PROCEEDINGS OF THE SYMPOSIUM
TO HONOUR JERZY NEYMAN Warszawa PWN, 1977

ON SOME CLASSES OF DISTRIBUTION FUNCTIONS DETERMINED BY AN ORDER RELATION

TOMASZ ROLSKI

Polish Academy of Sciences, Institute of Mathematics, Wrocław, Poland

1. Introduction

In many recent papers the concept of order relation in the set of distribution functions (d.f.'s) is applied in queueing theory (for example [3], [8], [9], [10]). Two order relations, namely $\prec_{st}$ (stochastic ordering relation) and $\prec_c$ (for the definition see Section 3) appear to be especially useful. Relation $\prec_{st}$ was investigated, in particular in [4] and relation $\prec_c$ in [3], [11], [12] (in [11], [12] the symbol $\prec_{st}$ is denoted by $\overset{(1)}{\leqslant}$ and $\prec_c$ by $\overset{(3)}{\leqslant}$). The relations $\prec_{st}$ and $\prec_c$ both have the property that the mean value of any d.f., provided its mean value exists, is an isotonic function with respect to these relations.

The following consideration shows how one can use relation $\prec_c$ to obtain bounds in queueing theory. It is known (see [3], [10]) that a d.f. of the waiting time in $GI/G/1$ system is an isotonic function of the d.f. of the service time. For a queueing system Σ with the negative exponentially distributed service time with mean m it is easy to compute the mean waiting time EW by the well-known Pollaczek formula (see for example [6]). If Σ' is a queueing system such that a d.f. of the inter-arrival time is earlier with respect to relation $\prec_c$ (to be abbreviated hereafter as "earlier ($\prec_c$)") than the negative exponential d.f. with a mean m, then the stationary waiting time W' in Σ' is earlier ($\prec_c$) than the stationary waiting time W in Σ. Hence we obtain a bound for the waiting time in Σ': $EW' \leqslant EW$.

The essential point here is that the distribution function of the service time is earlier ($\prec_c$) than a negative exponential d.f. For some other examples see [8].

The aim of this paper is a characterization of the class of d.f.'s that are earlier ($\prec_c$) than some negative exponential d.f. and a characterization of the class of d.f.'s that are earlier ($\prec_c$) than the negative exponential d.f. with the same mean. A relationship between the former class of d.f.'s and the class NBUE of d.f.'s is investigated. The case of discrete d.f.'s which are earlier ($\prec_c$) than a geometrical d.f. with the same mean is also considered.

2. Preliminaries

We restrict our attention to d.f.'s of nonnegative random variables which are left continuous. For a d.f. F we denote the survival function by $\bar{F} \equiv 1-F$. We say that the sequence F_n, $n = 1, \ldots,$ of d.f.'s converges weakly to the d.f. F, which we indicate by writing $F_n \Rightarrow F$, if $F_n \to F$ for all continuity points of F.

We shall use the following symbols. Let

$$M_m(x) = 1-\exp[(-x)/m], \quad x \geq 0,$$

and let

$$G_m(x) = \sum_{k=0}^{\infty} (1-p)p^k \delta_k(x), \quad m = p/(1-p),$$

where

$$\delta_a(x) = \begin{cases} 0, & \text{if} \quad x \leq a, \\ 1, & \text{if} \quad x > a. \end{cases}$$

Moreover, let

$$m_{F,n} = \int_0^{\infty} x^n dF(x).$$

Instead of $m_{F,1}$ we write the symbol m_F.

A d.f. F is said to have an Increasing Failure Rate (IFR) if $\log \bar{F}$ is concave on the support of F (see [1]).

