A note on transient Gaussian fluid models

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Abstract

In this paper we study the distribution of the supremum over interval $[0,T]$ of a centered Gaussian process with stationary increments with a general negative drift function. This problem is related to the distribution of the buffer content in a transient Gaussian fluid queue $Q(T)$ at time T, provided that at time 0 the buffer is empty. The general theory is illustrated by detailed considerations of different cases for the integrated Gaussian process and the fractional Brownian motion. We give asymptotic results for $\mathbb{P}(Q(T) > x)$ and $\mathbb{P}(\sup_{0\le t\le T} Q(t) > x)$ as $x \to \infty$.

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1 Introduction

Fluid models have been used in modeling and analysis of high-speed telecommunication networks. For example in ATM networks the data flows in the form of 53-byte packets or cells. The high-speed of the ATM network implies that it can transmit millions of cells per second. This makes fluid models useful in describing the flow of cells.

Suppose that $\xi(t)$ is the cumulative amount of traffic (data, work) generated by the input process during the time interval $[0,t]$ and that the service mechanism deterministically removes the fluid from the system with rate $\mu > 0$. Assuming that there is $Q(0)$ fluid in the system at $t = 0$, the amount of fluid at time t (or the *fluid queue*) is given by

$$Q(t) = \sup_{0 \le s \le t} \left(Q(0) + \xi(t) - \mu t, \xi(t) - \xi(s) - \mu(t-s) \right) . \tag{1.1}$$

Different assumptions have been made in literature concerning the input process. Originally it was supposed that $\xi(t)$ is a sum of independent and identically distributed processes. That is $\xi(t) = \sum_{j=1}^{N} \int_0^t X_j(s) ds$, where $X_j(t)$ are i.i.d. stationary processes modeling the input rate of the fluid to the buffer from the $j-$th stream; see e.g. [1] or [17].

Central limit theorem type arguments yield that the standardized version of the process $\xi(t)$ can be approximated by a Gaussian process with stationary increments and the same covariance function as the covariance function of the cumulative amount of traffic $\int_0^t X_j(s)ds$ generated by separate j-th stream. Details can be found in [12] and [5], where there were given limiting theorems with a Gaussian fluid flow model in the limit under heavy traffic environment parameterization conditions. ¿From these considerations we have that (1.1) describes evolution of the fluid queue, where now $\xi(t)$ is a Gaussian process. Due to normalizations μ loses the output rate interpretation in (1.1), however the exact form of μ may be obtained by results from [12, 5].

In this paper we study

$$\psi_T(u) := \mathbb{P}(Q(T) > u)$$

and

$$\psi_T^{\sup}(u) := \mathbb{P}(\sup_{0 \le t \le T} Q(t) > u)$$

as $u \to \infty$, assuming that T is fixed and $Q(0) = 0$.

In the classical models of Gaussian fluid queues it is assumed that $\xi(t)$ is a Gaussian process with stationary increments. In this work we extend this model in order to describe some deterministic perturbations caused by nonstationarity of the input process. Namely we suppose that $\xi(t) = \zeta(t) + \tilde{d}(t)$, where

- $\zeta(t)$ is a centered Gaussian process with stationary increments, continuous sample paths a.s., $\zeta(0) = 0$, and strictly increasing variance function $\sigma_\zeta^2(t) = \mathbf{Var}(\zeta(t))$;
- $\tilde{d}(t) = \delta(t) + \lambda t$, where $\delta(t)$ represents some deterministic perturbation caused by nonstationarity of the input process.

Then

$$\mathbb{P}(Q(T) > u) = \mathbb{P}(\sup_{0 \le t \le T} (\eta_T(t) + \tilde{d}(T) - \tilde{d}(T-t) - \mu t) > u) ,$$

where $\eta_T(t) = \zeta(T) - \zeta(T-t), 0 \le t \le T$. Since $\{\eta_T(t), 0 \le t \le T\}$ has the same distribution as $\{\zeta(t), 0 \le t \le T\}$ we can write

$$\psi_T(u) \quad = \quad \mathbb{P}(\sup_{0 \le t \le T} \zeta(t) - d_T(t) > u),$$

where $d_T(t) = \mu t - (\tilde{d}(T) - \tilde{d}(T-t))$. Since T is fixed, we simply write $d_T(t) = d(t)$. Suppose now $\delta(t) = 0$, so $d_T(t) = (\mu - \lambda)t$ does not depend on T. Then

$$\psi_T^{\sup}(u) = \mathbb{P}(\sup_{0\le v\le t\le T} \zeta(t) - \zeta(v) - (\mu-\lambda)(t-v) > u). \tag{1.2}$$

In this note we are interested in the following special cases of process $\zeta(t)$:

- $\zeta(t) = B_H(t)$ is the fractional Brownian motion (**FBM**) with Hurst parameter H; $0 < H \le 1$. That is $\zeta(t)$ is a centered Gaussian process with stationary increments, $\zeta(0) = 0$, continuous sample paths a.s., and variance function $\sigma_\zeta^2(t) = t^{2H}$, $H \in (0,1]$.
- $\zeta(t)$ is an integrated Gaussian (**IG**) process, that is $\zeta(t) = \int_0^t Z(s)\,ds$, where $Z(t)$ is a centered stationary process with covariance function $R(t) = \mathbb{E}Z(0)Z(t)$. We assume that $R(t)$ is continuous at zero (so is uniformly continuous on $\mathbb{R}_+$).

Motivations for studying of such problems stem from the theory of fluid queues; see [6] and references therein. Thus an assumption that $\zeta(t)$ is an **FBM** process appeared in queuing theory in [16], and it was motivated by some statistical studies of Ethernet traffic in [13]. Assumption that $\zeta(t)$ is an **IG** process is motivated by studies [12, 5]. Assuming the function $d(t)$ to be general we include in our considerations transient fluid queues with some nonstationary effects. Transient buffer content $Q(T)$ under Gaussian input was also studied in [11, 10] but our approach is different. Furthermore, **FBM** case with $H > 1/2$ was studied in [14] with $d(t) = ct$, see also [20].

Our main results, stated in Theorems 2.1, 2.2, are in some sense a generalization (to allow noncentered processes) and a modification of Theorem D.3 from [18]. Note that in the literature there exist related papers. For example in [18], [19] and [8] there are considered centered Gaussian processes, but results presented there cannot be directly applied to get the asymptotics for Gaussian fluid queues analyzed in this paper.

Results presented in this paper contribute to discussions on the old question how *the long range dependence* of the input data affects the performance characteristics of the system. In the stationary case, the presence of the long range dependence dramatically influences the distribution of the stationary buffer content. For example, if $\zeta(t) = B_H(t)$ with $H > 1/2$ (that is the long range dependent case), then it is known (see [15, 9]) that for $u \to \infty$

$$\psi_\infty(u) := \mathbb{P}(\sup_{t\ge 0}(B_H(t) - ct) > u) \sim \mathrm{Const}_1\, u^{(1-H)/H}\Psi(\mathrm{Const}_2\, u^{1-H}),$$

where

$$\Psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{v^2}{2}}\,dv$$

is the tail function of the standard normal random variable. In contrast to this result, where the impact of the long range dependence structure of $B_H(t)$ manifests in the heavy tailed distribution of the stationary buffer content, we show that for the transient buffer content

$$\psi_T(u) = \mathbb{P}(\sup_{0\le t\le T}(B_H(t) - ct) > u) \sim \Psi\left(\frac{u+cT}{T^H}\right), \qquad u \to \infty.$$

It is interesting to point out that for $H < 1/2$ the asymptotics of $\psi_T(u)$ is different than for $H > 1/2$; see Proposition 3.1. Similar effect can be observed in the **IG** case, that is the asymptotics of $\psi_T(u)$ is determined by the knowledge of the covariance function in interval

$[0, T]$. However the long range dependence is determined by the behavior of the covariance function at infinity.

