s+\theta(s))\mathbb{E}\,[\mathrm{e}^{sU_i};U_i\in B_i]\,\hat{m}_U(s)\mathbb{E}\,[\mathrm{e}^{-(\beta s-\theta(s))T_i};T_i\in B_i']\,.\end{aligned}$$

This are the desired distributions.

11.5.11 Under the measure $\mathbb{P}^{(\gamma)}$ the ruin probability can be expressed as

$$\psi(u)=\hat{l}_T(\beta\gamma)\mathbb{E}^{(\gamma)}\exp(\beta\gamma T(\tau)+\gamma R(\tau))$$

where we used that $\mathbb{P}^\gamma(\tau<\infty)=1$. $T(\tau)$ is the time till the next claim after ruin and therefore independent of $R(\tau)$ and has distribution $\tilde{F}_T$. A simple calculation gives $\mathbb{E}^{(\gamma)}\mathrm{e}^{\beta\gamma T_1}=\hat{m}_U(\gamma)$. Thus

$$\psi(u)=\mathbb{E}^{(\gamma)}\exp(\gamma R(\tau))\,.$$

This gives immediately Lundberg's inequality. The Cramér-Lundberg approximation follows as in Exercise 11.5.10.

12

Point Processes

12.1 Stationary Point Processes

Exercises

12.1.1 For an ergodic point process $\{\sigma_n\}$ show that for every fixed $\varepsilon, \varepsilon'$ there exists $c > 0$ such that $\mathbb{P}(\cap_{n\geq 1}\{\sigma_n \leq n(\lambda^{-1} + \varepsilon) + c\}) > 1 - \varepsilon'$.

12.2 Mixtures and Compounds of Point Processes

Exercises

12.2.1 Let $\{\sigma_n\}$ be a cluster process with cluster centres $\{\sigma'_n\}$ and clusters $\{N_n\}$. Let λ' be the intensity of the cluster center process. Show $\lambda = \lambda' \mathbb{E}\, N_n(\mathbb{R})$.

12.3 The Markov-Modulated Risk Model via PDMP

Exercises

12.3.1 Consider the Markov-modulated risk model. Show that the stochastic process $\{(J(t), R(t))\}$ is a PDMP and determine its characteristics.

12.3.2 Assume that $\boldsymbol{Q}$ is irreducible and let $V_i(t) = \int_0^t \mathbb{I}(J(v) = i)\, \mathrm{d}v$ denote the amount of time in $(0, t]$ which the Markov process $\{J(t)\}$ spends in state i. Show that $t^{-1}V(t)$ tends to π_i where $\boldsymbol{\pi} = (\pi_i)_{i=1,\ldots,\ell}$ is the stationary initial distribution of $\{J(t)\}$. [Hint. Consider the cycles between two consecutive visits to state i and use the law of large numbers.]

12.3.3 Let X be an arbitrary real-valued random variable. Assume that the generating function $\hat{m}_X(s)$ exists in a certain interval I different from the singleton $\{0\}$. Show that $\hat{m}_X(s)$ is convex on I, and strictly convex on I provided X is not deterministic. [Hint. Consider the random variable Y with distribution $F_Y(y) = \int_{-\infty}^y \mathrm{e}^{sx}\,\mathrm{d}F_X(x)/\hat{m}_X(s)$ and determine its variance.]

12.3.4 Consider the Markov-modulated risk model and let $\tau_1 = \inf\{t \ge 0 : J(t) = 1\}$. Show that $\lim_{u\to\infty} P^{(\gamma)}(\tau(u) < \tau_1) = 0$.

12.4 Periodic Risk Models

Exercises

12.4.1 In the periodic model specified by $(\lambda(t), F_t(x), \beta)$ show that the process $\{X(t)\}$, where $X(t) = (R(t), N(t), t)$ is a PDMP with generator

$$\begin{aligned}\boldsymbol{A}g(y,n,t) &= \frac{\partial}{\partial t}g(y,n,t) + \beta\frac{\partial}{\partial y}g(y,n,t)\\ &\quad + \lambda(t)\Big(\int_0^\infty g(n+1, y-z, t)\,\mathrm{d}F_t(z) - g(y,n,t)\Big)\,.\end{aligned}$$

provided $g : \mathbb{R}\times\mathbb{N}\times\mathbb{R}_+ \to \mathbb{R}$ fulfils conditions of Theorem 11.2.2.

12.4.2 Consider the periodic model specified by $(\lambda(t), F_t, \beta(t))$, where the premium rate $\beta(t) > 0$ depends on time and is a periodic function with period 1. Show that by the change of time this model can be reduced to a periodic model with constant premium rate $\beta^* = \int_0^1 \beta(s)\,\mathrm{d}s$.

12.5 The Björk-Grandell Model via PDMP

Exercises

12.5.1 Let $\{R(t), \lambda(t), A(t)\}$ be a Björk-Grandell model with an arbitrary distribution of (Λ_1, S_1). Assume that the distribution of S is nonlattice. Show that

$$\lim_{t\to\infty} \mathbb{P}(\lambda(t)\in A, A(t)\in B) = \frac{1}{\mathbb{E}\,S_2}\int_B \mathbb{P}(\Lambda_2 \in A, S_2 > s)\,\mathrm{d}s\,.$$

[Hint. Use the key renewal theorem.]

12.5.2 Prove Theorem 12.5.5.

12.5.3 Let $\{(R(t), \lambda(t), A(t))\}$ be a Björk-Grandell model. Assume that there exists a constant C such that in the ordinary case

$$\lim_{u\to\infty} \mathbb{E}^{(\gamma)}(\mathrm{e}^{(\beta\gamma-\lambda(\tau)(\hat{m}_U(\gamma)-1))A(\tau)}\mathrm{e}^{\gamma R(\tau)}) = C\,. \qquad (12.5.1)$$

Show that for any distribution of (Λ_1, S_1) such that $\mathbb{E}^{(0)} g(\Lambda_1, S_1) < \infty$ relation (12.5.1) holds. [Hint. Compare with the proof of Theorem 12.3.6.]

12.6 Subexponential Claim Sizes

Exercises

12.6.1 Consider a Poisson cluster process and let $\sigma''_n = \inf\{x : N_n(\{x\}) > 0\}$ denote the most far left point of the n-th cluster. Show that $\sigma'_n + \sigma''_n$ is a Poisson process.

12.6.2 Show that for any point process $\{\sigma_n\}$ with $N(t) = N((0,t])$ and for Z_ε defined in Section 12.6.1

$$\mathbb{P}(Z_\varepsilon > u) \le \mathbb{P}\Big(\bigcup_{n \ge 1} \{N(n(\lambda^{-1} - \varepsilon) - u) \ge n\} \Big) .$$

12.6.3 Let $\{J(t)\}$ be a Markov process with finite state space and assume $\{J(t)\}$ is irreducible. Let $J(0) = 1$. Show that $\zeta = \inf\{t > 0 : J(t) = 1, J(t-0) \ne 1\}$ is light-tailed. Conclude that also $N(\zeta)$ is light-tailed. [Hint. Represent the distribution of ζ as phase-type distribution.]

12.6.4 Complete the proof of Theorem 12.6.6..

13

Diffusion Models

In this chapter $\{W(t)\}$ is always a Brownian motion. A local martingale $\{M(t)\}$ is a stochastic process such that there exists stopping times $\tau_1 \le \tau_2 \le \ldots \to \infty$, called *localization sequence*, such that $\{M(\tau_n \wedge t)\}$ are martingales. Let $\{X(t)\}$ be a stochastic process. We say $\{X(t)\} \in L^2$ if $\mathbb{E} \int_0^t X(v)^2 \,\mathrm{d}v < \infty$ for all $t \ge 0$. We say $\{X(t)\} \in L^2_{\mathrm{loc}}$ if $\int_0^t X(v)^2 \,\mathrm{d}v < \infty$ for all $t \ge 0$. Note that $\{X(t)\}$ càdlàg implies $\{X(t)\} \in L^2_{\mathrm{loc}}$. For $\{X(t)\} \in L^2_{\mathrm{loc}}$ we can define the stochastic integral $\{\int_0^t X(v) \,\mathrm{d}W(v)\}$. This process is a local martingale. If $\{X(t)\} \in L^2$ the the stochastic integral is a martingale.

13.1 Stochastic Differential Equations

Exercises

13.1.1 Let $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = t$ be partitions of the interval $[0,t]$, such that $\sup\{t_i^{(n)} - t_{i-1}^{(n)}\} \to 0$ as $n \to \infty$.
(a) Show that

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big(\sum_{i=1}^{n}(W(t_i^{(n)}) - W(t_{i-1}^{(n)}))^2 - t\Big)^2\Big] = 0\,.$$

Conclude that, if $\liminf_{n\to\infty} \sum_{i=1}^n (W(t_i^{(n)}) - W(t_{i-1}^{(n)}))^2 - t = 0$.
(b) Show that

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big|\sum_{i=1}^{n}(W(t_i^{(n)}) - W(t_{i-1}^{(n)}))^2 - t\Big|\Big] = 0\,.$$

(c) Let $\{n_k\}$ be a subsequence such that

$$\sum_{k=1}^{\infty} \mathbb{E}\Big[\Big|\sum_{i=1}^{n_k}(W(t_i^{(n_k)}) - W(t_{i-1}^{(n_k)}))^2 - t\Big|\Big] < \infty\,.$$

Show that with probability one $\sum_{i=1}^{n_k}(W(t_i^{(n_k)})-W(t_{i-1}^{(n_k)}))^2$ is a Cauchy sequence and conclude that $\sum_{i=1}^{n_k}(W(t_i^{(n_k)}) - W(t_{i-1}^{(n_k)}))^2$ converges to t with probability one.
(d) Show that with probability one

$$\lim_{n\to\infty}\sum_{i=1}^{n}(W(t_i^{(n)}) - W(t_{i-1}^{(n)}))^2 = t\,.$$

13.1.2 Let $\{Y(t)\}$ be a piecewise constant process, i.e. there exists stopping times $0 = \tau_0 \le \tau_1 \le \tau_2 \le \ldots \to \infty$ such that $Y(t) = Y(\tau_n)$ if $\tau_n \le t < \tau_{n+1}$. Define

$$\begin{aligned}&\int_0^t Y(v)\,\mathrm{d}W(v) \qquad (13.1.1)\\ &= \sum_{i=0}^{N(t)-1} Y(\tau_i)(W(\tau_{i+1}) - W(\tau_i)) + Y(t)(W(t) - W(\tau_{N(t)}))\end{aligned}$$

where $N(t) = \max\{i : \tau_i \le t\}$. Let $\tau_n' = \inf\{t \ge 0 : |\int_0^t Y(v)\,\mathrm{d}W(v)| \ge n\}$. Show that $\{\int_0^{\tau_n'\wedge t} Y(v)\,\mathrm{d}W(v)\}$ is a uniformly integrable martingale.