A d.f. F is said to have Increasing Failure Rate Average (IFRA) if $-[\log \bar{F}(t)]/t$ is increasing with respect to t (see [2]). Another class of d.f.'s known to be important in reliability consists of the d.f.'s which are New Better than Used (NBUE) (see [7]). A d.f. F is said to be NBUE if for every $x > 0$

$$\int_x^{\infty} \bar{F}(t)\,dt/\bar{F}(x) \leq m_F < \infty.$$

We write {IFR} for the class of IFR d.f.'s, {IFRA} for the class of IFRA d.f.'s and {NBUE} for the class of NBUE d.f.'s. We shall use the well-known fact that

$$\{\mathrm{IFR}\} \subset \{\mathrm{IFRA}\} \subset \{\mathrm{NBUE}\}.$$

Throughout the paper we adopt the convention $1/(1/0) = 0$.

3. Relations $\prec_{st}$, $\prec_c$

DEFINITION 1. For a pair of d.f.'s F, G we write

(i) $F \prec_{st} G$ if, for every nonnegative, increasing function φ

$$\int_0^{\infty} \varphi\, dF \leq \int_0^{\infty} \varphi\, dG.$$

(ii) $F \prec_c G$ if, for every increasing, convex and nonnegative function φ such that $\int_0^\infty \varphi\, dG < \infty$,

$$\int_0^\infty \varphi\, dF \leqslant \int_0^\infty \varphi\, dG.$$

The relations $\prec_{st}$, $\prec_c$ are natural generalizations of the linear ordering relation in the set of all nonnegative real numbers $\mathscr{R}$. This follows by noting that

(I) $$a \leqslant b \text{ if and only if } \delta_a \prec_c \delta_b,$$

(II) $$F \prec_c G,\ a > 0 \text{ implies } F\left(\frac{\cdot}{a}\right) \prec_c G\left(\frac{\cdot}{a}\right),$$

(III) $$F_i \prec_c G_i,\ i = 1, \ldots, n, \text{ implies } \prod_{i=1}^n F_i \prec_c \prod_{i=1}^n G_i,$$

(IV) $$F_i \prec_c G_i,\ i = 1, \ldots, n, \text{ implies } F_1 * F_2 * \ldots * F_n \prec_c G_1 * G_2 * \ldots * G_n,$$

(V) $$F \prec_c G, \int_0^\infty x\, dF(x) < \infty, \int_0^\infty x\, dG(x) < \infty, \text{ implies } \int_0^\infty x\, dF(x) \leqslant \int_0^\infty x\, dG(x).$$

If (I) holds, then the set $\langle \mathscr{R}, \leqslant \rangle$ is isomorphically imbedded in the set $\langle$all d.f.'s, $\prec_c \rangle$. It may be worthwhile to mention that the relation $\prec_{st}$ has also properties (I)–(V). The proofs are given in [3], [8], [11]. One can also find there the following theorem:

THEOREM 1. (i) *$F \prec_{st} G$ if and only if for every $x > 0$:*

$$\bar{F}(x) \leqslant \bar{G}(x).$$

(ii) *$F \prec_c G$ if and only if $\int_0^\infty x\, dG(x) < \infty$ and for every $x > 0$:*

$$\int_x^\infty \bar{F}(t)\, dt \leqslant \int_x^\infty \bar{G}(t)\, dt.$$

One can see that if $F \prec_{st} G$ and $\int_0^\infty x\, dG(x) < \infty$, then $F \prec_c G$.

Other order relations in the set of d.f.'s which are weaker than $\prec_c$ but have properties (I)–(V), are investigated in [8], [9].

4. Mean residual life function

In various problems of the reliability theory the following function appears:

$$e_{F_X}(x) = E(X - x | X \geqslant x) = \begin{cases} \int_x^\infty \bar{F}_X(t)\, dt / \bar{F}_X(x) & \text{for } x \text{ such that } F(x) \neq 1, \\ 0 & \text{for } x \text{ such that } F(x) = 1. \end{cases}$$

This function is called the *mean residual life* of a component of age x. In the sequel the following lemma will be useful:

LEMMA 1. *We have, for every $x > 0$,*

$$\int_0^x \frac{1}{e_F(t)}\,dt = -\log \frac{\int_x^\infty \bar{F}(t)\,dt}{m_F}. \tag{1}$$

Proof. For $x = 0$ the left side is equal to the right side. Since both sides are absolutely continuous and their derivatives are equal a.e., we have (1).

