Some Problems in the Theory of Risk

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1 Lecture: Concepts from insurance

1.1 Individual Model

Consider a portfolio consisting of n policies with individual risks $U_1, \ldots, U_n$ over a given time period (one year, say). We assume that the nonnegative random variables $U_1, \ldots, U_n$ are independent, but not necessarily identically distributed. Let the distribution F_{U_i} of U_i be the mixture $F_{U_i} = (1-\theta_i)\delta_0 + \theta_i F_{V_i}$, where $0 < \theta_i \leq 1$ and where F_{V_i} is the distribution of a (strictly) positive random variable V_i; $i = 1, \ldots, n$. In actuarial applications, the probabilities θ_i are small and can be interpreted as the probability that the i-th policy produces a positive claim V_i. The *aggregate claim amount* in this model, which we call the *individual model*, is $X^{\text{ind}} = \sum_{i=1}^n U_i$ with distribution $F_{U_1} * \ldots * F_{U_n}$. A portfolio is called *homogeneous* if $F_{V_1} = \ldots = F_{V_n}$.

Example *Homogeneous portfolio* if $F_{V_1} = \ldots = F_{V_n} = F$. Let $I_1, \ldots, I_n$ be independent Bernoulli 0-1-variables with $\mathbb{P}(I_i = 1) = \theta_i$. Define $N' = \sum_{i=1}^n I_i$ with $p'_k = \mathbb{P}(N' = k)$. Then

$$F_{U_1} * \ldots * F_{U_n} = \sum_{i=1}^{n} p'_i F^{*i} \,. \tag{1.1}$$

Example of a portfolio In Table 1.1 we show an example of a portfolio, consisiting of 29 policies. In Table 1.2 we show $\mathbb{P}(X^{\text{ind}} \geq k)$ computed for this portfolio.

	Amount at Risk					
θ_i	1	2	3	4	5	6
.02	3	1	-	1	2	-
.03	-	-	2	3	-	2
.04	-	4	-	-	1	-
.05	-	3	1	1	-	1
.06	-	-	3	-	-	1

Table 1.1: A sample portfolio of 29 policies

1.2 Collective Model

We suppose that a portfolio consists of a number of anonymous policies which we do not observe separately. The total number N of claims occurring in a given period is random. Further, the claim sizes U_i are (strictly) positive and are

k	r_k	k	r_k	k	r_k
0	1.00000	7	0.19154	14	0.01866
1	0.67176	8	0.15469	15	0.01246
2	0.65166	9	0.11344	16	0.00807
3	0.53802	10	0.08019	17	0.00529
4	0.43065	11	0.05778	18	0.00336
5	0.35297	12	0.04053	19	0.00211
6	0.28675	13	0.02674	20	0.00133

Table 1.2: The distribution of aggregate claims

assumed to form a sequence $U_1, U_2, \ldots$ of independent and identically distributed random variables. We also assume that the sequence $U_1, U_2, \ldots$ of individual claim sizes is independent of the claim number N. Typically, N has a Poisson, binomial or negative binomial distribution, but other choices are possible. This model is called the *collective model*, and the *aggregate claim amount* is the random variable $X^{\mathrm{col}} = \sum_{i=1}^{N} U_i$. Here and throughout the whole lecture we use the convention that $\sum_{i=1}^{0} U_i = 0$.

The idea is to approximate the individual model by a suitably chosen collective model if the size n of the portfolio is large. This is done because the collective model is often mathematically easier to handle. In this connection, a crucial problem is how to specify the parameters of the collective model to have a good approximation.

1.3 Basic notions of risk theory

Suppose now that we register each claim separately; its occurence time and its size. We have then the following notions.

- The *epochs of the claims*; denote them by $\sigma_1, \sigma_2, \ldots$. In some cases we consider an additional claim arrival epoch at time zero denoted by $\sigma_0 = 0$. Apart from the fact that the epochs form a nondecreasing sequence we do not in general assume anything specific about their interdependence. The random variables defined by $T_0 = 0$ and $T_n = \sigma_n - \sigma_{n-1}$, $n \geq 1$, are called the *inter-occurrence times* in between successive claims.

- The *number of claims* up to time t is denoted by $N(t)$ where $N(t) = \sup\{n : \sigma_n \leq t\}$. The intrinsic relation between the sequence of claim arrivals $\{\sigma_0, \sigma_1, \sigma_2, \ldots\}$ and the counting process $\{N(t), t \geq 0\}$ is given by $\{N(t) = n\} = \{\sigma_n \leq t < \sigma_{n+1}\}$.

- The claim occurring at time σ_n has size U_n. The sequence $\{U_n, n = 1, 2, \ldots\}$ of consecutive *claim sizes* is often assumed to consist of independent and identically distributed random variables. However, other possibilities will show up in the text as well.

- The *aggregate claim amount* up to time t is given by $X(t) = \sum_{i=1}^{N(t)} U_i$ while $X(t) = 0$ if $N(t) = 0$. By its very definition, the aggregate claim amount is in general a *random sum* of random variables.

- The *premium income*. In the course of time 0 to t we assume that a total of $\Pi(t)$ has been received through premiums.

- The *risk reserve* at time t is then $R(t) = u + \Pi(t) - X(t)$, where u is *initial reserve*.

Examples of claim number processes
1. *Renewal process.* $T_1, T_2, \ldots,$ are iid. If the common distiribution is exponential with parameter λ ($\mathrm{Exp}(\lambda)$), then $N(t)$ is a *Poisson process* with rate λ. Then

$$\mathbb{P}(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}. \tag{1.2}$$

The *index of dispersion* $\mathrm{I}(t) = \mathrm{Var}\, N(t) / \mathbb{E}\, N(t)$ is constant and equal to 1.

2.*Mixed Poisson process.* Two step random mechanizm: first: choose λ according to distribution F_Λ, second: generate the counting Poisson process $N(t)$ with rate λ. Then

$$p_k(t) = \mathbb{P}(N(t) = k) = \int_0^\infty \frac{\mathrm{e}^{-\lambda t} (\lambda t)^k}{k!} \, dF_\Lambda(\lambda) \tag{1.3}$$

where $F_\Lambda(\lambda) = \mathbb{P}(\Lambda \le \lambda)$ is the distribution function of the *mixing random variable* Λ.

Proposition 1.1

$$I(t) = 1 + t \frac{\mathrm{Var}\, \Lambda}{\mathbb{E}\, \Lambda}.$$

Example We get *Polya process* if $\Lambda \sim \Gamma(a, b)$ with density

$$f_\Lambda(x) = \frac{1}{\Gamma(a)} b^a x^{a-1} e^{-bx}.$$

then

$$p_k(t) = \mathbb{P}(N(t) = k) = \binom{a+k-1}{k} \left(\frac{b}{t+b}\right)^a \left(\frac{t}{t+b}\right)^k.$$

Risk reserve process and ruin problem *Ruin theory* has always been a vital part of actuarial mathematics. At first glance, some of the theoretically derived results seem to have limited scope in practical situations. Nevertheless calculation of and approximation to ruin probabilities have been a constant source of inspiration and technique development in actuarial mathematics. Consider the process $R(t) = u + \Pi(t) - X(t)$ and define the random variable

$$\tau = \tau(u) = \inf\{t \geq 0 : R(t) < 0\}.$$

The instant τ gives us the *ruin time* of the portfolio, where we interpret ruin in a technical sense.

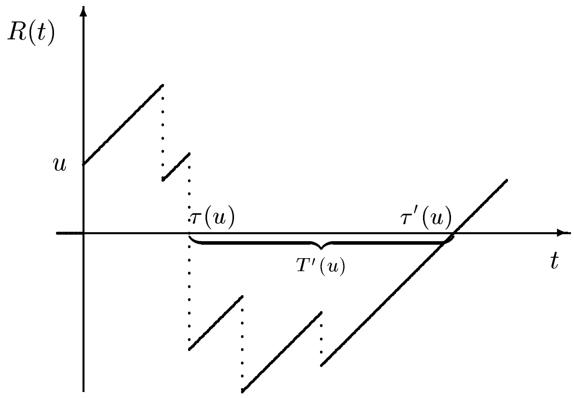

Figure 1.1: Risk reserve process

The *survival* or *non-ruin probability in finite time* will be defined and denoted by

$$\overline{\psi}(u;x) = \mathbb{P}\Big(\inf_{0 \leq t \leq x} R(t) \geq 0\Big) = \mathbb{P}(\tau(u) > x)$$

when we consider a *finite horizon* $x > 0$. The *survival probability* over an infinite time horizon is defined by the quantity

$$\overline{\psi}(u) = \mathbb{P}\Big(\inf_{t \geq 0} R(t) \geq 0\Big) = \mathbb{P}(\tau(u) = \infty)\ .$$

Alternative notations that are in constant use refer to the *ruin probabilities* which are defined by the equalities

$$\psi(u;x) = 1 - \overline{\psi}(u;x)\ , \qquad \psi(u) = 1 - \overline{\psi}(u)\,.$$

Exercises

1.1 (i) Show (1.1).
(ii) For $\theta_1 = \ldots = \theta_n$ compute $p_1^{'}, \ldots$.

1.2 Show (1.2).

2 Lecture: Probability distributions

2.1 Basic concepts and notations

Let X be a random variable. Its d.f. we denote by F or F_X. The density function (provided it exisits, is denoted by f or f_X. By $\overline{F}(x) = 1 - F(x)$ we denote the *tail* of F. In actuarial applications, a nonnegative random variable is frequently called a *risk*. $\mu = \mathbb{E}\, X$ is called the *mean*, the *expectation* or the *first moment* of X, $\mu^{(n)} = \mathbb{E}\,(X^n)$ is called the n-th *moment*. The *variance* of X is $\sigma^2 = \mathbb{E}\,(X - \mu)^2$ and $\sigma = \sqrt{\sigma^2}$ is the *standard deviation*. Sometimes the symbol $\operatorname{Var} X$ will be used instead of σ^2. The *moment generating function* $\hat{m} : I \to \mathbb{R}$ of X is defined by $\hat{m}(s) = \mathbb{E}\, \mathrm{e}^{sX}$. Note the difference between the moment generating function and the *Laplace-Stieltjes transform* $\hat{l}(s) = \mathbb{E}\, \mathrm{e}^{-sX} = \hat{m}(-s) = \int_{-\infty}^{\infty} \mathrm{e}^{-sx}\, dF(x)$ of X or of the distribution function F of X.

2.2 Parametrized families of distributions

- *Poisson distribution* $\mathrm{Poi}(\lambda)$ with $p_k = \mathrm{e}^{-\lambda}\lambda^k/k!$ for $k = 0, 1 \ldots;\ 0 < \lambda < \infty$.
- *Geometric distribution* $\mathrm{Geo}(p)$ with $p_k = (1-p)p^k$ for $k = 0, 1 \ldots;\ 0 < p < 1$.
- *Negative binomial distribution* or *Pascal distribution* $\mathrm{NB}(\alpha, p)$ with $p_k = \Gamma(\alpha + k)/(\Gamma(\alpha)\Gamma(k+1))(1-p)^\alpha p^k$ for $k = 0, 1, \ldots;\ \alpha > 0, 0 < p < 1$. With the notation

  $$\binom{x}{0} = 1, \qquad \binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!},$$

  for $x \in \mathbb{R}$, $k = 1, 2, \ldots$, we have

  $$p_k = \binom{\alpha + k - 1}{k}(1-p)^\alpha p^k = \binom{-\alpha}{k}(1-p)^\alpha(-p)^k.$$

  Moreover, for $\alpha = 1, 2, \ldots$, $\mathrm{NB}(\alpha, p)$ is the distribution of the sum of α independent and identically $\mathrm{Geo}(p)$-distributed random variables.
- *Uniform distribution* $\mathrm{U}(a, b)$ with $f(x) = (b-a)^{-1}$ for $a < x < b$; $-\infty < a < b < \infty$.
- *Normal distribution* $\mathrm{N}(\mu, \sigma^2)$ with $f(x) = (2\pi\sigma^2)^{-1/2}\mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}$ for all $x \in \mathbb{R}$; $\mu \in \mathbb{R}, \sigma^2 > 0$.
- *Exponential distribution* $\mathrm{Exp}(\lambda)$ with $f(x) = \lambda \mathrm{e}^{-\lambda x}$ for $x > 0$; $\lambda > 0$.
- *Erlang distribution* $\mathrm{Erl}(n, \lambda)$ with $f(x) = \lambda^n x^{n-1}\mathrm{e}^{-\lambda x}/(n-1)!$; $n = 1, 2, \ldots$.

- *Gamma distribution* $\Gamma(a,\lambda)$ with $f(x) = \lambda^a x^{a-1} \mathrm{e}^{-\lambda x}/\Gamma(a)$ for $x \geq 0$; with shape parameter $a > 0$ and scale parameter $\lambda > 0$: if $a = n \in \mathbb{N}$, then $\Gamma(n,\lambda) = \mathrm{Erl}(n,\lambda)$. If $\lambda = 1/2$ and $a = n/2$, then $\Gamma(n/2, 1/2) = \chi^2(n)$.

- *Logarithmic normal (or lognormal) distribution* $\mathrm{LN}(a,b)$ with

$$f(x) = (xb\sqrt{2\pi})^{-1} \exp\left(-(\log x - a)^2/(2b^2)\right)$$

  for $x > 0$; $a \in \mathbb{R}, b > 0$; if X is $\mathrm{N}(a,b)$-distributed, then e^X is $\mathrm{LN}(a,b)$-distributed.