The remainder of the paper is organized as follows. In Section 2 we state some preliminary definitions and assumptions and present the main results of the paper. In Section 3 we analyze two special fluid queues, when $\zeta(t)$ is either **FBM** or **IG**. Proofs of results from Section 2 are deffered to Section 4.

2 Main results

In this section we present the main theoretical results that enable us to obtain the exact asymptotic of $\psi_T(u)$ and $\psi_T^{\sup}(u)$ as $u \to \infty$.

We begin from introducing the basic notation and assumptions. Notice that for the process $\zeta(t)$ with stationary increments

$$\mathbf{Cov}(\zeta(t), \zeta(s)) = \frac{1}{2}(\sigma_\zeta^2(t) + \sigma_\zeta^2(s) - \sigma_\zeta^2(|t-s|)) . \tag{2.3}$$

In this note by bar we always denote standardized processes $\overline{\zeta}(t) = \zeta(t)/\sigma_\zeta(t)$ to have variance equal to 1. The first derivative of function $f(t)$ we denote by $\dot{f}(t)$. Note that $\dot{\sigma}^2(t)$ means $d/dt\ \sigma^2(t)$. The following function plays an important technical role:

$$g_u(t) = \frac{\sigma_\zeta(t)}{u + d(t)} .$$

Moreover we denote

$$m(u) = \frac{u + d(T)}{\sigma_\zeta(T)} .$$

We now show a list of assumptions for the variance function $\sigma_\zeta^2(t)$, covariance function $\mathbf{Cov}(\overline{\zeta}(s), \overline{\zeta}(t))$, and function $d(t)$. Different theoretical results will be stated under validity of subsets of these assumptions.

A For some $\alpha > 0$ and constant $a > 0$

$$1 - \mathbf{Cov}(\overline{\zeta}(s), \overline{\zeta}(t)) = a|t-s|^\alpha(1 + o(1)),$$

where $o(1)$ means a quantity converging to zero when $s, t \to T$.

B Function $g_u(t)$ is increasing on $[0, T]$ ultimately (that is an increasing function for u big enough) as $u \to \infty$ and for some $b > 0$

$$\sup_{T-t<o(1)} \left|\frac{1 - g_u(t)/g_u(T)}{b(T-t)} - 1\right| \to 0$$

as $u \to \infty$, where $o(1)$ means a quantity that goes to 0 as $u \to \infty$.

C For some constants $A_0 > 0$ and $\gamma > 0$

$$\sigma_\zeta^2(t)/t^\gamma \to A_0 \quad \text{as} \quad t \downarrow 0.$$

D For some constant $C_d > 0$

$$|d(s) - d(t)| \le C_d|s-t|, \qquad 0 \le s, t \le T .$$

It appears that if $\alpha > 1$ in **A**, then in order to obtain the exact asymptotic of $\psi_T(u)$ as $u \to \infty$ it is possible to weaken assumptions **A** and **B** as follows:

A1 There exists $\alpha > 1$ and $a > 0$ such that

$$1 - \mathbf{Cov}(\overline{\zeta}(T-s), \overline{\zeta}(T-t)) \leq a|s-t|^{\alpha}(1+o(1)) \tag{2.4}$$

as $s, t \to 0$.

B1 Function $g_u(t)$ is increasing on $[0, T]$ ultimately as $u \to \infty$ and for some $b > 0$ and $t_0 > 0$

$$1 - \frac{g_u(t)}{g_u(T)} \geq b(T-t), \qquad t_0 \leq t \leq T$$

ultimately as $u \to \infty$.

Theorem 2.1 *Let the variance function $\sigma_\zeta^2(t)$ of a centered Gaussian stochastic process with stationary increments $\zeta(t)$ and a function $d(t)$ satisfy conditions* **A1,B1,C**. *Then*

$$\psi_T(u) = \Psi(m(u))(1+o(1)) \quad \text{as} \quad u \to \infty.$$

Proof. The complete proof of Theorem 2.2 is presented in Section 4.

□

In the statement of the next theorem we need the notion of Pickands constants $\mathcal{H}_\alpha$ defined via the following limit

$$\mathcal{H}_\alpha = \lim_{T\to\infty} \frac{\mathcal{H}_\alpha(T)}{T},$$

where

$$\mathcal{H}_\alpha(T) = \mathbb{E}e^{\sup_{0\leq t\leq T}(\sqrt{2}B_{\alpha/2}(t)-t^\alpha)}.$$

It is known (see e.g. [18]) that $0 < \mathcal{H}_\alpha < \infty$.

Theorem 2.2 *Let the variance function $\sigma_\zeta^2(t)$ of a centered Gaussian stochastic process with stationary increments $\zeta(t)$ and a function $d(t)$ satisfy conditions* **A-C**.
(i) If $\alpha < 1$, then

$$\psi_T(u) = \mathcal{H}_\alpha \frac{a^{1/\alpha}}{b} m(u)^{\frac{2-2\alpha}{\alpha}} \Psi(m(u))(1+o(1)) \quad \text{as} \quad u \to \infty.$$

(ii) If $\alpha = 1$, then

$$\psi_T(u) = \frac{a+b}{b}\ \Psi(m(u))(1+o(1)) \quad \text{as} \quad u \to \infty.$$

Proof. The complete proof of Theorem 2.2 is presented in Section 4.

□

Finally we study asymptotics for $\psi_T^{\sup}(u) = \mathbb{P}(\sup_{0\leq t\leq T} Q(t) > u)$, where $d(t) = ct$. We analyze only the case $\alpha > 1$, which covers the most interesting cases in the fluid queues.

Theorem 2.3 *If conditions* **A**, **B1**, **C** *hold with $\alpha = \gamma > 1$ and $d(t) = ct$, then*

$$\psi_T^{\sup}(u) = \Psi(m(u))(1+o(1))$$

as $u \to \infty$.

Proof. The complete proof is presented in Section 4.3.

□

3 Special cases

In this section we give the exact asymptotics for $\psi_T(u)$ and $\psi_T^{\sup}(u)$ as $u \to \infty$ for two special cases, when $\zeta(t) \in \mathbf{FBM}$ or $\zeta(t) \in \mathbf{IG}$.

3.1 FBM case

Fluid queues with the **FBM** input play an important role in recent research since they possess *long range dependence* and *self-similarity* structure. In the following proposition we give the exact asymptotic of $\psi_T(u)$ as $u \to \infty$ for $\zeta(t) \in \mathbf{FBM}$. Notice that in this case

$$m(u) = \frac{u + d(T)}{T^H}.$$

Proposition 3.1 *Let $\zeta(t) \in \mathbf{FBM}$ and function $d(t)$ fulfills $\mathbf{D}$.*
(i) *If $H < 1/2$, then*

$$\begin{aligned}\psi_T(u) &= \frac{\mathcal{H}_{2H}}{2^{\frac{1}{2H}} H} m(u)^{\frac{1-2H}{H}} \Psi(m(u))(1+o(1)) \\ &= \frac{\mathcal{H}_{2H}}{2^{\frac{1}{2H}} H} \frac{u^{\frac{1-3H}{H}}}{T^{1-3H}} e^{-\frac{(u+d(T))^2}{T^{2H}}} (1+o(1)) \quad \text{as} \quad u \to \infty.\end{aligned}$$

(ii) *If $H = 1/2$, then*

$$\psi_T(u) = 2\Psi(m(u))(1+o(1)) \quad \text{as} \quad u \to \infty.$$

(iii) *If $H > 1/2$, then*

$$\psi_T(u) = \Psi(m(u))(1+o(1)) \quad \text{as} \quad u \to \infty.$$

(iv) *If $H = 1$, then*

$$\psi_T(u) = \Psi(m(u)) \quad \text{as} \quad u \to \infty.$$

In the following lemma, which proceeds the proof of the above proposition, we give conditions under which assumption $\mathbf{A}$ holds.

