Layering of the Poisson Process in the Quadrant

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Abstract

We consider the increasing sequence of non-intersecting monotone decreasing step processes $Y_n^*(t), n = 1, 2, \ldots$ $(t > 0)$ whose jump points cover all the points of the homogeneous rate 1 Poisson process on the quadrant R_+^2. We derive properties of these processes, in particular the marginal distributions $\mathbb{P}(Y_n^*(t) > x)$, in terms of a Toeplitz determinant of some modified Bessel functions. Our system provides a new view of the Hammersley interacting particle system discussed by Aldous and Diaconis, and the distributions we derive are related to the distribution of the length of the longest ascending sequence in a random permutation.

KEYWORDS: planar Poisson process, kth layer process, modified Bessel function, Hammersley interacting particle system, longest increasing subsequence, Ulam problem, random permutation

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1 Layering of the planar Poisson process in the quadrant

In a previous paper [11] we have studied a stochastic process $Y^*(t)$ which we called the Poisson hyperbolic staircase. One way to define $Y^*(t)$ is to consider a homogeneous planar Poisson process in the positive quadrant, and let $Y^*(t)$ be the supremum of all decreasing (we use the term in the weak sense of non-increasing) functions which have no points of the Poisson process below them. In fact, $Y^*(t)$ defined in this way will be an right continuous with left limits (RCLL) decreasing step function, which passes through a two sided infinite sequence of Poisson points (which are its values wherever it jumps down), $\ldots,(x_{-1},y_{-1}),(x_0,y_0),(x_1,y_1),\ldots$, with x_j increasing, y_j decreasing, and these points are exactly all the points of the planar Poisson process in the quadrant which have no other points in the rectangle between them and the origin. In this paper we study a sequence of processes, $Y_n^*(t), n=1,2,\ldots$, associated with the same Poisson process on the quadrant, which are defined inductively as follows: We let $Y_1^*(t)=Y^*(t)$, and let $Y_n^*(t)$ be the supremum of all decreasing functions which are above $Y_{n-1}^*(t)$, with no points of the Poisson process below them and above $Y_{n-1}^*(t)$. Again $Y_n^*(t)$ is an RCLL decreasing step function which passes through a two sided infinite sequence of Poisson points, ordered by increasing x and decreasing y, such that these points are exactly those points of the process for which the rectangle between them and the origin contains only points of the lower lines $Y_j^*(t), j<n$. We can think of these step functions as dividing the points of the planar Poisson process into layers, where layer n contains all the Poisson points (x,y) which are 'just above' the previous layer. Clearly this layering covers all the points of the planar Poisson process in the quadrant. Following terminology from [11] we call the process $Y_n^*(t)$ the nth Poisson hyperbolic staircase, or alternatively the nth layer process; see Fig. 1.1

Consider the time t and the height interval $(0,x)$, and let $0<Y_1^*(t)<\cdots<Y_n^*(t)<x<Y_{n+1}^*(t)$. Then $(Y_1^*(t),\ldots,Y_n^*(t))$ are the locations of particles at the time t, on the interval $(0,x)$, in the Hammersley interacting particle system, starting from empty at time 0 as defined by Aldous and Diaconis [2, 3].

In Section 2 we consider the vector Markov process $(Y_1(t),\ldots,Y_n(t)), t\geq 0$ conditional on the starting values $(Y_1(0),\ldots,Y_n(0))=(y_1,\ldots,y_n)$, or equivalently (in terms of distribution) $(Y_1^*(s+t),\ldots,Y_n^*(s+t))$ conditional on $(Y_1^*(s),\ldots,Y_n^*(s))=(y_1,\ldots,y_n)$. For this process we derive the conditional probability that $Y_n(t)$ remains at the level y_n until time t. This probability satisfies a renewal type equation, and most of the section is devoted to obtaining its solution. The solution is expressed in terms of determinants involving modified Bessel functions.

The probability derived in Section 2 can be used to express various conditional joint probabilities involving $Y_1(\cdot),\ldots,Y_n(\cdot),\ldots$. In particular it is the key to obtain the distribution of $Y_n^*(t)$, the nth layer process, for which we have, loosely speaking, initial state $Y_1^*(0)=\cdots=Y_n^*(0)=\infty$. The marginal distribution of $Y_n^*(t)$ is found in Section 3. We obtain $\mathbb{P}(Y_n^*(t)>x)$ as a determinant of a Toeplitz matrix of modified Bessel functions, obtained from our previous formula of Bessel determinants through substituting $y_1=\cdots=y_n=x$. This formula is a special case of a formula derived by Gessel [7], through combinatorial methods. It appears implicitly as formula (1.6) in [9]; see also formula (11) in [3].

One motivation to study the layer process $Y_n^*(t)$, $n=1,2,\ldots$ is its relation to an old

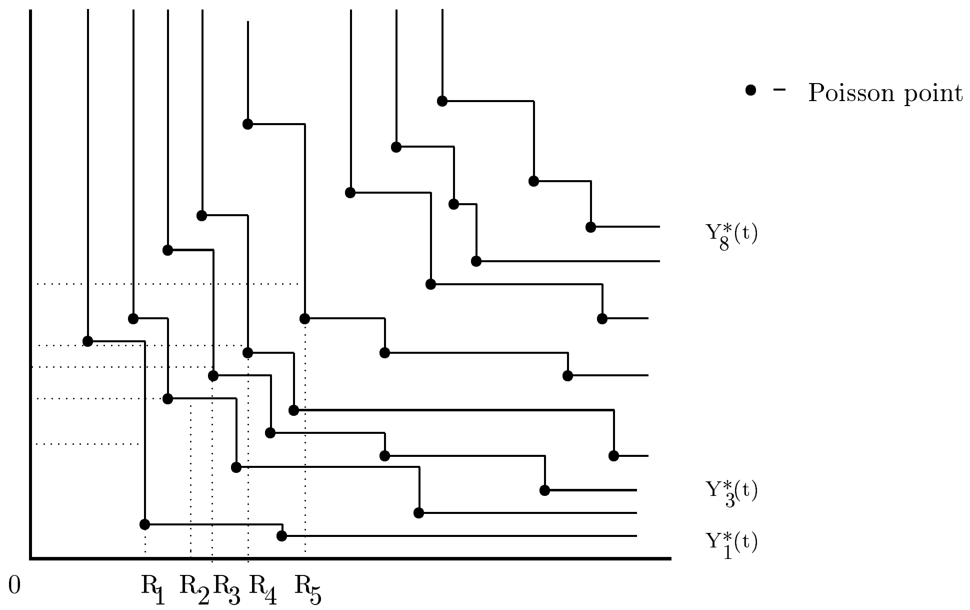

Figure 1.1: Layering of a Poisson point process on the quadrant.

combinatorial problem of Ulam and Hammersley; see e.g. Hammersley [8]. Let L_k denote the lenght of longest ascending subsequence in a random permutation of $1, \ldots, k$. Random here means that all permutations π are equally probable with probability $1/k!$; set $L_0 = 0$. Hammersley noted that L_k also equals the length of the longest ascending seqeence of points among k points uniformly and independently thrown onto rectangle $[0, t] \times [0, x]$. On the other hand, $\mathbb{P}(Y_n^*(t) > x)$ is the probability that the longest ascending sequence of points of the planar Poisson process in the rectangle $[0, t] \times [0, x]$ has length $< n$. From this observation one immediately obtains:

$$\mathbb{P}(Y_n^*(t) > x) = \sum_{k=0}^{\infty} \mathbb{P}(L_k < n) \frac{(tx)^k}{k!} e^{-tx} \ . \tag{1.1}$$

Hammersley has conjectured that $\mathbb{E}\, L_n \sim 2\sqrt{n}$. This conjecture was proven using analytic methods by Vershik and Kerov [14] and Logan and Shepp [12]. Aldous and Diaconis [2] obtained a probabilistic proof by defining the Hammersley interacting particle system and deriving its asymptotics. A different approach was used by Johansson [9], who has expressed $\mathbb{P}(Y_n^*(t) > x)$ in terms of eigenvalues of random unitary matrices and used it in a study of the asymptotic properties of $\mathbb{P}(L_k = n)$. Some further results about L_n appeared in [4, 5, 2].

We return now to equation (1.1). We see immediately that $Y_n^*(\cdot), n = 1, 2, \ldots$ have the scaling property:

$$\mathbb{P}(Y_n^*(t) > x) = \mathbb{P}(Y_n^*(1) > tx)$$

and in particular:

$$\mathbb{E}\,(Y_n^*(t)^l) = \frac{\mathbb{E}\,(Y_n^*(1))^l}{t^l}\ .$$

We can also easily derive from (1.1) the distributions of the first and second layer processes: Clearly $\mathbb{P}(Y_1^*(t) > x) = e^{-tx}$ and since $\mathbb{P}(L_k \le 1) = 1/k!$ we get for $n = 2$

$$\begin{aligned}
\mathbb{P}(Y_2^*(t) > x) &= \sum_{k=0}^{\infty} \frac{(tx)^k}{k!k!} e^{-tx} \\
&= e^{-tx} I_0(2\sqrt{tx})\ ,
\end{aligned}$$

where $I_0(x) = \sum_{s=0}^{\infty} \frac{(x/2)^{2s}}{s!s!}$ is the modified Bessel function of the first kind, of order 0.

In Section 4 we use the layer processes to study maximal ascending subsequences in random permutations. Our results provide a way to calculate $\mathbb{P}(L_k = n)$ which is an alternative to the method of Schensted [13] which used Young tableaus. We also do some numerical calculations.

