A note on tfie ificfi monotonicity in fifiefies

fiomas

variables. Typically $M(t)$ is a cumulative service time up to t. For example in [1, 5, 10, 11, 12, 14, 15, 17] and the book [13], $M(t) = \sum_{j=1}^{N(t)} S_j$, where $\{N(t)\}$ is a counting Cox process and $\{S_j\}$ a sequence of independent random variables, independent of the process $\{N(t)\}$, $C(t) = t$ and the comparisons are done with respect to different Cox processes. The seminal was a question of Ross [17], about the optimality of the constant arrival rate in the class of arrival rates with the same asymptotic rate, which was solved in [14]. The worst case, among all the stationary arrival rates with the same asymptotic intensity was detected in Rolski [16]. In this paper the use of the so called idcx functions was proposed to derive the result. In his paper Ross [17] posed a problem on the monotonicity of $\mathbb{E}V(M_a)$ with respect to a class $\{M_a\}$ parametrized by a. This problem was consider in many papers and the direct question of Ross for two state background CTMCs was solved in [5]. The case of more states of the background CTMC was addressed in the paper [1]. Recently in [12], it was given quite a general result in terms of a concept based on $\leq_{\rm idcx}$-ordering. The paper [2] is addressed to find insensitive bounds in some wireless data network by the use of $\leq_{\rm idcx}$-ordering.

The study of $\leq_{\rm idcv}$ ordering was motivated by two problems from the theory of queues, posed to the author by Sem Borst and Ono Boxma. For these problems we keep $\{M(t)\}$ fixed and study comparison theorems with respect to $\{C(t)\}$. While in the former studies the orderings $\leq_{\rm sm}$ and $\leq_{\rm idcx}$ were of importance, here we use the dual ordering $\leq_{idcv}$ to the ordering $\leq_{\rm idcx}$ defined by the class of increasing directionally concave functions. These are multivariate extensions of icx and icv orderings of univariate random variables. We also define the notion $\leq_{\rm idcv}$-regularity, a dual one to $\leq_{\rm idcx}$-regularity studied in [12]. We apply the theory to M/G/1 with varying instantaneous service speed, in particular priority queues and M/G/1 queues with positive and negative customers.

2 Orderings

We give definitions of needed integral orderings and classes of functions defining these orderings. It is standard to denote by cx and cv the class of convex and concave functions $f : \mathbb{R} \to \mathbb{R}$. For multivariable functions we have the following fundamental notion.

Definition 1 A function $f\colon \mathbb{R}^k \to \mathbb{R}$ is said to be supermodular or short sm if for any x and $y \in \mathbb{R}^k$,

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y),$$

where the operators $\wedge$ and $\vee$ denote, respectively, coordinate-wise minimum and maximum.

Suitable extensions of cx and cv classes for the multivariate case are:

Definition 2 A function $f\colon \mathbb{R}^k \to \mathbb{R}$ is said to be directionally convex or short dcx (directionally concave or short dcv) if for any x_1, x_2 and $y \in \mathbb{R}^k$ such that $x_1 \leq x_2$

and $y \geq 0$,

$$f(x_1+y)-f(x_1) \leq (\geq) f(x_2+y)-f(x_2).$$

We write "i" before "sm", "dcx" or "dcv" for restrictions to increasing functions respectively. Note also that the directional convexity is not the same as ordinary convexity. Important properties of dcx and dcv functions are as follows. If f is dcx (dcv) and twice differentiable, then $\partial^2 f(x)/\partial x_i \partial x_j \geq (\leq) 0$ for all i, j. Note that for sm, we have only $\partial^2 f(x)/\partial x_i \partial x_j \geq (\leq) 0$ for all $i \neq j$. More on sm and dcx function can be found in Müller and Stoyan (2002). Functions dcv and idcv appeared in the doctoral thesis of Meester [10]. It was pointed out there that the composition $f \circ d(y)$ of an icv function f with idcv function d on $\mathbb{R}^k$ is also idcv function of y (Lemma 2.2.6 of [10]). Notice that $f(x)$ is idcx if and only if $-f(-x)$ is idcv. We will use the fact that idcx functions form a cone; see e.g. Müller and Stoyan (2002).