Lemma 3.2 *Let the variance function $\sigma_\zeta^2(t)$ of a centered Gaussian stochastic process with stationary increments $\zeta(t)$ be strictly increasing and for some constant $A_T > 0$*

$$|\sigma_\zeta^2(t) - \sigma_\zeta^2(s)|/|t-s| \to A_T, \tag{3.5}$$

as $s, t \uparrow T$ (for $s < t \le T$ and $s \to T$).
(i) If $\mathbf{C}$ holds with $\gamma < 2$ then

$$1 - \mathbf{Cov}(\overline{\zeta}(T-s), \overline{\zeta}(T-t)) = \frac{A_0}{2\sigma_\zeta^2(T)} |t-s|^\gamma (1+o(1)),$$

(ii) If $\mathbf{C}$ holds with $\gamma = 2$ and $\frac{A_T^2}{4\sigma_\zeta^2(T) A_0} < 1$, then there exists a constant $C > 0$ such that

$$1 - \mathbf{Cov}(\overline{\zeta}(T-s), \overline{\zeta}(T-t)) = C|t-s|^2(1+o(1)),$$

where $o(1)$ means a quantity converging to zero as $s, t \to 0$.

Proof. Using (2.3) we write

$$\begin{aligned}
1-\mathbf{Cov}(\overline{\zeta}(T-s),\overline{\zeta}(T-t)) &= \frac{\sigma_\zeta^2(|t-s|)-(\sigma_\zeta(T-t)-\sigma_\zeta(T-s))^2}{2\sigma_\zeta(T-t)\sigma_\zeta(T-s)}\\
&= \frac{\sigma_\zeta^2(|t-s|)}{2\sigma_\zeta(T-t)\sigma_\zeta(T-s)}-\frac{(\sigma_\zeta(T-t)-\sigma_\eta(T-s))^2}{2\sigma_\zeta(T-t)\sigma_\zeta(T-s)}\\
&= W_1-W_2.
\end{aligned}$$

We will show that $\lim_{0<t,s\to 0} W_2/W_1 = q < 1$ from which both (i) and (ii) follow. We have

$$\begin{aligned}
\frac{W_2}{W_1} &= \frac{(\sigma_\eta(T-t)-\sigma_\zeta(T-s))^2}{\sigma_\zeta^2(|t-s|)}\\
&= \frac{(\sigma_\zeta^2(T-t)-\sigma_\zeta^2(T-s))^2}{(\sigma_\zeta(T-t)+\sigma_\zeta(T-s))^2\sigma_\zeta^2(|t-s|)}\\
&= \frac{(\sigma_\zeta^2(T-t)-\sigma_\zeta^2(T-s))^2}{(\sigma_\zeta(T-t)+\sigma_\zeta(T-s))^2|t-s|^2}\frac{|t-s|^2}{\sigma_\zeta^2(|t-s|)}
\end{aligned}$$

Thus, if $\gamma < 2$, then $q = 0$ and the proof of (i) is completed.

If $\gamma = 2$, then

$$\lim_{0<s,t\to 0} W_2/W_1 = \frac{A_T^2}{4\sigma_\zeta^2(T)A_0} < 1,$$

which completes the proof of (ii).

□

Remark 3.1 *Note that* (3.5) *is equivalent to*

$$|\sigma_\zeta(t)-\sigma_\zeta(s)|/|t-s| = a_T,$$

for some constant $a_T > 0$, as $s,t \uparrow T$ (for $s < t \le T$ and $s \to T$), where $A_T = 2a_T\sigma_\zeta(T)$.

The following lemma links assumption **D** with **B1** and it will be useful in the proof of Proposition 3.1.

Lemma 3.3 *Let the variance function $\sigma_\zeta^2(t)$ of a centered Gaussian stochastic process with stationary increments $\zeta(t)$ be strictly increasing and assumption* **D** *holds. If there exists constant $D > 0$ such that*

$$|\sigma_\zeta(t)-\sigma_\zeta(s)| \ge D|t-s|$$

for $s,t \in [0,T]$, then
(i) $g_u(t)$ is ultimately increasing as $u \to \infty$ for $t \in [0,T]$;
(ii) for some $b > 0$ and $t_0 > 0$

$$1-\frac{g_u(t)}{g_u(T)} \ge b(T-t), \qquad t_0 \le t \le T$$

ultimately as $u \to \infty$.

Proof. First we prove (i). Let $0 \le s < t \le T$ and write:

$$
\begin{aligned}
g_u(t) - g_u(s) &= \frac{\sigma_\zeta(t)}{u + d(t)} - \frac{\sigma_\zeta(s)}{u + d(s)} \\
&= \frac{(u + d(t))(\sigma_\zeta(t) - \sigma_\zeta(s))}{(u + d(t))(u + d(s))} + \frac{\sigma_\zeta(t)(d(s) - d(t))}{(u + d(t))(u + d(s))} = V_1 + V_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
\frac{|V_1|}{|V_2|} &= \frac{(u + d(t))(\sigma_\zeta(t) - \sigma_\zeta(s))}{\sigma_\zeta(t)|d(s) - d(t)|} \\
&\ge \frac{D(u + d(t))}{C_d \sigma_\zeta(T)} \to \infty
\end{aligned}
\tag{3.6}
$$

as $u \to \infty$ for $0 \le s < t \le T$. Hence, having in mind that $V_1 > 0$, we obtain that $g_u(t)$ is ultimately increasing for $t \in [0, T]$.

In order to prove (ii) let $0 \le t \le T$ and write:

$$
1 - \frac{g_u(T - t)}{g_u(T)} = \frac{(ud(T))(\sigma_\zeta(T) - \sigma_\zeta(T - t)) + \sigma_\zeta(T)(d(T - t) - d(T))}{(u + d(T - t))\sigma_\zeta(T)} .
$$

Using analogous argumentation to presented in (3.6) we have

$$
1 - \frac{g_u(T - t)}{g_u(T)} \ge \frac{1}{2}\frac{(u + d(T))(\sigma_\zeta(T) - \sigma_\zeta(T - t))}{(u + d(T - t))\sigma_\zeta(T)} \ge \frac{1}{2}\frac{D}{\sigma_\zeta(T)} t
$$

for sufficiently large u.

□

Remark 3.2 Note that Lemma 3.3 implies condition **B1**.

Proof of Proposition 3.1. The proof of (i), (ii) is based on the application of Theorem 2.2. Thus we have to check that **A-C** hold. Note that due to the mean value theorem

$$
|\sigma_\zeta^2(t) - \sigma_\zeta^2(s)| = |\dot{\sigma}_\zeta^2(\theta)||t - s|
$$

for some $\theta \in [s, t]$. Thus

$$
\dot{\sigma}_\zeta^2(\theta) \to \dot{\sigma}_\zeta^2(T) > 0 ,
$$

as $t, s \uparrow T$ and (3.5) is satisfied with $A_T = 2HT^{2H-1}$. Now using (i) of Lemma 3.2 we conclude that **A** is satisfied with $a = \frac{1}{2T^{2H}}$. In order to prove that **B** holds let $t \uparrow T$ and

$$
\frac{t^H}{T^H} = 1 - H\frac{T - t}{T} + o(T - t).
$$

Hence, recalling assumption **D**, we have

$$
\begin{aligned}
\frac{g_u(t)}{g_u(T)} &= \frac{u + d(T)}{u + d(t)} \frac{t^H}{T^H} \\
&= (1 + O(T - t)\frac{1}{u + d(t)})(1 - H\frac{T - t}{T} + o(T - t))
\end{aligned}
$$

from which **B** follows with $b = H/T$. Assumption **C** follows immediately from properties of $\sigma_\zeta^2(t)$.

The proof of (iii) is based on the application of Theorem 2.1. Namely if **D** holds, then due to Lemma 3.3 **B1** is satisfied. Moreover from Lemma 3.2 we have that **A1** is fulfilled with $\alpha = 2H > 1$.