2 The time to the first jump of the n'th line

In this section we consider the jointly distributed vector process $(Y_1(t), \ldots, Y_n(t))$, which is a continuous time Markov jump process in R_+^n. Each $Y_i(t)$ is a decreasing RCLL step process and $Y_1(t) \le Y_2(t) \le \ldots \le Y_n(t)$. If $Y_1(t) = y_1, \ldots, Y_n(t) = y_n$, then the nearest transition of the vector process $(Y_1(\cdot), \ldots, Y_n(\cdot))$ is with rate y_n and this transition is a jump downwards of the kth coordinate with probability $(y_k - y_{k-1})/y_n$ (for convenience we write $y_0 = 0$), and the jump of $Y_k(\cdot)$ is to a value u which is uniformly distributed over $y_{k-1} < u < y_k$. Equivalently, the time to the next transition is a random variable $\tau \sim \exp(y_n)$, and the transition generates a random value $u \sim U(0, y_n)$, which is the new level of $Y_k(\cdot)$ where $y_{k-1} < u < y_k$ (neglecting the null event $u = y_k$).

It is easily seen that the process $(Y_1(t), \ldots, Y_n(t))$ with the initial condition $Y_1(s) = y_1, \ldots, Y_n(s) = y_n$ can also be generated as the first n layers of a planar Poisson process of rate 1 in the strip $(s, \infty) \times (0, y_n)$, which is by definition $(Y_1^*(s+t), \ldots, Y_n^*(s+t))$ conditional on $(Y_1^*(s), \ldots, Y_n^*(s)) = (y_1, \ldots, y_n)$.

We use the notation $\mathbb{P}_{y_1,\ldots,y_n}(\cdot)$ for the probability of an event defined by $Y_1(t), \ldots, Y_n(t)$ with the initial condition $Y_1(0) = y_1, \ldots, Y_n(0) = y_n$ and we denote by $\mathbb{E}_{y_1,\ldots,y_n}(\cdot)$ the corresponding expectation operator.

We wish to calculate the probability $\mathbb{P}_{y_1,\ldots,y_n}(Y(t) = y_n)$ that $Y_n(\cdot)$ is still at the height y_n by time t. If we denote by T_0^n the time to the first downwards jump of the n'th line, this event is equivalent to $\{T_0^n > t\}$. We denote the probability of these equivalent events by

$$\begin{aligned}
\bar{F}_n(t, y_1, \ldots, y_{n-1}, y_n) &\stackrel{\text{def}}{=} \mathbb{P}_{y_1,\ldots,y_n}(Y_n(t) = y_n) \\
&= \mathbb{P}_{y_1,\ldots,y_n}(T_0^n > t)
\end{aligned}$$

2.1 An integral renewal type equation

We begin with a lemma with shows that $\bar{F}_n(t, y_1, \ldots, y_n)$ fulfills a renewal type integral equation.

Lemma 2.1

$$\bar{F}_n(t, y_1, \ldots, y_{n-1}, y_n) = \\ e^{-y_n t}\left(1 + \int_0^t e^{y_n s} \sum_{k=1}^{n-1} \int_{y_{k-1}}^{y_k} \bar{F}_n(s, y_1, \ldots, v, y_{k+1}, \ldots, y_{n-1}, y_n)\, dv\, ds\right). \tag{2.1}$$

Proof Consider the process $Y_1(t), \ldots, Y_n(t)$ starting at time 0 from $y_1, \ldots, y_n$. They will remain at those levels until the first (leftmost) Poisson point in the strip $(0, \infty) \times (0, y_n)$. Let (τ, u) denote the coordinates of this first point. Recall that τ, u are independent, with $\tau \sim exp(y_n)$, $u \sim U(0, y_n)$ that is

$$\mathbb{P}(\tau \in ds, u \in dv) = y_n e^{-y_n s}\, ds\, \frac{1}{y_n}\, dv.$$

If $\tau > t$, then at time t we will still have $Y_1(t) = y_1, \ldots, Y_{n-1}(t) = y_{n-1}$ and $Y_n(t) = y_n$, and so $\{T_0^n > t\}$. If $\tau \leq t$, and $y_{n-1} < u < y_n$, then $Y_n(\tau) = u$ and certainly $\{T_0^n \leq t\}$. If $\tau \leq t$, and $y_{k-1} < u < y_k$ for $k = 1, \ldots, n-1$, then $Y_k(\tau) = u$ while $Y_j(\tau) = Y_j(0), j \neq k$, and $\{T_0^n > \tau\}$. In fact, conditional on such values of (τ, u) the probability that $Y_n(t) = y_n$ is the probability that, starting with initial values $y_1, \ldots, y_{k-1}, u, y_{k+1}, \ldots, y_{n-1}, y_n$, at time τ, the n'th line will not drop below y_n for the remaining $t - \tau$ time, which is $\bar{F}_n(t-\tau, y_1, \ldots, y_{k-1}, u, y_{k+1}, \ldots, y_{n-1}, y_n)$. Hence:

$$\bar{F}_n(t, y_1, \ldots, y_{n-1}, y_n) = \\ e^{-y_n t} + \int_0^t y_n e^{-y_n s} \sum_{k=1}^{n-1} \int_{y_{k-1}}^{y_k} \bar{F}_n(t-s, y_1, \ldots, v, y_{k+1}, \ldots, y_{n-1}, y_n)\, \frac{1}{y_n}\, dv\, ds\,.$$

Cancel y_n with $\frac{1}{y_n}$, and substitute s for $t - s$, to obtain the renewal type integral equation (2.1). $\square$

Define $\Delta = \{(t, x_1, \ldots, x_n) : 0 \leq t,\ 0 \leq x_1 \leq \ldots \leq x_n\}$ and $\Delta_T = \{(t, x_1, \ldots, x_n) : 0 \leq t \leq T,\ 0 \leq x_1 \leq \ldots \leq x_n \leq T\}$. Let $C(\Delta_T)$ and $C(\Delta)$ be the set of bounded and continuous functions $f : \Delta_T \to \mathbb{R}$ and $f : \Delta \to \mathbb{R}$ respectively. Define mapping Φ

$$\Phi(f)(t, x_1, \ldots, x_n) = e^{-x_n t}\left(1 + \int_0^t e^{x_n s} \sum_{k=1}^{n-1} \int_{x_{k-1}}^{x_k} f(s, x_1, \ldots, v, x_{k+1}, \ldots, x_{n-1}, x_n)\, dv\, ds\right).$$

Notice that $\Phi : C(\Delta_T) \to C(\Delta_T)$. In $C(\Delta_T)$ we introduce the supremum metric $\rho_T(f, g)$ defined by

$$\rho_T(f, g) = \max_{(t, x_1, \ldots, x_n) \in \Delta_T} |f(t, x_1, \ldots, x_n) - g(t, x_1, \ldots, x_n)|\,.$$

The space $C(\Delta_T)$ with metric ρ_T is complete and therefore we can use Banach's fixed point theorem.

Lemma 2.2 *The mapping $\Phi : C(\Delta) \to C(\Delta)$ has a unique fixed point.*

Proof We first show that $\Phi : C(\Delta_T) \to C(\Delta_T)$ is a contraction. We have for $f, g \in C(\Delta_T)$

$$\begin{aligned}
&\rho_T(\Phi(f), \Phi(g)) \leq \\
&\quad \leq \max_{(t,x_1,\dots,x_n)\in\Delta_T} |e^{-x_n t} \int_0^t e^{x_n s} \sum_{k=1}^{n-1} \int_{x_{k-1}}^{x_k} \rho_T(f,g)\, dv\, ds| \leq \\
&\leq \ (1 - e^{-T^2})\rho_T(f,g) \ .
\end{aligned}$$

Hence, by Banach's fixed point theorem (see e.g. Kolmogorov and Fomin [10]), the mapping Φ has a unique fixed point $f_T(t, x_1, \dots, x_n)$ in $C(\Delta_T)$. Note that for $T < T'$

$$f_T(t, x_1, \dots, x_n) = f_{T'}(t, x_1, \dots, x_n), \qquad \text{for} \quad (t, x_1, \dots, x_n) \in C(\Delta_T) \ . \tag{2.2}$$

We now show that there exists a unique $f \in C(\Delta)$ such that $\Phi(f) = f$. Let $f \in C(\Delta)$ be defined as follows. For $(t, x_1, \dots, x_n) \in \Delta)$ we take $T \geq \max(t, x_n)$ and set

$$f(t, x_1, \dots, x_n) = f_T(t, x_1, \dots, x_n) \ .$$

By the above consistency statement (2.2) f is well defined, and it is clearly a fixed point of Φ. Suppose now that there exists another fixed point g of Φ. Then its restriction g_T to Δ_T is also a fixed point of $\Phi : C(\Delta_T) \to C(\Delta_T)$. Thus $g_T = f_T$ for all $T \geq 0$. This means that f is the unique fixed point of Φ. $\square$

From Proposition 2.2 we immediately conclude that the integral equation (2.1) has a unique solution.

Notice that from the definition of $Y_1(t), \dots, Y_n(t)$ we have

$$\begin{aligned}
\bar{F}_n(t, y_1, \dots, y_n) &= \mathbb{P}_{y_1,\dots,y_n}(Y_n(t) = y_n) \\
&= \mathbb{P}_{y_1 t,\dots,y_n t}(Y_n(1) = y_n t) \ .
\end{aligned} \tag{2.3}$$

To understand the next substitution we show in the following lemma another probabilistic representation for $\bar{F}_n(t, y_1, \dots, y_n)$.