- *Weibull distribution* $\mathrm{W}(r,c)$ with $f(x) = rcx^{r-1}\exp(-cx^r)$ for $x > 0$ where the shape parameter $r > 0$ and the scale parameter $c > 0$; $\overline{F}(x) = \exp(-cx^r)$; if $r \geq 1$, then $\mathrm{W}(r,c)$ has a light tail, otherwise $\mathrm{W}(r,c)$ is heavy-tailed.

- *Pareto distribution* $\mathrm{Par}(\alpha,c)$ with $f(x) = (\alpha/c)(c/x)^{\alpha+1}$ for $x > c$; with exponent or shape parameter $\alpha > 0$ and scale parameter $c > 0$; $\overline{F}(x) = (c/x)^{\alpha}$.

In the sequel it will be essential to distinguish two types of distributions:

- having *light tail* when $\hat{m}_X(s) < \infty$ for $0 < s \leq s_0$, where $s_0 > 0$,

- having *heavy tail* when $\hat{m}_X(s) = \infty$ for $s > 0$.

Examples of light tailed distribution: distributions of all bounded random variables, $\mathrm{N}(\mu,\sigma^2)$, $\mathrm{W}(r,c)$ for $r \geq 1$.

Examples of heavy tailed distributions: $\mathrm{LN}(a,b)$, $\mathrm{P}(\alpha,c)$, $\mathrm{W}(r,c)$ for $r < 1$.

2.3 Hazard rate function

X is non-negative with distribution F and density function f. We define the *hazard rate function* $m(t)$ by

$$m(t) = \frac{f(t)}{1 - F(t)} \qquad \text{if } F(t) < 1\,. \tag{2.1}$$

The formal infinitesimal interpretation

$$m(t)\,dt = \frac{f(t)\,dt}{1 - F(t)} = \mathbb{P}(X - t \in dt \mid X > t)$$

explains this terminology. We say that F is *IHR* if $m(t)$ is increasing, DHR if $m(t)$ is decreasing. Examples of IHR distributions are: $\mathrm{U}(0,b)$, $\mathrm{W}(r,c)$ for $c \geq 1$, $\mathrm{Erl}(n,\lambda)$. Examples of DHR distributions are: $\mathrm{W}((r,c)$, for $c \geq 1$, mixtures of exponential distributions. Other names: hazard rate = mortality rate = failure rate.

Define the *residual hazard distribution* F_t at t by

$$F_t(x) = \mathbb{P}(X - t \leq x \mid X > t) \tag{2.2}$$

if $F(t) < 1$. Note that $F_t(x) = (F(t+x) - F(t))(1 - F(t))^{-1}$. Consequently, for the expectation μ_{F_t} of F_t we get

$$\mu_{F_t} = (1 - F(t))^{-1} \int_t^\infty \overline{F}(x)\, dx \qquad \text{if } F(t) < 1\,. \tag{2.3}$$

Since $\mu_{F_t} = \mathbb{E}\,(X - t \mid X > t)$, the function $\mu_F(t) = \mu_{F_t}$, $t \geq 0$ is called *the mean residual hazard function.* Mean residual function for various distributions are shown on Fig. 2.1

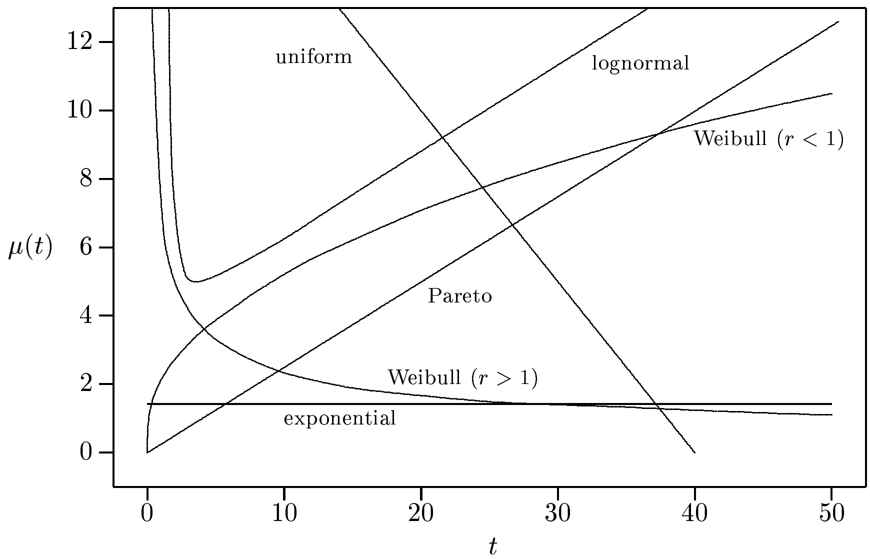

Figure 2.1: Mean residual hazard functions

A distribution F is called *stochastically smaller (larger)* than a distribution G if

$$\overline{F}(x) \leq (\geq)\, \overline{G}(x) \tag{2.4}$$

We write $X \leq_{\mathrm{st}} Y$ if $F_X \leq_{\mathrm{st}} F_Y$.

Theorem 2.1 *The distribution F is* IHR (DHR) *if and only if, for all $t_1 \leq t_2$,*

$$F_{t_1} \geq_{\mathrm{st}} (\leq_{\mathrm{st}}) F_{t_2}\ . \tag{2.5}$$

Exercises

2.1 Show that for the *Cauchy distribution* with density function

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}\,, \qquad x \in \mathbb{R}\,,$$

the moment generating function $\hat{m}(s)$ is finite only at $s = 0$.

2.2 Suppose that $F(x)$ is the distribution function of a nonnegative random variable X, i.e. $F(0-) = 0$. Show that

$$\hat{m}(s) = \int_0^\infty \mathrm{e}^{sx}\,\mathrm{d}F(x) = 1 + s\int_0^\infty \mathrm{e}^{sx}\overline{F}(x)\,\mathrm{d}x$$

for all $s \in \mathbb{R}$ with $\hat{m}(s) < \infty$. Derive a similar formula for the Laplace–Stieltjes transform $\hat{l}(s)$ of X.

2.3 Let $F = \mathrm{Exp}(\lambda)$. Prove that

$$\overline{F^{*n}}(x) = \mathrm{e}^{-\lambda x}\Big(1 + \frac{\lambda x}{1!} + \ldots + \frac{(\lambda x)^{n-1}}{(n-1)!}\Big)\,. \tag{2.6}$$

[Hint. Differentiate both sides of (2.6).]

2.4 Consider the mixture $F = \int_0^\infty F_\theta\,\mathrm{d}G(\theta)$, where $F_\theta = \mathrm{Poi}(\theta)$ and G is a mixing distribution, which is concentrated on $[0,\infty)$. Prove that $\sigma_F^2 \geq \mu_F$. Furthermore, show that $\mathrm{cv}_F \geq 1$ if $\mu_F \leq 1$, where the coefficient of variation is given by $\mathrm{cv} = \sigma/\mu$.

2.5 Show that the tail $\overline{F}(x)$ of the mixture $F = \int_0^\infty F_\theta\,\mathrm{d}G(\theta)$ with $F_\theta = \mathrm{Exp}(\theta)$ and $G = \Gamma(a,\lambda)$ is $\overline{F}(x) = (\lambda^{-1}x+1)^{-a}$.

2.6 Suppose that the duration T of a fire is a random variable with distribution $\mathrm{Exp}(\lambda)$. The loss $w(t)$ associated with a fire of a fixed duration t is assumed to be given by $w(t) = a\mathrm{e}^{bt}$ for some $a, b > 0$. Show that the random variable $w(T)$ has the Pareto distribution $\mathrm{Par}(\lambda/b, a)$.

2.7 Discuss whether or not for the Pareto, lognormal or Weibull $(0 < r < 1)$ distributions, their moment generating functions are finite on $(0,\infty)$.

2.8 Compute the hazard rate functions for the uniform distribution $\mathrm{U}(a,b)$, the exponential distribution $\mathrm{Exp}(\lambda)$, the Erlang distribution $\mathrm{Erl}(n,\lambda)$, the Weibull distribution $\mathrm{W}(r,c)$. Find out when these functions are increasing or decreasing.

2.9 Show that, for the Pareto distribution $\mathrm{Par}(\alpha,c)$, the hazard rate function $m(t)$ is given by $m(t) = \alpha/t$ for all $t \geq c$.

2.10 Let F, G be two distributions on $\mathbb{R}$ whose the first order absolute moments are finite. Show that $\mu_F \le \mu_G$ if $F \le_{\rm st} G$.

2.11 Let F be an absolutely continuous distribution on $\mathbb{R}_+$. Show that the mean residual hazard function $\mu_F(t)$ is decreasing (increasing) if the hazard rate function $m(t)$ is increasing (decreasing). [Hint. Use Theorem 2.2.1 and the result of Exercise 2.10.]

2.12 Let F be the Weibull distribution $\mathrm{W}(r, c)$. Show that the mean residual hazard function $\mu_F(t)$ is decreasing if $r > 1$, increasing if $0 < r < 1$, and constant if $r = 1$.

Other exercises

2.13 Let $F(x)$ be the distribution function of a nonnegative random variable X. Show that $\hat{m}(s_0) < \infty$ for some $s_0 > 0$ if and only if for some $a, b > 0$ the inequality $\overline{F}(x) \le a\,\mathrm{e}^{-bx}$ holds for all $x \ge 0$. Conclude from this that X has all moments finite if $\hat{m}(s_0) < \infty$ for some $s_0 > 0$. Give an example of a distribution F of a nonnegative random variable such that $\hat{m}(s) = \infty$ for all $s > 0$.

2.14 Show that if the density function f is decreasing, then its distribution is DHR.

2.15 Consider two independent absolutely continuous risks X_1 and X_2 with hazard rate functions $m_1(t)$ and $m_2(t)$, respectively. Show that the hazard rate function of $\min\{X_1, X_2\}$ is $m_1(t) + m_2(t)$. Find an interpretation for the distribution (so-called *Makeham's law*) defined by the hazard rate function $A + Cc^x$. [Comment. In life insurance the distribution with $A = 0$ is called *Gompertz' law*.]

2.16 Let $F = (1-\theta)\delta_0 + \theta G$ for some $\theta \in (0, 1)$, where G is an absolute continuous distribution from DHR. Show that $F_{t_1} \le_{\rm st} F_{t_2}$ for $t_1 \le t_2$.

2.17 Show by examples that the following cases are possible for the moment generating function $\hat{m}(s)$:

(a) $\hat{m}(s) < \infty$ for all $s \in \mathbb{R}$

(b) there exists $s_0 > 0$ such that $\hat{m}(s) < \infty$ for $s < s_0$ and $\hat{m}(s) = \infty$ for $s \ge s_0$,

(c) there exists $s_0 > 0$ such that $\hat{m}(s) < \infty$ for $s \le s_0$ and $\hat{m}(s) = \infty$ for $s > s_0$.

[Hint. Use the inverse Gaussian distribution as an example for (c).]

2.18 Let F be a distribution on $\mathbb{R}_+$ with $0 < \mu_F < \infty$. Prove the following relationship between the tail function $\overline{F}(x)$ and the corresponding mean residual hazard function $\mu_F(t)$:

$$\overline{F}(x) = \frac{\mu_F}{\mu_F(x)} \exp\Big(-\int_0^x \frac{1}{\mu_F(t)}\,\mathrm{d}t\Big)\,, \qquad x \geq 0\,. \tag{2.7}$$

2.19 (Continuation) Show that F is an exponential distribution if and only if the mean residual hazard function $\mu_F(t)$ is constant. [Hint. Use the result of Exercise 2.18 for the sufficiency part.]

2.20 (Continuation) Assume that $\sup\{\mu_F(t) : t \geq 0\} < \infty$. Show that then $\hat{m}_F(t_0) < \infty$ for some $t_0 > 0$.

3 Lecture: Heavy tailed distributions

In this lecture we cansider only non-negative random variables. We say that F on $\mathbb{R}_+$ is *heavy-tailed* if

$$\hat{m}_F(s) = \infty, \qquad s > 0.$$

In Exercise 2.13 it is stated: F is light-tailed if and only if for some $a, b > 0$

$$\overline{F}(x) \le a\mathrm{e}^{-bx}, \qquad x \ge 0.$$

From this we get immedialtely that if F is heavy-tailed, then

$$\limsup_{x\to\infty} \mathrm{e}^{sx}\overline{F}(x) = \infty,$$

for all $s > 0$.

3.1 Subexponential distributions

A distribution F on $\mathbb{R}_+$ is said to be *subexponential* if and only if

$$\lim_{x\to\infty} \frac{1 - F^{*2}(x)}{1 - F(x)} = 2\,. \tag{3.1}$$

Let $\mathcal{S}$ denote the class of all subexponential distributions. We show later that the following important (parameterized) families of distributions are in $\mathcal{S}$: the lognormal distributions, Pareto distributions, and Weibull distributions with shape parameter smaller than 1.