To each class of functions considered above we relate the stochastic ordering, which in terminology of Müller and Stoyan (2002) are integral orderings. Let $\mathcal{F}$ be a class of functions like cx, icx, cv, icv, sm, ism, dcx, idcx, dcv, idcv. We say that two random vectors (in case of cx, icx, cv, icv random variables) X and Y are $\prec_{\mathcal{F}}$ ordered if $\mathbf{E}[f(X)] \leq \mathbf{E}[f(Y)]$ for any function $f \in \mathcal{F}$, such that the expectations are finite. In this way we define orderings $\leq_{\mathrm{icx}}, \leq_{\mathrm{ism}}, \leq_{\mathrm{idcx}}, \leq_{\mathrm{icv}}, \leq_{\mathrm{idcv}}$ used in this note. For ordering of stochastic processes we have:

Definition 3

1. For two $\mathbb{R}$-valued stochastic processes $\{X(t)\}_{t\in\mathbb{R}}$ and $\{Y(t)\}_{t\in\mathbb{R}}$, we say that $\{X(t)\}_{t\in\mathbb{R}}$ is smaller than $\{Y(t)\}_{t\in\mathbb{R}}$ in the $\mathcal{F}$ ordering and write $\{X(t)\}_{t\in\mathbb{R}} \leq_{\mathcal{F}}$, if for any positive integer k and any $-\infty < t_1 \leq \cdots \leq t_k < +\infty$, $\big(X(t_1),\ldots,X(t_k)\big) \leq_{\mathcal{F}} \big(Y(t_1),\ldots,Y(t_k)\big)$.
2. Let $\{C(t)\}_{t\in\mathbb{R}_+}$ and $\{C^{'}(t)\}$ be two increasing stochastic processes (defining random measures $\mathrm{d}C(t)$ and $\mathrm{d}C^{'}(t)$ respectively). It is said that $C \prec_{\mathcal{F}} C^{'}$, or $\mathrm{d}C(t) \prec_{\mathcal{F}} \mathrm{d}C^{'}(t)$, if for any k and $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq \ldots \leq s_k \leq t_k$
$$(C(s_1,t_1],\ldots,C(s_k,t_k]) \prec_{\mathcal{F}} (C^{'}(s_1,t_1],\ldots,C^{'}(s_k,t_k]),$$
where for we denote $C(a,b] = C(b)-C(a)$.
3. We say that a stationary stochastic process $\{Y_t\}_{t\in T}$, where $T = \mathbb{Z}$ or $\mathbb{R}$ is $\leq_{\mathrm{idcv}}$-regular ($\leq_{\mathrm{idcx}}$-regular) if for each k and all idcv function $f : T^{k+1} \to \mathbb{R}$, function ϕ defined by
$$\phi(t_1,\ldots,t_k) = \mathbf{E} f(Y_0, Y_{t_1}, Y_{t_1+t_2}, \ldots, Y_{t_1+\ldots+t_k})$$
is increasing (decreasing) in $t_1, t_2, \ldots, t_k \in T$.

We first recall that from the Lorenz inequality (see e.g. Müller & Stoyan (2002)) we have:

Lemma 2.1 **fifi** $Z_0 =_{\mathrm{st}} Z_1 =_{\mathrm{st}} \ldots =_{\mathrm{st}} Z_n\gamma\ t\gamma\gamma n\ (Z_0,\ldots,Z_k) \le_{\mathrm{sm}} (Z_0\ldots,Z_0)$.

The notion of $\le_{\mathrm{idcx}}$-regularity, which function also under the name of monotonicity in lag, was studied in Miyoshi & Rolski (2004). For the later use we need the following quite an obvious lemma.