Lemma 2.3

$$\bar{F}_n(t, y_1, \dots, y_n) = e^{-y_n t} \mathbb{E}_{y_1,\dots,y_{n-1}} e^{\int_0^t Y_{n-1}(s)\, ds} \ .$$

Proof Let A be the event that the rectangle $[0, t] \times [0, y_n]$ contains no points apart from those belonging to layers $Y_1(t), \dots, Y_{n-1}(t)$. The conditional probability of this event upon the whole realization of Y_{n-1} is

$$e^{-\int_0^t (y_n - Y_{n-1}(s))\, ds} \ .$$

Taking the expectation completes the proof. $\square$

Define now

$$H_n(y_1, \dots, y_n) = \mathbb{E}_{y_1,\dots,y_n}(e^{\int_0^1 Y_n(s) ds}) \ .$$

By Lemma 2.3 it is easy to see that $\bar{F}_n$ is of the form:

$$\bar{F}_n(t, y_1, \ldots, y_{n-1}, y_n) = e^{-y_n t} H_{n-1}(y_1 t, \ldots, y_{n-1} t). \tag{2.4}$$

Substituting (2.4) into (2.1) we obtain an integral equation for H_m:

$$H_m(x_1, \ldots, x_m) = 1 + \int_0^1 \sum_{k=1}^m \int_{x_{k-1}}^{x_k} H_m(x_1\tau, \ldots, x_{k-1}\tau, u\tau, x_{k+1}\tau, \ldots, x_m\tau)\, du\, d\tau \tag{2.5}$$

Proposition 2.4 *Equation (2.5) has a unique solution.*

Proof The proof follows in view of substitution (2.4), from the fact that (2.1) has a unique solution. $\square$

2.2 A multivariate partial differential equation related to a Bessel equation

Before we present the solution H_m, we do some preliminary exploration which helps us to guess the solution. Manipulation of Equation (2.1) leads to a partial differential equation for $\bar{F}$. The identity $\bar{F}_n(t; 0, y_2, \ldots, y_{n-1}, y_n) = \bar{F}_{n-1}(t; y_2, \ldots, y_{n-1}, y_n)$ provides one initial condition for that equation, and taking derivatives with respect to y_1 at $y_1 = 0$ provides another initial condition. Substitution of H instead of $\bar{F}$ finally yields the following equations. We skip the details.

The equations are:

$$\sum_{k=1}^m \frac{\partial^{m-1}}{\prod_{j=1, j\neq k}^m \partial x_j} \left(x_k \frac{\partial^2}{\partial x_k^2} H_m(x_1, \ldots, x_m) + \frac{\partial}{\partial x_k} H_m(x_1, \ldots, x_m) - H_m(x_1, \ldots, x_m) \right) = 0, \tag{2.6}$$

and we seek solutions in the region:

$$0 \leq x_1 \leq \cdots \leq x_m.$$

which satisfy initial conditions in terms of the lower dimensional functions (here the 0 dimensional H_0 is 1):

$$H_m(0, x_2, \ldots, x_m) = H_{m-1}(x_2, \ldots, x_m), \tag{2.7}$$

and

$$\left.\frac{\partial}{\partial x_1} H_m(x_1, \ldots, x_m)\right|_{x_1=0} = H_{m-1}(x_2, \ldots, x_m) - \sum_{j=2}^m \frac{\partial}{\partial x_j} H_{m-1}(x_2, \ldots, x_m) \tag{2.8}$$

We try to solve this by separation of variables. We look for solutions of the form: $H_m(x_1, \ldots, x_m) = \prod_{j=1}^m \xi_j(x_j)$, for which the equation reads:

$$\sum_{k=1}^m \prod_{j\neq k} \xi_j'(x_j) \left(x_k \xi_k''(x_k) + \xi_k'(x_k) - \xi_k(x_k) \right) = 0$$

This leads to the Bessel equation:

$$x_k \xi_k''(x_k) + (1-\theta_k)\xi_k'(x_k) - \xi_k(x_k) = 0$$

whose general solution is of the form:

$$\xi_k(x_k) = x_k^{\frac{\theta_k}{2}} \mathcal{L}_{\pm\theta_k}(\pm 2\sqrt{x_k}),$$

where $\mathcal{L}_\nu$ is any linear combination of I_ν and K_ν, the modified Bessel functions of the first and second kind, of order ν.

To continue the solution we now should consider linear combinations of such special solutions. One can then see that any candidate solution of the form:

$$\begin{aligned} \hat{H}_m(x_1,\ldots,x_m) &= I_0(2\sqrt{x_1})H_{m-1}(x_2,\ldots,x_m) - x_1^{\frac{1}{2}} I_1(2\sqrt{x_1}) \sum_{k=2}^{m} \frac{\partial}{\partial x_k} H_{m-1}(x_2,\ldots,x_m) \\ &+ \sum_{\theta_1=2}^{\infty} x_1^{\frac{\theta_1}{2}} I_{\theta_1}(2\sqrt{x_1}) \sum_{\substack{\theta_2,\ldots,\theta_m : \\ \theta_2+\cdots+\theta_m = -\theta_1}} \prod_{k=2}^{m} x_k^{\frac{\theta_k}{2}} \mathcal{L}_{\theta_k}(2\sqrt{x_k}). \end{aligned} \tag{2.9}$$

satisfies equation (2.6) and the initial conditions, where the first term guarantees the initial condition (2.7) and the second term guarantees the initial condition (2.8). It turns out that for $m = 1, 2$ the first two terms of (2.9) give the correct solution. However, for $m = 3$ the first two terms give a wrong solution, which results in $\bar{F}_n$ which is not a probability distribution (it becomes negative). Thus we need more information to choose the correct linear combinations $\mathcal{L}_{\theta_k}(2\sqrt{x_k})$, and the equation and initial conditions (2.6, 2.7, 2.8) are not sufficient for that. Nevertheless, the candidate solutions (2.9) do suggest a guess at the form of the general solution to (2.5), and the uniqueness of the solution enables us to verify this guess. We do so in the following section.

2.3 The solution

For convenience of notation we introduce:

$$\iota_\theta(x) \stackrel{\text{def}}{=} x^{\frac{\theta}{2}} I_{|\theta|}(2\sqrt{x}) = \begin{cases} x^\theta \sum_{s=0}^{\infty} \frac{x^s}{s!(s+\theta)!} & \theta = 1, 2, \ldots \\ \sum_{s=0}^{\infty} \frac{x^s}{s!(s+|\theta|)!} & \theta = 0, -1, -2, \ldots \end{cases} \tag{2.10}$$

where $I_\theta(x)$ is the modified Bessel function of the first kind, of order θ:

$$I_\theta(x) = (\frac{1}{2}x)^\theta \sum_{s=0}^{\infty} \frac{(\frac{1}{4}x^2)^s}{s!(s+\theta)!}.$$

With that notation, the asymptotics of $\iota_\theta(x)$ at $x \sim 0$ are:

$$\iota_m(x) \sim \frac{x^m}{m!}, \quad \iota_{-m}(x) \sim \frac{1}{m!}, \quad m = 0, 1, \ldots \tag{2.11}$$

and

$$\iota_m(x)' = \iota_{m-1}(x), \quad m = \ldots, -1, 0, 1, \ldots. \tag{2.12}$$

We write down first the solutions of (2.5), for $m = 1, 2, 3$ ($m = 0$ is written for completeness). These can be verified by substitution in (2.5). Note that the last two terms of H_3 correspond to the third, undetermined, part of the r.h.s. of (2.9).

$$\begin{aligned}
H_0 &= 1, \\
H_1(x) &= \iota_0(x), \\
H_2(x_1, x_2) &= \iota_0(x_1)\iota_0(x_2) - \iota_1(x_1)\iota_{-1}(x_2), \\
H_3(x_1, x_2, x_3) &= \iota_0(x_1)\iota_0(x_2)\iota_0(x_3) - \iota_0(x_1)\iota_1(x_2)\iota_{-1}(x_3) - \iota_1(x_1)\iota_{-1}(x_2)\iota_0(x_3) \\
&\quad + \iota_1(x_1)\iota_1(x_2)\iota_{-2}(x_3) + \iota_2(x_1)\iota_{-1}(x_2)\iota_{-1}(x_3) - \iota_2(x_1)\iota_0(x_2)\iota_{-2}(x_3)
\end{aligned}$$

The above special cases suggest the general solution:

Theorem 2.5 *The solution to (2.5) is:*

$$H_m(x_1, \ldots, x_m) = \begin{vmatrix} \iota_0(x_1) & \iota_{-1}(x_2) & \ldots & \iota_{1-m}(x_m) \\ \iota_1(x_1) & \iota_0(x_2) & \ldots & \iota_{2-m}(x_m) \\ \vdots & \vdots & & \vdots \\ \iota_{m-1}(x_1) & \iota_{m-2}(x_2) & \ldots & \iota_0(x_m) \end{vmatrix}. \tag{2.13}$$

Note that to calculate $H(x_1, \ldots, x_m)$ we evaluate at x_i the functions ι_j with indices $j = -i+1, \ldots, m-i$. We can expand the determinant H_m in (2.13) as sum over permutations:

$$H_m(x_1, \ldots, x_m) = \sum_\pi \operatorname{sign}(\pi) \prod_{i=1}^m \iota_{\pi(i)-i}(x_i) \tag{2.14}$$

We shall use the following notations:

- The summation π runs over all the $m!$ permutations of $1, \ldots, m$.
- $\operatorname{sign}(\pi)$ is the sign of the permutation, 1 for even, -1 for odd.
- $\pi(i)$ is the i'th element of the permutation π.
- π^{-1} is the inverse of π, i.e. $k = \pi^{-1}(j) \Leftrightarrow \pi(k) = j$.
- π^{id} is the identity permutation.
- Denote by $\zeta(\pi) = \sum_{i=1}^m (\pi(i) - i)^+$. Note that

$$\zeta(\pi) \stackrel{\text{def}}{=} \sum_{i=1}^m (\pi(i) - i)^+ = \sum_{i=1}^m (\pi(i) - i)^- = \sum_{i \neq \pi^{-1}(m)} (\pi(i) - i)^- . \tag{2.15}$$

Note also that for all permutations except π^{id}, $\zeta(\pi) > 0$.

In the remainder of this section we prove Theorem 2.5. We need to verify that (2.13) satisfies (2.5), in other words, we need to verify that the right hand side of (2.13) is a fixed point of the mapping Ψ, related to the integral equation (2.5), which is defined by:

$$\Psi(f)(x_1,\ldots,x_m) \stackrel{\mathrm{def}}{=} 1+\int_0^1\left(\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k} f(x_1\tau,\ldots,x_{k-1}\tau,u\tau,x_{k+1}\tau,\ldots,x_m\tau)\,du\right)d\tau. \tag{2.16}$$

To do so we will substitute the determinant at the right hand side of (2.13), in (2.16).