Lemma 3.1

$$\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 1 + \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)}\, dF(y)\,, \tag{3.2}$$

from which we obtain that always

$$\liminf_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \ge 2\,. \tag{3.3}$$

Lemma 3.2 *If $F \in \mathcal{S}$, then for all $x' > 0$,*

$$\lim_{x\to\infty} \frac{\overline{F}(x-x')}{\overline{F}(x)} = 1 \tag{3.4}$$

and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)}\, dF(y) = 1\,. \tag{3.5}$$

Theorem 3.1 *Each $F \in \mathcal{S}$ is heavy-tailed.*

Theorem 3.2 *Let F be a distribution on $\mathbb{R}_+$. Then, $F \in \mathcal{S}$ if and only if for each $n = 2, 3, \ldots$*

$$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n\,. \tag{3.6}$$

The proof relies on the lemma [1]

Lemma 3.3 *Let $F \in \mathcal{S}$ and F' a distribution with $F'(0) = 0$ such that $\lim_{x\to\infty} \overline{F'}(x)/\overline{F}(x) = c$ for some $c \in [0, \infty)$. Then*

$$\lim_{x\to\infty} \frac{\overline{F * F'}(x)}{\overline{F}(x)} = 1 + c\,. \tag{3.7}$$

Lemma 3.4 *[Kosten] If $F \in \mathcal{S}$, then for each $\varepsilon > 0$ there exists a constant $c < \infty$ such that for all $n \geq 2$*

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \leq c(1+\varepsilon)^n, \qquad x \geq 0\,. \tag{3.8}$$

Theorem 3.3 *Let $F(x) = \sum_{k=0}^{\infty} p_k G^{*k}(x)$, where $\{p_0, p_1, \ldots\}$ is a probability function and $G \in \mathcal{S}$. If $\sum_{n=1}^{\infty} p_n(1+\varepsilon)^n < \infty$ for some $\varepsilon > 0$, then*

$$\lim_{x\to\infty} \frac{\overline{F}(x)}{\overline{G}(x)} = \sum_{k=0}^{\infty} k p_k\,. \tag{3.9}$$

We say that an ulimately [2] positive function L is a *slowly varying function of x at ∞* if for all $y > 0$, $L(xy)/L(x) \to 1$ as $x \to \infty$. Examples of such functions are $|\log^p x|$, and functions converging to a positive limit as $x \to \infty$. Note that (3.4) gives in particular that if $F \in \mathcal{S}$, then $\overline{F}(\log x)$ is a slowly varying function of x at ∞.

Lemma 3.5 *If $L(x)$ is slowly varying function, then for all $\alpha > 0$*

$$\lim_{x\to\infty} x^{-\alpha} L(x) = 0.$$

We now say that the distribution F is *Pareto-type* with exponent $\alpha > 0$ if $\overline{F}(x) \sim L(x)x^{-\alpha}$ as $x \to \infty$ for a slowly varying function $L(x)$. In the literature, Pareto-type distributions are also called *distributions with regular varying tail.*

Theorem 3.4 *If F is Pareto-type, then $F \in \mathcal{S}$.*

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[1] In these set of lectures by prime we denote always another object, never the first derivate. For denoting the first derivative always use $^{(1)}$, eg. $F^{(1)}(x) = f(x)$.

[2] ultimately – from some place on.

Exercises

3.1 Show that for F being the distribution function of a bounded random variables, or being Erlang $\mathrm{Erl}(n, c)$

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = \infty;.$$

3.2 Show that $|\log^p x|$, $\log(\log(1+x))$, functions with positive and finite limits are slowly varying functions. Show that $2 + \sin x$ is not a slowly varying function.

3.3 Show that the following distribution functions are Pareto-type:

- Pareto $\mathrm{Par}(\alpha, c)$,
- loggamma with density function

$$f(x) = \frac{\lambda^a}{\Gamma(a)} (\log x)^{a-1} x^{-\lambda-1}, \qquad x > 1,$$

- *Burr distribution* with

$$\overline{F}(x) = \left(\frac{b}{b + x^r}\right)^{\alpha}, \qquad x \geq 0,$$

where $\alpha, b, r > 0$.

3.4 Let X_1 and X_2 be two independent nonnegative random variables such that $\lim_{x\to\infty} \overline{F_{X_i}}(x)/\overline{F}(x) = c_i$ for some $F \in \mathcal{S}$ and $c_i \in [0, \infty)$, where either $c_1 > 0$ or $c_2 > 0$. Show that $\mathbb{P}(X_1 + X_2 > x)/\overline{F}(x)$ tends to $c_1 + c_2$ as $x \to \infty$.

3.5 Show that if L_1 and L_2 are slowly varying functions, then $L_1 + L_2$, $L_1 L_2$, L_1^r are slowly varying function too.

3.6 Show that

$$L(x) = c(x) \exp(\int_z^x \frac{\delta(u)}{u} \, \mathrm{d}u)$$

when $c(x) \to c_0 \in (0, \infty)$, $\delta(x) \to 0$ as $x \to \infty$, is a slowly varying function.

4 Criterion for subexponentiality

For a distribution F of a nonnegative random variable with finite expectation $\mu > 0$, the *integrated tail distribution* F^{s} is given by

$$F^{\mathrm{s}}(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \mu^{-1} \displaystyle\int_0^x \overline{F}(y)\,dy & \text{if } x > 0. \end{cases} \tag{4.1}$$

We give a criterion for a distribution F and the corresponding integrated tail distribution F^{s} are in $\mathcal{S}$. Throughout this section we only consider distributions F on $\mathbb{R}_+$ such that $F(0) = 0, F(x) < 1$ for all $x \in \mathbb{R}_+$.

Definition 4.1 (a) *We say that F belongs to the class $\mathcal{S}^*$ if F has finite expectation μ and*

$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y)\,dy = 2\mu\,. \tag{4.2}$$

(b) *We say that F belongs to $\mathcal{L}$ if for all $y \in \mathbb{R}$*

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1\,. \tag{4.3}$$

Lemma 4.1 *If $\lim_{x\to\infty} m_F(x) = 0$, then $F \in \mathcal{L}$.*

Lemma 4.2 *$F \in \mathcal{S}^*$ if and only if*

$$\lim_{x\to\infty} \int_0^{x/2} \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y)\,dy = \mu\,. \tag{4.4}$$

In this lecture we study the relationship between $\mathcal{S}^*$ and $\{F : F \in \mathcal{S} \text{ and } F^{\mathrm{s}} \in \mathcal{S}\}$.

Lemma 4.3 *Let F, G be two distributions on $\mathbb{R}_+$ and assume that there exists a constant $c \in (0, \infty)$ such that*

$$\lim_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} = c\,. \tag{4.5}$$

Then, $F \in \mathcal{S}$ if and only if $G \in \mathcal{S}$.

The above lemma justifies the following definition. Two distribution functions F and G on $\mathbb{R}_+$ are said to be *tail-equivalent* if $\lim_{x\to\infty} \overline{G}(x)/\overline{F}(x) = c$ for some $0 < c < \infty$. This will be denoted by $G \sim^{\mathrm{t}} F$.

Theorem 4.1 *If $F \in \mathcal{S}^*$, then $F \in \mathcal{S}$ and $F^{\mathrm{s}} \in \mathcal{S}$.*

Corollary 4.1 *Assume that the hazard rate function* $m_F(x)$ *of* F *exists and* $\mu < \infty$. *If* $\limsup_{x\to\infty} x m_F(x) < \infty$, *then* $F \in \mathcal{S}$ *and* $F^{\mathrm{s}} \in \mathcal{S}$.

In the case that $\limsup_{x\to\infty} x m_F(x) = \infty$, one can use the following criterion for $F \in \mathcal{S}^*$.

Theorem 4.2 *Assume that the hazard rate function* $m_F(x)$ *of* F *exists and is ultimately decreasing to* 0. *If* $\int_0^\infty \exp(x m_F(x))\overline{F}(x)\,dx < \infty$ *then* $F \in \mathcal{S}^*$.

Examples 1) For the Weibull distribution $F = \mathrm{W}(r, c)$ with $0 < r < 1, c > 0$ and lognormal $\mathrm{LN}(a, b)$ we can use Theorem 4.2 to demonstrate that both the distributions belong to $\mathcal{S}^*$. For showing that the integrated tail distribution of Pareto-type distributions is subexponential, we need the following result called *Karamata's theorem.* We state this theorem without proof, for which we refer to book of Feller (1971), *An Introduction to Probability Theory and its Applications, vol 2.*

Theorem 4.3 *If* $L_1(x)$ *is a slowly varying function and locally bounded in* $[x_0, \infty)$ *for some* $x_0 > 0$, *then for* $\alpha > 1$

$$\int_x^\infty y^{-\alpha} L_1(y)\,dy = x^{-\alpha+1} L_2(x)\,, \tag{4.6}$$

where $L_2(x)$ *is also a slowly varying function of* x *at* ∞ *and moreover* $\lim_{x\to\infty} L_1(x)/L_2(x) = \alpha - 1$. *If* $L_1(y)/y$ *is integrable, then the result also holds for* $\alpha = 1$.

Exercises

4.1 Show that F with the first finite moment is heavy tailed if and only if the corresponding integrated tail distribution F^{s} is heavy tailed.

4.2 Let F be an absolutely continuous distribution on $\mathbb{R}_+$ with hazard rate function $m(x)$. Show that $F \in \mathcal{L}$ if $\lim_{x\to\infty} m(x) = 0$. [Hint. $\overline{F}(x) = \exp(-\int_0^x m(v)\,dv)$]

4.3 Show that the Weibull distribution $F = \mathrm{W}(r, c)$ with $0 < r < 1, c > 0$ belongs to $\mathcal{S}^*$.

4.4 Let F be the distribution on $\mathbb{R}_+$ defined by

$$\overline{F}(x) = \left(\frac{b}{b + x^r}\right)^{\alpha}, \qquad x \geq 0\,, \tag{4.7}$$

where $\alpha, b, r > 0$. Show that $F \in \mathcal{S}$ and, if $\alpha r > 1$, $F^{\mathrm{s}} \in \mathcal{S}$.

Other exercises

4.5 Let F be the discrete distribution on $\mathbb{N} \setminus \{0\}$ with atoms of mass $3 \cdot 4^{-n}$ at the points 2^n for $n \in \mathbb{N} \setminus \{0\}$. Show that $\mu = 3$ and $F^{\mathrm{s}} \in \mathcal{S}$, but $F \notin \mathcal{S}$.

5 Distributions of aggregate claim amount

Let N be a nonnegative integer-valued random variable and $U_1, U_2, \ldots$ a sequence of nonnegative random variables. Then the random variable

$$X = \begin{cases} \sum_{i=1}^{N} U_i & \text{if } N \geq 1, \\ 0 & \text{if } N = 0, \end{cases} \tag{5.1}$$

is called *compound* and describes the aggregate claim amount in the individual model as well as in the collective model. We assume throughout this chapter that the random variables $N, U_1, U_2, \ldots,$ are independent. If not stated otherwise, we also assume that $U_1, U_2, \ldots$ are identically distributed. We say that X has a *compound distribution* determined by the (compounding) probability function $\{p_k,\ k \in \mathbb{N}\}$ of N and by the distribution F_U of U_i, if the distribution of X is given by

$$F_X = \sum_{k=0}^{\infty} p_k F_U^{*k}, \tag{5.2}$$

where F_U^{*k} denotes the k-fold convolution of F_U.

We first show how to express the Laplace-Stieltjes transform $\hat{l}_X(s)$ in terms of the generating function $\hat{g}_N(s)$ and the Laplace-Stieltjes transform $\hat{l}_U(s)$.

Theorem 5.1 *For each $s \geq 0$,*

$$\hat{l}_X(s) = \hat{g}_N(\hat{l}_U(s)). \tag{5.3}$$

Corollary 5.1 *Assume that the relevant moments exist. Then*

$$\mathbb{E}\, X = \mathbb{E}\, N \mathbb{E}\, U, \quad \mathrm{Var}\, X = \mathrm{Var}\, N (\mathbb{E}\, U)^2 + \mathbb{E}\, N \mathrm{Var}\, U. \tag{5.4}$$

In actuarial applications three cases are of special interest:

- *Poisson compounds* where N has a Poisson distribution; in this case the distribution of the compound, determined by $\lambda = \mathbb{E}\, N$ and by the distribution F_U, is called a *compound Poisson distribution* with characteristics (λ, F_U).
- *Pascal* (or *negative binomial*) *compounds* with compounding distribution $\mathrm{NB}(\alpha, p)$.
- *Geometric compounds* where N has a geometric distribution; in this case the distribution of the compound is determined by $p = 1 - \mathbb{P}(N = 0)$ and by the distribution F_U and is called a *compound geometric distribution.*

Theorem 5.2 *For some $n \in \mathbb{N}$, let $X = X_1 + \ldots + X_n$ be the sum of the independent Poisson compounds $X_1, \ldots, X_n$ with characteristics (λ_j, F_j) $(j =$*

$1,\ldots,n$). *Then X has a compound Poisson distribution with characteristics (λ, F) given by*

$$\lambda = \sum_{j=1}^{n} \lambda_j \qquad \textit{and} \qquad F = \sum_{j=1}^{n} \frac{\lambda_j}{\lambda} F_j \,. \tag{5.5}$$

Example 1) Consider the following special case of a Poisson compound. If the compounding probability function $\{p_k\}$ is Poisson with parameter λ and if the claim size distribution F_U is exponential, i.e. $F_U = \mathrm{Exp}(\delta)$ for some $\delta > 0$, then the distribution function F_X of the aggregate claim amount has the form $F_X(x) = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda}(\lambda^n/n!) F_U^{*n}(x)$ where

$$F_U^{*n}(x) = \frac{\delta^n}{(n-1)!} \int_0^x \mathrm{e}^{-v\delta} v^{n-1}\, dv$$

for $n = 1, 2, \ldots$. Note that the distribution function $F_X(x)$ has a jump of size $\mathrm{e}^{-\lambda}$ at the origin and is differentiable in the interval $(0, \infty)$ with

$$F_X^{(1)}(x) = \mathrm{e}^{-(\lambda + x\delta)} \sum_{n=1}^{\infty} \frac{1}{n!(n-1)!} \frac{1}{x} (x\lambda\delta)^n \,, \qquad x > 0 \,.$$

Using of the *modified Bessel function* $I_1(z)$ [3] we see that

$$F_X^{(1)}(x) = \mathrm{e}^{-(\lambda + x\delta)} \sqrt{\lambda\delta/x}\, I_1(2\sqrt{\lambda x \delta}) \,, \qquad x > 0 \,. \tag{5.6}$$

5.1 Geometric Compounds

Let N be a geometric distribution with parameter $p \in (0,1)$. Then the compound geometric distribution F_X is given by

$$F_X(x) = \sum_{k=0}^{\infty} (1-p) p^k F_U^{*k}(x) \,. \tag{5.7}$$

Writing the first summand in (5.7) separately, we get

$$F_X = (1-p)\delta_0 + p\, F_U * F_X \,. \tag{5.8}$$

Moreover, replacing the distribution F_X on the right-hand side of (5.8) by the term $(1-p)\delta_0 + pF_U * F_X$ and iterating this procedure, we obtain

$$F_X(x) = \lim_{n\to\infty} F_n(x) \,, \qquad x \geq 0 \,, \tag{5.9}$$

where F_n is defined by the recursion

$$F_n = (1-p)\delta_0 + pF_U * F_{n-1} \tag{5.10}$$

[3] $I_1(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{k!(k+1)!}$

for all $n \geq 1$ and F_0 is an arbitrary (initial) distribution on $\mathbb{R}_+$.