13.1.3 Construct a local martingale that is not a martingale.

13.1.4 Let $\{M(t)\}$ be a càdlàg local martingale adapted to $\{\mathcal{F}_t\}$. Show that it is possible to choose a localization sequence $\{\tau_n\}$ such that for all n the martingale $\{M(\tau_n \wedge t)\}$ is uniformly integrable.

13.1.5 Let $\{M(t)\}$ be a local martingale and assume that $\mathbb{E}\,M(t)^2 < \infty$ for all $t \ge 0$.
(a) Show that $\{M(t)^2\}$ is a submartingale.
(b) For $v < t$ show that $\mathbb{E}\,M(v)^2 \le \mathbb{E}\,M(t)^2$.
(c) Fix $t > 0$. Show that the family $\{M(v) : 0 \le v \le t\}$ is uniformly integrable.
(d) Show that $\{M(t)\}$ is a martingale.

13.1.6 Show that the geometric Brownian motion $\{X(t)\}$,

$$X(t) = X(0)\exp((\mu - \sigma^2/2)t + \sigma W(t)) \qquad (13.1.2)$$

is the solution to the stochastic differential equation $\mathrm{d}X(t) = \delta X(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)$.

13.1.7 Show that for the geometric Brownian motion $\{X(t)\}$ defined in (13.1.2),

$$\begin{aligned}\mathbb{E}\,X(t) &= X(0)\mathrm{e}^{\delta t}\\ \operatorname{Var} X(t) &= X(0)\mathrm{e}^{(2\delta+\sigma^2)t}\,.\end{aligned}$$

13.1.8 Show that for a deterministic càdlàg function $h \in L^2$, the random variable $\int_0^t h(v)\,\mathrm{d}W(v)$ is normally distributed $\mathrm{N}(0, \int_0^t h^2(v)\,\mathrm{d}v)$.

13.1.9 For the Ornstein-Uhlenbeck process $\{X(t)\}$ with $\mathrm{d}X(t) = -\alpha(X(t) - \bar{\delta})\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)$, where $\alpha, \sigma > 0$ derive the formulae

$$\begin{aligned} \mathbb{E}\, X(t) &= \bar{\delta} + \mathrm{e}^{-\alpha t}(\mathbb{E}\, X(0) - \bar{\delta})\,, \\ \mathrm{Cov}\,(X(t), X(t+h)) &= \frac{\sigma^2}{2\alpha}\big(\mathrm{e}^{-\alpha h} - \mathrm{e}^{-\alpha(2t+h)}\big) \end{aligned}$$

where $t, h \geq 0$.

13.1.10 Show that

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} W\Big(\frac{(i-\frac{1}{2})t}{2^n}\Big)\Big(W\Big(\frac{it}{2^n}\Big) - W\Big(\frac{(i-1)t}{2^n}\Big)\Big) = \tfrac{1}{2}W(t)^2\,.$$

13.1.11 Let X be a Markov process and assume that g and $\boldsymbol{A}g$ are bounded for a function g in the domain $\mathcal{D}(\boldsymbol{A})$ of the extended generator. Show that (13.1.33) is a martingale, i.e. g is in the domain of the full generator.

Solutions

13.1.1 (a) Using the independent increment property of the Brownian motion and that $(W(t_i^{(n)}) - W(t_{i-1}^{(n)}))/\sqrt{t_i^{(n)} - t_{i-1}^{(n)}}$ has the same distribution as W_1 it follows that

$$\mathbb{E}\,\Big((W(t_i^{(n)}) - W(t_{i-1}^{(n)}))^2 - (t_i^{(n)} - t_{i-1}^{(n)})\Big) = 0$$

and therefore

$$\mathbb{E}\,\Big(\Big(\sum_{i=1}^{n}(W(t_i^{(n)}) - W(t_{i-1}^{(n)}))^2 - t\Big)^2\Big) = \sum_{i=1}^{n}(t_i^{(n)} - t_{i-1}^{(n)})^2\,\mathbb{E}\,((W_1^2 - t)^2)\,.$$

The latter tends to zero as $n \to \infty$. The lim inf follows from Fatou's lemma.
(b) This follows from Jensen's inequality.
(c) We have

$$\begin{aligned} &\mathbb{E}\,\Big(\sum_{k=1}^{\infty}\Big|\sum_{i=1}^{n_k}(W(t_i^{(n_k)}) - W(t_{i-1}^{(n_k)}))^2 - \sum_{i=1}^{n_{k+1}}(W(t_i^{(n_{k+1})}) - W(t_{i-1}^{(n_{k+1})}))^2\Big|\Big) \\ &\leq\; 2\sum_{k=1}^{\infty}\mathbb{E}\,\Big(\Big|\sum_{i=1}^{n_k}(W(t_i^{(n_k)}) - W(t_{i-1}^{(n_k)}))^2 - t\Big|\Big) < \infty\,. \end{aligned}$$

Thus with probability one

$$\sum_{k=1}^{\infty}\Big|\sum_{i=1}^{n_k}(W(t_i^{(n_k)})-W(t_{i-1}^{(n_k)}))^2-\sum_{i=1}^{n_{k+1}}(W(t_i^{(n_{k+1})})-W(t_{i-1}^{(n_{k+1})}))^2\Big|<\infty$$

and $\sum_{i=1}^{n_k}(W(t_i^{(n_k)})-W(t_{i-1}^{(n_k)}))^2$ is a Cauchy sequence. Thus the limit $\lim_{k\to\infty}\sum_{i=1}^{n_k}(W(t_i^{(n_k)})-W(t_{i-1}^{(n_k)}))^2$ exists, and by (a) the limit must be t.

(d) By (c) any subsequence of $\sum_{i=1}^{n}(W(t_i^{(n)})-W(t_{i-1}^{(n)}))^2$ has a convergent subsequence and any convergent subsequence converges to t. Thus $\sum_{i=1}^{n}(W(t_i^{(n)})-W(t_{i-1}^{(n)}))^2$ converges to t.

13.1.2 By Lemma 13.1.1 the processes $\{\int_0^{\tau_n'\wedge t}Y(v)\,\mathrm{d}W(v)\}$ are local martingales. Thus for each n there exist stopping times $\{\tau_m''\}$ converging to infinity such that the processes $\{\int_0^{\tau_n'\wedge\tau_m''\wedge t}Y(v)\,\mathrm{d}W(v)\}$ are martingales. Because $|\int_0^{\tau_n'\wedge\tau_m''\wedge t}Y(v)\,\mathrm{d}W(v)|\le n$ one gets by bounded convergence that $\{\int_0^{\tau_n'\wedge t}Y(v)\,\mathrm{d}W(v)\}$ is a martingale. The uniform integrability follows from the boundedness.

13.1.3 In order to construct a local martingale that is not a martingale we can for instance find a piecewise constant process that is not integrable. For example let $\{Y_i\}$ be a sequence of independent and identically distributed random variables, independent of $\{W(t)\}$ such that $\mathbb{E}\,|Y_i|=\infty$. Let $\{Y(t)\}$ be defined as $Y(t)=Y_{\lceil t\rceil}$. Then for $0\le t<1$ we have by (13.1.1)

$$\mathbb{E}\Big|\int_0^t Y(v)\,\mathrm{d}W(v)\Big|=\mathbb{E}\,|Y_0W(t)|=\mathbb{E}\,|Y_0|\mathbb{E}\,|W(t)|=\infty\,.$$

Thus $\int_0^t Y(v)\,\mathrm{d}W(v)$ is not a martingale.

13.1.4 Let $\{\tau_n'\}$ be a localization sequence and let $\tau_n=\tau_n'\wedge n$. Clearly $\{\tau_n\}$ is a localization sequence by the stopping theorem. Let $X(t)=M(\tau_n\wedge t)$. Then $X(t)=\mathbb{E}\,(X(n)\mid\mathcal{F}_t)$. This gives

$$\mathbb{E}\,[|\mathbb{E}\,(X(n)\mid\mathcal{F}_t)|;|X(t)|>m]\le\mathbb{E}\,[|X(n)|;|X(t)|>m]\,.$$

Because $\sup_{0\le t\le n}|X(n)|\mathbb{I}(|X(t)|>m)\le|X(n)|$ is integrable it follows by dominated convergence that $\{X(t)\}$ is uniformly integrable.

13.1.5 (a) Let $\{\tau_n\}$ be a localization sequence. Using Jensen's inequality and Fatou's lemma one gets for $v\le t$

$$\mathbb{E}\,(M(t)^2\mid\mathcal{F}_v)\quad\ge\quad\mathbb{E}\,(-|M(t)|\mid\mathcal{F}_v)^2=\mathbb{E}\,(\lim_{n\to\infty}-|M(\tau_n\wedge t)|\mid\mathcal{F}_v)^2$$

$$\begin{aligned}
&\geq \Big(\limsup_{n\to\infty} -\mathbb{E}\left(|M(\tau_n \wedge t)| \mid \mathcal{F}_v\right)\Big)^2 \\
&= \liminf_{n\to\infty} \mathbb{E}\left(|M(\tau_n \wedge t)| \mid \mathcal{F}_v\right)^2 \\
&\geq \liminf_{n\to\infty} |\mathbb{E}\left(M(\tau_n \wedge t) \mid \mathcal{F}_v\right)|^2 = \lim_{n\to\infty} M(\tau_n \wedge v)^2 \\
&= M(v)^2 .
\end{aligned}$$

Thus $\{M(t)^2\}$ is a submartingale.
(b) This follows immediately from (a).
(c) We find

$$\begin{aligned}
\mathbb{E}\left[|M(v)|; |M(v)| > x\right] &\leq \frac{\mathbb{E}\left[M(v)^2; |M(v)| > x\right]}{x} \\
&\leq \frac{\mathbb{E}\, M(v)^2}{x} \leq \frac{\mathbb{E}\, M(t)^2}{x}
\end{aligned}$$

and therefore that $\{M(v) : 0 \leq v \leq t\}$ is uniformly integrable.
(d) This follows from the uniform integrability and

$$\begin{aligned}
\mathbb{E}\left(M(t) \mid \mathcal{F}_v\right) &= \mathbb{E}\left(\lim_{n\to\infty} M(\tau_n \wedge t) \mid \mathcal{F}_v\right) = \lim_{n\to\infty} \mathbb{E}\left(M(\tau_n \wedge t) \mid \mathcal{F}_v\right) \\
&= \lim_{n\to\infty} M(\tau_n \wedge v) = M(v) ,
\end{aligned}$$

where $\{\tau_n\}$ is a localization sequence.