In order to proof (iv) we notice that since $d(t)$ is continuous we have

$$m(u) = \min_{0 \le t \le T} \frac{u + d(t)}{t} = \frac{u + d(T)}{T} .$$

Recall (see e.g. [2]) that the **FBM** with $H = 1$ has the form $\zeta(t) = \mathcal{N}t$ where $\mathcal{N}$ is the standard normal random variable. Thus

$$\begin{aligned}
\Psi_T(u) = \mathbb{P}(\sup_{t\in[0,T]} Nt - d(t) > u) &= \mathbb{P}(\sup_{t\in[0,T]} N \frac{t}{u + d(t)} > 1) \\
= \mathbb{P}(N > \min_{t\in[0,T]} \frac{u + d(t)}{t}) &= \Psi(m(u)) .
\end{aligned}$$

□

Remark 3.3 For $H = 1/2$ the **FBM** is the standard Brownian motion and in this case $\psi_T(u)$ is know explicitly; see e.g. Harrison [7]. We included this case for the completeness of considerations.

Proposition 3.4 *Let $\zeta(t) \in$ **FBM**, function $d(t) = ct$. If $H > 1/2$, then*

$$\psi_T^{\sup}(u) = \Psi\left(m(u)\right)(1 + o(1)) \quad \text{as} \quad u \to \infty.$$

Proof. Due to Theorem 2.3, in order to proof the proposition it is enough to check that **A, B1, C** hold and $\alpha = \gamma = 2H > 1$, which straightforwardly follows from the proof of Proposition 3.1.

□

3.2 IG case

Input flows modeled by a stationary Gaussian processes appear as limits of the superposition of large number of i.i.d. *on-off* sources under heavy traffic environment parameterization (see Dębicki & Palmowski [5]). In this case $\sigma_\zeta^2(t) = 2\int_0^t ds \int_0^s R(v)\,dv$ and since $\sigma_\zeta^2(t)$ is strictly increasing, then $\int_0^t R(v)dv > 0$, $t > 0$.

In the following proposition we give the exact asymptotic of $\psi_T(u)$ as $u \to \infty$ for $\zeta(t) \in$ **IG**. Note that in this case

$$m(u) = \frac{u + d(T)}{\sqrt{2\int_0^T \int_0^s R(v)\,dv\,ds}}.$$

Proposition 3.5 *If $\zeta(t) \in$ **IG**, $\sigma_\zeta^2(t)$ is strictly increasing and function $d(t)$ fulfills condition **D**, then*

$$\psi_T(u) = \Psi(m(u))(1 + o(1))$$

as $u \to \infty$.

Proof of Proposition 3.5. ¿From Theorem 2.1, it is sufficient to prove that **A1,B1,C** hold. Since $\lim_{0<s,t\to 0}\sigma_\zeta^2(|t-s|)/|t-s|^2 = R(0)$, then condition **C** is satisfied with $\gamma = 2$ and $A_0 = R(0)$. Moreover from straightforward calculations it follows that (3.5) is satisfied with $A_T = 2\int_0^T R(s)\,ds$. Thus, in order to check that **A** holds we need to show that $\frac{A_T^2}{4\sigma_\zeta^2(T)A_0} < 1$. Since

$$\frac{(\int_0^T R(s)\,ds)^2}{2\int_0^T\int_0^s R(v)\,dv\,ds} < R(0),$$

then

$$\frac{\int_0^T \phi(s)\int_0^s \phi(v)\,dv\,ds}{\int_0^T\int_0^s \phi(v)\,ds\,dv} < 1$$

where $\phi(t) = R(t)/R(0)$. Let

$$i(s) = \begin{cases} 1 & \text{for } \phi(s) > 0, \\ 0 & \text{for } \phi(s) \le 0. \end{cases}$$

Then we have

$$\int_0^T \phi(s)\int_0^s \phi(v)\,dv\,ds < \int_0^T i(s)\int_0^s \phi(v)\,dv\,ds \le \int_0^T\int_0^s \phi(v)\,dv\,ds,$$

which shows that **A** is satisfied (and hence **A1**). Straightforward use of Lemma 3.3 implies that **B1** holds.

□

Proposition 3.6 *If $\zeta(t) \in$ **IG**, $\sigma_\zeta^2(t)$ is strictly increasing and function $d(t) = ct$, then*

$$\psi_T^{\sup}(u) = \Psi\left(m(u)\right)(1 + o(1))$$

as $u \to \infty$.

Proof. Due to Theorem 2.3, in order to prove the proposition it suffices to check that **A-C** hold and $\alpha = \gamma = 2$, which straightforwardly follows from the proof of Proposition 3.5.

□

4 Proofs

Before presenting the proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3 we need some lemmas that are also of independent interest.

Let $\overline{X}_{\alpha_i}(t)$ $(i = 1,\ldots,n)$ be independent centered Gaussian processes with $\mathbf{Var}(\overline{X}_{\alpha_i}(t)) = 1$ and

$$\overline{X}_{\alpha_1,\ldots,\alpha_n}(t_1,\ldots t_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \overline{X}_{\alpha_i}(t_i)\ .$$

The following lemma can be deduced from a result in [3, 4] (see also [18]).

Lemma 4.1 *If for $i = 1,\ldots,n$*

$$1 - \mathbf{Cov}(\overline{X}_{\alpha_i}(s), \overline{X}_{\alpha_i}(t)) = |s-t|^{\alpha_i}(1 + o(1)), \tag{4.7}$$

as $s,t\to 0$, *then*

$$\mathbb{P}\Big(\sup_{\substack{t_1\in[0,T/u^{2/\alpha_1}]\\ \vdots \\ t_n\in[0,T/u^{2/\alpha_n}]}} \overline{X}_{\alpha_1,\dots,\alpha_n}(t_1,\dots,t_n)>u\Big)=\mathcal{H}_{\alpha_1,\dots,\alpha_n}(T)\Psi(u)(1+o(1)),$$

where

$$\mathcal{H}_{\alpha_1,\dots,\alpha_n}(T)=\mathbb{E}\exp\left[\sup_{(t_1,\dots,t_n)\in[0,T]^n}\left(\frac{\sqrt{2}}{\sqrt{n}}\sum_{i=1}^n B_{\alpha_i/2}(t_i)-\frac{1}{n}\sum_{i=1}^n t_i^{\alpha_i}\right)\right]$$

Proof. We will use Theorem 2.1 from [4]. Let

$$\overline{X}_{\alpha_i,u}(t)=\overline{X}_{\alpha_i}\left(\frac{t}{u^{2/\alpha_i}}\right).$$

Clearly we have

$$\sup_{s,t\in[0,T/u^{2/\alpha_i}]}\left|\frac{1-\mathbf{Cov}(\overline{X}_{\alpha_i}(s),\overline{X}_{\alpha_i}(t))}{|t-s|^{\alpha_i}}-1\right|\to 0$$

for $i=1,\dots,n$. Hence

$$\sup_{s,t\in[0,T]}\left|\frac{1-\mathbf{Cov}(\overline{X}_{\alpha_i;u}(s),\overline{X}_{\alpha_i;u}(t))}{|t-s|^{\alpha_i}/u^2}-1\right|\to 0$$

for $i=1,\dots,n$. This means that condition D.0 from [4] is fulfilled with $f(u)=u$ and hence the lemma follows.

□

The following result is a corollary to Lemma 4.1.