We divide the proof of the theorem into three lemmas. In the first lemma we perform integration with respect to u.

Lemma 2.6

$$\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\begin{vmatrix} \iota_0(x_1\tau) & \ldots & \iota_{1-k}(u\tau) & \ldots & \iota_{1-m}(x_m\tau) \\ \iota_1(x_1\tau) & \ldots & \iota_{2-k}(u\tau) & \ldots & \iota_{2-m}(x_m\tau) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \iota_{m-2}(x_1\tau) & \ldots & \iota_{m-k-1}(u\tau) & \ldots & \iota_{-1}(x_m\tau) \\ \iota_{m-1}(x_1\tau) & \ldots & \iota_{m-k}(u\tau) & \ldots & \iota_0(x_m\tau) \end{vmatrix} du \tag{2.17}$$

$$= \frac{1}{\tau}\begin{vmatrix} \iota_0(x_1\tau) & \iota_{-1}(x_2\tau) & \ldots & \iota_{1-m}(x_m\tau) \\ \iota_1(x_1\tau) & \iota_0(x_2\tau) & \ldots & \iota_{2-m}(x_m\tau) \\ \vdots & \vdots & \vdots & \vdots \\ \iota_{m-2}(x_1\tau) & \iota_{m-3}(x_2\tau) & \ldots & \iota_{-1}(x_m\tau) \\ \iota_m(x_1\tau) & \iota_{m-1}(x_2\tau) & \ldots & \iota_1(x_m\tau) \end{vmatrix} \tag{2.18}$$

$$= \frac{1}{\tau}\sum_{\pi}\mathrm{sign}(\pi)\iota_{m-\pi^{-1}(m)+1}(x_{\pi^{-1}(m)}\tau)\prod_{i\neq\pi^{-1}(m)}\iota_{\pi(i)-i}(x_i\tau)\ . \tag{2.19}$$

Proof We rewrite (2.17) according to the expansion (2.14) and integrate using (2.12):

$$\begin{aligned} &\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\sum_{\pi}\mathrm{sign}(\pi)\iota_{\pi(k)-k}(u\tau)\prod_{i\neq k}\iota_{\pi(i)-i}(x_i\tau)du \\ &= -\frac{1}{\tau}\sum_{k=1}^{m}\sum_{\pi}\mathrm{sign}(\pi)\iota_{\pi(k)-k+1}(x_{k-1}\tau)\prod_{i\neq k}\iota_{\pi(i)-i}(x_i\tau) \\ &\quad +\frac{1}{\tau}\sum_{k=1}^{m}\sum_{\pi}\mathrm{sign}(\pi)\iota_{\pi(k)-k+1}(x_k\tau)\prod_{i\neq k}\iota_{\pi(i)-i}(x_i\tau) \end{aligned}$$

where we use for convenience of notation $x_0 = 0$.

We consider first the negative terms, corresponding to the lower limit of the integrals, and show that they are zero for each $k = 1,\ldots,m$. For $k = 1$ we have

$$\sum_{\pi}\mathrm{sign}(\pi)\iota_{\pi(1)}(x_0\tau)\prod_{i\neq 1}\iota_{\pi(i)-i}(x_i\tau) = 0$$

since $\pi(1) > 0$, $x_0 = 0$, and so by (2.11) $\iota_{\pi(1)}(x_0\tau) = 0$.

For $k = 2, \dots, m$, we have:

$$\sum_\pi \mathrm{sign}(\pi)\iota_{\pi(k)-k+1}(x_{k-1}\tau)\prod_{i\neq k}\iota_{\pi(i)-i}(x_i\tau) = 0$$

because if we rewrite this as a determinant the columns $k-1$ and k are the same.

We now consider all the positive terms, corresponding to the upper limit of the integral. The generic term (ignoring the $1/\tau$ factor) is:

$$\mathrm{sign}(\pi)\iota_{\pi(k)-k+1}(x_k\tau)\prod_{i\neq k}\iota_{\pi(i)-i}(x_i\tau)$$

and this is summed over all π and $k = 1, \dots, m$. We now exclude from consideration for each permutation π the term of $k = \pi^{-1}(m)$, and show that without those excluded terms the sum of all the terms is zero.

For any permutation π and any k such that $k \neq \pi^{-1}(m)$, let $j = \pi(k) < m$ and let $l = \pi^{-1}(j+1)$. Define another permutation, $\tilde{\pi}$ by: $\tilde{\pi}(k) = \pi(l) = j+1, \tilde{\pi}(l) = \pi(k) = j, \tilde{\pi}(i) = \pi(i), i \neq k, l$. The mapping which assigns to the pair (π, k) the pair $(\tilde{\pi}, l)$ is one to one onto (since applying it twice is the identity). Note that π and $\tilde{\pi}$ differ by a single transposition, and therefore have opposite parities. Hence we have a one to one onto mapping from all pairs (π, k) with π even and $k \neq \pi^{-1}(m)$ to all pairs $(\tilde{\pi}, l)$ with $\tilde{\pi}$ odd and $l \neq \tilde{\pi}^{-1}(m)$. Furthermore, for such a pair:

$$\begin{aligned}
&\iota_{\pi(k)-k+1}(x_k\tau)\prod_{i\neq k}\iota_{\pi(i)-i}(x_i\tau)\\
&= \iota_{\pi(k)-k+1}(x_k\tau)\iota_{\pi(l)-l}(x_l\tau)\prod_{i\neq k,l}\iota_{\pi(i)-i}(x_i\tau)\\
&= \iota_{j+1-k}(x_k\tau)\iota_{j+1-l}(x_l\tau)\prod_{i\neq k,l}\iota_{\pi(i)-i}(x_i\tau)\\
&= \iota_{\tilde{\pi}(k)-k}(x_k\tau)\iota_{\tilde{\pi}(l)-l+1}(x_l\tau)\prod_{i\neq k,l}\iota_{\tilde{\pi}(i)-i}(x_i\tau)\\
&= \iota_{\tilde{\pi}(l)-l+1}(x_l\tau)\prod_{i\neq l}\iota_{\tilde{\pi}(i)-i}(x_i\tau)
\end{aligned}$$

and therefore the terms corresponding to (π, k) and $(\tilde{\pi}, l)$ cancel, and

$$\sum_\pi \mathrm{sign}(\pi)\sum_{k\neq\pi^{-1}(m)}\iota_{\pi(k)-k+1}(x_k\tau)\prod_{i\neq k}\iota_{\pi(i)-i}(x_i\tau) = 0.$$

The only remaining non-zero terms are those of the form:

$$\mathrm{sign}(\pi)\iota_{m-\pi^{-1}(m)+1}(x_{\pi^{-1}(m)}\tau)\prod_{i\neq\pi^{-1}(m)}\iota_{\pi(i)-i}(x_i\tau)$$

which establishes (2.19). The determinant form (2.18) is immediate. □

Our substitution of (2.13), in (2.16) has by lemma 2.6 yielded so far that:

$$\begin{aligned}\Psi(H_m)(x_1,\ldots,x_m) &= 1+\int_0^1 \frac{1}{\tau}\sum_\pi \operatorname{sign}(\pi)\iota_{m-\pi^{-1}(m)+1}(x_{\pi^{-1}(m)}\tau)\\ &\qquad \prod_{i\neq\pi^{-1}(m)} \iota_{\pi(i)-i}(x_i\tau)\ d\tau.\end{aligned}$$

In the next lemma we perform the integration with respect to τ.

For any π, $k\neq\pi^{-1}(m)$, let

$$D(\pi,k)=\operatorname{sign}(\pi)$$
$$\begin{cases}\sum_{s_1,\ldots,s_m\geq 0}\frac{x_k^{s_k+\pi(k)-k+1}}{(s_k)!(s_k+\pi(k)-k+1)!}\prod_{i\neq k}\frac{x_i^{s_i+(\pi(i)-i)^+}}{(s_i)!(s_i+|\pi(i)-i|)!}\frac{1}{\sum_{i\leq m}s_i+\zeta(\pi)+1}, & \pi(k)\geq k\\ \sum_{s_1,\ldots,s_m\geq 0}\frac{x_k^{s_k}}{(s_k)!(s_k-\pi(k)+k-1)!}\prod_{i\neq k}\frac{x_i^{s_i+(\pi(i)-i)^+}}{(s_i)!(s_i+|\pi(i)-i|)!}\frac{1}{\sum_{i\leq m}s_i+\zeta(\pi)}, & \pi(k)<k\end{cases}$$

Lemma 2.7

$$\begin{aligned}1+\int_0^1\frac{1}{\tau}\sum_\pi\operatorname{sign}(\pi)\iota_{m-\pi^{-1}(m)+1}(x_{\pi^{-1}(m)}\tau)\prod_{i\neq\pi^{-1}(m)}\iota_{\pi(i)-i}(x_i\tau)\,d\tau &=\\ = H_m(x_1,\ldots,x_m)+\sum_\pi\sum_{k\neq\pi^{-1}(m)}D(\pi,k)\ .\end{aligned}$$

Proof We perform the integration for each permutation π. To be able to integrate with respect to τ we expand each of the functions ι_θ as a Taylor series using (2.10), take the product of these series, and integrate it term by term. The exchange of integration and summation is justified by the fact that all the terms of the series are positive.