Lemma 2.2 **fifi a stationary process** $\xi(t)$ is $\le_{\mathrm{idcv}}$((((l(((th((((((m(((t((((((c(ti(($f\ t\gamma\gamma\ \gamma\gamma\gamma c\gamma ss\ f(\xi(t))$ is $\le_{\mathrm{idcv}}$(((((l(((

3 General framework

Let $M(t)$ and $\{C(t)\}$ be two increasing process with stationary increments such that $M(0) = 0$ and $C(0) = 0$ and let $\mathbb{E}C(0,1] = \gamma$. Our aim is to study bounds and comparison theorems for

$$V = V(\infty; C) = \sup_{0\le t}\{M(t) - C(t)\} \tag{3.1}$$

or

$$V_T = V(T; C) = \sup_{0\le t\le T}\{M(t) - C(t)\}. \tag{3.2}$$

We have the following representations for V:

$$V = \lim_{i\to\infty} \sup\Big\{\sum_{k=1}^{j}(M(I_{i,k}) - C(I_{i,k})),\ j = 0,\ldots,i\,2^i\Big\}, \tag{3.3}$$

where $I_{i,k} = ((k-1)/2^i, k/2^i]$ for $k = 1,\ldots,i\,2^i$ and $\sum_{k=1}^{0}\cdot = 0$ conventionally. We have

$$V_T = \lim_{i\to\infty}(X + M(T) - C(T)) \vee ([\max\Big\{\sum_{k=j}^{2^i}(M(I^T_{i,k}) - C(I^T_{i,k})),\ j = 1,\ldots,2^i\Big\}, \tag{3.4}$$

where $I^T_{i,k} = ((k-1)\frac{T}{2^i}, k\frac{T}{2^i}]$ for $k = 1,\ldots,2^i$, and X,M,C are independent.

Let f: $\mathbb{R}_+ \to \mathbb{R}$ be icx, which is also assumed to be continuous on the whole $\mathbb{R}_+$. Similarly to Lemma 1 of Rolski [16], we have from the monotone convergence theorem and the continuity of f,

$$\begin{aligned} &\mathbf{E}[f(V(\infty;C))] && (3.5)\\ &= \lim_{i\to\infty}\mathbf{E}\Big[f\Big(\max\Big\{\sum_{k=1}^{j}M(I_{i,k}) - \sum_{k=1}^{j}C(I_{i,k}),\ j = 0,\ldots,i\,2^i\Big\}\Big)\Big]\\ &= \lim_{i\to\infty}\mathbf{E}\Big[f\circ g_{i2^i}(M(I_{i,1}) - C(I_{i,1}),\ldots,M(I_{i,i2^i}) - C(I_{i,i2^i}))\Big], && (3.6)\end{aligned}$$

where $g_k(x) = \max\{0, x_1,\ldots,x_1+\cdots+x_k\}$ for $x = (x_1,\ldots,x_k) \in \mathbb{R}^k$. Similarly we can represent $f(V_T)$, by the use of functions $x \to \max\{0, y + x_1 + \ldots + x_k, x_2 + \ldots + x_k,\ldots,x_1\}$. The above functions are idcx.

As in Lemma 2 of [16], $g_k(x)$ is idcx (and moreover it is a convex function too). Furthermore $d_x(y) = -f \circ g_k(x-y)$ is idcv function of y. This can be justified as follows. The composition $f \circ g_k(x+y)$ of an icx function f with an idcx function g_k on $\mathbb{R}^k$ is also idcx function of $y \in \mathbb{R}^k$ (Lemma 2.2.6 of [10]). Therefore $-f \circ g_k(x-y)$ is idcv. Hence we have the following lemma:

Lemma 3.1 Let f *bγ icxγ Tγγn*

$$\psi_k(y_1,\ldots,y_k)) = -\mathbf{E}[f \circ g_k(M(I_1)-y_1,\ldots,M(I_k)-y_k)] \tag{3.7}$$

is idcv and hence $-\psi_k(y_1,\ldots,y_k)$ is a decreasing sm function.

Further on, to a given process $\{C(t)\}$, we now associate two increasing processes $\{C^{'}(t)\}$ and $\{C^{''}(t)\}$ defined as follows:

- $C^{'}(\mathrm{d}t) = \gamma\,\mathrm{d}t$,
- $C^{''}(\mathrm{d}t) = c(0)\,\mathrm{d}t$, provided $\mathrm{d}C(t)$ to be absolute continuous and $c(t)$ is a stationary non-negative process, called intensity process; we always assume that intensity processes have Riemann integrable sample paths.

For further reference we state without proof the following lemma.