13.1.8 Let $n \in \mathbb{N}$ and $t_i^{(n)} = it/n$. Because we only consider càdlàg functions we have that

$$\lim_{n\to\infty} \frac{t}{n} \sum_{i=0}^{n-1} h^2(t_i) = \int_0^t h^2(v)\,\mathrm{d}v .$$

Hence $Y_n(v) = h(t_i)$ for $t_i \leq v < t_{i+1}$ approximates the process $Y(v) = h(v)$ in the sense of the definition of the stochastic integral. By definition $\int_0^t Y_n(v)\,\mathrm{d}W(v)$ has a normal distribution with mean zero and variance $t\sum_{i=0}^{n-1} h^2(t_i)/n$. We get that the variance converges to $\int_0^t h^2(v)\,\mathrm{d}v$. It is left to the reader to show that a sequence of $\mathrm{N}(\mu_n, \sigma_n^2)$ distributed random variables such that $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$ converges weakly to a $\mathrm{N}(\mu, \sigma^2)$ distributed random variable, see Chapter 2.[1] Thus $\int_0^t h(v)\,\mathrm{d}W(v)$ is normally distributed $\mathrm{N}(0, \int_0^t h^2(v)\,\mathrm{d}v)$.

13.1.10 Consider first

$$\sum_{i=1}^{2^n} \Big(W\Big(\frac{(i-\frac{1}{2})t}{2^n}\Big) - W\Big(\frac{(i-1)t}{2^n}\Big)\Big)\Big(W\Big(\frac{it}{2^n}\Big) - W\Big(\frac{(i-1)t}{2^n}\Big)\Big) .$$

[1] Do we have an exercise?

This sum consists of independent random variables with mean value $2^{-(n+1)}t$ and variance $2^{-2n}t^2$. This gives readily that the above sum converges almost surely to $t/2$. Moreover, we know that

$$\sum_{i=1}^{2^n} W\Big(\frac{(i-1)t}{2^n}\Big)\Big(W\Big(\frac{it}{2^n}\Big) - W\Big(\frac{(i-1)t}{2^n}\Big)\Big)$$

converges to $(W^2(t) - t)/2$. Putting these results together yields the result.

13.1.11 The process $\{M(t)\}$ defined as $M(t) = g(X(t)) - g(X(0)) - \int_0^t \boldsymbol{A}g(X(v))\,\mathrm{d}v$ is a local martingale. Let $\{\tau_n\}$ be a localization sequence and note that $\sup_{0\le w\le t}|M(w)| \le 2\overline{g} + \overline{\boldsymbol{A}g}\,t$ where $\overline{g} = \sup_{0\le w\le t}|g(w)|$ and $\overline{\boldsymbol{A}g} = \sup_{0\le w\le t}|\boldsymbol{A}g(w)|$. Thus by bounded convergence we have for $w < t$

$$\begin{aligned}\mathbb{E}\,(M(t) \mid \mathcal{F}_w) &= \mathbb{E}\,(\lim_{n\to\infty} M(t\wedge\tau_n) \mid \mathcal{F}_w) = \lim_{n\to\infty}\mathbb{E}\,(M(t\wedge\tau_n) \mid \mathcal{F}_w)\\ &= \lim_{n\to\infty} M(w\wedge\tau_n) = M(w)\,.\end{aligned}$$

13.2 Perturbed Risk Processes

Let $\{R(t)\}$ be a risk model, called the unperturbed risk model. Let $\{W(t)\}$ be independent of $\{R(t)\}$. The process $\{X(t)\}$ with $X(t) = R(t) + \varepsilon W(t)$ is called a perturbed risk model. The claim sizes are denoted by $\{U_i\}$ and the occurrence times by $\{\sigma_i\}$. $\{N(t)\}$ is the claim number process $N(t) = \sup\{n \in \mathbb{N} : \sigma_n \le t\}$. In particular, $\{R(t)\}$ is modelled as

$$R(t) = u + \beta t - \sum_{i=1}^{N(t)} U_i\,.$$

In a perturbed Sparre Andersen model the adjustment coefficient is the strictly positive number such that $\mathbb{E}\,\mathrm{e}^{-\gamma(X(\sigma_n)-u)} = 1$, if such an exponent exists. We let $T = \sigma_2 - \sigma_1$ and $\lambda = 1/\mathbb{E}\,T$.

Exercises

13.2.1 Let $\{M(t) : t \ge 0\}$ be a non-negative local martingale. Show that $\{M(t)\}$ is a supermartingale. Conclude that if $\mathbb{E}\,M(t) = \mathbb{E}\,M(0)$ then $\{M(t)\}$ is a martingale.

13.2.2 Let $X(t) = \sigma W(t) + \mu t$ be (μ, σ^2)-Brownian motion and $M(t) = \sup_{0\le v\le t} X(v)$. Show that for $x \le m$

$$\mathbb{P}(M(t) > m, X(t) \le x) \quad = \quad \frac{1}{\sqrt{2\pi\sigma^2 t}}\int_{-\infty}^{x} \exp(-(z-\mu t)^2/(2\sigma^2 t))$$

$$\times \exp(-2m(m-z)/(\sigma^2 t))\,\mathrm{d}z$$

and conclude that the joint distribution of $(M(t), X(t))$ has density

$$\begin{aligned} f(m,x) &= \mathbb{I}(m \geq x)\frac{2(2m-x)}{\sqrt{2\pi(\sigma^2 t)^3}} \\ &\quad \times \exp(-((x-\mu t)^2 + 4m(m-x))/(2\sigma^2 t)) \end{aligned}$$

[Hint. Use that $\{vW(1/v)\}$ is a standard Brownian motion (Exercise [**??e10.3.10??**]).]

13.2.3 Let $\{X(t)\}$ be the perturbed compound Poisson model. Define

$$\tau^+ = \inf\{\sigma_i : X(\sigma_i) < \inf\{X(t) : 0 \leq t < \sigma_i\}\}$$

and if $\tau^+ < \infty$

$$Y_{\mathrm{c}}^+ = -\inf\{X(t) : 0 < t < \tau^+\}, \qquad Y_{\mathrm{d}}^+ = -X(\tau^+) - Y_{\mathrm{c}}^+ .$$

Assume that the adjustment coefficient γ exists. Define the new measure $\mathbb{P}^{(\gamma)}$ via the Radon-Nikodym derivative $\mathrm{d}\mathbb{P}^{(\gamma)} = \mathrm{e}^{-\gamma(X(t)-u)}\,\mathrm{d}\mathbb{P}$. Verify

$$\mathbb{E}^{(\gamma)}(Y_d^+ + Y_c^+) = \frac{\lambda \hat{m}_U^{(1)}(\gamma) - \beta + \varepsilon^2\gamma}{\gamma(\beta - \gamma\varepsilon^2/2)} .$$

13.2.4 Let $\{X(t)\}$ be the perturbed compound Poisson model. Let $\psi_2(u) = \mathbb{P}(\tau < \infty, X(\tau) = 0)$. Assume that the adjustment coefficient γ exists and that $\hat{m}_U^{(1)}(\gamma) < \infty$. Verify

$$\lim_{u\to\infty} \psi_2(u)\mathrm{e}^{\gamma u} = \frac{\gamma\varepsilon^2/2}{\lambda\hat{m}_U^{(1)}(\gamma) - \beta + \varepsilon^2\gamma} .$$

13.2.5 Consider the perturbed Sparre Andersen risk model $\{X(t)\}$. Let $T(t) = \sigma_{N(t)+1} - t$ be the time till the next jump. Suppose $g(y,w)$ is twice continuously differentiable with respect to y and continuously differentiable with respect to w such that

$$g(y,0) = \int_0^\infty \int_0^\infty g(y-z,w)\,\mathrm{d}F_T(w)\,\mathrm{d}F_U(z)$$

and

$$\mathbb{E}\Big(\sum_{i=1}^{N(t)\wedge n} |g(X(\sigma_i), T(\sigma_i)) - g(X(\sigma_i - 0), 0)|\Big) < \infty ,$$

$t \geq 0, n \in \mathbb{N}$. Show that g is in the domain of the extended generator $\mathcal{D}(\boldsymbol{A})$ and the generator has the form

$$\boldsymbol{A}g(y,w) = \frac{\varepsilon^2}{2}\frac{\partial^2}{\partial y^2}g(y,w) + \beta\frac{\partial}{\partial y}g(y,w) - \frac{\partial}{\partial w}g(y,w) .$$

13.2.6 Let U and T be positive random variables and β and ε be positive constants. Consider the equation

$$\hat{m}_U(s)\hat{l}_T(\beta s - \varepsilon^2 s^2/2) = 1\,. \tag{13.2.1}$$

Show that there is at most one solution $\gamma \neq 0$ and that this solution is strictly positive if $\beta \mathbb{E}\, T > \mathbb{E}\, U$. [Hint. Define $\theta(s)$ as the solution to $\hat{m}_U(s)\hat{l}_T(\beta s + \theta(s) - \varepsilon^2 s^2/2) = 1$ and show that $\theta(s)$ is convex.]

