MEAN RESIDUAL LIFE

TOMASZ ROLSKI
University of Wrocław, Poland

Introduction. In reliability theory the Mean Residual Life

$$e_F(x) = \begin{cases} \int_x^{\infty}(1 - F(t))dt/(1 - F(x)), & \text{for } x \text{ such that } F(x) \neq 1, \\ 0 & \text{for } x \text{ such that } F(x) = 1, \end{cases}$$

is often used (see e.g. Barlow and Proschan (1965), Marshall and Proschan (1972)). The function e_F can be interpreted as follows. If X is a life time of a unit with a distribution function (d.f.) F then

$$e_F(x) = E(X - x \mid X \geqslant x) = \int_0^{\infty}(1 - F_x(t))\,dt$$

where

$$F_x(t) = \frac{F(x+t) - F(x)}{1 - F(x)} = P(X - x < t \mid X \geqslant x)$$

i.e. the function $e_F(x)$ is the expected remaining life of an used but unfailed unit.

For the function $e_F(x)$ (or $F_x(t)$) one can state theorems concerning

1) its limiting behaviour (see e.g. Balkema and de Haan (1974), Meilijson (1972)),
2) its shape.

The function $e_F(x)$ (or $F_x(t)$) is also applied to specify and characterize some classes of d.f.'s (see e.g. Azlarov et alli (1972), Shanbhag (1970)).

In this paper we shall investigate relationships between various kinds of classes of d.f.'s determined on the ground of e_F. We shall assume that d.f.'s are left continuous and have support $[0,\infty)$. We shall denote $\bar{F} \equiv 1 - F$ and $m_F = \int_0^{\infty} x\,dF(x)$.

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A d.f. F is said to have an Increasing Failure Rate ($F \in \{IFR\}$) if for every fixed $t > 0$ the function $\bar{F}_x(t)$ is an increasing function of x (see Barlow and Proschan (1965)).

A d.f. F is said to be New Better than Used in Expectation ($F \in \{NBUE\}$) if for every $x > 0$ the inequality $e_F(x) \leq m_F$ holds (see Marshall and Proschan (1972)).

A d.f. F is said to have Decreasing Mean Residual Life ($F \in \{DMRL\}$) if $e_F(x)$ is a decreasing function (see Barlow and Proschan (1965)).

A d.f. F is said to have Convex Decreasing Mean Residual Life ($F \in \{CDMRL\}$) if $e_F(x)$ is a decreasing, convex function of x.

<u>The class of HNBUE d.f.'s.</u> In queueing or reliability theory the following order relation in the set of d.f.'s

$$F <_c G \quad \text{iff} \quad \int_0^\infty x \, dG(x) < \infty \text{ and } \int_x^\infty \bar{F}(t)dt \leq \int_x^\infty \bar{G}(t)dt, x > 0$$

is used (see for example Stoyan and Stoyan (1969), Rolski (1975a)).

It is known that $F <_c G$ iff for every increasing convex function f

$$\int_0^\infty f \, dF \leq \int_0^\infty f \, dG$$

provided above integrals exist.

Rolski (1975a) has found bounds for some characteristics of queueing systems under the assumption that $F <_c M_m$, where $M_m = 1 - \exp[(-x)/m]$, $x > 0$. Also Rolski (1975b) has given the following characterization of the class $\{F : F <_c M_{m_F}\}$ in terms of the function e_F.

<u>Theorem</u>. Relation $F <_c M_{m_F}$ holds iff

(1) $$x / \int_0^x (e_F(t))^{-1} dt \leq m_F \quad \text{for every } x > 0.$$

The inequality (1) means that for every $x > 0$

the integral harmonic mean of e_F in the interval $\,,x)$ is less than the mean of F. Thus the class of d.f. $\{F: F \prec_c M_{m_F}\}$ can be called Harmonic New Better than Used in Expectation ($\{HNBUE\}$).

The relationships between classes of d.f.'s. The following theorem is known (see Barlow and Proschan (1965))

Theorem 2. $\{IFR\} \subset \{DMRL\}$.