Assume additionally that the distribution F_U is such that

$$\hat{m}_U(\gamma) = p^{-1} \tag{5.11}$$

has a solution $\gamma > 0$.

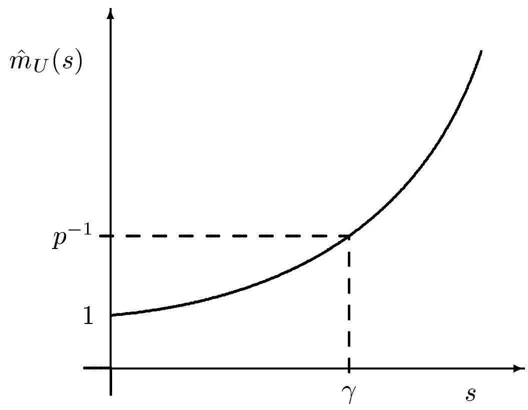

Figure 5.1: Moment generating function

We get the following lower and upper *Lundberg-type bounds* for the tail of the compound distribution F_X. Let $x_0 = \sup\{x : F_U(x) < 1\}$.

Theorem 5.3 *If X is a geometric compound with characteristics (p, F_U) such that* (5.11) *admits a positive solution γ, then*

$$a_- \mathrm{e}^{-\gamma x} \leq \overline{F}_X(x) \leq a_+ \mathrm{e}^{-\gamma x}, \qquad x \geq 0\,, \tag{5.12}$$

where

$$a_- = \inf_{x\in[0,x_0)} \frac{\mathrm{e}^{\gamma x}\overline{F}_U(x)}{\int_x^\infty \mathrm{e}^{\gamma y}\, dF_U(y)}\,, \qquad a_+ = \sup_{x\in[0,x_0)} \frac{\mathrm{e}^{\gamma x}\overline{F}_U(x)}{\int_x^\infty \mathrm{e}^{\gamma y}\, dF_U(y)}\,. \tag{5.13}$$

Corollary 5.2 *Suppose that F_U is exponential with parameter δ. Then $\gamma = \delta(1-p)$, $a_- = a_+ = p$ and, consequently,*

$$\overline{F}_X(x) = p\mathrm{e}^{-(1-p)\delta x}, \qquad x \geq 0\,. \tag{5.14}$$

Remark For the constants a_- and a_+ appearing in Theorem 5.3, we obviously have $a_- \leq a_+$. Moreover, since $\int_x^\infty \mathrm{e}^{\gamma y}\, dF_U(y) \geq \mathrm{e}^{\gamma x}\overline{F}_U(x)$, we conclude that $a_+ \leq 1$. To get more insight note that

$$\frac{\int_x^\infty \mathrm{e}^{\gamma y}\, dF_U(y)}{\mathrm{e}^{\gamma x}\overline{F}_U(x)} = \mathbb{E}\,(\mathrm{e}^{\gamma(U-x)} \mid U > x) = \int_0^\infty \mathrm{e}^{\gamma y}\, dF_{U,x}(y)\,,$$

where the residual hazard distribution $F_{U,x}$ at x was defined in (2.2). If F_U is DHR, then by Theorem 2.1, the value of $\mathbb{E}\left(e^{\gamma(U-x)} \mid U > x\right)$ is increasing in x. Analogously, for F_U being IHR the above is decreasing in x. Hence $a_+ = 1/\hat{m}_U(\gamma) = p$ for F_U being DHR (by our definition it cannot have an atom at zero) and, for F_U being IHR, $a_+ = \lim_{x\to\infty} e^{\gamma x}\overline{F}_U(x)(\int_x^\infty e^{\gamma y}\, dF_U(y))^{-1}$.

5.2 More General Compound Distributions

Generalizing the previous results, we now only assume the existence of a constant $\theta \in (0,1)$ such that the distribution of N is stochastically smaller than the $(1-p_0)$-modification of the truncated geometric distribution $\mathrm{TG}(\theta)$. More precisely,

$$\{p_k\} \leq_{\mathrm{st}} \{p'_k\} = p_0\delta_0 + (1-p_0)\mathrm{TG}(\theta)\,, \tag{5.15}$$

which is equivalent to $r_{j+1} \leq (1-p_0)\theta^j$ for all $j \in \mathbb{N}$, where $r_j = \sum_{k=j}^\infty p_k$. Note that the inequality (5.15) holds provided that

$$r_{j+1} \leq \theta r_j\,, \qquad j \geq 1\,, \tag{5.16}$$

because $r_1 = 1 - p_0 = \sum_{k=1}^\infty p'_k$ and because (5.16) yields

$$r_{j+1} \leq \theta^j r_1 = \sum_{k=j}^{\infty}(1-\theta)\theta^k r_1 = \sum_{k=j+1}^{\infty}(1-p_0)(1-\theta)\theta^{k-1} \tag{5.17}$$

for all $j \geq 1$. However, in general, (5.16) does not follow from (5.15).

Theorem 5.4 *If* (5.15) *holds for some* $\theta < 1$, *then*

$$\overline{F}_X(x) \leq \frac{1-p_0}{\theta} a_+ e^{-\gamma x}\,, \qquad x \geq 0\,, \tag{5.18}$$

where $\gamma > 0$ *fulfils* $1 = \theta\hat{m}_U(\gamma)$ *and* a_+ *is defined by* (5.13).

The logconcave probability functions $\{p_k\}$ form a subclass of $\mathrm{IHR_d}$ and lead to further improvements.

Corollary 5.3 *If* $\{p_k\}$ *is logconcave with* $p_0 + p_1 < 1$, *then*

$$\overline{F}_X(x) \leq \frac{(1-p_0)^2}{1-p_0-p_1}\, a_+ e^{-\gamma x}, \qquad x \geq 0\,, \tag{5.19}$$

where a_+ *is defined as in* (5.13) *and* $\gamma > 0$ *is the solution to the equation*

$$\hat{m}_U(\gamma) = \frac{1-p_0}{1-p_0-p_1}\,. \tag{5.20}$$

5.3 Classical Examples; Panjer's Recurrence Relation

There are at least three particular cases that are immediately applicable in insurance. The *Poisson distribution* $\mathrm{Poi}(\lambda)$ is by far the most celebrated example of a claim number distribution. Recall that then $p_k = \mathrm{e}^{-\lambda}\lambda^k/k!$ for $k \in \mathbb{N}$ where $\lambda = \mathbb{E}\, N$ is the mean number of claims. Moreover $\mathrm{Var}\, N = \lambda$ so that the index of dispersion $\mathrm{I}_N = \mathrm{Var}\, N/\mathbb{E}\, N$ equals 1.

In the next two examples we show an *overdispersed distribution* (index of dispersion greater than 1) and an *underdispersed distribution* (index of dispersion less than 1).

The *negative binomial* or *Pascal distribution* $\mathrm{NB}(\alpha, p)$ is another favoured claim number distribution. Recall that then

$$p_k = \binom{\alpha+k-1}{k}(1-p)^\alpha p^k\,, \qquad k \in \mathbb{N}\,,$$

where $\alpha > 0$ and $p \in (0,1)$. Now, $\mathbb{E}\, N = \alpha p/(1-p)$ and $\mathrm{Var}\, N = \alpha p/(1-p)^2$. Notice that $\mathrm{I}_N = p^{-1} > 1$. The overdispersion of the Pascal distribution is one reason for its popularity as an actuarial model.

The *binomial distribution* $\mathrm{Bin}(n,p)$ is an underdispersed distribution. Assume that an insurance policy covers the total breakdown of a vital component in a computer system. At the beginning of the year the portfolio covers n identically manufactured and newly installed components in a computer park. The probability that an arbitrary but fixed component will fail before time t is given by a distribution function $G(t)$. The probability p_k that the number N of breakdowns up to time t is equal to k, is given by

$$p_k = \binom{n}{k} p^k (1-p)^{n-k}\,, \qquad 0 \le k \le n\,,$$

where $p = G(t)$. For such a *binomial model* the mean number of claims equals $\mathbb{E}\, N = np$ while $\mathrm{Var}\, N = np(1-p)$. The index of dispersion is now $\mathrm{I}_N = 1-p$ which is less than 1. The latter result is intuitive since the larger t, the smaller $1-p = 1-G(t)$, hence the smaller the dispersion.

We can check that the probability functions $\{p_k\}$ of the Poisson, negative binomial and binomial distribution all fulfil *Panjer's recurrence relation*

$$p_k = \left(a + \frac{b}{k}\right) p_{k-1}\,, \qquad k = 1, 2, \ldots \tag{5.21}$$

where $a < 1$ and $b \in \mathbb{R}$ are some constants. The following result shows that there are no other distributions which satisfy (5.21).

Theorem 5.5 *Suppose that the probability function $\{p_k\}$ of the $\mathbb{N}$-valued random variable N satisfies* (5.21). *Then $\{p_k\}$ is the probability function of a binomial, Poisson or negative binomial distribution. More specifically:*

(a) *If* $a = 0$ *and* $b = \lambda > 0$*, then* N *has the Poisson distribution* $\mathrm{Poi}(\lambda)$.
(b) *If* $0 < a < 1$ *and* $a + b > 0$*, then* N *has the negative binomial distribution* $\mathrm{NB}(\alpha, p)$ *with* $p = a$ *and* $\alpha = 1 + bp^{-1}$.
(c) *If* $a < 0$ *and* $b = -a(n+1)$ *for some* $n \in \mathbb{N}$*, then* N *has the binomial distribution* $\mathrm{Bin}(n, p)$ *with* $p = a(a-1)^{-1}$ *and* $n = -1 - ba^{-1}$.

5.4 Discrete Compound Poisson Distributions

Assume that claims occur in bulk, where the number of bulks N' occurring in a given period of time follows a Poisson distribution with parameter λ. Each bulk consists of a random number of claims so that the total number of claims is of the form $N = \sum_{i=1}^{N'} Z_i$ where Z_i denotes the number of claims in the i-th bulk. Assume that $\{Z_i,\ i = 1, 2, \ldots\}$ is a sequence of independent random variables with values in $\{1, 2, \ldots\}$ and with common probability function $\{p_k^Z,\ k \geq 1\}$. If the claim numbers $Z_1, Z_2, \ldots$ are independent of the bulk number N' then $\hat{g}_N(s) = \exp(-\lambda(1 - \hat{g}_Z(s)))$. From Corollary 5.1 we have $\mathbb{E}\, N = \lambda\, \mathbb{E}\, Z$ and $\mathrm{Var}\, N = \lambda\, \mathbb{E}\,(Z^2)$. The total number of claims N henceforth follows a *(discrete) compound Poisson distribution.* The probability function $\{p_k\}$ of N is given by

$$p_k = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda} \frac{\lambda^n}{n!} \left(p^Z\right)_k^{*n}, \qquad k \in \mathbb{N}, \tag{5.22}$$

where $\{(p^Z)_k^{*n},\ k = 1, 2, \ldots\}$ is the n-fold convolution of $\{p_k^Z\}$.