Lemma 3.2 fifi ϕ *is sγ γf k vaγiablγsγ tγan tγγ fγnctiφ̂n* $\mathbb{R}^{l_1} \times \ldots \times \mathbb{R}^{l_k} \to \mathbb{R}$

Hence

$$\begin{aligned}
&\mathbb{E}\phi\left(\int_{I_1} c(s)\,\mathrm{d}s,\ldots,\int_{I_k} c(s)\,\mathrm{d}s\right)\\
&\quad= \lim_{n\to\infty}\mathbb{E}\phi\left(\sum_{j:j/n\in I_1} c(j/n)/n,\ldots,\sum_{j:j/n\in I_k} c(j/n)/n\right)\\
\leq &\quad \lim_{n\to\infty}\mathbb{E}\phi\left(\sum_{j:j/n\in I_1} c(0)/n,\ldots,\sum_{j:j/n\in I_k} c(0)/n\right) = \mathbb{E}\phi(c(0)|I_1|,\ldots,c(0)|I_k|).
\end{aligned}$$

□

Hence we may say that $-f(V(\infty;C))$ or $-f(V(T;C))$ are idcv representable, that is their expectations are limits of functions like in Lemma 3.3.

In the next Proposition we consider three stochastic functionals: $V(\infty;C)$ as defined in (3.1),

$$V(\infty;C^{''}) = \sup_{t\geq 0}\{M(t) - \int_0^t c(t)\}$$

and

$$V(\infty;C^{'}) = \sup_{t\geq 0}\{M(t) - \gamma t\}.$$

Similarly we define $V(T;C)$, $V(T;C^{'})$ and $V(T;C^{''})$.

In point (iv) of the proposition below, we assume that $C(\mathrm{d}t) = c(t)\mathrm{d}t$ admits an intensity process $c(t)$ and consider a family of random variables $V(T;C_a)$ defined by random measures $C_a(\mathrm{d}t) = c_a(t)\mathrm{d}t$, where $(a > 0)$, admitting an intensity process $c_a(t)$ respectively defined by $c_a(t) = c(at)$, where $\{c(t)\}$ is the intensity process of some random measure C. Similarly to Miyoshi & Rolski (2002) we can prove the following a monotonicity the family of $V(T;C_a)$ with respect to $a > 0$. In the next proposition T is finite or infinite and it is tacitly assumed that all stochastic functionals V are proper random variables.

Proposition 3.4 (i)

$$V(T;C^{'}) \leq_{\mathrm{icx}} V(T;C).$$

(ii) If C $is\ abs\gamma l\gamma t\gamma\ c\gamma ntin\gamma\gamma\gamma s\ \gamma it\gamma\ int\gamma nsity$ $c(t)$, then

$$V(T;C) \leq_{\mathrm{icx}} V(T;C^{''}).$$

(iii) If $C_1 \leq_{\mathrm{idcv}} C_{2(}$ th(($V(T;C\ _1) \geq_{\mathrm{icx}} V(T;C_2)$.
(iv) Suppose that $C_a(\mathrm{d}t) = c_a(t)\mathrm{d}t\gamma$ $(a > 0)$ is a family of random measures. If $\{c(t)\}$ –

Proofi. (i) Assume T infinite. Use (3.6) and conditional Jensen inequality to obtain

$$\begin{aligned}\mathbf{E}[f(V(\infty;C))] &= \lim_{i\to\infty}\mathbf{E}\Big[g_{i2^i}(M(I_{i,1})-C(I_{i,1}),\ldots,M(I_{i,i2^i})-C(I_{i,i2^i}))\Big]\\ &\geq \lim_{i\to\infty}\mathbf{E}\Big[g_{i2^i}(M(I_{i,1})-\gamma|I_{i,1}|,\ldots,M(I_{i,i2^i})-\gamma|I_{i,i2^i}|)\Big]\\ &= \mathbf{E}[f(V(\infty;C^{'}))].\end{aligned}$$

(ii) Use Lemma 3.3 and formula (3.6).
(iii) In view of (3.6) and Lemma 3.1

$$\mathbb{E}\psi_k(C_1,\ldots,C_1)\leq\mathbb{E}\psi_k(C_2,\ldots,C_2),$$

where ψ_k was defined in (3.7). Since $C_1\leq_{\text{idcv}}C_2$ we have

$$\mathbb{E}\psi_k(C_1(I_1),\ldots,C_1(I_k))\leq\mathbb{E}\psi_k(C_2(I_1),\ldots,C_2(I_k)),$$

which yields

$$\begin{aligned}&\mathbf{E}[f\circ g_k(M(I_1)-C_1(I_1),\ldots,M(I_k)-C_1(I_k))]\\ &\quad\geq\ \mathbf{E}[f\circ g_k(M(I_1)-C_2(I_1),\ldots,M(I_k)-C_2(I_k))].\end{aligned}$$