13.2.7 Let $\{X(t)\}$ be the perturbed Sparre Andersen model introduced in Exercise 13.2.5.
(a) Assume the the adjustment coefficient γ exists. Show that γ solves (13.2.1).
(b) Suppose that $\gamma > 0$ solves (13.2.1). Show that γ is the adjustment coefficient.

13.2.8 Let $\{X(t)\}$ be the perturbed Sparre Andersen risk model introduced in Exercise 13.2.5. Assume the adjustment coefficient γ exists, that is a positive solution to (13.2.1). Show that the process $\{M(t)\}$ defined via

$$M(t) = \mathrm{e}^{-(\beta\gamma - \varepsilon^2\gamma^2/2)T(t)}\mathrm{e}^{-sX(t)}$$

is a $\{\mathcal{F}_t^{X,T}\}$-martingale, where $\{\mathcal{F}_t^{X,T}\}$ is the natural filtration of the process $\{(X(t), T(t))\}$.

13.2.9 Assume that the adjustment coefficient in the unperturbed Sparre Andersen risk model exists. Show that γ, defined as the positive solution to (13.2.1) in the perturbed Sparre Andersen risk model exists.

Solutions

13.2.1 Let $\{\tau_n\}$ be a localization sequence. Note that $\mathbb{E}\, M_0 = \mathbb{E}\, M_{0\wedge\tau_n} < \infty$. Then by Fatou's lemma

$$\begin{aligned}
\mathbb{E}\,(M(t) \mid \mathcal{F}_v) &= \mathbb{E}\,(\lim_{n\to\infty} M(t\wedge\tau_n) \mid \mathcal{F}_v) \le \liminf_{n\to\infty} \mathbb{E}\,(M(t\wedge\tau_n) \mid \mathcal{F}_v) \\
&= \lim_{n\to\infty} M(v\wedge\tau_n) = M(v)
\end{aligned}$$

for $0 \le v < t$. In particular, $\mathbb{E}\, M(t) \le \mathbb{E}\, M(0) < \infty$. Thus $\{M(t)\}$ is a supermartingale. Since

$$\begin{aligned}
\mathbb{E}\, M(0) &= \mathbb{E}\, M(t) = \mathbb{E}\,(\mathbb{E}\,(M(t) \mid \mathcal{F}_v)) \\
&= \mathbb{E}\,(\mathbb{E}\,(M(t) \mid \mathcal{F}_v)\mathbb{I}(\mathbb{E}\,(M(t) \mid \mathcal{F}_v) < M(v))) \\
&\quad + \mathbb{E}\,(\mathbb{E}\,(M(t) \mid \mathcal{F}_v)\mathbb{I}(\mathbb{E}\,(M(t) \mid \mathcal{F}_v) = M(v))) \\
&= \mathbb{E}\,(\mathbb{E}\,(M(t) \mid \mathcal{F}_v)\mathbb{I}(\mathbb{E}\,(M(t) \mid \mathcal{F}_v) < M(v))) \\
&\quad + \mathbb{E}\,(M(v)\mathbb{I}(\mathbb{E}\,(M(t) \mid \mathcal{F}_v) = M(v))) \\
&\le \mathbb{E}\, M(v) = \mathbb{E}\, M(0)
\end{aligned}$$

we have

$$\begin{aligned}&\mathbb{E}\,(\mathbb{E}\,(M(t)\mid\mathcal{F}_v)\mathbb{I}(\mathbb{E}\,(M(t)\mid\mathcal{F}_v)<M(v)))\\ &=\ \mathbb{E}\,(M(v)\mathbb{I}(\mathbb{E}\,(M(t)\mid\mathcal{F}_v)<M(v)))\,.\end{aligned}$$

This implies $\mathbb{P}(\mathbb{E}\,(M(t)\mid\mathcal{F}_v)<M(v))=0$.

13.2.2 It follows readily that

$$\begin{aligned}&\mathbb{P}(M(t)>m, X(t)\le x)\\ &=\ \mathbb{P}\Big(\sup_{v\ge t^{-1}}\sigma W(v)+\mu-mv>0, \sigma W(1/t)+\mu\le x/t\Big)\\ &=\ \mathbb{P}\Big(\sup_{v\ge t^{-1}}\sigma(W(v)-W(\frac{1}{t}))-m(v-\frac{1}{t})>\frac{m}{t}-\sigma W(\frac{1}{t})-\mu,\\ &\qquad\qquad \sigma W(\frac{1}{t})\le\frac{x}{t}-\mu\Big)\,.\end{aligned}$$

The process $\{\sigma(W(v)-W(t^{-1}))-m(v-t^{-1}):v\ge t^{-1}\}$ is a Brownian motion with drift and therefore

$$\begin{aligned}&\mathbb{P}(M(t)\le m, X(t)\le x)\\ &=\ \mathbb{E}\,[\mathrm{e}^{-2m(mt^{-1}-\sigma W(t^{-1})-\mu)/\sigma^2};\sigma W(t^{-1})\le x/t-\mu]\,.\end{aligned}$$

Straightforward calculation gives now the desired results.

13.2.5 Considering the process $\{X(t)\}$ up to time σ_n we have

$$\begin{aligned}X(\sigma_n\wedge t)\ &=\ u+\sum_{i=1}^{N(t)\wedge n}\Big(\int_{\sigma_{i-1}}^{\sigma_i}(\beta\,\mathrm{d}v+\varepsilon\,\mathrm{d}W(v))-U_i\Big)\\ &\quad+\int_{\sigma_{N(t)\wedge n}}^{\sigma_n\wedge t}(\beta\,\mathrm{d}v+\varepsilon\,\mathrm{d}W(v))\,.\end{aligned}$$

Applying Itô's formula to the process $\{M_n(t)\}$ with

$$M_n(t)=g(X(\sigma_n\wedge t),T(\sigma_n\wedge t))-g(u,T(0))-\int_0^{\sigma_n\wedge t}\boldsymbol{A}g(X(v),T(v))\,\mathrm{d}v\,,$$

yields

$$\begin{aligned}M_n(t)\ &=\ \int_0^{\sigma_n\wedge t}\varepsilon\frac{\partial}{\partial y}g(X(v),T(v))\,\mathrm{d}W(v)\\ &\quad+\sum_{i=1}^{N(t)\wedge n}g(X(\sigma_i),T(\sigma_i))-g(X(\sigma_i-0),0)\,.\end{aligned}$$

The stochastic integral is by construction a local martingale. Consider an increment of the sum. We have

$$\begin{aligned}
&\mathbb{E}\left(g(X(\sigma_i), T(\sigma_i)) \mid X(\sigma_i - 0) = y\right) \\
&\quad = \int_0^\infty \int_0^\infty g(y-z, w)\,\mathrm{d}F_T(w)\,\mathrm{d}F_U(z) = g(y, 0)
\end{aligned}$$

and therefore the sum is a martingale. Note that the sum is integrable by the conditions stated. Let σ_n' be a localization sequence of the local martingale $\{\int_0^t \varepsilon \frac{\partial}{\partial y} g(X(v), T(v))\,\mathrm{d}W(v)\}$ then $\{M_n(t \wedge \sigma_n')\}$ is a martingale. This implies that $\{M(t)\}$ defined as

$$M(t) = g(X(t), T(t)) - g(u, T(0)) - \int_0^t \boldsymbol{A} g(X(v), T(v))\,\mathrm{d}v$$

is a local martingale. Thus $g \in \mathcal{D}(\boldsymbol{A})$.

13.2.6 The function $\hat{l}_T(s')$ is monotone and therefore there is at most one solution s_0' to $\hat{m}_U(s)\hat{l}_T(s')$ for every fixed s. This solution exists in any case if $s \ge 0$. Thus $\theta(s) = s_0' - \beta s + \varepsilon^2 s^2/2$ is uniquely defined. Because $\hat{m}_U(s)$ and $\hat{l}_T(s)$ are continuous functions there is an interval containing zero on which $\theta(s)$ is well-defined. A solution γ to (13.2.1) is equivalent to $\theta(\gamma) = 0$. Without loss of generality we can assume that $\theta(s)$ is defined on an interval different from $[0, 0]$.

Let $m(s) = \log \hat{m}_U(s)$ and $l(s) = \log \hat{l}_T(s)$. The equation can be written as $m(s) + l(\beta s + \theta(s) - \varepsilon^2 s^2/2) = 0$. By the implicit function theorem $\theta(s)$ is differentiable and

$$m^{(1)}(s) + (\beta + \theta^{(1)}(s) - \varepsilon^2 s) l^{(1)}(\beta s + \theta(s) - \varepsilon^2 s^2/2) = 0\,.$$

It follows that $\theta(s)$ is infinitely often differentiable and

$$\begin{aligned}
m^{(2)}(s) &+ (\beta + \theta^{(1)}(s) - \varepsilon^2 s)^2 l^{(2)}(\beta s + \theta(s) - \varepsilon^2 s^2/2) \\
&\qquad + (\theta^{(2)}(s) - \varepsilon^2) l^{(1)}(\beta s + \theta(s) - \varepsilon^2 s^2/2) = 0\,.
\end{aligned}$$

Because $m(s)$ and $l(s)$ are convex functions[2] and $l(s)$ is decreasing it follows that $\theta''(s) - \varepsilon^2 \ge 0$, i.e. $\theta(s)$ is strictly convex. Since clearly $\theta(0) = 0$ there can at most be another solution $\gamma \ne 0$ to $\theta(s) = 0$. Note that $m^{(1)}(0) = \mathbb{E}U$ and $l^{(1)}(0) = -\mathbb{E}T$. This yields $\theta^{(1)}(0) = (\mathbb{E}U - \beta\mathbb{E}T)/\mathbb{E}T$. Thus if $\beta\mathbb{E}T > \mathbb{E}U$ we have $\theta^{(1)}(0) < 0$ and $\theta(s) > 0$ for all $s < 0$.

[2] Add an Exercise in Chapter 2.