Corollary 4.2 *(i) If*

$$1-\mathbf{Cov}(\overline{X}_\alpha(s),\overline{X}_\alpha(t))=|s-t|^\alpha(1+o(1)), \tag{4.8}$$

as $s,t\to 0$, *then*

$$\mathbb{P}\left(\sup_{t\in[0,T/u^{2/\alpha}]}\overline{X}_\alpha(t)>u\right)=\mathcal{H}_\alpha(a^{1/\alpha}T)\Psi(u)(1+o(1))$$

as $u\to\infty$.
(ii) If $\overline{X}^{(1)}_\alpha(t),\overline{X}^{(2)}_\alpha(t)$ *are independent and for* $i=1,2$

$$1-\mathbf{Cov}(\overline{X}^{(i)}_\alpha(s),\overline{X}^{(i)}_\alpha(t))=a|s-t|^\alpha(1+o(1)), \tag{4.9}$$

as $s,t\to 0$, *then*

$$\mathbb{P}\left(\sup_{(t_1,t_2)\in[0,T/u^{2/\alpha}]^2}\frac{1}{\sqrt{2}}\left(\overline{X}^{(1)}_\alpha(t_1)+\overline{X}^{(2)}_\alpha(t_2)\right)>u\right)=\mathcal{H}_{\alpha,\alpha}(a^{1/\alpha}T)\Psi(u)(1+o(1))$$

as $u\to\infty$.

Proof. First we prove (i). Consider process $\{\overline{X}_\alpha(t/a^{1/\alpha}), t \geq 0\}$ which fulfills conditions of Lemma 4.1 and note that

$$\mathbb{P}\Big(\sup_{0\leq t\leq T/u^{2/\alpha}} \overline{X}_\alpha(t) \geq u\Big) = \mathbb{P}\Big(\sup_{0\leq t\leq a^{1/\alpha}T/u^{2/\alpha}} \overline{X}_\alpha(t/a^{1/\alpha}) \geq u\Big)\ .$$

The proof of (ii) goes analogously.

□

Lemma 4.3 *If condition* **B1** *and* **C** *hold, then*

$$\mathbb{P}\Big(\sup_{t\in[0,T-(\log m(u)/m(u))^2]} \zeta(t) - d(t) > u\Big) = o(\Psi(m(u)))$$

as $u \to \infty$.

Proof. Let t_0 be such that **B1** holds. We have

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,T-(\log m(u)/m(u))^2]} \zeta(t) - d(t) > u\Big) \leq \\
&\quad\leq \quad \mathbb{P}\left(\sup_{t\in[0,s_0]} \zeta(t) - d(t) > u\right) \qquad (4.10)\\
&\qquad + \mathbb{P}\Big(\sup_{t\in[s_0,T-(\log m(u)/m(u))^2]} \zeta(t) - d(t) > u\Big)\ , \qquad (4.11)
\end{aligned}$$

where s_0 is such that $t_0 \leq s_0 < T$ and (using condition **C**)

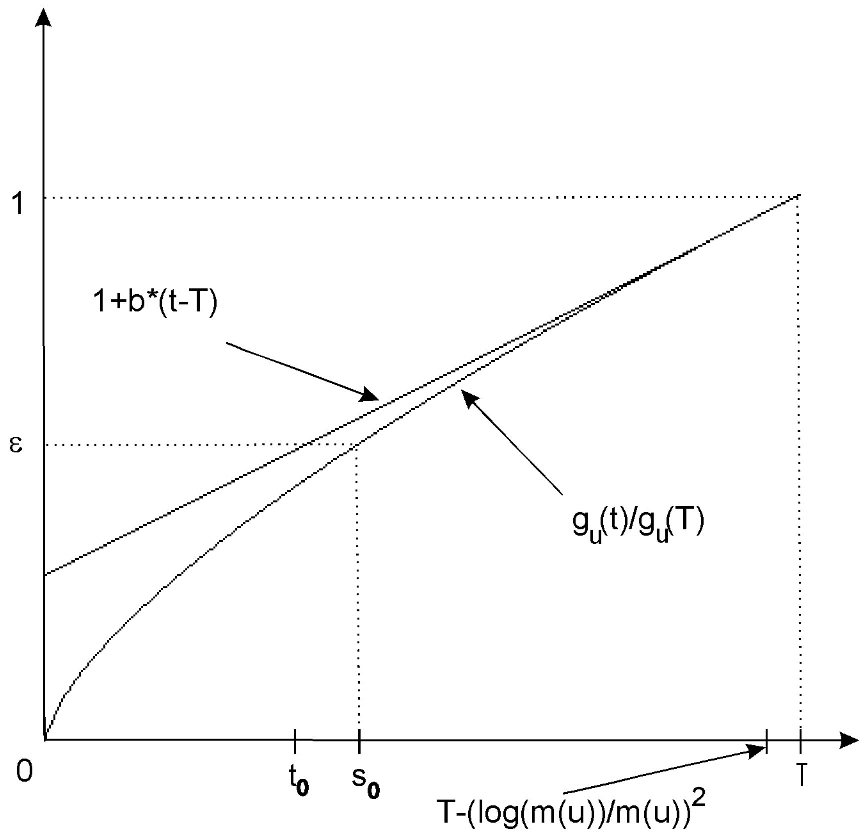
$$\sigma_\zeta^2(t) \leq 2A_0 t^\gamma \qquad 0 < t \leq T - s_0. \tag{4.12}$$

We first bound probability in line (4.10). In view of **B1** we have for some $0 < \epsilon < 1$

$$\frac{g_u(t)}{g_u(T)} < \epsilon$$

for $t \leq s_0$ (see Figure 1).

Figure 1:



Hence from Borell inequality (see [18], Theorem D.1)

$$
\begin{aligned}
\mathbb{P}\Big(\sup_{t\in[0,s_0]}\zeta(t)>u\Big) &= \mathbb{P}\Big(\sup_{t\in[0,s_0]}\overline{\zeta}(t)\frac{\sigma_\zeta(t)}{u+d(t)}m(u)>m(u)\Big)\\
&\le \mathbb{P}(\sup_{t\in[0,s_0]}\overline{\zeta}(t)>\epsilon^{-1}m(u))=o(\Psi(m(u))\ .
\end{aligned}
$$

In order to bound (4.11) note that due to **B1**

$$
\frac{g_u(t)}{g_u(T)}<1-b\left(\frac{\log m(u)}{m(u)}\right)^2, \qquad t\in[s_0,T-\left(\frac{\log m(u)}{m(u)}\right)^2]
$$

ultimately as $u\to\infty$. Thus

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[s_0,T-(\log m(u)/m(u))^2]}\zeta(t)-d(t)>u\Big)=\\
&= \mathbb{P}\Big(\sup_{t\in[s_0,T-(\log m(u)/m(u))^2]}\overline{\zeta}(t)\frac{g_u(t)}{g_u(T)}>m(u)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathbb{P}\Big(\sup_{t\in[s_0,T-(\log m(u)/m(u))^2]} \overline{\zeta}(t)\left(1-b\left(\frac{\log m(u)}{m(u)}\right)^2\right) > m(u)\Big)\\
&\leq \mathbb{P}(\sup_{t\in[s_0,T-(\log m(u)/m(u))^2]} \overline{\zeta}(t)\frac{1}{1+b\left(\frac{\log m(u)}{m(u)}\right)^2} > m(u)) \leq\\
&\leq \mathbb{P}(\sup_{t\in[s_0,T-(\log m(u)/m(u))^2]} \overline{\zeta}(t) > m(u)(1+b\left(\frac{\log m(u)}{m(u)}\right)^2))\,.
\end{aligned}
$$

Hence, in view of Theorem D.4 in Piterbarg [18], in order to complete the proof it is enough to show that

$$
\mathbb{E}(\overline{\zeta}(t)-\overline{\zeta}(s))^2 \leq \mathrm{C}_0\ |t-s|^\gamma
$$

for some constant C_0 and $t,s\in[s_0,T]$, where γ was defined in **C**; see also (4.12). It follows from the fact that

$$
\begin{aligned}
&\mathbb{E}(\overline{\zeta}(t)-\overline{\zeta}(s))^2 =\\
&= 2(1-\mathbf{Cov}(\overline{\zeta}(s),\overline{\zeta}(t)))\\
&= 2\frac{\sigma_\zeta^2(|t-s|)-(\sigma_\zeta(t)-\sigma^2(s))^2}{2\sigma_\zeta(t)\sigma_\zeta(s)}\\
&\leq \frac{1}{\sigma^2(T-s_0)}\sigma_\zeta^2(|t-s|)\\
&\leq \frac{2A_0\ |t-s|^\gamma}{\sigma^2(T-s_0)} \qquad (4.13)
\end{aligned}
$$

where (4.13) is a consequence of the fact that due to (4.12)

$$
\sigma_\zeta^2(|t-s|)\leq 2A_0\ |t-s|^\gamma, \qquad |t-s|\leq T-s_0\,.
$$

This completes the proof.