We start with the term corresponding to the identity permutation, for which:

$$\begin{aligned}&1+\int_0^1\frac{1}{\tau}\iota_1(x_m\tau)\prod_{i<m}\iota_0(x_i\tau)\,d\tau=\\ &= 1+\int_0^1\left(\sum_{r=0}^\infty\frac{x_m^{r+1}\tau^r}{r!(r+1)!}\right)\prod_{i<m}\left(\sum_{s_i=0}^\infty\frac{x_i^{s_i}\tau^{s_i}}{s_i!s_i!}\right)d\tau\\ &= 1+\sum_{r\geq 0,s_1\geq 0,\ldots,s_{m-1}\geq 0}\frac{x_m^{r+1}}{r!(r+1)!}\prod_{i<m}\frac{x_i^{s_i}}{s_i!s_i!}\frac{1}{r+\sum_{i<m}s_i+1}\\ &= 1+\sum_{r,s_1,\ldots,s_{m-1}\geq 0}\frac{x_m^{r+1}}{(r+1)!(r+1)!}\prod_{i<m}\frac{x_i^{s_i}}{s_i!s_i!}\frac{r+1}{r+\sum_{i<m}s_i+1}\\ &= 1+\sum_{r\geq 1,s_1,\ldots,s_{m-1}\geq 0}\frac{x_m^r}{r!r!}\prod_{i<m}\frac{x_i^{s_i}}{s_i!s_i!}\frac{r}{r+\sum_{i<m}s_i}\\ &= 1+\sum_{s_1+\ldots+s_m>0}\prod_{i\leq m}\frac{x_i^{s_i}}{s_i!s_i!}\frac{s_m}{\sum_{i\leq m}s_i}\end{aligned}$$

$$
\begin{aligned}
&= 1 + \sum_{s_1+\ldots+s_m>0} \prod_{i\le m} \frac{x_i^{s_i}}{s_i!s_i!} - \sum_{s_1+\ldots+s_m>0}^{\infty} \prod_{i\le m} \frac{x_i^{s_i}}{s_i!s_i!} + \sum_{s_1+\ldots+s_m>0}^{\infty} \prod_{i\le m} \frac{x_i^{s_i}}{s_i!s_i!} \frac{s_m}{\sum_{i\le m} s_i} \\
&= \prod_{i=1}^{m} \iota_0(x_i) - \sum_{s_1+\ldots+s_m>0} \prod_{i\le m} \frac{x_i^{s_i}}{s_i!s_i!} \frac{\sum_{i<m} s_i}{\sum_{i\le m} s_i} \\
&= \prod_{i=1}^{m} \iota_0(x_i) - \sum_{k=1}^{m-1} \left(\sum_{s_1,\ldots\ge 0, s_k\ge 1,\ldots,s_m\ge 0} \frac{x_k^{s_k} s_k}{s_k!s_k!} \prod_{i\ne k} \frac{x_i^{s_i}}{s_i!s_i!} \frac{1}{\sum_{i\le m} s_i} \right) \\
&= \prod_{i=1}^{m} \iota_0(x_i) - \sum_{k=1}^{m-1} \left(\sum_{s_1,\ldots\ge 0, s_k\ge 1,\ldots,s_m\ge 0} \frac{x_k^{s_k}}{(s_k-1)!s_k!} \prod_{i\ne k} \frac{x_i^{s_i}}{s_i!s_i!} \frac{1}{\sum_{i\le m} s_i} \right) \\
&= \prod_{i=1}^{m} \iota_0(x_i) - \sum_{k=1}^{m-1} \left(\sum_{s_1,\ldots,s_k,\ldots,s_m\ge 0} \frac{x_k^{s_k+1}}{s_k!(s_k+1)!} \prod_{i\ne k} \frac{x_i^{s_i}}{s_i!s_i!} \frac{1}{\sum_{i\le m} s_i + 1} \right) \\
&= \prod_{i=1}^{m} \iota_0(x_i) - \sum_{k=1}^{m-1} D(\pi^{\mathrm{id}}, k).
\end{aligned}
$$

We now consider a general permutation $\pi \neq \pi^{\mathrm{id}}$. Recall the definition and properties of $\zeta(\pi)$, in (2.15). Excluding the identity permutation, the integral equals:

$$
\int_0^1 \frac{1}{\tau} \iota_{m-\pi^{-1}(m)+1}(x_{\pi^{-1}(m)}\tau) \prod_{i\ne\pi^{-1}(m)} \iota_{\pi(i)-i}(x_i\tau)\, d\tau =
$$

$$
\begin{aligned}
&= \int_0^1 \sum_{r=0}^{\infty} \frac{x_{\pi^{-1}(m)}^{r+m-\pi^{-1}(m)+1} \tau^{r+m-\pi^{-1}(m)}}{r!(r+m-\pi^{-1}(m)+1)!} \prod_{i\ne\pi^{-1}(m)} \sum_{s_i=0}^{\infty} \frac{x_i^{s_i+(\pi(i)-i)^+} \tau^{s_i+(\pi(i)-i)^+}}{s_i!(s_i+|\pi(i)-i|)!}\, d\tau \\
&= \sum_{r,s_1,\ldots,s_{\pi^{-1}(m)-1},s_{\pi^{-1}(m)+1},\ldots,s_m\ge 0} \frac{x_{\pi^{-1}(m)}^{r+m-\pi^{-1}(m)+1}}{r!(r+m-\pi^{-1}(m)+1)!} \\
&\qquad \prod_{i\ne\pi^{-1}(m)} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i+|\pi(i)-i|)!} \frac{1}{r+\sum_{i\ne\pi^{-1}(m)} s_i + \zeta(\pi) + 1} \\
&= \sum_{r,s_1,\ldots,s_{\pi^{-1}(m)-1},s_{\pi^{-1}(m)+1},\ldots,s_m\ge 0} \frac{x_{\pi^{-1}(m)}^{r+m-\pi^{-1}(m)+1}}{(r+1)!(r+m-\pi^{-1}(m)+1)!} \\
&\qquad \prod_{i\ne\pi^{-1}(m)} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i+|\pi(i)-i|)!} \frac{r+1}{r+\sum_{i\ne\pi^{-1}(m)} s_i + \zeta(\pi) + 1} \\
&= \sum_{r\ge 1,s_1,\ldots,s_{\pi^{-1}(m)-1},s_{\pi^{-1}(m)+1},\ldots,s_m\ge 0} \frac{x_{\pi^{-1}(m)}^{r+m-\pi^{-1}(m)}}{r!(r+m-\pi^{-1}(m))!} \\
&\qquad \prod_{i\ne\pi^{-1}(m)} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i+|\pi(i)-i|)!} \frac{r}{r+\sum_{i\ne\pi^{-1}(m)} s_i + \zeta(\pi)}
\end{aligned}
$$

$$= \sum_{s_1,\ldots,s_m\geq 0} \prod_{i\leq m} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i+|\pi(i)-i|)!} \frac{s_{\pi^{-1}(m)}}{\sum_{i\leq m} s_i + \zeta(\pi)}$$

$$= \prod_{i=1}^{m} \iota_{\pi(i)-i}(x_i) - \sum_{s_1,\ldots,s_m\geq 0} \prod_{i\leq m} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i+|\pi(i)-i|)!} \frac{\sum_{i\neq\pi^{-1}(m)} s_i + \zeta(\pi)}{\sum_{i\leq m} s_i + \zeta(\pi)}$$

$$= \prod_{i=1}^{m} \iota_{\pi(i)-i}(x_i) - \sum_{k\neq\pi^{-1}(m)} \sum_{s_1,\ldots,s_m\geq 0} \frac{x_k^{s_k+(\pi(k)-k)^+}}{s_k!(s_k+|\pi(k)-k|)!}$$

$$\prod_{i\neq k} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i+|\pi(i)-i|)!} \frac{s_k+(\pi(k)-k)^-}{\sum_{i\leq m} s_i + \zeta(\pi)} \ .$$

We now split the negative term into two parts and the above equals to:

$$\prod_{i=1}^{m} \iota_{\pi(i)-i}(x_i) - \sum_{k:k\neq\pi^{-1}(m),\pi(k)\geq k} \sum_{s_1,\ldots\geq 0, s_k\geq 1,\ldots,s_m\geq 0} \frac{x_k^{s_k+(\pi(k)-k)^+}}{(s_k-1)!(s_k+|\pi(k)-k|)!}$$

$$\prod_{i\neq k} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i+|\pi(i)-i|)!} \frac{1}{\sum_{i\leq m} s_i + \zeta(\pi)}$$

$$- \sum_{k:k\neq\pi^{-1}(m),\pi(k)<k} \sum_{s_1,\ldots,s_k,\ldots,s_m\geq 0} \frac{x_k^{s_k+(\pi(k)-k)^+}}{s_k!(s_k+|\pi(k)-k|-1)!}$$

$$\prod_{i\neq k} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i+|\pi(i)-i|)!} \frac{1}{\sum_{i\leq m} s_i + \zeta(\pi)}$$

$$= \prod_{i=1}^{m} \iota_{\pi(i)-i}(x_i) - \sum_{k:k\neq\pi^{-1}(m),\pi(k)\geq k} \sum_{s_1,\ldots\geq 0, s_k\geq 0,\ldots,s_m\geq 0} \frac{x_k^{s_k+\pi(k)-k+1}}{s_k!(s_k+\pi(k)-k+1)!}$$

$$\prod_{i\neq k} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i+|\pi(i)-i|)!} \frac{1}{\sum_{i\leq m} s_i + \zeta(\pi)+1}$$

$$- \sum_{k:k\neq\pi^{-1}(m),\pi(k)<k} \sum_{s_1,\ldots,s_k\geq 0,\ldots,s_m\geq 0} \frac{x_k^{s_k}}{s_k!(s_k-\pi(k)+k-1)!}$$

$$\prod_{i\neq k} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i+|\pi(i)-i|)!} \frac{1}{\sum_{i\leq m} s_i + \zeta(\pi)}$$

$$= \prod_{i=1}^{m} \iota_{\pi(i)-i}(x_i) + \sum_{k:k\neq\pi^{-1}(m)} |D(\pi,k)| \ .$$

Hence the lemma follows. $\square$

To complete the proof of Theorem 2.5 we need to show:

Lemma 2.8

$$\sum_{\pi} \sum_{k:k\neq\pi^{-1}(m)} D(\pi,k) = 0 \ .$$

Proof We use the same $\tilde{\pi}$ and l as before, where the mapping $(k,\pi) \to (l,\tilde{\pi})$ is one to one. Recall that π and $\tilde{\pi}$ have opposite parity, and $\pi(k) = \tilde{\pi}(l) = j, \tilde{\pi}(k) = \pi(l) = j+1$. Since π, k and $\tilde{\pi}, l$ play symmetric roles we can assume, without loss of generality, that $k < l$. It is easy to see that:

$$\zeta(\tilde{\pi}) = \begin{cases} \zeta(\pi), & j < k \text{ or } l \leq j \\ \zeta(\pi)+1, & k \leq j < l \end{cases}$$