THEOREM 2. *There is one-to-one correspondence between a d.f. F and the function e_F.*

Proof. Let us assume that there exists a d.f. G such that for every $x > 0$

$$e_F(x) = e_G(x) = \int_x^\infty \bar{F}(t)\,dt / \bar{F}(x).$$

Hence by Lemma 1 we obtain

$$\frac{d}{dx} \log \int_x^\infty \bar{G}(t)\,dt = \frac{d}{dx} \log \int_x^\infty \bar{F}(t)\,dt,$$

which implies for every $x > 0$

$$\int_x^\infty \bar{G}(t)\,dt = c \int_x^\infty \bar{F}(t)\,dt.$$

Since $e_F(0) = e_G(0) = m_F = m_G$, we have $c = 1$ and the proof is completed.

5. D.f.'s earlier ($\prec_c$) than M

In [12] Stoyan has shown that all d.f.'s F such that $F(b+) - F(a) = 1$ $(a < b)$ and $b \leqslant v$, where v is the root of the equation $v = 1 - \log[(1-a)(v-a)^{-1}]$, are earlier ($\prec_c$) than M_{m_F}.

It is also known (see [7]) that

$$\{\mathrm{NBUE}\} \subset \{F\colon F \prec_c M_{m_F}\}.$$

Notice that the inequality $F \prec_c M_m$ is equivalent to the inequality

$$\frac{1}{m} \int_x^\infty \bar{F}(t)\,dt \leqslant \exp[(-x)/m], \qquad x \geqslant 0, \tag{2}$$

which means, for a non-lattice d.f. F with a mean equal to m, that the equilibrium d.f. of the d.f. F is stochastically less ($\prec_{st}$) than the negative exponential d.f. with the mean equal to m.

THEOREM 3. *$F \prec_c M_m$ if and only if for every $x > 0$*

$$\frac{1}{x}\int_0^x \frac{1}{e_F(t)}\,dt \geq \frac{1}{m} + \frac{1}{x}\log\frac{m_F}{m}. \tag{3}$$

Proof. The assertion follows from the fact that (2) is equivalent to

$$\frac{m_F}{m}\frac{1}{m_F}\int_x^\infty \bar{F}(t)\,dt \leq \exp[(-x)/m], \qquad x > 0,$$

and this, by Lemma 1, is equivalent to

$$-\frac{1}{x}\log\frac{m_F}{m} + \frac{1}{x}\int_0^x \frac{1}{e_F(t)}\,dt \geq \frac{1}{m}, \qquad x > 0,$$

which in turn is equivalent to (3).

COROLLARY 1. *The class of d.f.'s $\{F\colon F \prec_c M_{m_F}\}$ is equal to*

$$\left\{F\colon x\Big/\int_0^x \frac{1}{e_F(t)}\,dt \leq m_F;\ x > 0\right\},$$

which contains the class of NBUE *d.f.'s.*

The class of d.f.'s $\{F\colon F \prec_c M_{m_F}\}$ can be called Harmonic New Better than Used in Expectation.

COROLLARY 2.

$$\left\{F\colon x\Big/\int_0^x \frac{1}{e_F(t)}\,dt \leq m,\ x > 0\right\} \subset \{F\colon F \prec_c M_m\}$$

and

$$\bigcup_{m_F \leq m} \{F\colon F \prec_c M_{m_F}\} \subset \left\{F\colon x\Big|\int_0^x \frac{1}{e_F(t)}\,dt \leq m,\ x > 0\right\}.$$

Proof. Since the condition $x\Big/\int_0^x \frac{1}{e_F(t)}\,dt \leq m,\ x > 0$, implies $m_F \leq m$ (the right side limit of the left side in 0 is equal to m_F), we have for every $x > 0$

$$-\log\frac{m_F}{m} + \int_0^x \frac{1}{e_F(t)}\,dt \geq \frac{x}{m},$$

which is equivalent, by Theorem 3, to $F \prec_c M_m$.