(iv) In view of (3.6) and Lemma 3.1 it suffices to demonstrate that for an idcv function ψ, $\mathbb{E}\psi(C_a(I_1,\ldots,C_a(I_k))$ is decreasing in $a>0$. Without loss of generality we may assume that intervals I are of forms $I_k=[a_i,b_i)$, where $a_1=0$ and $b_i=a_{i+1}$. Then for each $n=1,2,\ldots$ we define the function

$$\hat{\psi}(c(0),c(1/n),\ldots,c((J-1)/n))=\mathbb{E}\psi\left(\sum_{j:j/n\in I_1}c(j/n)/n\ \ldots,\ \sum_{j:j/n\in I_k}c(j/n)/n\right)$$

where $J=l_1\times\ldots\times l_k$ and l_i is the number j such that $j/n\in I_i$. Now notice that

$$\begin{aligned}&\hat{\psi}(c_a(0),c(1/n),\ldots,c((J-1)/n)=\\ &\qquad\hat{\psi}(c(0),c(a/n),c((a/n)+(a/n)),\ldots,c((a/n)+\ldots c((a/n)+\ldots(a/n))).\end{aligned}$$

Hence $\mathbb{E}\hat{\psi}(c_a(0),c_a(1/n),\ldots,c_a((J-1)/n)$ is increasing in $a>0$ because $c(t)$ is assumed to be idcv-regular and $\hat{\phi}$ is idcv. $\square$

4 Examples

In this section we show two applications of the theory to single server queues with a Poisson arrivals.

4.1 M/G/1 queue with varying instantaneous service speed

Let $\{N(t)\}$ be the Poisson process with rate $\lambda > 0$, $\{S_i\}$ a sequence of nonnegative i.i.d. r.v.s. with common distribution B. We denote by

$$M(s,t] = \sum_{j=N(s)}^{N(t)} S_j$$

the cumulative service requirement arriving within interval $(s,t]$.

Suppose now that the service rate is given by a process $c^*(t)$, where $0 \le c^*(t) \le 1$ for all t. We assume that $c^*(t)$ is stationary (and ergodic) and that $\{N(t)\}$, $\{S_i\}$ $\{c^*(t)\}$ are independent.

Let $V(t) = V(t; C^*)$ be a stochastic process defined by

$$\mathrm{d}V(t-) = \begin{cases} \mathrm{d}M(t) - c^*(t) & ,\text{for} \quad V(t-) > 0 \quad \text{or} \quad \mathrm{d}M(t) - c^*(t) \ge 0 \\ 0, & \text{otherwise.} \end{cases} \tag{4.8}$$

The solution of (4.8) is

$$V(t) = (V(0-) + M(t) - \int_0^t c^*(v)\,dv) \vee \sup_{0\le s\le t} \{M(s,t] - \int_s^t c^*(v)\mathrm{d}v\}, \qquad t \ge 0.$$

The process $\{V(t)\}$ represents the workload at time t in the work-conserving single server queues with varying instantaneous service speed. Since we wish to study the stationary work-load, without loss of generality we may assume $V(0-) = 0$. Otherwise we have to assume that $V(-0)$ is independent from (M, C). Then

$$\begin{aligned} V(t) \quad &= \quad \sup_{0\le s\le t} \{M[s,t] - \int_s^t c^*(v)\mathrm{d}v\} \qquad (4.9)\\ &=_{\mathrm{d}} \quad \sup_{-t\le s\le 0} \{M[s,0] - \int_s^0 c^*(v)\mathrm{d}v\} \end{aligned}$$

and hence the stationary workload

$$\begin{aligned} V(\infty) \quad &=_{\mathrm{d}} \quad \sup_{-\infty< s\le 0} \{M(s,0] - \int_s^0 c^*(v)\mathrm{d}v\} \\ &=_{\mathrm{d}} \quad \sup_{0\ge s}\{M(0,s] - \int_0^s c(v)\mathrm{d}v\}) = V(\infty; C), \qquad (4.10) \end{aligned}$$