Exercises

5.1 Show that mean and variance of a Poisson compound X with characteristics (λ, F) are given by $\mu_X = \lambda\mu_F$, $\sigma_X^2 = \lambda\mu_F^{(2)}$, the Laplace–Stieltjes transform and the moment generating function by

$$\hat{l}_X(s) = \mathrm{e}^{-\lambda(1-\hat{l}_F(s))}, \qquad \hat{m}_X(s) = \mathrm{e}^{\lambda(\hat{m}_F(s)-1)}.$$

5.2 Show that for a geometric compound X with characteristics (p, F)

$$\mu_X = \frac{p\mu_F}{1-p}, \ \sigma_X^2 = \frac{p}{(1-p)^2}(\mu_F^2 + (1-p)\sigma_F^2), \ \hat{l}_X(s) = \frac{1-p}{1-p\hat{l}_F(s)}.$$

5.3 Show that the distribution of a zero-truncated geometric compound with characteristics (p, F) (that is $(1-p)\sum_{j=1}^{\infty} p^{j-1}F^{*j}$) is exponential with parameter $\delta(1-p)$ provided that F is the exponential distribution with parameter δ.

5.4 Consider the tail $\overline{F}_X(x)$ of the compound $X = \sum_{i=1}^{N} U_i$, where N has the negative binomial distribution $\mathrm{NB}(2, p)$ and U is $\mathrm{Exp}(\delta)$-distributed. Show that for all $x \geq 0$

$$\overline{F}_X(x) = \mathrm{e}^{-(1-p)\delta x}(1 - (1-p)^2 + p^2(1-p)\delta x). \tag{5.23}$$

5.5 Consider the tail $\overline{F}_X(x)$ of the compound $X = \sum_{i=1}^{N} U_i$, where N has the negative binomial distribution $\mathrm{NB}(2, p)$ and U is $\mathrm{Exp}(\delta)$-distributed. Show that if $\gamma > 0$ is computed from (5.20), then

$$\gamma = \frac{2(1-p)^2\delta}{2-p} < (1-p)\delta.$$

Conclude that in this case the Lundberg bound (5.19) is not optimal. [Hint. Use the result of Exercise 5.4.]

Other exercises

5.6 Show that if N is $\mathrm{NB}(\alpha, p)$-distributed and U_i has distribution F_U, then the distribution F_X of $X = \sum_{i=1}^{N} U_i$ can be represented as

$$F_X = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}\tilde{F}^{*k}, \tag{5.24}$$

where $\lambda = -\alpha\log(1-p)$ and

$$\tilde{F} = \frac{1}{-\log(1-p)}\sum_{k=1}^{\infty}\frac{p^k}{k}F_U^{*k}.$$

5.7 Let $X = \sum_{i=1}^{N} U_i$ be a compound with characteristics $(\{p_k\}, F_U)$. Show that for the tail $\overline{F}_X(x) = \mathbb{P}(X > x)$,

$$\overline{F}_X(x) = \sum_{k=1}^{\infty} p_k \overline{F_U^{*k}}(x)\,, \qquad x \geq 0$$

and

$$\int_0^\infty \mathrm{e}^{-sx} \overline{F}_X(x) \mathrm{d}x = \frac{1 - \hat{g}_N\big(\hat{l}_U(s)\big)}{s}\,, \qquad s > 0\,. \tag{5.25}$$

5.8 Consider the tail $\overline{F}_X(x)$ of the compound $X = \sum_{i=1}^{N} U_i$ and let N have the binomial distribution $\mathrm{Bin}(n, p)$. Show that if $\gamma > 0$ is the unique solution to

$$\hat{m}_{F_U}(\gamma) = \frac{1 - (1-p)^n}{1 - (1-p)^n - np(1-p)^{n-1}}\,,$$

then

$$\overline{F}_X(x) \leq \frac{(1 - (1-p)^n)^2}{1 - (1-p)^n - np(1-p)^{n-1}} a_+ \mathrm{e}^{-\gamma x}, \qquad x \geq 0\,,$$

where a_+ is defined as in (5.13).

6 Lecture: Computational methods for compound distributions

6.1 Panjer's Algorithm

Consider a compound $X = \sum_{i=1}^{N} U_i$ satisfying the following assumptions. Let the claim sizes $U_1, U_2, \ldots$ be (discrete) independent and identically distributed random variables taking their values on a lattice. We also assume that the sequence $U_1, U_2, \ldots$ of claim sizes is independent of the claim number N. Without further loss of generality we can and will assume that $\mathbb{P}(U_i \in \mathbb{N}) = 1$. We denote the probability function of $U_1, U_2, \ldots$ by $\{q_k, k \in \mathbb{N}\}$. Besides this we suppose that the probability function $\{p_k, k \in \mathbb{N}\}$ of the number of claims N satisfies Panjer's recurrence relation (5.21).

We need the following auxiliary result.

Lemma 6.1 *For any $j, k \in \mathbb{N}$ and $n = 1, 2, \ldots$,*

$$\mathbb{E}\Big(U_1 \,\Big|\, \sum_{i=1}^{n} U_i = j\Big) = \frac{j}{n} \tag{6.1}$$

and

$$\mathbb{P}\Big(U_1 = k \,\Big|\, \sum_{i=1}^{n} U_i = j\Big) = \frac{q_k q_{j-k}^{*(n-1)}}{q_j^{*n}}\,, \tag{6.2}$$

where $\{q_k^{*n}\}$ denotes the n-fold convolution of $\{q_k\}$.

In the following theorem we state a recursive method which is called *Panjer's algorithm* and which can be used to calculate the probabilities $p_k^X = \mathbb{P}(X = k)$ of the compound $X = \sum_{i=1}^{N} U_i$ provided that (5.21) holds.

Theorem 6.1 *Assume that* (5.21) *is fulfilled. Then*

$$p_j^X = \begin{cases} \hat{g}_N(q_0), & \text{for } j = 0, \\ (1 - aq_0)^{-1} \sum_{k=1}^{j} \left(a + bkj^{-1}\right) q_k p_{j-k}^X, & \text{for } j = 1, 2, \ldots. \end{cases} \tag{6.3}$$

6.2 A Continuous Version of Panjer's Algorithm

Panjer's algorithm stated in Theorem 6.1 has the disadvantage that the individual claim sizes need to take their values on a lattice. In order to overcome this drawback we derive an integral equation for the distribution F_X of the aggregate claim amount.

Theorem 6.2 *If $\{p_k\}$ is governed by recursion* (5.21) *with parameters a, b and p_0, and F_U is absolutely continuous with bounded density $f(x)$, then the density*

$\tilde f_X(x)$ of the absolutely continuous part of the compound distribution $F_X = \sum_{k=0}^\infty p_k F_U^{*k}$ satisfies

$$\tilde f_X(x) - \frac{1}{x}\int_0^x (ax+by) f_U(y) \tilde f_X(x-y)\,dy = p_1 f_U(x)\,, \qquad x>0\,. \tag{6.4}$$

Moreover, the function $\tilde f_X(x)$ is the only solution to (6.4) *in the set of all integrable functions on* $(0,\infty)$.

6.3 Inverting the Fourier Transform

We can use a formula for inverting the Fourier transform when calculating the probability function of a discrete random variable. In particular, this inversion formula can be used to calculate the probability function of the compound $\sum_{k=1}^N U_k$ when the claim sizes $U_1, U_2, \ldots$ are lattice. In comparison to Panjer's algorithm, this method has the advantage that no special assumptions on the distribution of N are needed. Recall that the characteristic function of an $\mathbb{N}$-valued random variable with probability function $\{p_k\}$ is given by the Fourier transform

$$\hat\varphi(s) = \sum_{k=0}^\infty \mathrm{e}^{\mathrm{i}sk} p_k\,, \qquad s\in\mathbb{R}\,. \tag{6.5}$$

It is useful to introduce the Fourier transform in a more general situation. For some fixed $n\in\mathbb{N}$, consider a sequence $p_0, p_1, \ldots, p_{n-1}$ of arbitrary real numbers. The Fourier transform $\hat\varphi_{(n)}(s)$ of $\{p_0,\ldots,p_{n-1}\}$ is defined by

$$\hat\varphi_{(n)}(s) = \sum_{k=0}^{n-1} \mathrm{e}^{\mathrm{i}sk} p_k\,, \qquad s\in\mathbb{R}\,. \tag{6.6}$$

Clearly, if $\{p_0,\ldots,p_{n-1}\}$ is a probability function, then $\hat\varphi_{(n)}(s)$ is the characteristic function of $\{p_0,p_1,\ldots,p_{n-1}\}$. We now show that the row vector $\boldsymbol{p} = (p_0,p_1,\ldots,p_{n-1})$ can be calculated from limited information on the Fourier transform $\hat\varphi_{(n)}(s)$. Assume that the values $\hat\varphi_{(n)}(s_j)$ of $\hat\varphi_{(n)}$ are given at the points $s_j = 2\pi j/n$ for $j=0,1,\ldots,n-1$. Write these values of the Fourier transform $\hat\varphi_{(n)}(s)$ as a row vector $\boldsymbol{\varphi} = (\hat\varphi_{(n)}(s_0),\ldots,\hat\varphi_{(n)}(s_{n-1}))$. Then

$$\boldsymbol{\varphi}^\top = \boldsymbol{F}\boldsymbol{p}^\top, \tag{6.7}$$

where $\boldsymbol{F}$ is the $n\times n$ matrix given by

$$\boldsymbol{F} = \left(\mathrm{e}^{\mathrm{i}s_j k}\right)_{j,k=0,\ldots,n-1} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \mathrm{e}^{\mathrm{i}2\pi/n} & \ldots & \mathrm{e}^{\mathrm{i}2\pi(n-1)/n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{e}^{\mathrm{i}s_j 0} & \mathrm{e}^{\mathrm{i}s_j 1} & \ldots & \mathrm{e}^{\mathrm{i}s_j(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \mathrm{e}^{\mathrm{i}2\pi(n-1)/n} & \ldots & \mathrm{e}^{\mathrm{i}2\pi(n-1)^2/n} \end{pmatrix}$$

and $\boldsymbol{x}^\top$ is the transpose of $\boldsymbol{x}$. Note that the rows of $\boldsymbol{F}$ form orthogonal vectors because

$$\sum_{k=0}^{n-1} \mathrm{e}^{\mathrm{i}s_j k}\overline{\mathrm{e}^{\mathrm{i}s_\ell k}} = \sum_{k=0}^{n-1} \mathrm{e}^{\mathrm{i}(s_j - s_\ell)k} = \begin{cases} \dfrac{\mathrm{e}^{\mathrm{i}(s_j-s_\ell)n}-1}{\mathrm{e}^{\mathrm{i}(s_j-s_\ell)}-1} = 0 & \text{for } j \neq \ell, \\ n & \text{for } j = \ell. \end{cases}$$

Therefore the inverse matrix $\boldsymbol{F}^{-1}$ of $\boldsymbol{F}$ is given by

$$\boldsymbol{F}^{-1} = \left(\frac{1}{n}\mathrm{e}^{-\mathrm{i}s_k j}\right)_{j,k=0,\ldots,n-1}.$$

Thus, from (6.7) we get $\boldsymbol{p}^\top = \boldsymbol{F}^{-1}\boldsymbol{\varphi}^\top$, i.e. for all $k = 0,\ldots,n-1$.

$$p_k = \frac{1}{n}\sum_{j=0}^{n-1} \hat{\varphi}_{(n)}(s_j)\mathrm{e}^{-\mathrm{i}s_j k}. \tag{6.8}$$

Note that computing $\boldsymbol{p}$ directly from $\boldsymbol{\varphi}$ requires n^2 operations. However, if $n = 2^m$ for some nonnegative integer m, then there exist the FFT algorithm with complexity of order $n \log_2 n$.

Suppose now that we know the Fourier transform $\hat{\varphi}(s) = \sum_{k=0}^\infty \mathrm{e}^{\mathrm{i}sk}p_k$ of an infinite, summable sequence $p_0, p_1, \ldots$ (that is $\sum_{k=0}^\infty |p_k| < \infty$). Then it is possible to obtain an approximation to the first n terms $p_0,\ldots,p_{n-1}$ of the sequence $p_0, p_1, \ldots$ by sampling the Fourier transform $\hat{\varphi}(s)$ at the points $s_j = 2\pi j/n$; $j = 0,1,\ldots,n-1$. Since for each $k = 1,2,\ldots$ the function $\{\mathrm{e}^{\mathrm{i}sk},\ s \geq 0\}$ has the period 2π, we obtain

$$\hat{\varphi}(s_j) = \sum_{k=0}^\infty p_k \mathrm{e}^{\mathrm{i}s_j k} = \sum_{\ell=0}^\infty \sum_{k=0}^{n-1} p_{k+n\ell}\mathrm{e}^{\mathrm{i}s_j(k+n\ell)} = \sum_{k=0}^{n-1} \tilde{p}_k \mathrm{e}^{\mathrm{i}s_j k},$$

where $\tilde{p}_k = \sum_{\ell=0}^\infty p_{k+n\ell}$ for $k = 0,1,\ldots,n-1$. The values $\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_{n-1}$ can be calculated from $\{\hat{\varphi}(s_j); j = 0,1,\ldots,n-1\}$ by the same argument which led to (6.8), i.e.

$$\tilde{p}_k = \frac{1}{n}\sum_{j=0}^{n-1} \hat{\varphi}(s_j)\mathrm{e}^{-\mathrm{i}s_j k}. \tag{6.9}$$

Note that $\tilde{p}_k$ approximates p_k for each $k = 0,\ldots,n-1$, since by the assumed summability of $p_0, p_1, \ldots$, the error $\tilde{p}_k - p_k = \sum_{\ell=1}^\infty p_{k+n\ell}$ becomes arbitrarily small if n tends to infinity. When bounds on the tail of $\{p_k\}$ are available, we can even estimate the error $\tilde{p}_k - p_k$.