In order to prove the second inclusion let us consider a d.f.

$$F \in \bigcup_{m_F \leq m} \{F\colon F \prec_c M_{m_F}\}.$$

Then, by Corollary 1, we have

$$\frac{1}{x}\int_0^x \frac{1}{e_F(t)}\,dt \geqslant \frac{1}{m_F} \geqslant \frac{1}{m},$$

and the proof is completed.

The following theorem gives some properties of the class $\{F\colon F \prec_c M_m\}$:

THEOREM 4. *Let* $F_i \in \{F\colon F \prec_c M_m\}$, $i = 1, \ldots, n$. *Then*

(i) $$\prod_{i=1}^{n} F_i \in \{F\colon F \prec_c M_\mu\} \quad \textit{where} \quad \mu = m\sum_{i=1}^{n}\frac{1}{i},$$

(ii) $$1-\prod_{i=1}^{n}(1-F_i) \in \{F\colon F \prec_c M_m\},$$

(iii) $$F_1 * F_2 * \ldots * F_n \in \{F\colon F \prec_c M_{nm}\},$$

(iv) $$\sum_{i=1}^{n} p_i F_i \in \{F\colon F \prec_c M_m\}, \quad p_i \geqslant 0, \quad \sum_{i=1}^{n} p_i = 1.$$

Proof. (i) From property (III) of the relation $\prec_c$ it follows that

$$\prod_{i=1}^{n} F_i \prec_c (1-\exp[(-\,\cdot\,)/m])^n.$$

In [1] it has been proved that

$$H(\cdot) = (1-\exp[(-\,\cdot\,)/m])^n \in \{\mathrm{IFR}\}.$$

The class $\{\mathrm{IFR}\}$ is contained in $\{\mathrm{NBUE}\}$ and hence, by Corollary 1, $H \prec_c M_{m_H}$. In order to finish the proof it suffice to notice that the mean value of the d.f. H is equal to

$$\int_0^\infty \sum_{i=1}^{n} (-1)^{i+1}\binom{n}{i}\exp[(-it)/m]\,dt = m\sum_{i=1}^{n}\frac{(-1)^{i+1}}{i}\binom{n}{i} = m\sum_{i=1}^{n}\frac{1}{i}.$$

(ii) We have for every $x > 0$

$$\int_x^\infty \prod_{i=1}^{n} \bar{F}_i(t)\,dt \leqslant \int_x^\infty \bar{F}_1(t)\,dt \leqslant \int_x^\infty M_m(t)\,dt,$$

which completes the proof.

(iii) Let X_i, $i = 1, \ldots, n$, be independent random variables such that $P(X_i < x) = F_i(x)$. Then, for every nondecreasing convex function φ,

$$\int_0^\infty \varphi(x)\,dF_1 * \ldots * F_n(nx) = E\varphi\left(\frac{X_1+\ldots+X_n}{n}\right) \leqslant \sum_{i=1}^n \frac{1}{n}\int_0^\infty E\varphi(X_i)$$
$$= \sum_{i=1}^n \frac{1}{n}\int_0^\infty \varphi(x)\,dF_i(x) \leqslant \int_0^\infty \varphi(x)\,dM_m(x),$$

(iv) For every nonnegative increasing and convex function φ, we have

$$\int_0^\infty \varphi(x)\,d\sum_{i=1}^n p_i F_i(x) = \sum_{i=1}^n p_i \int_0^\infty \varphi(x)\,dF_i(x) \leqslant \int_0^\infty \varphi(x)\,dM_m(x),$$

which means that $\sum_{i=1}^n p_i F_i \prec_c M_m$.