We compare $D(\pi,k)$ and $D(\tilde{\pi},l)$, and show them to be equal except for the opposite sign. We consider the three cases $j < k, k \leq j < l, l \leq j$. For $j < k$:

$$\begin{aligned} &|D(\pi,k)| = \\ &= \sum_{s_1=0}^{\infty} \cdots \sum_{s_m=0}^{\infty} \frac{x_k^{s_k}}{s_k!(s_k - j + k - 1)!} \frac{x_l^{s_l}}{s_l!(s_l + l - (j+1))!} \\ &\quad \prod_{i\neq k,l} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i + |\pi(i)-i|)!} \frac{1}{\sum_{i=1}^m s_i + \zeta(\pi)} \\ &= |D(\tilde{\pi},l)| \end{aligned}$$

For $l \leq j$:

$$\begin{aligned} &|D(\pi,k)| = \\ &= \sum_{s_1=0}^{\infty} \cdots \sum_{s_m=0}^{\infty} \frac{x_k^{s_k+j-k+1}}{s_k!(s_k + j - k + 1)!} \frac{x_l^{s_l+j+1-l}}{s_l!(s_l + j + 1 - l)!} \\ &\quad \prod_{i\neq k,l} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i + |\pi(i)-i|)!} \frac{1}{\sum_{i=1}^m s_i + \zeta(\pi) + 1} \\ &= |D(\tilde{\pi},l)| \end{aligned}$$

For $k \leq j < l$:

$$\begin{aligned} &|D(\pi,k)| = \\ &= \sum_{s_1=0}^{\infty} \cdots \sum_{s_m=0}^{\infty} \frac{x_k^{s_k+j-k+1}}{s_k!(s_k + j - k + 1)!} \frac{x_l^{s_l}}{s_l!(s_l + l - j - 1)!} \\ &\quad \prod_{i\neq k,l} \frac{x_i^{s_i+(\pi(i)-i)^+}}{s_i!(s_i + |\pi(i)-i|)!} \frac{1}{\sum_{i=1}^m s_i + \zeta(\pi) + 1} \\ &= |D(\tilde{\pi},l)| \end{aligned}$$

This completes the proof. $\square$

2.4 Calculation of some joint probabilities

Formula (2.4) can be used to calculate some more complicated probabilites as we do now. For $n_1 < \cdots < n_m$ and $t > 0$ we calculate $\mathbb{P}(Y_{n_1}(t) = y_{n_1}, \ldots, Y_{n_m}(t) = y_{n_m} | Y_1(0) = y_1, \ldots, Y_{n_m}(0) = y_{n_m})$, in words, none of the layers $n_1, \ldots, n_m$ has a downwards jump over the interval $(0, t)$. We denote this by $\bar{F}_{n_1,\ldots,n_m}(t, y_1, \ldots, y_{n-1}, y_n)$

Proposition 2.9

$$\begin{aligned}
&\bar{F}_{n_1,\ldots,n_m}(t, y_1, \ldots, y_{n-1}, y_n) \\
&= e^{-y_{n_m} t} \prod_{k=1}^{m} H_{n_k - n_{k-1} - 1}\left((y_{n_{k-1}+1} - y_{n_{k-1}})t, \ldots, (y_{n_k - 1} - y_{n_{k-1}})t\right)
\end{aligned}$$

where we use for convenience of notation $n_0 = 0$ and $y_0 = 0$.

Proof The proof is by induction on m, where the case $m = 1$ was covered in 2.4.

$$\begin{aligned}
&\bar{F}_{n_1,\ldots,n_m}(t, y_1, \ldots, y_{n_m}) \\
&= \mathbb{P}(Y_{n_1}(t) = y_{n_1}, \ldots, Y_{n_m}(t) = y_{n_m} | Y_1(0) = y_1, \ldots, Y_{n_m}(0) = y_{n_m}) \\
&= \mathbb{P}(Y_{n_m}(t) = y_{n_m} | Y_1(0) = y_1, \ldots, Y_{n_m}(0) = y_{n_m}, Y_{n_1}(t) = y_{n_1}, \ldots, Y_{n_{m-1}}(t) = y_{n_{m-1}}) \\
&\quad \mathbb{P}(Y_{n_1}(t) = y_{n_1}, \ldots, Y_{n_{m-1}}(t) = y_{n_{m-1}} | Y_1(0) = y_1, \ldots, Y_{n_m}(0) = y_{n_m}) \\
&= \mathbb{P}(Y_{n_m}(t) = y_{n_m} | Y_{n_{m-1}}(0) = y_{n_{m-1}}, \ldots, Y_{n_m}(0) = y_{n_m}, Y_{n_{m-1}}(t) = y_{n_{m-1}}) \\
&\quad \mathbb{P}(Y_{n_1}(t) = y_{n_1}, \ldots, Y_{n_{m-1}}(t) = y_{n_{m-1}} | Y_1(0) = y_1, \ldots, Y_{n_{m-1}}(0) = y_{n_{m-1}}) \\
&= \mathbb{P}(Y_{n_m - n_{m-1}}(t) = y_{n_m} - y_{n_{m-1}} | Y_1(0) = y_{n_{m-1}+1} - y_{n_{m-1}}, \ldots, Y_{n_m - n_{m-1}}(0) = y_{n_m} - y_{n_{m-1}}) \\
&\quad \mathbb{P}(Y_{n_1}(t) = y_{n_1}, \ldots, Y_{n_{m-1}}(t) = y_{n_{m-1}} | Y_1(0) = y_1, \ldots, Y_{n_{m-1}}(0) = y_{n_{m-1}}) \\
&= \bar{F}_{n_m - n_{m-1}}(t, y_{n_{m-1}+1} - y_{n_{m-1}}, \ldots, y_{n_m} - y_{n_{m-1}}) \bar{F}_{n_1,\ldots,n_{m-1}}(t, y_1, \ldots, y_{n_{m-1}}) \\
&= e^{-(y_{n_m} - y_{n_{m-1}})t} H_{n_m - n_{m-1} - 1}((y_{n_{m-1}+1} - y_{n_{m-1}})t, \ldots, (y_{n_m - 1} - y_{n_{m-1}})t) \\
&\quad \bar{F}_{n_1,\ldots,n_{m-1}}(t, y_1, \ldots, y_{n_{m-1}})
\end{aligned}$$

and we now use the induction hypothesis. □

3 Distribution of the n'th Poisson hyperbolic staircase

3.1 Conditional distribution of the n'th Poisson hyperbolic staircase.

We first calculate $\mathbb{P}_{y_1,\ldots,y_n}(Y_n(t) > x)$ the marginal distribution of the n'th line height at time t conditional on the initial values of all the lines $1, \ldots, n$ at time 0. In the following proposition we consider several cases, according to the value of x relative to $y_1, \ldots, y_n$.

Proposition 3.1

$$\mathbb{P}_{y_1,\ldots,y_{n-1},y_n}(Y_n(t) > x) = \begin{cases} 0, & y_n \le x \\ \bar{F}_n(t, y_1, \ldots, y_{n-1}, x), & y_{n-1} \le x < y_n \\ \bar{F}_n(t, y_1, \ldots, y_{n-2}, x, x), & y_{n-2} \le x < y_{n-1} \\ \vdots & \vdots \\ \bar{F}_n(t, y_1, \ldots, y_{k-1}, x, \ldots, x), & y_{k-1} \le x < y_k \\ \vdots & \vdots \\ \bar{F}_n(t, x, \ldots, x), & 0 \le x < y_1 \end{cases} \tag{3.1}$$

Proof *Case n*: $x \ge y_n$. In this case, $Y_n(t) \le x$, so,

$$\mathbb{P}_{y_1,\ldots,y_{n-1},y_n}(Y_n(t) > x) = 0, \quad x \ge y_n.$$

Case $n-1$: $y_{n-1} \le x < y_n$. Whether $Y_n(t)$ is still above the level x, or is below it, depends only on whether there are any Poisson points in the area bounded on the left and right by $0, t$, above by the height x, and below by the line $Y_{n-1}(s), 0 \le s \le t$. However, the event that there are no points in this region is also exactly the event that $Y_n(t) = x$ if the initial state was $Y_1(0) = y_1, \ldots, Y_{n-1}(0) = y_{n-1}, Y_n(0) = x$. Hence:

$$\mathbb{P}_{y_1,\ldots,y_{n-1},y_n}(Y_n(t) > x) = \bar{F}_n(t, y_1, \ldots, y_{n-1}, x), \quad y_{n-1} \le x < y_n.$$

Case $n-2$: $y_{n-2} \le x < y_{n-1}$. Whether $Y_n(t)$ is above the level x, or is below it, is determined entirely by Poisson points in the area bounded above by the height x, and below by the line $Y_{n-2}(s), 0 \le s \le t$, and the result then is:

$$\mathbb{P}_{y_1,\ldots,y_{n-1},y_n}(Y_n(t) > x) = \bar{F}_n(t, y_1, \ldots, y_{n-2}, x, x), \quad y_{n-2} \le x < y_{n-1}.$$