where $c(t) = c^*(-t)$. The above is well defined for $\mathbb{E}M(0,1] < \mathbb{E}c(0) = \gamma$. Note that $V(\infty; C')$ is the work-load in the standard M/G/1 system, with arrival rate λ/γ and service time distribution B. Under the condition that $C(\mathrm{d}t) = c(t)\,\mathrm{d}t$ admits stationary intensity and $\lambda\mathbb{E}S < c(0)$ a.s. we consider

$$V(\infty; C'') = \sup_{t\ge 0}\{\sum_{j=1}^{N(t)} S_i - c(0)t\}.$$

Note that the above is the workload in the M/G/1 system, with arrival rate $\lambda/c(0)$ and service time distribution B, of course provided $\lambda\mathbb{E}Sc(0)$, a.s. Unfortunately, in many cases the later assumption is a difficult requirement.

Proposition 4.1 **fifi** *T is γnitγγ γγγ inγnitγ and* $\lambda \mathbb{E}B <$ *γγ tγγn*

$$V(T;C^{'}) \le_{\mathrm{icx}} V(T;C)$$

and if *T is γnitγγ γγγ inγnitγ and* $\lambda \mathbb{E}B < c(0)$ a.s., then

$$V(T;C) \le_{\mathrm{icx}} V(T;C^{''}).$$

γγγtγγγγγγγ

$$\frac{(\lambda/\gamma)\mathbb{E}S^2}{1-(\lambda \mathbb{E}S)/\gamma} \le \mathbb{E}V(\infty;C)$$

and under the assumption that $\lambda \mathbb{E}S < c(0)$ a.s.

$$\mathbb{E}V(\infty;C) \le \mathbb{E}\frac{(\lambda/c(0))\mathbb{E}S^2}{1-(\lambda \mathbb{E}S)/c(0)}.$$

γγγγfγ The first part follows directly from Proposition 3.4. For the second part we use that in the standard M/G/1 queue with arrival rate λ and generic service time S, the mean workload is $(\lambda \mathbb{E}S^2)/(1-\lambda \mathbb{E}S)$.

□

Consider now a preemptive resume priority M/G/1 system with two types of customers: low and high priority. Customers of the low (high) priority arriving at the system according to the Poisson process with rate λ_{l} (λ_{h}) and with i.i.d. service times with distribution B_{l} (B_{h}). Then the workload process for low priority customers is the workload process in the M/G/1 system with instantaneous service speed

$$c^*(t) = \begin{cases} 1 & \text{no high priority customers at } t \\ 0 & \text{otherwise.} \end{cases}$$

Thus $c^*(t)$ is an on-off process with on and off times being distributed like the busy and idle periods respectively in the M/G/1 system with arrival rate λ_{h} and service time distribution B_{h}. Now γ is the stationary probability of no high priority customers. Let V^{l} be the steady-state work-load for the low priority customers.

Corollary 4.2

$$\frac{\frac{\lambda^{\mathrm{l}}}{\gamma}\mathbb{E}(S^{\mathrm{l}})^2}{1-\frac{\lambda}{\gamma}\mathbb{E}S^{\mathrm{l}}} \le \mathbb{E}V^{\mathrm{l}}.$$

γγγγfγ Use Lemma 4.10 and Proposition 3.4. □

We may also ask for the monotonicity of $V(T;C_a)$ with respect to $a > 0$. In this case we must know the answer for the following question.

Question. Is the stationary work-load process $V(t)$ in the M/G/1 system $\le_{idcx}$-regular or $\le_{idcv}$-regular ? From this we would have that $c(t) = \mathbb{I}(V(t) = 0)$ has the same property.