The following theorem gives a necessary condition for $F \prec_c M_m$:

THEOREM 5. *If* $F \prec_c M_m$ *then*

$$\frac{m_{F,n}}{n!} \leqslant m^n, \qquad n = 1, 2, \ldots$$

Proof. The simple proof is given in [8].

THEOREM 6. *Let* F_n, F *be d.f.'s such that*

$$F_n \Rightarrow F, \qquad F_n \prec_c M_m,$$

and

$$\lim_{n\to\infty} m_{F_n,2} = \lim_{n\to\infty} m_{F_n}^2 = m^2.$$

Then

$$F = M_m.$$

Proof. It is known that $F_n \Rightarrow F$ is equivalent to the inequality $\varliminf_{n\to\infty} \bar{F}_n(x+) \geqslant \bar{F}(x+)$, $x \geqslant 0$. Hence and from Fatou's lemma we have for every $x > 0$

$$\int_x^\infty \bar{F}(t)\,dt = \int_x^\infty \bar{F}(t+)\,dt \leqslant \int_x^\infty \varliminf_{n\to\infty} \bar{F}_n(t+)\,dt \leqslant \varliminf_{n\to\infty} \int_x^\infty \bar{F}_n(t+)\,dt \leqslant m\exp[(-x)/m].$$

Now we show that the only d.f. such that

$$F \prec_c M_m \quad \text{and} \quad m_{F,2} = 2m_F^2 = 2m^2 \tag{4}$$

is $F = M_m$.

It is easy to verify that the d.f. M_m fulfils (4).

Let us assume, to the contrary, that there exists a $G \neq M_m$ such that $G \prec_c M_m$ and $2m_G^2 = m_{G,2}$. Then for every $x > 0$

$$\frac{1}{m_G}\int_x^\infty \bar{G}(t)\,dt \leqslant \exp[(-x)/m_G]$$

and the inequality is strict at least at one point x_0. Since both sides are continuous we have

$$\frac{m_{G,2}}{2m_G} = \frac{1}{m_G}\int_0^\infty\int_x^\infty \bar{G}(t)\,dt\,dx < \int_0^\infty \exp[(-x)/m_G] = m_G,$$

which contradicts (4).

For the class of d.f.'s $\{F\colon F \prec_c M_{m_F}\}$ we have the following theorems:

THEOREM 7. *If* F, $G \in \{F\colon F \prec_c M_{m_F}\}$, *then*

$$F*G \in \{F\colon F \prec_c M_{m_F}\}.$$

Proof. By property (IV) we have $F*G \prec_c M_{m_F}*M_{m_G}$. Since $M_{m_F}*M_{m_G} \in \{\text{IFRA}\}$ (see [2]), we have $M_{m_F}*M_{m_G} \prec_c M_{m_F+m_G}$. The statement follows from the transitivity property of the relation $\prec_c$.

THEOREM 8. *Let* F_n, F *be d.f.'s such that*

$$F_n \Rightarrow F, \quad F_n \prec_c M_{m_F},$$

$$\sup_n m_{F_n} = N < \infty, \quad \lim_{n\to\infty}(m_{F_n,2} - 2m_{F_n}^2) = 0.$$

Then $F = M_{m_F}$.

Proof. Since, according to Theorem 5, $m_{F,2} \leqslant 2m_{F_n}^2 \leqslant 2N^2$, we have $m_{F_n} \to m_F$ (see [5], Theorem 11.4 B). As in Theorem 6, we can write

$$F \prec_c M_{m_F}. \tag{5}$$

Moreover, from the assumption of the theorem it follows that $m_{F,2} = 2m_F^2$. In turn, in view of (5) this implies $F = M_{m_F}$.