We prove this statement by induction in discussion of the next, more general case.
Case $k-1$, $k = 1, \ldots, n$: $y_{k-1} \le x < y_k$ (for convenience, we denote $Y_0(s) = y_0 = 0$). Consider the time T_k at which $Y_k(s)$ first drops below the level x. Apply the argument of case $n-1$ above, to the first k lines, that is to $n = k$, to see that the distribution of T_k, and the distribution of the line $Y_k(s), s \ge T_k$, is the same whether the initial value of $Y_k(0)$ is $y_k > x$ or whether it is x. Clearly, $P(Y_n(t) > x \mid T_k \ge t) = 1$. Else, for $s < t$, $P(Y_n(t) > x \mid T_k = s)$ will be dependent on the values of $Y_j(s)$. Let then $Y_j(s) = y'_j, j = 1, \ldots, n$. The distribution of $Y_j(s), j = 1, \ldots, k$ is the same whether $y_k > x$ or $y_k = x$, and $y'_j < x, j = 1, \ldots, k$. Using as induction hypothesis the cases $k, \ldots, n$,

$$\begin{aligned} &\mathbb{P}(Y_n(t) > x \mid T_k = s, Y_j(s) = y'_j, j = 1, \ldots, n) = \\ &= \mathbb{P}_{y'_1,\ldots,y'_n}(Y_n(t-s) > x) \\ &= \mathbb{P}_{y'_1,\ldots,y'_k,x,\ldots,x)}(Y_n(t-s) > x)\ . \end{aligned}$$

The last expression is again the same whether $y_k > x$ or $y_k = x$, and so we have shown:

$$\mathbb{P}_{y_1,\ldots,y_{k-1},y_k,y_{k+1},\ldots,y_n}(Y_n(t) > x) = \bar{F}_n(t, y_1, \ldots, y_{k-1}, x, y_{k+1}, \ldots, y_n), \quad y_{k-1} \le x < y_k$$

Furthermore, clearly $\bar{F}_n$ is continuous in y_k at $y_k = x$ (since the probability of Poisson points at the level x is 0), so we can write:

$$\mathbb{P}_{y_1,\ldots,y_{k-1},y_k,y_{k+1},\ldots,y_n}(Y_n(t) > x) = \mathbb{P}_{y_1,\ldots,y_{k-1},x-,y_{k+1},\ldots,y_n}(Y_n(t) > x), \quad y_{k-1} < x < y_k$$

where $x-$ denotes limit from the left as $y_k \uparrow x$. But for $x-$ we can use the induction hypothesis of case k, to get:

$$\mathbb{P}_{y_1,\ldots,y_{k-1},x-,y_{k+1},\ldots,y_n}(Y_n(t) > x) = \bar{F}_n(t, y_1, \ldots, y_{k-1}, x-, x, \ldots, x), \quad x- < x < y_k$$

and so we have shown, using the continuity of $\bar{F}$ again:

$$\mathbb{P}_{y_1,\ldots,y_{k-1},y_k,\ldots,y_n}(Y_n(t) > x) = \bar{F}_n(t, y_1, \ldots, y_{k-1}, x, \ldots, x), \quad y_{k-1} < x < y_k$$

Case 0: $0 < x \leq y_1$. For this case by what we have shown:

$$\mathbb{P}_{y_1,\ldots,y_{k-1},y_n}(Y_n(t) > x) = \bar{F}_n(t, x, \ldots, x), \quad 0 < x \leq y_1$$

□

3.2 Distribution of the canonical nth Poisson hyperbolic staircase

We return to the full quadrant, and consider the nth layer of the Poisson points in the plane. We define

$$h_n(x) = H_n(x, \ldots, x) \ .$$

Theorem 3.2

$$\mathbb{P}(Y_n^*(t) > x) = e^{-tx} h_{n-1}(tx) \ .$$

Proof The Poisson hyperbolic staircases start very high at times close to 0. Hence $\mathbb{P}(Y_n^*(t) > x)$ is like calculating $\mathbb{P}(Y_n(t) > x)$ with initial conditions $Y_1(0) = \cdots = Y_n(0) = \infty$. The formal argument is:

For any $0 < t_0 < t$,

$$\begin{aligned}
&\mathbb{P}(Y_n^*(t) > x) = \\
&\quad = \mathbb{P}(Y_1^*(t_0) > x)\mathbb{P}(Y_n^*(t) > x \,|\, Y_1^*(t_0) > x) + \mathbb{P}(Y_1^*(t_0) \leq x)\mathbb{P}(Y_n^*(t) > x \,|\, Y_1^*(t_0) \leq x) \\
&\quad = \mathbb{P}(Y_1^*(t_0) > x)\mathbb{P}_{x,\ldots,x}(Y_n(t - t_0) > x) + \mathbb{P}(Y_1^*(t_0) \leq x)P(Y_n^*(t) > x \,|\, Y_1^*(t_0) \leq x) \ ,
\end{aligned}$$

where the second equality follows from the last section. Recall $Y_1^*(t_0) \sim \exp(t_0)$, hence, letting $t_0 \to 0$, the second summand approaches 0, and the first approaches $\bar{F}_n(t, x, \ldots, x)$. We get:

$$P(Y_n^*(t) > x) = \mathbb{P}_{x,\ldots,x}(Y_n(t) > x) = \bar{F}_n(t, x, \ldots, x) = e^{-tx} H_{n-1}(tx, \ldots, tx) \ .$$

□

Remark 1 The calculation of Section 2 gave us $\bar{F}_n(t, x_1, \ldots, x_n) = e^{-tx_n} H_n(x_1, \ldots, x_{n-1})$, and this was the key quantity to compute the marginal distributions of $Y_n(t)$ conditional on arbitrary values at time 0, and from those we finally obtained the marginal distribution of $Y_n^*(t)$. While doing so it turns out that this distribution no longer depends on the full multivariate function $H_n(x_1, \ldots, x_{n-1})$ but only on the value of this function on its diagonal. However, there seems to be no direct way to reach the univariate result: The probabilistic arguments as well as the solution of the integral equation had to be done for the multivariate case, starting from initial conditions away from the diagonal (at $x_1 = 0$), given in terms of lower dimensional functions, and the collapse to the diagonal was possible only at the end.

Remark 2 The function $h_m(x)$ can be written in several different ways:

$$
\begin{aligned}
h_m(x) = & \\
= & \begin{vmatrix} \iota_0(x) & \iota_{-1}(x) & \ldots & \ldots & \iota_{1-m}(x) \\ \iota_1(x) & \iota_0(x) & \iota_{-1}(x) & \ldots & \iota_{2-m}(x) \\ \vdots & \ddots & \ddots & & \vdots \\ \iota_{m-1}(x) & \iota_{m-2}(x) & \ldots & \iota_1(x) & \iota_0(x) \end{vmatrix} \\
= & \text{ Wronskian } \left(\iota_{m-1}(x), \iota_{m-1}^{(1)}(x), \ldots, \iota_{m-1}^{(m-1)}(x)\right) \\
= & \begin{vmatrix} I_0(2\sqrt{x}) & I_1(2\sqrt{x}) & I_2(2\sqrt{x}) & \ldots & I_{m-1}(2\sqrt{x}) \\ I_1(2\sqrt{x}) & I_0(2\sqrt{x}) & I_1(2\sqrt{x}) & \ldots & I_{m-2}(2\sqrt{x}) \\ I_2(2\sqrt{x}) & I_1(2\sqrt{x}) & I_0(2\sqrt{x}) & \ldots & I_{m-3}(2\sqrt{x}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{m-1}(2\sqrt{x}) & I_{m-2}(2\sqrt{x}) & I_{m-3}(2\sqrt{x}) & \ldots & I_0(2\sqrt{x}) \end{vmatrix}
\end{aligned}
$$

The first expression is obtained when substituting $x_1 = \cdots = x_n = x$ in (2.13). The second follows from (2.12) where $^{(\ell)}$ denotes the ℓ'th derivative. The third follows from (2.10), and noting that the powers of x cancel. The third form is the determinant of a symmetric Toeplitz matrix.

4 Some applications and examples

4.1 Connection to Ulam-Hammersley problem

We have mentioned the connection of the Poisson hyperbolic staircases to the random variable L_n which is the length of the longest ascending subsequence in a permutation of $1, \ldots, n$ in the Introduction. By (1.1) and (3.2) we have

$$
\begin{aligned}
\mathbb{P}(Y_n^*(1) > x) &= e^{-x} h_{n-1}(x) \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!} e^{-x} \mathbb{P}(L_k < n).
\end{aligned}
$$

Hence:

$$
\sum_{k=0}^{\infty} \frac{x^k}{k!} \mathbb{P}(L_k \leq n) = h_n(x).
$$

and we obtain formulas for calculating the distribution and expectation of L_k:

$$\mathbb{P}(L_k \leq n) = \frac{d^k}{dx^k} h_n(x)|_{x=0} \tag{4.1}$$

$$\Pr(L_k = n) = \frac{d^k}{dx^k}(h_n(x) - h_{n-1}(x))|_{x=0} \tag{4.2}$$

$$\mathbb{E}\,(L_k) = \sum_{n=1}^{k} n\mathbb{P}(L_k = n) = -\frac{d^k}{dx^k}\sum_{n=1}^{k} h_n(x)|_{x=0}\ . \tag{4.3}$$

For $k = 1$ we have trivially $\mathbb{P}(L_k = 1) = 1/k!$. We now compute $\mathbb{P}(L_k = 2)$. Clearly for $k = 0, 1$ these probabilites are 0. Table 4.1 has values for $k = 2\ldots, 10$.