For a d.f. F let us denote by $\hat{F}(x) = \frac{1}{m_F}\int_0^x \bar{F}(t)\,dt$ the d.f. of the random variable X_1 in the equilibrium renewal process $S_n = \sum_{i=1}^{n} X_i$, $n \geqslant 1$, where X_i, $i \geqslant 2$ are independent random variables with the common d.f. F

Theorem 3. $F \in \{DMRL\}$ iff $\hat{F} \in \{IFR\}$.

Proof. Let $\hat{R}(x) = -\log \frac{1}{m_F}\int_x^\infty \bar{F}(t)\,dt = \int_0^x \frac{1}{e_F(t)}\,dt.$ If $\hat{R}$ is convex, then $\frac{1}{e_F}$ is increasing or e_F is decreasing and conversely.

Remark 1. The inclusion in Theorem 2 is strict. To see this it is sufficient to consider the d.f. $\hat{\bar{F}}(x) = \exp[-(x+1)^{1/2} - 1] \in IFR$. After easy computation we get

$$\bar{F}(x) = (x+1)^{(-1/2)}\exp[-(x+1)^{1/2} - 1] \notin \{IFR\}$$

Theorem 4. $\{CDMRL\} \subset \{IFR\}$.

Proof. We shall prove the assertion under an assumption that $F(x) = \int_0^x f(t)\,dt$. Then

$$\frac{d}{dx} e_F(x) = -1 + r(x)\, e_F(x), \tag{2}$$

where $r(x) = f(x)/\bar{F}(x)$ is the failure rate. Since $e_F(x)$ is convex, therefore $\frac{d}{dx} e_F(x)$ is increasing and hence

$$x \leqslant y \quad \text{implies} \quad r(x)\, e_F(x) \leqslant r(y)\, e_F(y) .$$

From the monotonicity assumption on e_F we get

$$1 \leqslant \frac{e_F(x)}{e_F(y)} \leqslant \frac{r(y)}{r(x)}$$

which completes the proof because if $r(x)$ is an increasing function then $F \in \{IFR\}$ (see Barlow and Proschan (1965)).

Remark 2. The inclusion in Theorem 4 is strict. To see this it is sufficient to consider a d.f. F with

$$r(x) = \begin{cases} 2, & 0 \leqslant x < 1, \\ 3, & 1 \leqslant x. \end{cases}$$

Then

$$e_F(x) = \begin{cases} \frac{1}{2} - \frac{1}{6} \exp(-2+2x), & 0 \leqslant x < 1, \\ \frac{1}{3} & 1 \leqslant x. \end{cases}$$

Now a question arises are there any interesting classes of d.f.'s contained in $\{CDMRL\}$.

Theorem 5. If $\bar{F}(x) = \exp[-R(x)]$, $x \geqslant 0$ is a d.f. and

$$\Psi_t(x) = R(x+t) - R(x)$$

is increasing and concave in x for every $t > 0$, then $F \in \{CDMRL\}$.

The easy proof of Theorem 5 is omitted.

Remark 3. If the assumptions of Theorem 5 are fulfilled then

$$R(x) = \int_0^x r(t)\, dt$$

and $r(t)$ is a decreasing nonnegative concave function.

Remark 4. If $F \in \{DMRL\}$ then, by (2),

$$e_F(x) \leqslant \frac{1}{r(x)}.$$

If $F \in \{CDMRL\}$ and $\frac{d}{dx} r(x) = r'(x)$ exists, then using the identity $\frac{d^2}{dx^2} e_F(x) = e_F(x)(r'(x) + r^2(x)) - r(x)$ we get $0 < r(x) \leqslant (r'(x) + r^2(x))\, e_F(x)$ and hence

$$\frac{r(x)}{r'(x) + r^2(x)} \leqslant e_F(x) , \quad x > 0.$$

The following sequence of inclusions summaries all the relations between discussed classes of d.f.'s. Here the first symbol stands for the class of d.f.'s F which fulfils the assumption of Theorem 5.

$$\{IFR_2\} \subset \{CDMRL\} \subset \{IFR\} \subset \{DMRL\} \subset \{NBUE\} \subset \{HNBUE\}$$

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Summary. In the paper the function

$$e_F(x) = \begin{cases} 0 , & x: F(x) = 1 \\ \int_x^{\infty} (1-F(t)dt/(1-F(x)) , & x: F(x) < 1 \end{cases}$$

is used to determine some classes of distribution functions. Then relations between these classes are investigated.