We turn to the compound $X = \sum_{k=1}^N U_k$ where $U_1, U_2, \ldots$ are independent and identically distributed $\mathbb{N}$-valued random variables. Assume that the generating function $\hat{g}_N$ of N and the values $\{\hat{\varphi}_U(s_j);, j = 0,1,\ldots,n-1\}$ of the characteristic function $\hat{\varphi}_U(t)$ are known. Then the values $\{\hat{\varphi}_X(s_j);\ j =$

$0, 1, \ldots, n-1\}$ of the characteristic function $\hat{\varphi}_X(s)$ can be computed from the formula $\hat{\varphi}_X(s) = \hat{g}_N(\hat{\varphi}_U(s))$, $(s \in \mathbb{R})$, which can be derived as in the proof of Theorem 5.1. Thus, the probability function $\{p_k\}$ of X can be calculated or, at least, approximated in the way given above.

Exercises

6.1 Show that for the Poisson, negative-binomial, binomial the probability function $\{p_k, k \in \mathbb{N}\}$ has the following property: There exist constants $a, b \in \mathbb{R}$ such that

$$p_k = \Big(a + \frac{b}{k}\Big) p_{k-1} \tag{6.10}$$

for all $k = 1, 2, \ldots$. Find the constants a, b for these distributions.

6.2 Let $\{p_k, k \in \mathbb{N}\}$ be a probability function which satisfies (6.10) for $k = 1, 2, \ldots$. Show that then

$$p_0 = \begin{cases} e^{-b} & \text{if } a = 0, \\ (1-a)^{1+b/a} & \text{if } a \neq 0. \end{cases}$$

6.3 Consider a Poisson compound $X = \sum_{k=1}^{N} U_i$ specified by (λ, F), where F is the geometric distribution $\mathrm{Geo}(1/2)$. Suppose that we want to compute the probability function $\{p_k\}$ of X for $k = 0, \ldots, 10$. In how many points one has to sample the characteristic function $\hat{\varphi}_X(s)$ to have an error less than 10^{-2}. Show the calculations for $\lambda = 1$ and $\lambda = 10$. [Hint. Use inequality (5.19).]

7 Lecture: Time-Dependent Risk Models

7.1 The Ruin Problem

We assume that X_n – claim size, takes values in $\mathbb{N}$ and that the common probability function is $\{p_k\} = \{\mathbb{P}(X_n = k)\}$. Apart from the sequence $X_1, X_2, \ldots$ we take into account that constant premiums are collected during each period. For simplicity we take the premium equal to 1. Finally, suppose that the initial reserve is equal to $u \in \mathbb{N}$. Then, the reserve R_n after the n-th period is

$$R_n = u + n - \sum_{i=1}^{n} X_i \,. \tag{7.1}$$

We will call R_n process the *discrete-time risk reserve process.* We search for the probability that the risk reserve process $\{R_n\}$ ever falls below a certain critical level.

Obviously, if $p_0 + p_1 < 1$, then the risk reserve R_n can be negative for some $n \in \mathbb{N}$. This event $\{R_1 < 0\} \cup \{R_2 < 0\} \cup \ldots$, is called the (technical) *ruin* of the portfolio. The probability of ruin is defined in the following way. Consider the epoch $\tau_{\mathrm{d}}(u) = \min\{n \geq 1 : \ R_n < 0\}$ when the risk reserve process becomes negative for the first time and where we put $\tau_{\mathrm{d}}(u) = \infty$ if $R_n \geq 0$ for all $n \in \mathbb{N}$. Thus, $\tau_{\mathrm{d}}(u)$ is an extended random variable and it is called the *time of ruin* (or *ruin time*). Furthermore,

$$\psi(u) = \mathbb{P}(\{R_1 < 0\} \cup \{R_2 < 0\} \cup \ldots) = \mathbb{P}(\tau_{\mathrm{d}}(u) < \infty)$$

is called the (infinite-horizon) *ruin probability* for the initial reserve $u \in \mathbb{N}$. If $\psi(u)$ is seen as a function of the variable u, then $\psi(u)$ is called the *ruin function.* Another characteristic is the so-called *finite-horizon ruin probability* $\psi(u;n) = \mathbb{P}(\tau_{\mathrm{d}}(u) \leq n)$ which is the probability that ruin occurs not later than after the n-th period, $n \in \mathbb{N}$. Unfortunately, in many cases it is difficult to express the finite-horizon ruin probabilities $\psi(u;n)$ in a closed form. The infinite-horizon ruin probabilities $\psi(u)$ are mathematically simpler. As will be seen in later sections of the book, bounds and approximations to $\psi(u)$ are often available, even for more general risk models.

Instead of the risk reserve process, it is sometimes preferable to consider the *claim surplus process* $\{S_n\}$ defined by

$$S_n = \sum_{i=1}^{n} X_i - n \,, \qquad n \in \mathbb{N} \,. \tag{7.2}$$

Then,

$$\psi(u) = \mathbb{P}(\{S_1 > u\} \cup \{S_2 > u\} \cup \ldots) = \mathbb{P}(\max\{S_1, S_2, \ldots\} > u) \,. \tag{7.3}$$

With the notation $Y_i = X_i - 1$, we have

$$S_n = \sum_{i=1}^{n} Y_i\,, \tag{7.4}$$

where the $Y_1, Y_2, \ldots$ are independent and identically distributed.

From the strong law of large numbers we have that $S_n/n \to \mathbb{E}\,Y$. This tells us that $S_n \to \infty$ provided that $\mathbb{E}\,Y > 0$. Thus, in this case (7.3) implies that $\psi(u) \equiv 1$ for each $u \in \mathbb{N}$. It can be shown that this result extends to the case where $\mathbb{E}\,Y = 0$ However, if $\mathbb{E}\,Y < 0$, then $S_n \to -\infty$, i.e. the maximum $\max\{S_1, S_2, \ldots\}$ is finite with probability 1 and therefore $\psi(u) < 1$ for all $u \in \mathbb{N}$.

7.2 Computation of the Ruin Function

In this section we consider the random walk $\{S_n\}$ with generic increment $Y = X-1$. We assume $\mathbb{E}\,Y < 0$. Note that for the maximum $M = \max\{0, S_1, S_2, \ldots\}$ of $\{S_n\}$ we have $\psi(u) = \mathbb{P}(M > u)$. Furthermore, $M = \max\{0, Y_1, Y_1 + \tilde{S}_1, Y_1 + \tilde{S}_2, \ldots\}$ where $\tilde{S}_1 = Y_2$, $\tilde{S}_2 = Y_2 + Y_3, \ldots$, i.e.

$$M \stackrel{\mathrm{d}}{=} (M + Y)_+\,. \tag{7.5}$$

Thus, with the notations $\hat{g}_M(s) = \mathbb{E}\,s^M, \hat{g}_X(s) = \mathbb{E}\,s^X$, and $\rho = \mathbb{E}\,X$ we get the following result.

Theorem 7.1 (a) *The generating function of M is*

$$\hat{g}_M(s) = \frac{(1-\rho)(1-s)}{\hat{g}_X(s) - s}\,, \qquad s \in (-1, 1)\,. \tag{7.6}$$

(b) *The maximum M of the random walk $\{S_n\}$ has the same distribution as the geometric compound $\sum_{i=1}^{N} U_i$ specified by $\rho = \mathbb{P}(N \geq 1)$ and $\mathbb{P}(U = k) = \mathbb{P}(X > k)/\mathbb{E}\,X$, $k \in \mathbb{N}$.*

It turns out that the simple risk model considered above can be used to analyse a problem of data transmission in a computer network. Suppose that at each time $n \in \mathbb{N}$ a random number of data packets arrives at a node of the network; in each time-slot $[n, n+1)$ one packet can be transmitted provided that at the beginning of this slot at least one packet is waiting. Let X_n be the number of packets arriving at $n-1$-st slot. As before, we assume that the $\mathbb{N}$-valued random variables $X_1, X_2, \ldots$ are independent and identically distributed with common probability function $\{p_k\}$. Suppose that just before time $n = 0$ there are L_0 packets waiting for transmission. We may assume that L_0 is random, but independent of the $X_1, X_2, \ldots$. After one unit of time the number of packets decreases by 1. But at the same time X_1 new packets arrive and so $(L_0 + X_1 - 1)_+$ packets are waiting just before the beginning of the next slot. In general, the number L_n of packets just before the beginning of the $(n+1)$-st slot fulfils the recurrence relation

$$L_n = (L_{n-1} + X_n - 1)_+ \,, \tag{7.7}$$

where L_n is called the *queue length* at time n. We leave it to the reader to show that the solution to (7.7) is

$$L_n = \max\{0, Y_n, Y_{n-1} + Y_n, \ldots, L_0 + Y_1 + \ldots + Y_n\} \,, \tag{7.8}$$

where $Y_i = X_i - 1$.

An important auxiliary result is the following *duality property* expressing a certain invariance property under *inversion of time*.

Lemma 7.1 *Let* $Y_1, Y_2, \ldots$ *be independent and identically distributed. Then,* $(Y_n, Y_{n-1} + Y_n, \ldots, Y_1 + \ldots + Y_n) \stackrel{\mathrm{d}}{=} (Y_1, Y_1 + Y_2, \ldots, Y_1 + \ldots + Y_n)$ *for all* $n = 1, 2, \ldots$.

Now we are ready to state a relationship between the queueing model of data transmission and the risk model considered above. It says that the limit distribution of the queue lengths L_n can be expressed by the ruin function $\psi(u)$. For simplicity, we assume that $L_0 = 0$.

Theorem 7.2 *Let* $L_0 = 0$, $\rho = \mathbb{E}\, X < 1$ *and* $\psi(u) = \mathbb{P}(M > u)$. *Then,*

$$\lim_{n\to\infty} \mathbb{P}(L_n \le u) = 1 - \psi(u) \,, \qquad u \in \mathbb{N} \,. \tag{7.9}$$

7.3 A Risk Model in Continuous Time

The risks $X_1, X_2, \ldots$ considered in the preceding sections of this chapter can take the value 0 with a positive probability p_0. This means that in the corresponding period no claim occurs. An equivalent model would be obtained by recording the (random) indices of the periods where claims occur, together with the corresponding (strictly positive) aggregate claim amounts. We make this statement more precise. Consider the 0-1 sequence $\{I_n\} = \{\mathbb{I}(X_n > 0),\ n = 1, 2, \ldots\}$ and define the positions of ones and the distances between them in the sequence $\{I_n\}$: $\sigma_0 = 0$, $\sigma_n = \min\{k > \sigma_{n-1} : I_k = 1\}$ and $T_n = \sigma_n - \sigma_{n-1}$; $n \ge 1$. The number of ones in the sequence $I_1, \ldots, I_n$ is $N(n) = \sum_{k=1}^{\infty} \mathbb{I}(\sigma_k \le n)$. Let $U_1, U_2, \ldots$ be a sequence of independent and identically distributed random variables with distribution being equal to the conditional distribution of X provided that $X > 0$. Moreover, let the sequences $\{N(n)\}$ and $\{U_n\}$ be independent. Then, for the risk reserve process $\{R_n\}$ defined in (7.1), we have $R_n \stackrel{\mathrm{d}}{=} R'_n$ where $R'_n = u + n - \sum_{i=1}^{N(n)} U_i$. Note that an even stronger property holds. Namely, for each $n = 1, 2, \ldots$ we have $(R_1, \ldots, R_n) \stackrel{\mathrm{d}}{=} (R'_1, \ldots, R'_n)$.

We can weaken the assumption that the random variables T and U are $\mathbb{N}$-valued assuming only that they are nonnegative. Then, a more general risk reserve model is defined as follows. We are given

- random epochs $\sigma_1, \sigma_2, \ldots$ with $0 < \sigma_1 < \sigma_2 < \ldots$ at which the claims occur, where the random variables σ_n can be discrete or continuous,
- the corresponding positive (individual or aggregate) claim sizes $U_1, U_2, \ldots$,
- the initial risk reserve $u \geq 0$, and
- the premiums which are collected at a constant rate $\beta > 0$, so that the premium income is a linear function of time.

Let $N(t)$ counts the number of claims by time t. The *risk reserve process* $\{R(t),\ t \geq 0\}$ is then given by

$$R(t) = u + \beta t - \sum_{i=1}^{N(t)} U_i\,, \tag{7.10}$$

while the *claim surplus process* $\{S(t),\ t \geq 0\}$ is

$$S(t) = \sum_{i=1}^{N(t)} U_i - \beta t\,. \tag{7.11}$$

The *time of ruin* $\tau(u) = \min\{t : R(t) < 0\} = \min\{t : S(t) > u\}$ is the first epoch when the risk reserve process becomes negative or, equivalently, when the claim surplus process crosses the level u. We will mainly be interested in the *ruin probabilities* $\psi(u;x) = \mathbb{P}(\tau(u) \leq x)$ and $\psi(u) = \lim_{x\to\infty} \psi(u;x) = \mathbb{P}(\tau(u) < \infty)$. Here $\psi(u;x)$ is called the *finite-horizon ruin probability* and $\psi(u)$ the *infinite-horizon ruin probability.* Alternatively, $\psi(u)$ can be called the *probability of ultimate ruin.* We will further need the notion of the *survival probability* $\overline{\psi}(u) = 1 - \psi(u)$.