13.2.8 The function $g(y,w) = \mathrm{e}^{-(\beta\gamma-\varepsilon^2\gamma^2/2)w}\mathrm{e}^{-\gamma y}$ fulfils the conditions of Exercise 13.2.5. Thus $\{M(t)\}$ is a local martingale. Since $\{W(t)\}$ and $\{R(t)\}$ are independent we find

$$\begin{aligned} \mathbb{E}\, M(t) &= \mathbb{E}\,\mathrm{e}^{-(\beta\gamma-\varepsilon^2\gamma^2/2)T(t)-\gamma R(t)-\gamma\varepsilon W(t)} \\ &= \mathbb{E}\,\mathrm{e}^{-(\beta\gamma-\varepsilon^2\gamma^2/2)T(t)-\gamma(R(t)-\varepsilon^2\gamma t/2)}\,. \end{aligned}$$

The process $\{R(t) - \varepsilon^2\gamma/2t\}$ is a unperturbed Sparre Andersen model with premium rate $\beta - \varepsilon^2\gamma/2$. The process $\{M'(t)\}$ defined via

$$M'(t) = \mathrm{e}^{-(\beta\gamma-\varepsilon^2\gamma^2/2)T(t)-\gamma(R(t)-\varepsilon^2\gamma t/2)}$$

is then by Theorem 11.5.2 a martingale. Thus

$$\mathbb{E}\, M(t) = \mathbb{E}\, M'(t) = \mathbb{E}\, M'(0) = \mathbb{E}\, M(0)$$

and $\{M(t)\}$ is a martingale by Exercise 13.2.1.

13.2.9 Let γ' be the adjustment coefficient in the unperturbed Sparre Andersen model. Then $\hat{m}_U(\gamma')\hat{l}_T(\beta\gamma') = 1$. This implies $\theta(\gamma') = \varepsilon^2\gamma'^2/2$ where $\theta(s)$ was defined in Exercise 13.2.6. Because $\theta(s)$ is convex and $\theta(s) < 0$ for $0 < s$ small enough, there must be a $\gamma \in (0,\gamma')$ such that $\theta(\gamma) = 0$. Thus the adjustment coefficient in the perturbed risk model exists.

13.3 Other Applications to Insurance and Finance

Let now $\{\mathcal{F}_t\}$ be the natural filtration of the Brownian motion $\{W(t)\}$. A Black Scholes model is an economy with two assets $\{X(t)\}$ and $\{I(t)\}$, where $\mathrm{d}X(t) = \mu X(t)\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}W(t)$ and $\mathrm{d}I(t) = \delta I(t)\,\mathrm{d}t$ with $I(0) = 0$. We work with a finite time horizon t_0. A $\mathcal{F}_{t_0}$-measurable claim Z has at time t the price $\tilde{\mathbb{E}}\,(ZI(t)/I(t_0) \mid \mathcal{F}_t)$ where $\tilde{\mathbb{P}}$ is an equivalent measure such that the process $\{W^*(t)\}$ defined as $W(t) + (\mu-\delta)t/\sigma$ is a Brownian motion under $\tilde{\mathbb{P}}$. Thus $\mathrm{d}X(t) = \delta X(t)\,\mathrm{d}t + \sigma\,\mathrm{d}\tilde{W}^*(t)$. A trading strategy is a two-dimensional adapted càdlàg process $\{(\alpha(t),\gamma(t))\}$. A trading strategy is called *self-financing* if the value process $\{V(t)\}$ with $V(t) = \alpha(t)X(t)+\gamma(t)I(t)$ has the property $\mathrm{d}V(t) = \alpha(t)\,\mathrm{d}X(t) + \gamma(t)\,\mathrm{d}I(t)$ and if $\{\int_0^t \alpha(v)X(v)/I(v)\,\mathrm{d}W^*(v)\}$ is a martingale. If a claim Z has the property that $\tilde{E}Z^2 < \infty$ then there exists always a self-financing trading strategy $\{(\alpha(t),\gamma(t))\}$, called *duplication strategy*, such that

$$Z = \tilde{\mathbb{E}}\,Z/I(t_0) + \int_0^{t_0} \alpha(v)\,\mathrm{d}X(v) + \int_0^{t_0} \gamma(v)\,\mathrm{d}I(v)\,.$$

Exercises

13.3.1 Let $\{M(t)\}$ be a bounded local martingale. Show that $\{M(t)\}$ is a martingale.

13.3.2 Fix $b < 0 < a$. Let $g : [0,1) \to \mathbb{R}_+$ be a continuous function such that $\int_0^1 g(t)^2 \,\mathrm{d}t = \infty$. Define the stopping times $\tau_a = \inf\{t \geq 0 : \int_0^t g(v)\,\mathrm{d}W(v) = a\}$ and $\tau_{a,b} = \inf\{t \geq 0 : \int_0^t g(v)\,\mathrm{d}W(v) \in \{a,b\}\}$. Let t_n be such that $\int_0^{t_n} g(t)^2\,\mathrm{d}t = n$. We consider the special case of a Black-Scholes model with $I(t) = 1$ and $\mathrm{d}X(t) = X(t)\,\mathrm{d}W(t)$.
(a) Show $\{\int_0^{t_n} g(t)\,\mathrm{d}W(t) : n \in \mathbb{N}\}$ is a random walk with standard normally distributed increments.
(b) Show that $\mathbb{P}(\tau_a \in (0,1)) = 1$.
(c) Show that there exists a trading strategy $\{(\alpha(t), \gamma(t))\}$ fulfilling $\alpha(t)X(t) + \gamma(t) = \int_0^t \alpha(v)\,\mathrm{d}X(v)$ with $\alpha(1)X(1) + \gamma(1) = a$. That is, it is possible to generate the capital a out of nothing. [Hint. Use (b).]
(d) Calculate $\mathbb{P}(\tau_{a,b} \neq \tau_a)$.
(e) Show $\mathbb{E}\left(-\inf_{0\leq t\leq 1}(\alpha(t)X(t) + \gamma(t))\right) = \infty$, i.e. the expected capital needed to play the above strategy is infinite.

13.3.3 Let $L(t) = \exp\{-\kappa W(t) - \kappa^2 t/2\}$. Define the new measure $\tilde{\mathbb{P}}$ on $\mathcal{F}_1$ via the Radon-Nikodym derivative $\mathrm{d}\tilde{\mathbb{P}} = L(t)\,\mathrm{d}\mathbb{P}$. Let $\{W^*(t) : 0 \leq t \leq 1\}$ be defined as $W^*(t) = W(t) + \kappa t$. Show that $\{W^*(t) : 0 \leq t \leq 1\}$ is a standard Brownian motion under the measure $\tilde{\mathbb{P}}$.

13.3.4 Show that the price at time t of the European call option with strike price K, $g(X(t_0)) = (X(t_0) - K)_+$, is $C(X(t), t_0 - t)$ where

$$
\begin{aligned}
C(x,y) \quad &= \quad x\Phi\left(\frac{\log(x/K) + (\delta + \sigma^2/2)y}{\sigma\sqrt{y}}\right) \\
&\qquad\qquad - K\mathrm{e}^{-\delta y}\Phi\left(\frac{\log(x/K) + (\delta - \sigma^2/2)y}{\sigma\sqrt{y}}\right).
\end{aligned}
$$

Show, moreover, that the duplicating strategy is given by

$$
\begin{aligned}
\alpha(t) \quad &= \quad \Phi\left(\frac{\log(X(t)/K) + (\delta + \sigma^2/2)(t_0 - t)}{\sigma\sqrt{t_0 - t}}\right), \\
\gamma(t) \quad &= \quad -K\mathrm{e}^{-\delta t_0}\Phi\left(\frac{\log(X(t)/K) + (\delta - \sigma^2/2)(t_0 - t)}{\sigma\sqrt{t_0 - t}}\right).
\end{aligned}
$$

13.3.5 Let for $x > 0$ and $0 \leq t < t_0$

$$
\begin{aligned}
f(x,t) \quad &= \quad x\Phi\left(\frac{\log(x/K) + (\delta + \sigma^2/2)(t_0 - t)}{\sigma\sqrt{t_0 - t}}\right) \\
&\qquad\qquad - K\mathrm{e}^{-\delta(t_0 - t)}\Phi\left(\frac{\log(x/K) + (\delta - \sigma^2/2)(t_0 - t)}{\sigma\sqrt{t_0 - t}}\right).
\end{aligned}
$$

Verify that $f(x,t)$ fulfils the differential equation

$$f_t(x,t) + \frac{1}{2} f_{xx}(x,t)\sigma^2 x^2 + f_x(x,t)\delta x - \delta f(x,t) = 0\,.$$

13.3.6 Suppose an equity linked insurance contract with guaranteed sum G is written with an a-year old. Let T_a denote the time of death of the customer. A term insurance contract gives then the payoff $\mathbb{I}(T_a \le t_0)(G \vee X(T_a))$ and a pure endowment contract gives the payoff $\mathbb{I}(T_a > t_0)(G \vee X(t_0))$. Let $F_a(t) = \mathbb{P}(T_a \le t)$ and assume that $F_a(t)$ is absolutely continuous with density $\lambda_{a+t}\overline{F}_a(t)$. We assume that the premium rate is proportional to the equity price, i.e. the (fair) premium rate at time t is $\delta X(t)$. It is assumed that T_a and $\{X(t)\}$ are independent. We use the equivalence principle for the calculation of premiums, that is expected premiums and expected outflow coincide under the measure $\tilde{\mathbb{P}}$, where T_a is not affected by the change of measure. Show that

$$\delta = \frac{\displaystyle\int_0^{t_0} \tilde{\mathbb{E}}\left((G \vee X(v))/I(v)\right)\lambda_{a+v}\left(1 - F_a(v)\right) \mathrm{d}v}{X(0)\,\mathbb{E}\int_0^{T_a \wedge t_0} I(w)^{-1}\,\mathrm{d}w}$$

in the term insurance case and

$$\delta = \frac{\tilde{\mathbb{E}}\left((G \vee X(t_0))/I(t_0)\right)\left(1 - F_a(t_0)\right)}{X(0)\,\mathbb{E}\int_0^{T_a \wedge t_0} I(w)^{-1}\,\mathrm{d}w}$$

in the pure endowment insurance case. Explain why $\delta X(0) = \beta$ where β is the (fair) premium rate in the case where the premium rate is constant.