□

Lemma 4.4 *Let $\{X(t),t\in[0,T]\}$ be a centered Gaussian process. If there exists a constant $\alpha>0$ such that*

$$
\sup_{s,t\in[0,(\log(u)/u)^{2/\alpha}]}\left|\frac{1-\mathbf{Cov}(\overline{X}(T-t),\overline{X}(T-s))}{a|t-s|^\alpha}-1\right| \to 0
$$

as $u\to\infty$ and

$$
\sup_{t\in[0,(\log(u)/u)^{2/\alpha}]}\left|\frac{1-\sigma_X(T-t)}{bt^\alpha}-1\right|\to 0
$$

as $u\to\infty$, then

$$
\mathbb{P}\Big(\sup_{t\in[0,(\log(u)/u)^2]} X(T-t)>u\Big) = \mathcal{H}_\alpha^{b/a}\Psi(n(u))(1+o(1))\,,
$$

where

$$
\mathcal{H}_\alpha^{b/a} = \mathbb{E}\exp(\max_{t\in[0,\infty)}\sqrt{2}B_\alpha(t)-(1+\frac{b}{a})|t|^\alpha).
$$

Proof. The proof is a straightforward consequence of Lemma 8.3 form Piterbarg [18].

□

The following lemma is a special case of Theorem 4.2. in [4].

Lemma 4.5 *Let $\{X_{\alpha;u}(t), t \in [0,T], u \geq 0\}$ be a family of centered Gaussian processes. If for some constants $a, b > 0$ and $\alpha < 1$*

$$\sup_{s,t\in[0,(\log(u)/u)^2]} \left| \frac{1 - \mathbf{Cov}(\overline{X}_{\alpha;u}(T-t), \overline{X}_{\alpha;u}(T-s))}{a|t-s|^{\alpha}} - 1 \right| \to 0$$

as $u \to \infty$ and

$$\sup_{t\in[0,(\log(u)/u)^2]} \left| \frac{1 - \sigma_{X_{\alpha;u}}(T-t)}{bt} - 1 \right| \to 0$$

as $u \to \infty$, then

$$\mathbb{P}(\sup_{t\in[0,(\log(u)/u)^2]} X_{\alpha;u}(T-t) > u) = \mathcal{H}_{\alpha} \frac{a^{1/\alpha}}{b} n(u)^{\frac{2-2\alpha}{\alpha}} \Psi(u)(1+o(1))$$

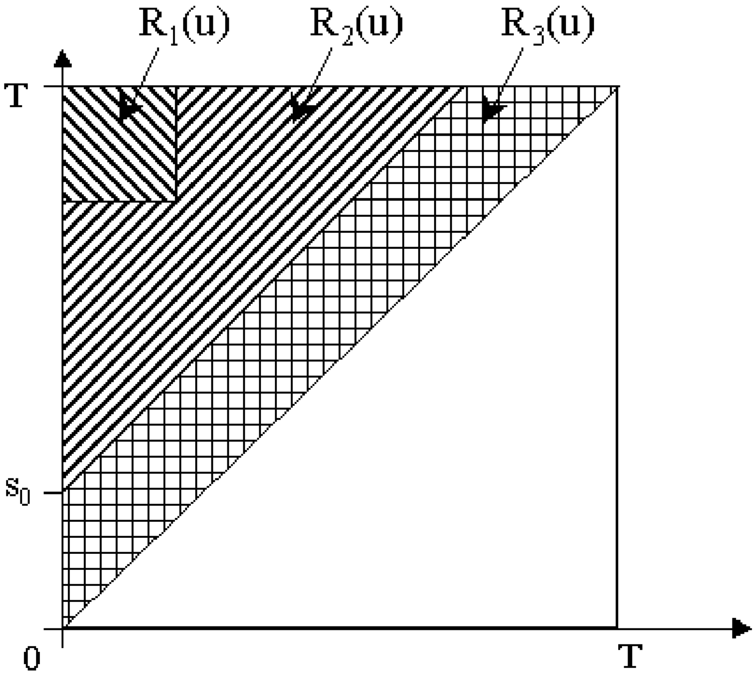
as $u \to \infty$, where $\mathcal{H}_{\alpha} = \lim_{T\to\infty} \frac{\mathcal{H}_{\alpha}(T)}{T}$ is the classical Pickands constant.

In the rest of this section we present the Lemmas that will be used in the proof of Theorem 2.3. For notational convenience we divide the set $\{(s,t) : \ 0 \leq s < t \leq T\}$ into disjoint sets $R_1(u) \cup R_2(u) \cup R_3(u)$, where

$$\begin{aligned} R_1(u) &= \{(s,t) : \ 0 \leq s \leq (\log m(u)/m(u))^2, T - (\log m(u)/m(u))^2 \leq t \leq T\} \\ R_3(u) &= \{(s,t) : \ t - s \leq s_0\} \\ R_2(u) &= \{(s,t) : \ 0 \leq s < t \leq T\} \setminus (R_1(u) \cup R_2(u)), \end{aligned}$$

see Figure 2.

Figure 2:



Lemma 4.6 *There exists $C_1 > 0$ such that*

$$\mathbf{Var}(\overline{\zeta}(s,t) - \overline{\zeta}(s_1,t_1)) \leq C_1(\mathbb{E}(\zeta(t) - \zeta(t_1))^2 + \mathbb{E}(\zeta(s) - \zeta(s_1))^2) \ ,$$

for $(s,t), (s_1,t_1) \in R_1(u) \cup R_2(u)$.

Proof. We have

$$\mathbf{Var}(\overline{\zeta}(s,t) - \overline{\zeta}(s_1,t_1)) = \mathbb{E}\left(\frac{\zeta(s,t) - \zeta(s_1,t_1)}{\sigma_\zeta(|t-s|)} - \zeta(s_1,t_1)\left(\frac{1}{\sigma_\zeta(t-s)} - \frac{1}{\sigma_\zeta(t_1-s_1)}\right)\right)^2 .$$

Since $(x-y)^2 \leq 2(x^2+y^2)$ the above is less or equal than

$$\frac{2}{\sigma^2_\zeta(t-s)}(\mathbb{E}(\zeta(s,t) - \zeta(s_1,t_1))^2 + (\sigma_\zeta(t_1-s_1) - \sigma_\zeta(t-s))^2).$$

Using now that for centered random variables $\mathbb{E}(X-Y)^2 \geq (\sigma_X - \sigma_Y)^2$ we have that the above is less or equal than

$$\frac{4}{\sigma_\zeta(t-s)}\mathbb{E}(\zeta(s,t) - \zeta(s_1,t_1))^2 \quad = \quad \frac{4}{\sigma_\zeta(t-s)}\mathbb{E}((\zeta(t) - \zeta(t_1)) - (\zeta(s) - \zeta(s_1)))^2$$

$$\leq \frac{8}{\sigma_\zeta(t-s)}(\mathbb{E}(\zeta(t)-\zeta(t_1))^2+\mathbb{E}(\zeta(s)-\zeta(s_1))^2)\ .$$

Since

$$\inf_{(s,t)\in R_1(u)\cup R_2(u)} \sigma_\zeta(t-s)>0$$

the proof of the lemma is completed.