There is a relationship between infinite-horizon ruin probabilities of risk models in discrete time and in continuous time. To get $\tau(u)$ it is sufficient to check the claim surplus process $\{S(t)\}$ at the embedded epochs σ_k $(k = 1, 2, \ldots)$; see Figure 7.1. Indeed, the largest value $M = \max_{t\geq 0} S(t)$ of the claim surplus process can be given by $M = \max_{n\geq 0} \sum_{k=1}^{n} (U_k - \beta T_k)$ and consequently

$$\psi(u) = \mathbb{P}(M > u)\,. \tag{7.12}$$

The representation formula (7.12) gives us the possibility to interpret the ruin function $\psi(u)$ as the tail function of the stationary waiting time in a *single-server system* of queueing theory.

Note, however, that one has to be careful when comparing finite-horizon ruin probabilities in discrete time with those in continuous time because in general

$$\mathbb{P}\Big(\max_{0\leq t\leq x} S(t) > u\Big) \neq \mathbb{P}\Big(\max_{0\leq n\leq x} \sum_{k=1}^{n} (U_k - \beta T_k) > u\Big).$$

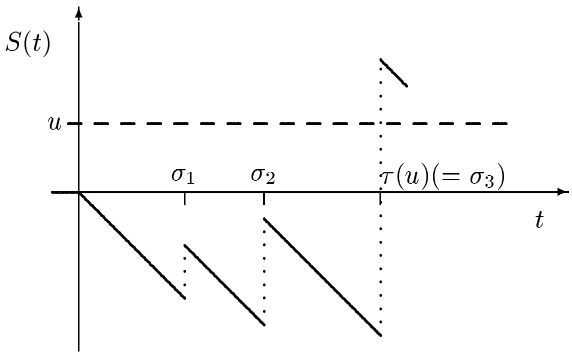

Figure 7.1: Claim surplus process

Anyhow, in order to keep the notation simple we will use the same symbol for the finite-horizon ruin function in the continuous-time risk model as in the discrete-time risk model, i.e.

$$\psi(u;x) = \mathbb{P}(\tau(u) \leq x) = \mathbb{P}\Big(\max_{0\leq t\leq x} S(t) > u\Big).$$

Exercises

7.1 Consider the risk model in discrete time. Assume that X_n only can assume two values: 0 with probability q and 2 with probability $p = 1-q$, where $p < q$. Show that $\psi(u) = (p/q)^{u+1}$. [Comment: In this case also $Y_n = X_n - 1$ only assumes two values: -1 with probability q and 1 with probability p as in the classical *gambler's ruin problem*, i.e. in each game the gambler can win one currency unit with probability p and lose one currency unit (that is to win -1 currency unit) with probability q, $p > q$. Assume that initially the gambler has u currency units and that the game is against an infinitely rich player. If by being ruined we mean that the gambler's capital reaches the level -1, then the probability of ruin is $(q/p)^{u+1}$.]

7.2 Consider the risk model in discrete time and assume that X_n is geometrically distributied. Compute $\hat{g}_M(s)$ and find the probability function $\{p_k^M\}$ of M.

7.3 Consider the risk model in discrete time and assume that X_n is negative-binomial $\mathrm{NB}(k,a)$ distributed for some $k = 1, 2, \ldots$. Compute $\hat{g}_M(s)$. Find Lundberg bounds for $\mathbb{P}(M > x)$. Make computations for $k = 2$ and $a = 1/4$.

7.4 Consider the risk model in continuous time. Let the inter-occurrence times T_n be independent and identically distributed with distribution $\mathrm{TG}(p)$ and let the claim sizes U_n be positive $\mathbb{N}$-valued random variables which are independent and identically distributed with probability function $\{p_k\}$, independent of the inter-occurrence times. We also assume that the premium rate β is equal to 1 and that $\sum_{n=1}^{\infty} np_n < 1/(1-p)$. Compute the generating function $\hat{g}_M(s)$ of the random variables M given in (7.12). [Hint. Represent this model as risk model in discrete time and use Theorem 8.1.a.]

8 Lecture: Classical Risk Model

8.1 Poisson Arrival Processes

Let $\{T_n\}$ be a sequence of independent random variables with exponential distribution $\mathrm{Exp}(\lambda)$; $\lambda > 0$. Then, the counting process $\{N(t)\}$ is called a *homogeneous Poisson process* with *intensity* λ.

Definition 8.1 *A real-valued stochastic process $\{X(t), t \geq 0\}$ is said to have*
(a) *independent increments if for all $n = 1, 2, \ldots$ and $0 \leq t_0 < t_1 < \ldots < t_n$, the random variables $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent,*
(b) *stationary increments if for all $n = 1, 2, \ldots,$ $0 \leq t_0 < t_1 < \ldots < t_n$ and $h \geq 0$, the distribution of $(X(t_1 + h) - X(t_0 + h), \ldots, X(t_n + h) - X(t_{n-1} + h))$ does not depend on h.*

We now give some equivalent definitions of a Poisson process.

Theorem 8.1 *Let $\{N(t), t \geq 0\}$ be a counting process. Then the following statements are equivalent:*
(a) *$\{N(t)\}$ is a Poisson process with intensity λ.*
(b) *For all $t \geq 0$, $n = 1, 2, \ldots$ the random variable $N(t)$ has distribution $\mathrm{Poi}(\lambda t)$ and, given $\{N(t) = n\}$, the random vector $(\sigma_1, \ldots, \sigma_n)$ has the same distribution as the order statistics of n independent uniformly on $[0, t]$ distributed points.*
(c) *$\{N(t)\}$ has independent increments such that $\mathbb{E}\, N(1) = \lambda$ and for all $t \geq 0$, $n = 1, 2, \ldots,$ given $\{N(t) = n\}$, the random vector $(\sigma_1, \ldots, \sigma_n)$ has the same distribution as the order statistics of n independent uniformly on $[0, t]$ distributed points.*
(d) *$\{N(t)\}$ has stationary and independent increments and satisfies as $h \downarrow 0$,*

$$\mathbb{P}(N(h) = 0) = 1 - \lambda h + o(h)\,, \qquad \mathbb{P}(N(h) = 1) = \lambda h + o(h)\,. \tag{8.1}$$

(e) *$\{N(t)\}$ has stationary and independent increments and for each fixed $t \geq 0$, the random variable $N(t)$ is $\mathrm{Poi}(\lambda t)$ distributed.*

Let the counting process $\{N(t)\}$ be a Poisson process with intensity λ. Let the claim sizes $\{U_n\}$ be independent and identically distributed with distribution F_U and let $\{U_n\}$ be independent of $\{N(t)\}$. Then the cumulative arrival process $\{X(t),\ t \geq 0\}$ defined by

$$X(t) = \sum_{j=1}^{N(t)} U_j$$

is called a *compound Poisson process* with characteristics (λ, F_U), i.e. with intensity λ and jump size distribution F_U.

Corollary 8.1 *Let $\{X(t)\}$ be a compound Poisson process with characteristics (λ, F_U). Then,*
(a) *the process $\{X(t)\}$ has stationary and independent increments,*
(b) *the moment generating function of $X(t)$ is given by*

$$\hat{m}_{X(t)}(s) = \mathrm{e}^{\lambda t(\hat{m}_U(s)-1)}, \tag{8.2}$$

and the mean and variance by

$$\mathbb{E}\, X(t) = \lambda t \mu_U\,, \qquad \operatorname{Var} X(t) = \lambda t \mu_U^{(2)}\,. \tag{8.3}$$

8.2 Ruin Probabilities: The Compound Poisson Model

In the sequel of this chapter we consider the compound Poisson model. The risk reserve process $\{R(t), t \geq 0\}$ is defined in (7.10) and claims occur according to a compound Poisson process with characteristics (λ, F_U). The most frequently used property of the process $\{R(t)\}$ is the independence and stationarity of its increments. Considering $\{R(t)\}$ from time t onwards is like restarting a risk reserve process with an identically distributed claim arrival process but with initial reserve $R(t)$. In particular, if $R(t) = y$ and ruin has yet not occurred by time t, then the (conditional) ruin probability is $\psi(y)$. Furthermore, considering $\{R(t)\}$ from the first claim occurrence epoch σ_1 on, is like starting a risk reserve process with initial reserve $R(t + \sigma_1 - U_1)$.

Let $\{S_n, n \geq 0\}$ be the random walk given by

$$S_n = \sum_{i=1}^{n} Y_i\,, \qquad Y_i = U_i - \beta T_i\,. \tag{8.4}$$

In can be shown that $\limsup_{n\to\infty} S_n = \infty$ if $\mathbb{E}\, Y \geq 0$. Thus, (7.12) implies that $\psi(u) \equiv 1$ in this case. Let us therefore assume that $\mathbb{E}\, Y < 0$, i.e. $\beta > \lambda\mu$, where $\mu = \mu_U$ denotes the expected claim size. Recall that β is the premium income in the unit time interval and that $\lambda\mu$ is the expected aggregate claim over the unit time interval (see (8.3)). The condition

$$\beta > \lambda\mu \tag{8.5}$$

is therefore called the *net profit condition*. In the sequel we will assume (8.5). Note that in this case $\lim_{n\to\infty} S_n = -\infty$, since from the strong law of large numbers we have $S_n/n \to \mathbb{E}\, Y < 0$. Thus, the maximum of $\{S_n\}$ is finite. Using (7.12) we get $\lim_{u\to\infty} \psi(u) = 0$. Moreover, we will see later in Theorem 8.5 that this implies $\psi(u) < 1$ for all $u \geq 0$.

8.3 An Integro-Differential Equation

We now study the survival probability $\overline{\psi}(u) = 1 - \psi(u)$. We show that $\overline{\psi}(u)$ is differentiable everywhere on $\mathbb{R}_+$ with the exception of an at most countably infinite set of points. Furthermore, we prove that $\overline{\psi}(u)$ fulfils an integro-differential equation.

Theorem 8.2 *The survival function $\overline{\psi}(u)$ is continuous on $\mathbb{R}_+$ with right and left derivatives $\overline{\psi}_+^{(1)}(u)$ and $\overline{\psi}_-^{(1)}(u)$, respectively. Moreover*

$$\beta\overline{\psi}_+^{(1)}(u) = \lambda\Big(\overline{\psi}(u) - \int_0^u \overline{\psi}(u-y)\,dF_U(y)\Big) \tag{8.6}$$

and

$$\beta\overline{\psi}_-^{(1)}(u) = \lambda\Big(\overline{\psi}(u) - \int_0^{u-0} \overline{\psi}(u-y)\,dF_U(y)\Big)\,. \tag{8.7}$$

8.4 An Integral Equation

Equation (8.6) is not easily solved because it involves both the derivative and an integral of $\overline{\psi}(u)$. It would be more convenient to get rid of the derivative. Indeed, integrating (8.6) we arrive at the following result.

Theorem 8.3 *The ruin function $\psi(u)$ satisfies the integral equation*

$$\beta\psi(u) = \lambda\Big(\int_u^\infty \overline{F}_U(x)\,dx + \int_0^u \psi(u-x)\overline{F}_U(x)\,dx\Big). \tag{8.8}$$

8.5 Laplace Transforms, Pollaczek-Khinchin Formula

In this section we compute the Laplace transforms

$$\hat{L}_\psi(s) = \int_0^\infty \psi(u)\mathrm{e}^{-su}\,du\,, \qquad \hat{L}_{\overline{\psi}}(s) = \int_0^\infty \overline{\psi}(u)\mathrm{e}^{-su}\,du\,.$$

Note that both integrals make sense for all $s > 0$. Furthermore, we have

$$\hat{L}_\psi(s) = \int_0^\infty (1-\overline{\psi}(u))\,\mathrm{e}^{-su}\,du = \frac{1}{s} - \hat{L}_{\overline{\psi}}(s)\,. \tag{8.9}$$

Theorem 8.4 *The Laplace transforms $\hat{L}_{\overline{\psi}}(s)$ and $\hat{L}_\psi(s)$ are given by*

$$\hat{L}_{\overline{\psi}}(s) = \frac{\beta-\lambda\mu}{\beta s - \lambda(1-\hat{l}_U(s))}\,, \qquad s > 0 \tag{8.10}$$

and

$$\hat{L}_\psi(s) = \frac{1}{s} - \frac{\beta-\lambda\mu}{\beta s - \lambda(1-\hat{l}_U(s))}\,, \qquad s > 0\,. \tag{8.11}$$

The representation formula for $\psi(u)$ derived in the next theorem is called the *Pollaczek-Khinchin formula*.

Theorem 8.5 *For each $u \geq 0$,*

$$\psi(u) = \Big(1 - \frac{\lambda\mu}{\beta}\Big) \sum_{n=1}^{\infty} \Big(\frac{\lambda\mu}{\beta}\Big)^n \overline{(F_U^{\rm s})^{*n}}(u)\,. \tag{8.12}$$

Proof Taking Laplace transforms on both the sides of (8.8) gives

$$\hat{L}_\psi(s) = \frac{\lambda\mu}{\beta} \hat{L}_{\overline{F_U^{\rm s}}}(s) + \frac{\lambda\mu}{\beta} \hat{L}_\psi(s) \cdot \hat{l}_{\overline{F_U^{\rm s}}}(s)\ .$$

Thus, $\hat{L}_\psi(s) = \lambda\mu\beta^{-1}\hat{L}_{\overline{F_U^{\rm s}}}(s)(1 - \lambda\mu\beta^{-1}\hat{l}_{\overline{U^{\rm s}}}(s))^{-1}$. Now using that

$$\hat{L}_{\overline{F_U^{\rm s}}}(s) = \frac{1 - \hat{\ell}_{F_U^{\rm s}}(s)}{s}$$

we have that

$$\begin{aligned} \hat{L}_\psi(s) &= \lambda\mu\beta^{-1}\frac{1 - \hat{\ell}_{F_U^{\rm s}}(s)}{s} \sum_{n=0}^{\infty} (\lambda\mu\beta^{-1}\hat{\ell}_{F_U^{\rm s}}(s))^n \\ &= \frac{1}{s} - \frac{\beta - \lambda\mu}{\beta s - \lambda(1 - \hat{l}_U(s))}\,, \qquad s > 0\,. \end{aligned}$$

Again, the one-to-one correspondence between functions and their Laplace transforms yields (8.12). □

Exercises

8.1 Assume that $\{X(t), t \geq 0\}$ is a process with independent increments and for each $t \geq 0$ the distribution of $X(t+h) - X(h)$ does not depend on $h \geq 0$. Show that then the process $\{X(t)\}$ has stationary increments.