Table 4.1: *Values of* $P(L_k = 2)$.

k	2	3	4	5	6	7	8	9	10
$P(L_k = 2)$	.5	.66667	.54167	.34167	.18194	.084921	.035441	.013396	.0046283

Proposition 4.1

$$P(L_k = 2) = \sum_{s=0}^{k-1}\binom{k}{s}\frac{2s-k+1}{(s+1)!(k-s)!} = \sum_{s=0}^{[\frac{k}{2}]-1}\frac{k!(k-2s-1)^2}{(s+1)!^2(k-s)!^2} \tag{4.4}$$

Proof We have:

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{x^k}{k!}P(L_k \leq 2) = \iota_0(x)\iota_0(x) - \iota_1(x)\iota_{-1}(x) \\
&= \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^s}{s!s!}\frac{x^t}{t!t!} - \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{x^{(s+1)}}{s!(s+1)!}\frac{x^t}{t!(t+1)!} \\
&= \sum_{k=0}^{\infty}x^k\sum_{s=0}^{k}\frac{1}{s!s!}\frac{1}{(k-s)!(k-s)!} - \sum_{k=0}^{\infty}x^{(k+1)}\sum_{s=0}^{k}\frac{1}{s!(s+1)!}\frac{1}{(k-s)!(k-s+1)!} \\
&= \sum_{n=0}^{\infty}x^k\sum_{s=0}^{k}\frac{1}{s!s!}\frac{1}{(k-s)!(k-s)!} - \sum_{k=1}^{\infty}x^k\sum_{s=0}^{k-1}\frac{1}{s!(s+1)!}\frac{1}{(k-s-1)!(k-s)!} \\
&= \sum_{n=0}^{\infty}x^k[\frac{1}{k!k!} + \sum_{s=0}^{k-1}\frac{(s+1)-(k-s)}{s!(s+1)!(k-s)!(k-s)!}] \\
&= \sum_{k=0}^{\infty}\frac{x^k}{k!}[\frac{1}{k!} + \sum_{s=0}^{k-1}\binom{k}{s}\frac{2s-k+1}{(s+1)!(k-s)!} \qquad (4.5)
\end{aligned}$$

and by equating terms, and recalling $P(L_k = 0 \cup L_k = 1) = \frac{1}{k!}$, we have (4.4). $\square$

4.2 Evaluation of the n'th staircase distribution

We now give some numerical examples of calculations. In addition to the variables $Y_n^*(1)$ we define R_n to be the length of the side of the largest square below the layer $Y_n(\cdot)$. Then

$$\begin{aligned}\mathbb{P}(R_n > r) &= \mathbb{P}(Y_n^*(r) > r) = e^{-r^2} h_{n-1}(r^2) \\ &= e^{-r^2} \begin{vmatrix} I_0(2r) & I_1(2r) & I_2(2r) & \dots & I_{n-2}(2r) \\ I_1(2r) & I_0(2r) & I_1(2r) & \dots & I_{n-3}(2r) \\ I_2(2r) & I_1(2r) & I_0(2r) & \dots & I_{n-4}(2r) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{n-2}(2r) & I_{n-3}(2r) & I_{n-4}(2r) & \dots & I_0(2r) \end{vmatrix}\end{aligned} \tag{4.6}$$

Calculation of expected values follows by integration of $\mathbb{P}(Y_n^*(1) > x), \mathbb{P}(R_n > r)$, which we carried out numerically. The following table gives some values obtained for the expectation of $Y_n^*(1), R_n$. Figure 4.2 plots these values. By the asymptotics of $L_k \to 2\sqrt{k}$ we would expect

Table 4.2: *Expectations of line height at time 1 and of side of the supporting square.*

n	1	2	3	4	5
$E(Y_n(1))$	1.	2.718281828	5.090678729	8.086508585	
$E(R_n)$	0.8862269255	1.55389935	2.172009063	2.766453430	3.346552361
n	6	7	8	9	10
$E(R_n)$	3.916841038	4.479911245	5.037401684	5.590422930	6.139767236

that $\mathbb{E}(R_n) \sim \frac{n}{2}$.

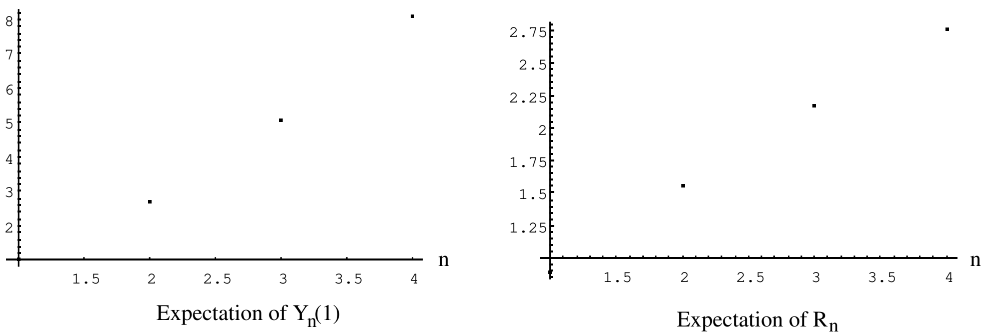

Figure 4.1: Expectations of line heights $Y_n(1)$ and square sides R_n

Figure 4.2 plots some of the distributions.

We now notice that

$$
\begin{aligned}
&\mathbb{P}(R_n > r) \\
&= P(\text{longest ascending sequence of points in } r \times r \text{ square is} < n) \\
&= \sum_{k=0}^{\infty} \frac{r^{2k}}{k!} e^{-r^2} P(L_k < n)
\end{aligned} \tag{4.7}
$$

We can easily check that this is consistent with our results for $n = 1, 2$:

$$
\begin{aligned}
\mathbb{P}(R_1 > r) &= \sum_{k=0}^{\infty} \frac{r^{2k}}{k!} e^{-r^2} P(L_k < 1) = e^{-r^2} \\
\mathbb{P}(R_2 > r) &= \sum_{k=0}^{\infty} \frac{r^{2k}}{k!} e^{-r^2} P(L_k = 0 \cup L_k = 1) = \sum_{k=0}^{\infty} \frac{r^{2k}}{k!k!} e^{-r^2} .
\end{aligned}
$$

Integrating (4.7) from 0 to ∞ we have

$$
\mathbb{E}\, R_n = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^{k+1} k!} \mathbb{P}(L_k < n)
$$

because

$$
\int_0^{\infty} r^{2k} e^{-r^2}\, dr = \sqrt{\pi} \frac{(2k-1)!!}{2^{k+1}}
$$

(see [1], formula 7.4.3)

For example for $k = 1$ we have

$$
\mathbb{E}\, R_1 = \sqrt{\pi} .
$$

For $k = 2$ we have

$$
\mathbb{E}\, R_2 = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k k! k!}
$$

and for $k = 3$

$$
\mathbb{E}\, R_3 = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{2k-1)!!}{2^k k!} [\frac{1}{k!} + \sum_{s=0}^{k-1} \binom{k}{s} \frac{2s-k+1}{(s+1)!(k-s)!}] .
$$

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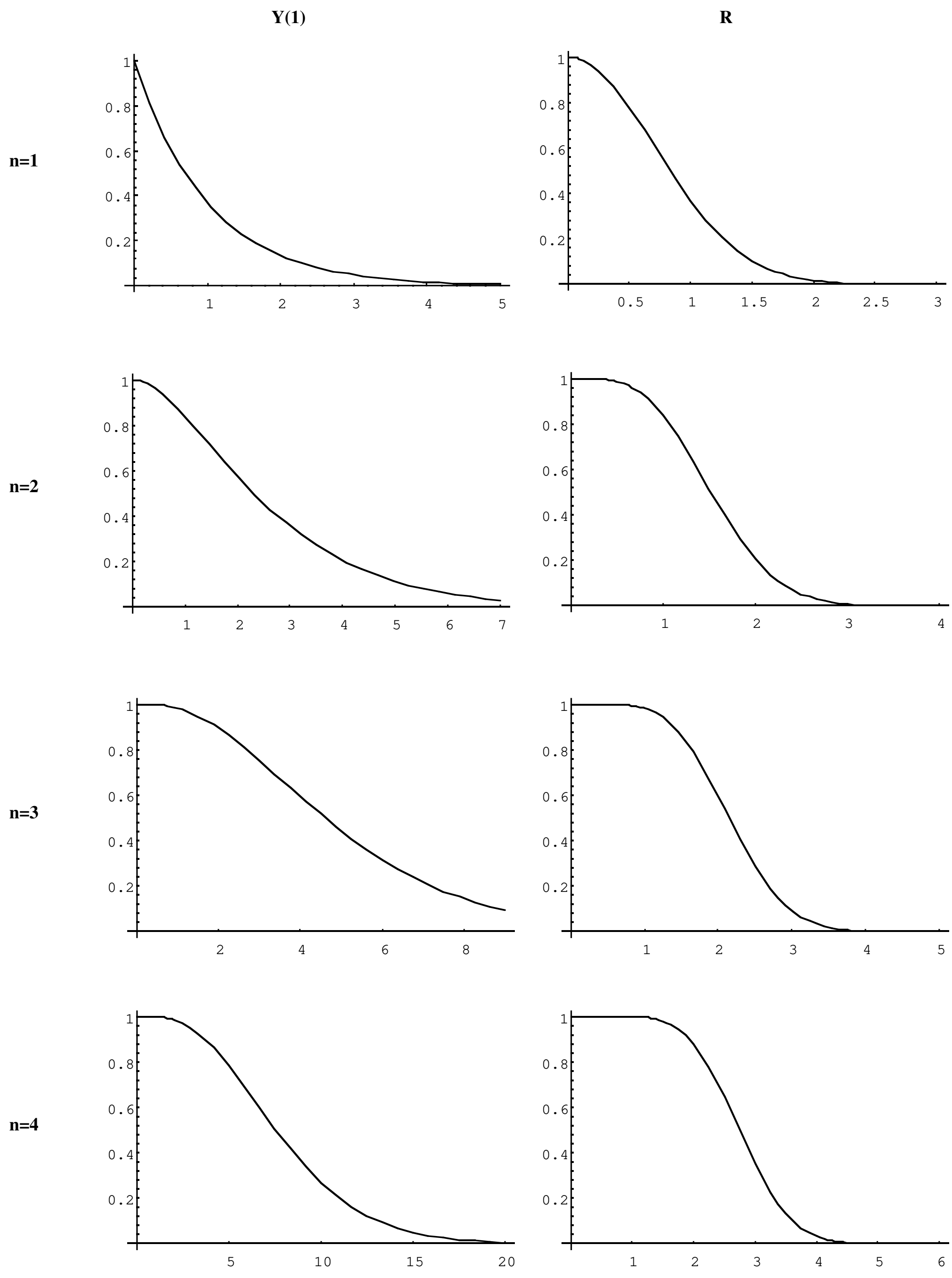

Figure 4.2: Tail distributions of heights $Y_n(1)$ and square sides R_n