□

Remark 4.1 From Lemma 4.6 we immediately conclude that there exists γ and G such that

$$\mathbb{E}(\overline{\zeta}(s,t)-\overline{\zeta}(s_1,t_1))^2\leq G(|s-s_1|^\gamma+|t-t_1|^\gamma), \tag{4.14}$$

for $(s,t),(s_1,t_1)\in R_1(u)\cup R_2(u)$.

Lemma 4.7

$$\mathbb{P}\Big(\sup_{(s,t)\in R_2(u)\cup R_3(u)} \overline{\zeta}(s,t)\frac{g_u(|t-s|)}{g_u(T)}>m(u)\Big)=o(\Psi(m(u)),$$

as $u\to\infty$.

Proof. Similarly as in the proof of Lemma 4.3 (s_0 and t_0 have the same meaning)

$$\begin{aligned}
\mathbb{P}(\sup_{(s,t)\in R_3(u)} \zeta(s,t)-c(t-s)>u) &= \mathbb{P}\Big(\sup_{(s,t)\in R_3(u)} \overline{\zeta}(s,t)\frac{g_u(|t-s|)}{g_u(T)}>m(u)\Big)\\
&= \mathbb{P}\Big(\sup_{(s,t)\in R_3(u)} \overline{\zeta}(s,t)>\epsilon^{-1}m(u)\Big)=o(\Psi(m(u))\ .
\end{aligned}$$

Next

$$\begin{aligned}
\mathbb{P}(\sup_{(s,t)\in R_2(u)} \zeta(s,t)-c(t-s)>u) &= \mathbb{P}(\sup_{(s,t)\in R_2(u)} \overline{\zeta}(s,t)\frac{g_u(|t-s|)}{g_u(T)}>m(u)\Big)\\
&\leq \mathbb{P}\left(\sup_{(s,t)\in R_2(u)} \overline{\zeta}(s,t)>m(u)\left(1+\left(\frac{\log m(u)}{m(u)}\right)\right)\right)\\
&= o(\Psi(m(u)),
\end{aligned}$$

where the last line, similarly as in the proof of Lemma 4.3, follows from Theorem 8.1 in [18], p. 119 which holds in view of (4.14).

□

Lemma 4.8 *If* **A** *and* **C** *hold and $\alpha=\gamma$, then for each $C_1>0$ there exists a constant $C_0>0$ such that*

$$C_1\mathbb{E}(\zeta(s)-\zeta(s_1))^2\leq\mathbb{E}(\overline{\zeta}(T-C_0s)-\overline{\zeta}(T-C_0s_1))^2$$

as $s,s_1\downarrow 0$.

Proof. Let $C_1 > 0$ be a given constant and $a > 0$ be the constant defined in **A**. Due to **C** we have

$$\begin{aligned} C_1 \mathbb{E}(\zeta(s) - \zeta(s_1))^2 &\leq 2C_1 A_0 |s - s_1|^\gamma \\ &= a\left(\left(\frac{2C_1A_0}{a}\right)^{1/\gamma} |s-s_1|\right)^\gamma . \end{aligned}$$

Using **A** and the assumption that $\alpha = \gamma$ we have

$$\begin{aligned} a\left(\left(\frac{2C_1A_0}{a}\right)^{1/\gamma} |s-s_1|\right)^\gamma &\leq 2(1 - \mathbf{Cov}(\overline{\zeta}(T - C_0 s), \overline{\zeta}(T - C_0 s_1)) = \\ &= \mathbb{E}(\overline{\zeta}(T - C_0 s) - \overline{\zeta}(T - C_0 s_1))^2, \end{aligned}$$

where $C_0 = \left(\frac{2C_1A_0}{a}\right)^{1/\gamma}$ and $s, s_1 > 0$ are sufficiently small. This completes the proof. $\square$

4.1 Proof of Theorem 2.1

Since

$$\psi_T(u) = \mathbb{P}(\sup_{0\leq t\leq T} \zeta(t) - d(t) > u) \geq \mathbb{P}(\zeta(T) - d(T) > u) \geq \Psi\left(\frac{u + d(T)}{\sigma_\zeta(T)}\right),$$

it suffices to obtain the asymptotical upper bound for $\psi_T(u)$. From Lemma 4.3 we have

$$\begin{aligned} \psi_T(u) &= \mathbb{P}(\sup_{0\leq t\leq T} \zeta(t) - d(t) > u) = \\ &= \mathbb{P}(\sup_{T-(\log m(u)/m(u))^2\leq t\leq T} \zeta(t) - d(t) > u)(1 + o(1)) = \\ &= \mathbb{P}(\sup_{T-(\log m(u)/m(u))^2\leq t\leq T} \frac{\zeta(t)}{u + d(t)} > 1)(1 + o(1)) \\ &= \mathbb{P}\Big(\sup_{T-(\log m(u)/m(u))^2\leq t\leq T} \frac{\zeta(t)}{u + d(t)} m(u) > m(u)\Big)(1 + o(1)) \\ &= \mathbb{P}\Big(\sup_{T-(\log m(u)/m(u))^2\leq t\leq T} \overline{\zeta}(t)\frac{g_u(t)}{g_u(T)} > m(u)\Big)(1 + o(1)) \end{aligned}$$

From condition **B1** there exists constant $b > 0$ such that

$$\begin{aligned} &\mathbb{P}\Big(\sup_{T-(\log m(u)/m(u))^2\leq t\leq T} \overline{\zeta}(t)\frac{g_u(t)}{g_u(T)} > m(u)\Big) \\ &\leq \mathbb{P}\Big(\sup_{0\leq t\leq(\log m(u)/m(u))^2} \overline{\zeta}(T - t)(1 - bt) > m(u)\Big) \leq \mathbb{P}(\sup_{0\leq t\leq \epsilon m(u)^{-2/\alpha}} \overline{X}_\alpha(t) > m(u)), \end{aligned}$$

for each $\epsilon > 0$ as $u \to \infty$, where $\overline{X}_\alpha(t) = \overline{\zeta}(T - t)$.

Let $\overline{Y}(t)$ be a centered Gaussian process with variance function $\sigma^2_{\overline{Y}}(t) = 1$ and covariance function

$$1 - \mathbf{Cov}(\overline{Y}(s), \overline{Y}(t)) = 2a|s - t|^\alpha(1 + o(1)) \tag{4.15}$$

as $s,t\to 0$, where a was defined in (2.4). From Slepian inequality (see [18], Theorem C.1) we have

$$\mathbb{P}(\sup_{0\le t\le \epsilon m(u)^{-2/\alpha}} \overline{X}_\alpha(t) > m(u)) \le \mathbb{P}(\sup_{0\le t\le \epsilon m(u)^{-2/\alpha}} \overline{Y}(t) > m(u))\ ,$$

for each $u>0$. Moreover $\overline{Y}(t)$ fulfills conditions of Lemma 4.2, and hence

$$\mathbb{P}\Big(\sup_{0\le t\le \epsilon m(u)^{-2/\alpha}} \overline{X}_\alpha(t) > m(u)\Big) \le \mathcal{H}_\alpha((2a)^{1/\alpha}\epsilon)\Psi\left(\frac{u+d(T)}{\sigma_\zeta(T)}\right)(1+o(1))\ .$$

Thus, letting $\epsilon\to 0$ and having in mind that $\mathcal{H}_\alpha((2a)^{1/\alpha}\epsilon)\downarrow 1$ as $\epsilon\to 0$, the proof of (i) of Theorem 2.2 is completed.