13.3.7 Suppose a company sells a classical life insurance, paying out the amount b upon survival up to time t_0. The premium is paid as a single premium. The force of interest $\{\delta(t)\}$ is assumed to follow an Ornstein-Uhlenbeck process

$$\mathrm{d}\delta(v) = -\alpha(\delta(v) - \bar{\delta})\,\mathrm{d}v + \sigma\,\mathrm{d}W(v).$$

Let T be the time of death and

$$\mu_1(t,z) = \mathbb{E}\Big(b\exp\Big(-\int_t^{t_0} \delta(v)\,\mathrm{d}v\Big)\mathbb{I}(T > t_0)\;\Big|\; T > t, \delta(t) = z\Big)$$

denote the reserve if the customer still is alive at time t.

(a) Calculate $\mu_1(t,z)$ directly.

(b) Verify that $\mu_1(t,z)$ fulfils the differential equation

$$-\mu_1(t,z)q_{12}(t) - z\mu_1(t,z) + \frac{\partial \mu_1}{\partial t}(t,z) - \alpha(\delta(v)-\bar{\delta})\frac{\partial \mu_1}{\partial z}(t,z) + \frac{\sigma^2}{2}\frac{\partial^2 \mu_1}{\partial z^2}(t,z) = 0$$

with the boundary condition $\mu_1(t_0,z) = b$. Here $q_{12}(t)$ is the force of mortality, i.e. $\mathbb{P}(T > t) = \exp(-\int_0^t q_{12}(v)\,\mathrm{d}v)$.

13.3.8 Let $\{\delta(t)\}$ follow a Merton model $\delta(t) = \alpha t + \sigma W(t)$. Calculate

$$\delta'(t) = -\frac{\partial}{\partial t}\log \mathbb{E}\exp\Big(-\int_0^t \delta(v)\,dv\Big).$$

Solutions

13.3.1 Let $\{\tau_n\}$ be a localization sequence. Then $\mathbb{E}(M(\tau_n \wedge t) \mid \mathcal{F}_v) = M(\tau_n \wedge v)$. The result follows by bounded convergence.

13.3.2 (a) This follows from Exercise 13.1.8.
(b) Because $\limsup_{n\to\infty}\int_0^{t_n} g(t)\,\mathrm{d}W(t) = \infty$ (Theorem 6.3.1c) we get $\tau_a < 1$.
(c) Such a strategy is obtained by choosing $\alpha(t) = \mathbb{I}(t < \tau_a)g(t)/X(t)$ and $\gamma(t) = \int_0^t \alpha(v)\,\mathrm{d}X(v) - \alpha(t)X(t)$. Then

$$\alpha(1)X(1) + \gamma(1) = \int_0^1 \alpha(v)\,\mathrm{d}X(v) = \int_0^{\tau_a} g(v)\,\mathrm{d}W(v) = a\,.$$

(d) The process $\{\int_0^{\tau_{a,b}\wedge t} g(v)\,\mathrm{d}W(v)\}$ is bounded and therefore a uniformly integrable martingale. The optional stopping theorem yields

$$0 = \mathbb{E}\int_0^{\tau_{a,b}} g(v)\,\mathrm{d}W(v) = a\mathbb{P}(\tau_{a,b} = \tau_a) + b\mathbb{P}(\tau_{a,b} \neq \tau_a)$$

from which $\mathbb{P}(\tau_{a,b} \neq a) = a/(a-b)$ follows.
(e) Note that

$$\alpha(t)X(t) + \gamma(t) = \int_0^{\tau_a\wedge t} g(v)\,\mathrm{d}W(v)\,.$$

Since $\{\inf_{0\le t\le \tau_a}\int_0^t g(v)\,\mathrm{d}W(v) \le b\} = \{\tau_{a,b} \neq \tau_a\}$ we have

$$\mathbb{E}\,(-\inf_{0\le t\le 1}(\alpha(t)X(t) + \gamma(t))) = \int_{-\infty}^0 \mathbb{P}(\tau_{a,b} \neq \tau_a)\,\mathrm{d}b = \infty\,.$$

13.3.3 It is clear that $W^*(0) = 0$ and that $\{W^*(t)\}$ has continuous paths. From

$$\begin{aligned}
&\tilde{\mathbb{E}}\left(\exp(s(W^*(t) - W^*(v))) \mid \mathcal{F}_v\right) \\
&= \frac{\mathbb{E}\left(\exp(s(W^*(t) - W^*(v)))L(t) \mid \mathcal{F}_v\right)}{L(v)} \\
&= \mathbb{E}\left(\exp(s(W(t) - W(v) + \kappa(t-v)) - \kappa(W(t) - W(v))\right. \\
&\qquad\qquad \left. - \kappa^2(t-v)/2) \mid \mathcal{F}_t\right) \\
&= \mathbb{E}\exp((s-\kappa)(W(t) - W(v)) + (s\kappa - \kappa^2/2)(t-v)) \\
&= \exp(s^2(t-v)/2)\,.
\end{aligned}$$

It follows that $\{W^*(t)\}$ has stationary and independent increments that are normally distributed with mean zero and the same variance as a Brownian motion. Thus $\{W^*(t)\}$ is a Brownian motion under $\tilde{\mathbb{P}}$.

13.4 Simple Interest Rate Models

In this section we use the natural filtration of the Brownian motion $\{W(t)\}$. The force of interest $\{\delta(t)\}$ is modelled as a diffusion. That is, the value at time t of a monetary unit invested at time zero is $I(t) = \exp(\int_0^t \delta(v)\,\mathrm{d}v)$. A zero coupon bond with maturity t is a contract that pays out one monetary unit at time t. Its value at time $v \le t$ is denoted by $D(v,t)$. We assume that there is an equivalent measure $\tilde{\mathbb{P}}$ such that

$$D(v,t) = \tilde{\mathbb{E}}\left(\exp\Bigl(-\int_v^t \delta(w)\,\mathrm{d}w\Bigr) \;\Big|\; \mathcal{F}_v\right).$$

Exercises

13.4.1 Let $\{W^*(t)\}$ be the Brownian motion under the measure $\tilde{\mathbb{P}}$. Show that $D(t,t_0)$ is a diffusion process of the form

$$\mathrm{d}D(t,t_0) = \delta(t)D(t,t_0)\,\mathrm{d}t + \sigma_D(t)D(t,t_0)\,\mathrm{d}W^*(t)\,. \tag{13.4.1}$$

13.4.2 Let $\{Z(t)\} \in L^2_{\mathrm{loc}}$ such that $\{L(t)\}$ with

$$L(t) = \exp\Bigl(-\int_0^t Z(v)\,\mathrm{d}W(v) - \tfrac{1}{2}\int_0^t Z(v)^2\,\mathrm{d}v\Bigr)$$

is a martingale. Define the measure $\tilde{\mathbb{P}}$ on the σ-algebra $\mathcal{F}_{t_0}$ as $\mathrm{d}\tilde{\mathbb{P}}/\mathrm{d}\mathbb{P} = L(t_0)$. Let $W^*(t) = W(t) + \int_0^t Z(v)\,\mathrm{d}v$. Show that $\{W^*(t)\}$ is a Brownian motion under $\tilde{\mathbb{P}}$.

13.4.3 Let $\{Z(t)\} \in L^2_{\rm loc}$ and assume that

$$\mathbb{E} \exp\Big(\tfrac{1}{2} \int_0^t Z(v)^2 \, \mathrm{d}v\Big) < \infty$$

for all $t \geq 0$. Define the process $\{L(t)\}$ as

$$L(t) = \exp\Big(\int_0^t Z(v) \, \mathrm{d}W(v) - \int_0^t Z(v)^2 \, \mathrm{d}v\Big)$$

and the process $\{M(t)\}$ as

$$M(t) = \exp\Big(\tfrac{1}{2} \int_0^t Z(v) \, \mathrm{d}W(v)\Big) .$$

(a) Show that $\{L(t)\}$ is a supermartingale.
(b) Show that $\{\int_0^t Z(v) \, \mathrm{d}W(v)\}$ is a martingale.
(c) Show that $M(t)$ is integrable for all $t \geq 0$ and conclude that $\{M(t)\}$ is a submartingale.
(d) Let for $0 < a < 1$ the process $\{L_a(t)\}$ be defined as

$$\begin{aligned} L_a(t) &= \exp\Big(a \int_0^t Z(v) \, \mathrm{d}W(v) - \frac{a^2}{2} \int_0^t Z(v)^2 \, \mathrm{d}v\Big) \\ &= (L(t))^{a^2} \Big(\exp\Big(\frac{2a}{1+a} \frac{1}{2} \int_0^t Z(v) \, \mathrm{d}W(v)\Big)\Big)^{1-a^2} . \end{aligned}$$

Let $x \geq 0$. For $0 < t \leq t_0$ show the inequalities

$$\begin{aligned} \mathbb{E}\,[L_a(t); L_a(t) > x] &\leq (\mathbb{E}\, L(t))^{a^2} (\mathbb{E}\,[M(t); L_a(t) > x])^{2a(1-a)} \\ &\leq (\mathbb{E}\,[M(t_0); \sup_{v \leq t_0} L_a(v) > x])^{2a(1-a)} . \quad (13.4.2) \end{aligned}$$

(e) Show that $\{L_a(t)\}$ is a martingale.
(f) Show that $\{L(t)\}$ is a martingale. [Hint. Let $a \uparrow 1$ in (13.4.2) for an appropriately chosen x.]