4.2 Queues with positive and negative customers

Two types of customers are arriving at the system: according to a Poisson process with rate λ^+ i.i.d. customers with service times $\{S_i^+\}$ (so called positive ones), and according to a Poisson process with rate λ^- i.i.d. customers with service times $\{S_i^-\}$ (so called negative). The system works as follows. As before we assume a work-conserving discipline. Negative customers require no service, but at their arrival a stochastic work is instantly removed from the system. Such the systems were introduced by Gelenbe [8] and subsequently studied among others by Boucherie, Boxma and Sigman [4]. In this later system it was demonstrated the stationary workload is

$$\begin{aligned} V &= \sup_{t\geq 0}\{\sum_{j=0}^{N^+(t)} S_j^+ - \sum_{j=0}^{N^-(t)} S_j^- - t\} \\ &= \sup_{t\geq 0}\{\sum_{j=0}^{N^+(t)} S_j^+ - C(t)\}, \end{aligned}$$

where

$$C(t) = \sum_{j=0}^{N^-(t)} S_j^- + t. \tag{4.11}$$

Proposition 4.3

$$\mathbb{E}V \geq \frac{\frac{\lambda^+}{1+\lambda^-\mathbb{E}S^-}\mathbb{E}(S^+)^2}{1-\frac{\lambda^+}{1+\lambda^-\mathbb{E}S^-}\mathbb{E}S^+}.$$

$\gamma\gamma\gamma f\gamma$ From Proposition (3.4). $\square$

To use Proposition 3.4 (iii) we have to derive a criterion for idcv comparison of stochastic processes like (4.11).

Proposition 4.4 fifi $S_j^{'} \leq_{icv} S_j^{-''}$ *and* $\lambda^{-'} \leq \lambda^{-''}$–

References

[1] Bäuerle, N. and Rolski, T. (1998) A monotonicity result for the workload in Markov-modulated queues. J. Appl. Probab. **35**, 741–747.

[2] Bonald, T., Borst, S. and Proutière, A. (2003) How mobility impacts the flow-lavel performance of wireless data networks. MS.

[3] Boucherie, R.J. and Boxma, O.J. (1996) The workload in the M/G/1 queue with work removal. Probability in the Engineering and Information Sciences **10**, 261–277.

[4] Boucherie, R.J., Boxma, O.J. and Sigman, K. (1997) A note on negative customers, GI/G/1 workload, and risk processes. Probability in the Engineering and Information Sciences **11**, 305–311.

[5] Chang, C.-S., Chao, X. and Pinedo, M. (1991) Monotonicity results for queues with doubly stochastic Poisson arrivals: Ross's conjecture. Adv. Appl. Probab. **23**, 210–228.

[6] Daley, D.J. (1968) Stochastically monotone Markov chains. ff. Wahrsch. Verw. Gebiete **10**, 305–317.

[7] Daley, D.J. and Vere-Jones, D. (1988). An Introduction to the Theory of Point Processes. New York: Springer-Verlag.

[8] Gelembe, E. (1991) Product–form queueing networks with negative and positive customers. J. Appl. Probab. **28**, 656–663.

[9] Grandell, J. (1976) Double Stochastic Poisson Processes Springer-Verlag, New York.

[10] Meester, L.E. (1990) Contribution to the Theory and Applications of Stochastic Convexity, PhD thesis, University of California in Berkely.

[11] Meester, L.E. and Shanthikumar, J.G. (1993) Regularity of stochastic processes: A theory based on directional convexity. Probab. Eng. and Inf. Sci. **7**, 343–360.

[12] Miyoshi, N. and Rolski, T. (2004) Ross type conjectures on monotonicity of queues. Festschrift for Daryl Daley, Phil Pollet and Peter Taylor Eds. Australian & New ffealand Journal of Statistics **46**, 121–132.

[13] Müller, A. and Stoyan, D. (2002) Comparison Methods for Stochastic Models and Risks. Wiley, Chichester.

[14] Rolski, T. (1981) Queues with non-stationary input stream: Ross's conjecture. Adv. Appl. Probab., **13**, 603–618.

[15] Rolski, T. (1984) Comparison theorems for queues with dependent inter-arrival times. in Modeling and Performance Evaluation Methodology, Eds. F. Baccelli and G. Fayolle. Lecture Notes in Control and Information Sciences **60**, Springer-Verlag, 42–67.

[16] Rolski, T. (1986) Upper bounds for single server queues with doubly stochastic Poisson arrivals. Math. Oper. Res. **11**, 442–450.

[17] Ross, S.M. (1978) Average delay in queues with non-stationary Poisson arrivals. J. Appl. Probab. **15**, 602–609.