□

4.2 Proof of Theorem 2.2

Proof of (i)
Using the same argumentation as in the proof of Theorem 2.1 we have

$$\begin{aligned}\psi_T(u) &= \mathbb{P}\Big(\sup_{t\in[T-(\log m(u)/m(u))^2,T]} \overline{\zeta}(t)\frac{g_u(t)}{g_u(T)} > m(u)\Big)(1+o(1))\\ &= \mathbb{P}\Big(\sup_{t\in[T-(\log m(u)/m(u))^2,T]} Y_u(t) > m(u)\Big)(1+o(1))\end{aligned}$$

as $u\to\infty$ and $Y_u(t)=\overline{\zeta}(t)\frac{g_u(t)}{g_u(T)}$. Notice that from **B**

$$\sigma_{Y_u}(T-t) = 1-b|t|(1+o(1))$$

as $t\to 0$ and from **A**

$$\mathbf{Cov}(\overline{Y}_u(T-t),\overline{Y}_u(T-s)) = 1-a|t-s|(1+o(1))$$

as $t,s\to 0$. For sufficiently small $\varepsilon>0$ define

$$Y_{+\varepsilon}(t) = \frac{\xi_{+\varepsilon}(t)}{1+(b-\varepsilon)|t|}$$

$$Y_{-\varepsilon}(t) = \frac{\xi_{-\varepsilon}(t)}{1+(b+\varepsilon)|t|},$$

where $\xi_{+\varepsilon}(t)$ is a Gaussian stationary process with covariance function $R(t)=\exp(-(a+\varepsilon)|t|)$ and $\xi_{-\varepsilon}(t)$ is a Gaussian stationary process with covariance function $R(t)=\exp(-(a-\varepsilon)|t|)$. From Lemma 4.4 we have

$$\mathbb{P}\Big(\sup_{t\in[T-(\log m(u)/m(u))^2,T]} Y_{+\varepsilon}(t) > m(u)\Big) = \mathcal{H}_1^{(b-\varepsilon)/(a+\varepsilon)}\Psi(m(u))(1+o(1)) \qquad (4.16)$$

and

$$\mathbb{P}\Big(\sup_{t\in[T-(\log m(u)/m(u))^2,T]} Y_{-\varepsilon}(t) > m(u)\Big) = \mathcal{H}_1^{(b+\varepsilon)/(a-\varepsilon)}\ \Psi(m(u))(1+o(1))\ . \qquad (4.17)$$

Moreover using that the distribution of $\sup_{t\geq 0}(B_{1/2}(t)-ct)$ is exponential with parameter $2c$ we get

$$\mathcal{H}_1^c = \mathbb{E}\exp(\max_{t\in[0,\infty)}\sqrt{2}B_{1/2}(t)-(1+c)|t|) = \frac{1+c}{c}$$

for each $c>0$. Since from Slepian inequality for sufficiently large u

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[T-(\log m(u)/m(u))^2,T]} Y_{+\varepsilon}(t) > m(u)\right) &\geq \mathbb{P}\left(\sup_{t\in[T-(\log m(u)/m(u))^2,T]} Y_u(t) > m(u)\right) \\ &\geq \mathbb{P}\left(\sup_{t\in[T-(\log m(u)/m(u))^2,T]} Y_{-\varepsilon}(t) > m(u)\right),\end{aligned}$$

then passing in (4.16) and (4.17) with $\varepsilon\to 0$ we obtain

$$\psi_T(u) = \frac{a+b}{b}\,\Psi(m(u))(1+o(1))\ .$$

This completes the proof of (i).

Proof of (ii)
We have

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in[T-(\log m(u)/m(u))^2,T]}\zeta(t)-d(t)>u\Big) &\leq \psi_T(u) = \mathbb{P}\Big(\sup_{t\in[0,T]}\zeta(t)-d(t)>u\Big) \\ \leq\ \mathbb{P}\Big(\sup_{t\in[0,T-(\log m(u)/m(u))^2]}\zeta(t)-d(t)>u\Big) &+ \mathbb{P}\Big(\sup_{t\in[T-(\log m(u)/m(u))^2,T]}\zeta(t)-d(t)>u\Big).\end{aligned}$$

Hence by Lemma 4.3 it suffices to consider

$$\mathbb{P}\Big(\sup_{t\in[T-(\log m(u)/m(u))^2,T]}\zeta(t)-d(t)>u\Big) = \mathbb{P}\Big(\sup_{t\in[T-(\log m(u)/m(u))^2,T]}\overline{\zeta}(t)\frac{g_u(t)}{g_u(T)}>m(u)\Big).$$

Using **A** and **B** we are in the setting of Theorem 4.5. Hence the proof is completed.

□

4.3 Proof of Theorem 2.3

Let

$$\zeta(s,t) = \zeta(t)-\zeta(s)\ . \tag{4.18}$$

For the standardized process we write

$$\overline{\zeta}(s,t) = \frac{\zeta(t)-\zeta(s)}{\sigma_\zeta(|s-t|)}\ . \tag{4.19}$$

Using (1.2) we can write for the case $Q(0)=0$

$$\psi_T^{\sup}(u) = \mathbb{P}\Big(\sup_{0\leq s<t\leq T}\overline{\zeta}(s,t)\frac{g_u(t-s)}{g_u(T)}>m(u)\Big)\ , \tag{4.20}$$

where $g_u(t)$ takes form $g_u(t)=\sigma_\zeta/(u+ct)$.

Analogously as in the proof of Theorem 2.1 the lower bound asymptotic is immediate. Thus in the rest of the proof we focus on the upper bound asymptotic. We have

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\le s<t\le T}\overline{\zeta}(s,t)\frac{g_u(t-s)}{g_u(T)} > m(u)\Big) &\le \mathbb{P}\Big(\sup_{0\le s<t\le T}\overline{\zeta}(s,t) > m(u)\Big)\\ &= \mathbb{P}\Big(\sup_{(s,t)\in R_1(u)}\overline{\zeta}(s,t) > m(u)\Big)(1+o(1))\end{aligned} \quad (4.21)$$

where (4.21) is justified by Lemma 4.7. Let $\zeta_1(t)$ and $\zeta_2(t)$ be independent copies of $\zeta(t)$. Combining Lemma 4.6 with Lemma 4.8 we have that there exists a constant $C_0 > 0$ such that

$$\mathbb{E}(\overline{\zeta}(s,t)-\overline{\zeta}(s_1,t_1))^2 \le \frac{1}{2}\left(\mathbb{E}(\overline{\zeta}(C_0 s)-\overline{\zeta}(C_0 s_1))^2 + \mathbb{E}(\overline{\zeta}(C_0 t)-\overline{\zeta}(C_0 t_1))^2\right)$$

for $(s,t),(s_1,t_1)\in R_1(u)$. Hence from Slepian inequality we have

$$\begin{aligned}\mathbb{P}(\sup_{(s,t)\in R_1(u)}\overline{\zeta}(s,t)\ge m(u)) &\le \mathbb{P}\Big(\sup_{s,t\in[T-(\log m(u)/m(u))^2,T]}\frac{1}{\sqrt{2}}(\overline{\zeta}_1(C_0 s)+\overline{\zeta}_2(C_0 t)) > m(u)\Big)\\ &\le \mathbb{P}\Big(\sup_{s,t\in[0,\epsilon/m(u)^{2/\alpha}]}\frac{1}{\sqrt{2}}(\overline{\zeta}_1(s)+\overline{\zeta}_2(t)) > m(u)\Big),\end{aligned}$$

where $\epsilon > 0$. Thus from Corollary 4.2, using the same argumentation as in the proof of Theorem 2.1

$$\begin{aligned}&\mathbb{P}\Big(\sup_{s,t\in[0,\epsilon/m(u)^{2/\alpha}]}\frac{1}{\sqrt{2}}(\overline{\zeta}_1(s)+\overline{\zeta}_2(t)) > m(u)\Big) =\\ &= \mathcal{H}_{\alpha,\alpha}(a^{1/\alpha}\epsilon)\Psi(m(u))(1+o(1))\end{aligned}$$

as $u\to\infty$. Since $\lim_{\epsilon\to 0}\mathcal{H}_{\alpha,\alpha}(a^{1/\alpha}\epsilon) = 1$, then the proof is completed.

□

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