13.4.4 The value $I(t)$ at time t of a monetary unit invested at time zero fulfils $\mathrm{d}I(t) = \delta(t) I(t) \, \mathrm{d}t$ with $I(0) = 1$. Assume that there is an equivalent measure $\tilde{\mathbb{P}}$ such that $D(t, t_i) = \tilde{\mathbb{E}}\,(I(t)/I(t_i) \mid \mathcal{F}_t)$. By Exercise 13.4.1 we have

$$\mathrm{d}D(t, t_i) = \delta(t) D(t, t_i) \, \mathrm{d}t + \sigma_i(t) D(t, t_i) \, \mathrm{d}W^*(t)$$

where $\{W^*(t)\}$ is a Brownian motion under $\tilde{\mathbb{P}}$. The trading strategy $\{(\alpha_1(t), \ldots, \alpha_n(t), \gamma(t))\}$ denotes the number $\alpha_i(t)$ of zero-coupon bonds with maturity t_i in the portfolio at time t and $\gamma(t)I(t)$ is the amount invested in the riskless bond. We assume $\alpha_i(t) = 0$ for $t > t_i$. Then

$$V(t) = \sum_{i=1}^{n} \alpha_i(t) D(t, t_i) + \gamma(t) I(t)$$

is the value of the portfolio at time t.
(a) Formulate the condition for a trading strategy to be self-financing. [Hint. For the technical condition needed solve (b) first.]
(b) Let $V^*(t) = V(t)/I(t)$ and assume that $\{(\alpha_1(t), \ldots, \alpha_n(t), \gamma(t))\}$ is self-financing. Show that $\{V^*(t)\}$ is a martingale under $\tilde{\mathbb{E}}$.
(c) Show that there is no arbitrage in a market with securities $\{D(t, t_i)\}$ $(i \le n)$ and $\{I(t)\}$, that is for a self-financing trading strategy such that $V(0) \le 0$ and $\mathbb{P}(V(t_0) \ge 0) = 1$ we must have $\mathbb{P}(V(t_0) = V(0) = 0) = 1$.

13.4.5 We assume, see Exercise 13.4.1, that $\{D(t, t_0)\}$ fulfils the stochastic differential equation (13.4.1). Assume that the process $\{\tilde{L}(t) : 0 \le t \le t_0\}$ defined as

$$\tilde{L}(t) = \exp\Bigl(\int_0^t \sigma_D(v)\, \mathrm{d}W^*(v) - \tfrac{1}{2} \int_0^t \sigma_D(v)^2\, \mathrm{d}v\Bigr)$$

is a martingale under $\tilde{\mathbb{P}}$. We define the measure $\mathbb{P}_{t_0}$ via the Radon-Nikodym derivative $\mathrm{d}\mathbb{P}_{t_0}/\mathrm{d}\tilde{\mathbb{P}} = \tilde{L}(t_0)$. The process $\{W_{t_0}(t)\}$ with $W_{t_0}(t) = W^*(t) - \int_0^t \sigma_D(v)\, \mathrm{d}v$ is then a Brownian motion under $\mathbb{P}_{t_0}$. The measure $\mathbb{P}_{t_0}$ is called *forward adjusted risk measure.* Let $\{X(t)\}$ be a diffusion process such that $\{X(t)\exp(-\int_0^t \delta(v)\, \mathrm{d}v)\}$ is a martingale under $\tilde{\mathbb{P}}$. For example, $X(t)$ is the price of a financial claim as for instance the price of a zero-coupon bond with maturity $t_1 \le t_0$.
(a) Solve the stochastic differential equation (13.4.1).
(b) Show that $X(t)/D(t, t_0)$ is a local martingale under $\mathbb{P}_{t_0}$.
(c) Show that $X(t) = D(t, t_0)\mathbb{E}_{t_0}(X(t_0) \mid \mathcal{F}_t)$ for $t \le t_0$. Conclude that $X(t)/D(t, t_0)$ is in fact a martingale.

13.4.6 Let $\{\delta(t)\}$ follow a Vasicek model, i.e. $\mathrm{d}\delta(t) = -\alpha(\delta(t) - \bar{\delta})\, \mathrm{d}t + \sigma\, \mathrm{d}W^*(t)$ where $\{W^*(t)\}$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$. The arbitrage-free time t price of an call option with exercise time t_1 and strike price K on a zero-coupon bond with maturity $t_0 > t_1$ is then

$$p(t) = \tilde{\mathbb{E}}\left(D(t_1, t_0)\exp\Bigl(-\int_t^{t_1} \delta(v)\, \mathrm{d}v\Bigr) \;\Big|\; \mathcal{F}_t\right).$$

Calculate $p(t)$.

13.4.7 Consider the following Cox-Ingersoll-Ross type model $\{X(t)\}$ satisfying $\mathrm{d}X(t) = \sqrt{X(t)}\,\mathrm{d}W(t)$ with $X(0) = x \geq 0$. Suppose there exists a solution to the above stochastic differential equation. In fact, it is possible to show that there is a unique solution.
(a) Show that $\{X(t)\}$ is a supermartingale.
(b) Show that $\mathbb{E}\,X(t) < \infty$ for all t and conclude that $\{X(t)\}$ is a martingale. Conclude that for $x = 0$ we have $X(t) = 0$ for all t.
(c) Find $\mathbb{E}\,X(t)^2$.

13.4.8 Let $\{\delta(t)\}$ follow a Cox-Ingersoll-Ross model, i.e. $\mathrm{d}\delta(t) = -\alpha(\delta(t) - \bar{\delta})\,\mathrm{d}t + \sigma\sqrt{\delta(t)}\,\mathrm{d}W^*(t)$ where $\{W^*(t)\}$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$. Find the Laplace transform

$$\tilde{\mathbb{E}}\left(-s\delta(t_0) - s'\int_0^{t_0} \delta(v)\,\mathrm{d}v\right).$$

[Hint. Try a function of the form $\exp(-\bar{\delta}g_1(t_0; s, s') - \delta(0)g_2(t_0; s, s'))$. If $f(y, x)$ fulfils

$$\frac{\sigma^2 x}{2}\frac{\partial^2 f(y,x)}{\partial x^2} - \alpha(x - \bar{\delta})\frac{\partial f(y,x)}{\partial x} - \frac{\partial f(y,x)}{\partial y} - s'xf(y,x) = 0$$

with boundary condition $f(0, x) = \mathrm{e}^{-sx}$, then $\{M(t)\}$ defined as

$$M_t = f(t_0 - t, \delta(t))\exp\Bigl(-s'\int_0^t \delta(v)\,\mathrm{d}v\Bigr)$$

is a martingale.]

Solutions

13.4.1 The process $D^*(t, t_0) = D(t, t_0)\exp(-\int_0^t \delta(v)\,\mathrm{d}v)$ is a martingale under the measure $\tilde{\mathbb{E}}$. By the martingale representation theorem (Lemma 13.4.1) there is a process $\{Y(t)\}$ such that

$$D^*(t, t_0) = D(0, t_0) + \int_0^t Y(v)\,\mathrm{d}W^*(v)\,.$$

From Itô's formula we find

$$\begin{aligned}
\mathrm{d}D(t, t_0) &= \exp\Bigl(\int_0^t \delta(v)\,\mathrm{d}v\Bigr)Y(t)\,\mathrm{d}W^*(t) \\
&\qquad + D^*(t, t_0)\delta(t)\exp\Bigl(\int_0^t \delta(v)\,\mathrm{d}v\Bigr)\,\mathrm{d}t \\
&= \delta(t)D(t, t_0)\,\mathrm{d}t + \exp\Bigl(\int_0^t \delta(v)\,\mathrm{d}v\Bigr)Y(t)\,\mathrm{d}W^*(t)\,.
\end{aligned}$$

Thus $\sigma_D(t) = Y(t)/D^*(t, t_0)$.

13.4.2 The process $\{W^*(t)\}$ has continuous paths and $W^*(0) = 0$. Let $t > v$ and $s \in \mathbb{R}$. Then

$$\begin{aligned}
&\tilde{\mathbb{E}}\left(\exp(\mathrm{i}s(W^*(t) - W^*(v))) \mid \mathcal{F}_v\right) \\
&= \frac{\mathbb{E}\left(\exp(\mathrm{i}s(W^*(t) - W^*(v)))L(t) \mid \mathcal{F}_v\right)}{L(v)} \\
&= \mathbb{E}\Bigl(\exp\Bigl(\mathrm{i}s\Bigl(W(t) - W(v) + \int_v^t Z(w)\,\mathrm{d}w\Bigr) - \int_v^t Z(w)\,\mathrm{d}W(w) \\
&\qquad\qquad - \tfrac{1}{2}\int_v^t Z(w)^2\,\mathrm{d}w\Bigr) \Bigm| \mathcal{F}_v\Bigr) \\
&= \mathbb{E}\Bigl(\exp\Bigl(-\int_v^t (Z(w) - \mathrm{i}s)\,\mathrm{d}W(w) - \tfrac{1}{2}\int_v^t (Z(w) - \mathrm{i}s)^2\,\mathrm{d}w\Bigr) \Bigm| \mathcal{F}_v\Bigr) \\
&\qquad\qquad \times \exp(-s^2(t - v)/2)\,.
\end{aligned}$$

Suppose the expected value is one. Then $W^*(t) - W^*(v)$ has the same distribution as a Brownian motion and is independent of $\mathcal{F}_v$, that is $\{W^*(t)\}$ is a Brownian motion. Consider the process

$$M(t) = \exp\Bigl(-\int_0^t (Z(w) - \mathrm{i}s)\,\mathrm{d}W(w) - \tfrac{1}{2}\int_0^t (Z(w) - \mathrm{i}s)^2\,\mathrm{d}w\Bigr)\,.$$

From Itô's formula it follows that $\mathrm{d}M(t) = (Z(w) - \mathrm{i}s)\,\mathrm{d}W(w)$ and therefore $M(t)$ is a complex valued local martingale, i.e. both the real part and the imaginary part are local martingales. Thus there exists a localization sequence $\{\tau_n\}$, $\tau_n \to \infty$, such that $\{M(\tau_n \wedge t)\}$ are martingales. This gives

$$\mathbb{E}\Bigl(\exp\Bigl(-\int_{\tau_n\wedge v}^{\tau_n\wedge t}(Z(w)-\mathrm{i}s)\,\mathrm{d}W(w)-\tfrac{1}{2}\int_{\tau_n\wedge v}^{\tau_n\wedge t}(Z(w)-\mathrm{i}s)^2\,\mathrm{d}w\Bigr) \Bigm| \mathcal{F}_v\Bigr) = 1\,.$$

The absolute value of the integrand is bounded by

$$\exp\Bigl(-\int_{\tau_n\wedge v}^{\tau_n\wedge z} Z(w)\,\mathrm{d}W(w) - \tfrac{1}{2}\int_{\tau_n\wedge v}^{\tau_n\wedge z} Z(w)^2\,\mathrm{d}w\Bigr)\,.$$