6. The discrete case

In this section only discrete d.f.'s will be considered. Write

$$P(n) = \sum_{j=n}^\infty \sum_{i=j}^\infty p_i, \quad \Pi(n) = \sum_{j=n}^\infty \sum_{i=j}^\infty \pi_i.$$

Notice that if $F(x) = \sum_{i=0}^\infty p_i \delta_i(x)$, then

$$\bar{F}(x) = \sum_{i\geqslant x} p_i, \quad \int_x^\infty \bar{F}(t)\,dt = \sum_{k=i}^\infty p_k(i-x) + P(i+1), \quad j-1 < x \leqslant i,$$

and

$$\int_0^\infty \bar{G}_{\frac{p}{1-p}}(t)\,dt = \frac{p}{1-p}.$$

In the sequel the following theorem will be useful.

THEOREM 9. *Relation*

$$F = \sum_{i=0}^{\infty} p_i \delta_i \prec_c \sum_{i=0}^{\infty} \pi_i \delta_i = H$$

holds if and only if $P(i) \leqslant \Pi(i)$ *for* $i = 1, 2, \ldots$

Proof. Since functions $\int_x^\infty \bar{F}(t)dt$, $\int_x^\infty \bar{G}(t)dt$ are convex and linear in $[i, i+1)$, $i = 0, 1, \ldots,$ relation

$$\int_i^\infty \bar{F}(t)dt = P(i+1) \leqslant \Pi(i+1) = \int_i^\infty \bar{G}(t)dt$$

holds if and only if

$$\int_x^\infty \bar{F}(t)dt = P(i+2) + \sum_{k=i+1}^{\infty} p_k(i-x) \leqslant \Pi(i+2) + \sum_{k=i+1}^{\infty} \pi_k(i-x) = \int_x^\infty \bar{G}(t)dt,$$

$$i-1 < x \leqslant i.$$

This completes the proof of the assertion.

From Theorem 9 we infer that

$$F = \sum_{i=0}^{\infty} p_i \delta_i \prec_c G_{\frac{p}{1-p}}$$

is equivalent to

$$P(i+1) \leqslant \frac{p}{1-p} p^i, \quad i = 0, 1, \ldots, \tag{6}$$

which under the condition $P(1) = \frac{p}{1-p}$ means that the equilibrium d.f. of a d.f. F is stochastically less ($\prec_{st}$) than the geometrical d.f. with the same mean value. The function of the mean residual life is equal to

$$e_F(i) = P(i+1) \Big/ \sum_{k=i}^{\infty} p_k, \quad i = 0, 1, \ldots$$

THEOREM 10. *If* $F = \sum_{i=0}^{\infty} p_k \delta_i \prec_c G_{m_F}$, *then for* $n = 1, 2, \ldots$

$$\frac{n}{\sum_{i=0}^{n-1} \frac{1}{e_F(i)}} \leqslant \frac{p}{1-p}, \quad \textit{where} \quad m_F = \frac{p}{1-p}.$$

Proof. Using the identity

$$1 + \frac{1}{e_F(i)} = \frac{P(i)}{P(i+1)}, \quad i = 0, 1, \ldots,$$

we obtain

$$1+\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{e_F(i)}=\frac{1}{n}\sum_{i=0}^{n-1}\frac{P(i)}{P(i+1)}\geqslant\sqrt[n]{\prod_{i=0}^{n-1}\frac{P(i)}{P(i+1)}}\geqslant\frac{1}{p},$$

and the theorem is proved.

7. Remarks

Other classes of d.f.'s can be obtained in the following way. One can invert the inequalities in Theorems 3, 5, 6 and in Corollaries 1, 2. If we invert the inequalities in Corollary 1, then we find that

$$\{F\colon M_{m_F}\prec_c F\}=\left\{F\colon x\int_0^x\frac{1}{e_F(t)}\,dt\geqslant m_F\colon x>0\right\}.$$

Hence one can see that the class of NWUE of d.f.'s (see [7]) is contained in $\{F\colon M_{m_F}\prec_c F\}$.

If we invert the inequalities in Theorem 4, then (iv) is true and (i), (ii), (iii) are not true. To see this it is sufficient to consider a negative exponential d.f. which belongs to $\{F\colon M_{m_F}\prec_c F\}$.

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