Because $\{L(t)\}$ is a martingale the latter is uniformly integrable. It follows therefore that

$$\mathbb{E}\Bigl(\exp\Bigl(-\int_v^t (Z(w) - \mathrm{i}s)\,\mathrm{d}W(w) - \tfrac{1}{2}\int_v^t (Z(w) - \mathrm{i}s)^2\,\mathrm{d}w\Bigr) \Bigm| \mathcal{F}_v\Bigr) = 1\,.$$

13.4.3 (a) From Itô's formula we find $\mathrm{d}L(t) = Z(t)L(t)\,\mathrm{d}W(t)$ and $\{L(t)\}$ is a local martingale. Because $L(t) > 0$ it follows that $\{L(t)\}$ is a

supermartingale.
(b) From Jensen's inequality it follows that $\mathbb{E}\int_0^t Z(v)^2\,\mathrm{d}v < \infty$ and therefore that $\{\int_0^t Z(v)\,\mathrm{d}W(v)\}$ is a martingale.
(c) We have $M(t) = (L(t)\exp(\frac{1}{2}\int_0^t Z(v)^2\,\mathrm{d}v))^{1/2}$ and it follows from Schwartz' inequality

$$\mathbb{E}\,M(t) = \sqrt{\mathbb{E}\,L(t)\mathbb{E}\,\exp\Big(\tfrac{1}{2}\int_0^t Z(v)^2\,\mathrm{d}v\Big)} < \infty\,.$$

That $\{M(t)\}$ is a submartingale follows now from Jensen's inequality.
(d) By Jensen's inequality we have

$$\begin{aligned}
&\mathbb{E}\,[L_a(t); L_a(t) > x] \\
&= \mathbb{E}\Big[(L(t))^{a^2}\Big(\exp\Big(\frac{2a}{1+a}\frac{1}{2}\int_0^t Z(v)\,\mathrm{d}W(v)\Big)\mathbb{I}(L_a(t) > x)\Big)^{1-a^2}\Big] \\
&\le (\mathbb{E}\,L(t))^{a^2}\Big(\mathbb{E}\Big[\exp\Big(\frac{2a}{1+a}\frac{1}{2}\int_0^t Z(v)\,\mathrm{d}W(v)\Big)\mathbb{I}(L_a(t) > x)\Big]\Big)^{1-a^2} \\
&\le (\mathbb{E}\,L(t))^{a^2}\Big(\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^t Z(v)\,\mathrm{d}W(v)\Big); L_a(t) > x\Big]\Big)^{2a(1-a)} \\
&\le (\mathbb{E}\,[\mathbb{E}\,(M(t_0) \mid \mathcal{F}_t); L_a(t) > x])^{2a(1-a)} \\
&= (\mathbb{E}\,[M(t_0); L_a(t) > x])^{2a(1-a)} \\
&\le \Big(\mathbb{E}\Big[M(t_0); \sup_{v \le t_0} L_a(v) > x\Big]\Big)^{2a(1-a)}
\end{aligned}$$

where we used $2a < 1 + a$ in the second inequality. (e) Since the process $\{L_a(t) : 0 \le t \le t_0\}$ is uniformly integrable by (d), it is a martingale.
(f) For $x = 0$ we have

$$1 = \mathbb{E}\,L_a(t) \le (\mathbb{E}\,L(t))^{a^2}(\mathbb{E}\,M(t))^{2a(1-a)} \to \mathbb{E}\,L(t)$$

as $a \uparrow 1$. Since $\{L(t)\}$ is a supermartingale we have $\mathbb{E}\,L(t) = 1 = L(0)$. Thus $\{L(t)\}$ is a martingale. The latter follows analogously to Exercise 13.2.1.

13.4.4 (a) There should be no cash flow in the interval $(0, t_0)$ and therefore we need

$$V(t) = V(0) + \sum_{i=1}^{n}\int_0^t \alpha_i(v)\,\mathrm{d}D(v, t_i) + \int_0^t \gamma(v)\,\mathrm{d}I(v)\,.$$

From (b) or from the definition in the Black-Scholes model one needs that $\{\int_0^t \alpha_i(v)\,\mathrm{d}(D(v, t_i)/I(v))\}$ are martingales under $\tilde{\mathbf{P}}$. This is for

instance the case if $\tilde{\mathbb{E}} \int_0^t \alpha_i(v)^2 \sigma_i(t)^2 D(v,t_i)^2 \,\mathrm{d}t < \infty$.
(b) We have by Itô's formula

$$\begin{aligned}
\mathrm{d}V^*(t) &= \frac{\mathrm{d}V(t)}{I(t)} - \frac{V(t)\,\mathrm{d}I(t)}{I(t)^2} \\
&= \sum_{i=1}^n \frac{\alpha_i(t)\,\mathrm{d}D(t,t_i)}{I(t)} + \frac{\gamma(t)\,\mathrm{d}I(t)}{I(t)} - \delta(t)\frac{V(t)}{I(t)}\,\mathrm{d}t \\
&= \sum_{i=1}^n \frac{\alpha_i(t)\,\mathrm{d}D(t,t_i)}{I(t)} - \delta(t)\frac{V(t)-\gamma(t)I(t)}{I(t)}\,\mathrm{d}t \\
&= \sum_{i=1}^n \Big(\frac{\alpha_i(t)\,\mathrm{d}D(t,t_i)}{I(t)} - \delta(t)\frac{\alpha_i(t)D(t,T_i)}{I(t)}\,\mathrm{d}t\Big) \\
&= \sum_{i=1}^n \alpha_i(t)\,\mathrm{d}\Big(\frac{D(t,t_i)}{I(t)}\Big).
\end{aligned}$$

This gives that $\{V^*(t)\}$ is a martingale under $\tilde{\mathbb{P}}$.
(c) Suppose $\{(\alpha_1(t),\ldots,\alpha_n(t),\gamma(t))\}$ is a self-financing trading strategy such that $V(0) \le 0$ and $V(t_0) \ge 0$. Then also $V^*(0) \le 0$ and $V^*(t_0) \ge 0$. Thus $0 \le \tilde{\mathbb{E}}\, V^*(t_0) = V^*(0) \le 0$ and $\tilde{\mathbb{P}}(V^*(0) = V^*(t_0) = 0) = 1$. Thus also $\mathbb{P}(V^*(0) = V^*(t_0) = 0) = 1$. Therefore there is no arbitrage.

13.4.5 (a) The solution is

$$D(t,t_0) = \exp\Big(\int_0^t \sigma_D(v)\,\mathrm{d}W(v) + \int_0^t (\delta(v) - \sigma_D(v)^2/2)\,\mathrm{d}v\Big).$$

(b) Analogously to Exercise 13.4.1 we find that $\mathrm{d}X(t) = \delta(t)X(t)\,\mathrm{d}t + \sigma_X(t)X(t)\,\mathrm{d}W^*(t)$ for some process $\{\sigma_X(t)\}$. By Itô's formula we obtain

$$\begin{aligned}
\mathrm{d}\Big(\frac{X(t)}{D(t,t_0)}\Big) = \frac{\mathrm{d}X(t)}{D(t,t_0)} &- \frac{X(t)\,\mathrm{d}D(t,t_0)}{D(t,t_0)^2} \\
+ \tfrac{1}{2}\frac{2X(t)\sigma_D(t)^2 D(t,t_0)^2\,\mathrm{d}t}{D(t,t_0)^3} &- \frac{\sigma_X(t)X(t)\sigma_D(t)D(t,t_0)\,\mathrm{d}t}{D(t,t_0)^2} \\
= (\sigma_X(t) - \sigma_D(t))&\frac{X(t)}{D(t,t_0)}(\mathrm{d}W^*(t) - \sigma_D(t)\,\mathrm{d}t) \\
= (\sigma_X(t) - \sigma_D(t))&\frac{X(t)}{D(t,t_0)}\mathrm{d}W_{t_0}(t)
\end{aligned}$$

and $\{X(t)/D(t,t_0)$ is a local martingale under $\mathbb{P}_{t_0}$.
(c) From the martingale property of $\{X(t)\exp(-\int_0^t \delta(v)\,\mathrm{d}v)\}$ it follows

that

$$
\begin{aligned}
X(t) &= \tilde{\mathbb{E}}\left(X(t_0)\exp\Big(-\int_t^{t_0}\delta(v)\,\mathrm{d}v\Big)\;\Big|\;\mathcal{F}_t\right)\\
&= \mathbb{E}_{t_0}\Big(X(t_0)\exp\Big(-\int_t^{t_0}\delta(v)\,\mathrm{d}v-\int_t^{t_0}\sigma_D(v)\,\mathrm{d}W(v)\\
&\qquad\qquad +\tfrac{1}{2}\int_t^{t_0}\sigma_D(v)^2\,\mathrm{d}v\Big)\;\Big|\;\mathcal{F}_t\Big)\\
&= \mathbb{E}_{t_0}(X(t_0)D(t,t_0)/D(t_0,t_0)\mid\mathcal{F}_t)=D(t,t_0)\mathbb{E}_{t_0}(X(t_0)\mid\mathcal{F}_t)
\end{aligned}
$$

by (a) and $D(t_0,t_0)=1$. Thus $X(t)/D(t,t_0)=\mathbb{E}_{t_0}(X(t_0)\mid\mathcal{F}_t)$ is a martingale under $\mathbb{P}_{t_0}$.

13.4.7 (a) $\{X(t)\}$ is positive and a local martingale. Hence it is a supermartingale.
(b) We have $\mathbb{E}\,X(t)\le\mathbb{E}\,X(0)=x$. Thus $\mathbb{E}\int_0^t(\sqrt{X(v)})^2\,\mathrm{d}v\le tx$ for all t implying that $\{\int_0^t\sqrt{X(v)}\,\mathrm{d}W(v)\}$ is a martingale. If $x=0$ we have $\mathbb{E}\,X(t)=0$ and therefore $\mathbb{P}(X(t)=0)=1$.
(c) We find

$$
\mathbb{E}\,X(t)^2=\mathbb{E}\left(x+\int_0^t\sqrt{X(v)}\,\mathrm{d}W(v)\right)^2=x^2+\int_0^t\mathbb{E}\,X(v)\,\mathrm{d}v=x^2+tx\,.
$$

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—– End of forwarded message from Volker Schmidt —–