8.2 Let $\{N(t), t \geq 0\}$ be a stochastic process which has stationary and independent increments and satisfies as $h \downarrow 0$,

$$\mathbb{P}(N(h) = 0) = 1 - \lambda h + o(h)\,, \qquad \mathbb{P}(N(h) = 1) = \lambda h + o(h)$$

for some $\lambda > 0$. Show that then the probability $p_n(t) = \mathbb{P}(N(t) = n)$ is given by

$$p_n(t) = \frac{(\lambda t)^n}{n!} {\rm e}^{-\lambda t}$$

for all $t \geq 0$ and $n \in \mathbb{N}$.

8.3 Let $\{N(t), t \geq 0\}$ be a Poisson process with intensity λ and let $c > 0$ be some constant. Show that $\{N(ct), t \geq 0\}$ is a Poisson process and determine its intensity.

8.4 Let $\{X_1(t)\}$ and $\{X_2(t)\}$ be two independent compound Poisson processes with characteristics (λ_1, F_1), (λ_2, F_2). Show that $\{X(t)\} = \{X_1(t) + X_2(t)\}$ is a compound Poisson process and determine its characteristics.

8.5 Let the claim sizes U_n be $\mathrm{Erl}(2, \delta)$-distributed. Show, that

$$\psi(u) = a\,\mathrm{e}^{-r_1 u} - b\,\mathrm{e}^{-r_2 u}\,, \tag{8.13}$$

where $r_1 < r_2$ are the solutions to the equation

$$\beta r^2 - (2\delta\beta - \lambda)r + \delta(\delta\beta - 2\lambda) = 0$$

and

$$a = \frac{\lambda(2\lambda - \beta\delta + 2\beta r_2)}{\beta^2\delta(r_2 - r_1)}\,, \qquad b = \frac{\lambda(\beta\delta - 2\lambda - 2\beta r_1)}{\beta^2\delta(r_2 - r_1)}\,.$$

[Hint. Differentiate (8.6) twice.]

8.6 Let the claim sizes U_n be $\mathrm{Erl}(2, \delta)$-distributed. Determine the Laplace transforms of $\overline{\psi}(u)$ and $\psi(u)$. Invert the Laplace transforms in order to verify (8.13).

8.7 Let the distribution F_U of claims sizes U_n be the exponential distribution $\mathrm{Exp}(\delta)$ with parameter $\delta > 0$. Determine the integrated tail distribution F_U^{s}. Use this result to recover formula for the ruin function with exponential claim sizes using the Pollaczek-Khinchin formula (8.12).

Other exercises

8.8 Consider a compound Poisson process with claim occurrence times σ_n and claim sizes U_n. Define, for some fixed $u \geq 0$,

$$X_1(t) = \sum_{k=1}^{\infty} U_k \mathbb{I}(U_k \leq u)\mathbb{I}(\sigma_k \leq t)$$

and

$$X_2(t) = \sum_{k=1}^{\infty} U_k \mathbb{I}(U_k > u)\mathbb{I}(\sigma_k \leq t)\,.$$

Show that $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ are two independent compound Poisson processes. Determine their characteristics.

8.9 Let $\beta = \lambda = 1$ and $F_U(x) = 1 - \frac{1}{3}(\mathrm{e}^{-x} + \mathrm{e}^{-2x} + \mathrm{e}^{-3x})$. Show that

$$\psi(u) = 0.550790\mathrm{e}^{-0.485131u} + 0.0436979\mathrm{e}^{-1.72235u} + 0.0166231\mathrm{e}^{-2.79252u}\,.$$

[Hint. Calculate the Laplace transform $\hat{L}_\psi(s)$ and invert it.]

8.10 Put $\rho = \lambda\mu\beta^{-1}$. Let $\rho = 0.75$, $\beta = 1$ and let the claim sizes have the distribution $F_U = p\mathrm{Exp}(a_1) + (1-p)\mathrm{Exp}(a_2)$, where $p = 2/3$, $a_1 = 2$ and $a_2 = 1/2$. Show that the ruin function $\psi(u)$ is given by

$$\psi(u) = 0.75 \left(0.935194\mathrm{e}^{-0.15693u} + 0.0648059\mathrm{e}^{-1.59307u}\right) .$$

9 Lecture: Some Properties of Classical Risk Model

9.1 Lundberg Bounds

Let $\theta(s) = \lambda(\hat{m}_U(s) - 1) - \beta s$. Consider the equation

$$\theta(s) = 0\,. \tag{9.1}$$

If a root $s \neq 0$ exists, then it is unique and strictly positive. We call this solution, if it exists, the *adjustment coefficient* or the *Lundberg exponent* and denote it by γ. Let $x_0 = \sup\{x : F_U(x) < 1\}$.

Theorem 9.1 *Assume that the adjustment coefficient $\gamma > 0$ exists. Then,*

$$a_- \mathrm{e}^{-\gamma u} \le \psi(u) \le a_+ \mathrm{e}^{-\gamma u} \tag{9.2}$$

for all $u \ge 0$, where

$$a_- = \inf_{x \in [0, x_0)} \frac{\mathrm{e}^{\gamma x} \int_x^\infty \overline{F}_U(y)\, dy}{\int_x^\infty \mathrm{e}^{\gamma y} \overline{F}_U(y)\, dy}\,, \qquad a_+ = \sup_{x \in [0, x_0)} \frac{\mathrm{e}^{\gamma x} \int_x^\infty \overline{F}_U(y)\, dy}{\int_x^\infty \mathrm{e}^{\gamma y} \overline{F}_U(y)\, dy}\,.$$

9.2 The Cramér-Lundberg Approximation

Lemma 9.1 *Assume that the function $z_1 : \mathbb{R}_+ \to (0, \infty)$ is increasing and let $z_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be decreasing, such that*

$$\int_0^\infty z_1(x) z_2(x)\, dx < \infty \tag{9.3}$$

and

$$\lim_{h \to 0} \sup \{z_1(x+y)/z_1(x) : x \ge 0, 0 \le y \le h\} = 1\,. \tag{9.4}$$

Then, for $z(x) = z_1(x) z_2(x)$ and for each absolute continuous distribution F on $\mathbb{R}_+$ (that is having a density function), the equation

$$g(u) = z(u) + \int_0^u g(u-v)\, dF(v)\,, \qquad u \ge 0\,, \tag{9.5}$$

admits a unique locally bounded solution such that

$$\lim_{u \to \infty} g(u) = \begin{cases} \mu_F^{-1} \displaystyle\int_0^\infty z(u)\, du & \text{if } \mu_F < \infty, \\ 0 & \text{if } \mu_F = \infty. \end{cases} \tag{9.6}$$

Lemma 9.1 is a version of the so-called *key renewal theorem*.

Theorem 9.2 *Assume that the adjustment coefficient $\gamma > 0$ exists. If $\hat{m}_U^{(1)}(\gamma) < \infty$, then*

$$\lim_{u\to\infty} \psi(u) e^{\gamma u} = \frac{\beta - \lambda\mu}{\lambda \hat{m}_U^{(1)}(\gamma) - \beta} . \tag{9.7}$$

If $m_U^{(1)}(\gamma) = \infty$, then $\lim_{u\to\infty} \psi(u) e^{\gamma u} = 0$.

The asymptotic result obtained in Theorem 9.2 for the ruin probability $\psi(u)$ gives rise to the so-called *Cramér-Lundberg approximation*

$$\psi_{\text{app}}(u) = \frac{\beta - \lambda\mu}{\lambda \hat{m}_U^{(1)}(\gamma) - \beta} e^{-\gamma u}. \tag{9.8}$$

The following numerical investigation shows that the above approximation works quite well even for small values of u.

Example Let $\beta = \lambda = 1$ and $F_U(x) = 1 - \frac{1}{3}(e^{-x} + e^{-2x} + e^{-3x})$. In this example we use the expected inter-occurrence time as time unit and the premium per unit time as monetary unit. The mean value of claim sizes is $\mu = 0.611111$, i.e. the net profit condition (8.5) is fulfilled. Furthermore, computing the Laplace transform $\hat{L}_\psi(s)$ and inverting it, we get

$$\psi(u) = 0.550790 e^{-0.485131 u} + 0.0436979 e^{-1.72235 u} + 0.0166231 e^{-2.79252 u} . \tag{9.9}$$

On the other hand, (9.8) implies that in this case the Cramér-Lundberg approximation to $\psi(u)$ is $\psi_{\text{app}}(u) = 0.550790 e^{-0.485131 u}$. By comparison to the exact formula given in (9.9), the accuracy of this approximation can be analysed. Table 9.1 shows the ruin function $\psi(u)$, its Cramér-Lundberg approximation $\psi_{\text{app}}(u)$ and the relative error $(\psi_{\text{app}}(u) - \psi(u))/\psi(u)$ multiplied by 100. Note that the relative error is below 1% for $u \geq 1.71358 = 2.8\mu$.

u	0	0.25	0.5	0.75	1
$\psi(u)$	0.6111	0.5246	0.4547	0.3969	0.3479
$\psi_{\text{app}}(u)$	0.5508	0.4879	0.4322	0.3828	0.3391
Er	-9.87	-6.99	-4.97	-3.54	-2.54
u	1.25	1.5	1.75	2	2.25
$\psi(u)$	0.3059	0.2696	0.2379	0.2102	0.1858
$\psi_{\text{app}}(u)$	0.3003	0.2660	0.2357	0.2087	0.1849
Er	-1.82	-1.32	-0.95	-0.69	-0.50

Table 9.1: Cramér-Lundberg approximation to ruin probabilities

9.3 Subexponential Claim Sizes

Theorem 9.3 *Let $\rho = \lambda\mu\beta^{-1}$ and assume that $F_U^s \in \mathcal{S}$. Then*

$$\lim_{u\to\infty} \frac{\psi(u)}{1 - F_U^s(u)} = \frac{\rho}{1-\rho} . \tag{9.10}$$

The above theorem suggests the approximation

$$\psi_{\text{app}}(u) = \frac{\rho}{1-\rho}(1 - F_U^s(u)) \ . \tag{9.11}$$

Note that the quantity ρ captures all the information on the claim number process one needs to know.

Examples 1) Assume that the claim sizes are $\mathrm{Par}(\alpha, c)$ distributed. In order to have a finite mean (which is necessary by the net profit condition (8.5)) we must have $\alpha > 1$. The integrated tail distribution F_U^{s} is readily obtained as

$$F_U^{\mathrm{s}}(x) = \begin{cases} (\alpha-1)x/\alpha c & \text{if } x \le c, \\ 1 - \alpha^{-1}\,(x/c)^{-(\alpha-1)} & \text{if } x > c. \end{cases}$$

By Theorem 3.4, F_U^{s} is subexponential. Thus, Theorem 9.3 leads to the following approximation to the ruin probability $\psi(u)$:

$$\psi_{\text{app}}(u) = \frac{\rho}{\alpha(1-\rho)}\left(\frac{u}{c}\right)^{-(\alpha-1)}$$

for $u > c$. Details are left to the reader.

2) Let $\beta = 1$, $\lambda = 9$ and $F_U(x) = 1 - (1+x)^{-11}$, where we use the premium as monetary unit. The integrated tail distribution F_U^{s} is readily obtained: $F_U^{\mathrm{s}}(x) = 1 - (1+x)^{-10}$. From Theorem 3.4 we conclude that both $F_U \in \mathcal{S}$ and $F_U^{\mathrm{s}} \in \mathcal{S}$. Approximation (9.11) reads then $\psi_{\text{app}}(u) = 9(1+u)^{-10}$. Table 9.2 gives some values of $\psi(u)$ and of the approximation $\psi_{\text{app}}(u) = 9(1+u)^{-10}$ as well as 100 times the relative error. The "exact values" of $\psi(u)$ were calculated using Panjer's algorithm described in Section 6.10. In order to get a discrete approximation to the claim size distribution, this distribution was discretized with bandwidth $h = 10^{-3}$, i.e. $q_k = \mathbb{P}(k/1000 \le U < (k+1)/1000)$. Consider

u	$\psi(u)$	$\psi_{\text{app}}(u)$	Er
1	0.364	$8.79 \cdot 10^{-3}$	-97.588
2	0.150	$1.52 \cdot 10^{-4}$	-99.898
3	$6.18 \cdot 10^{-2}$	$8.58 \cdot 10^{-6}$	-99.986
4	$2.55 \cdot 10^{-2}$	$9.22 \cdot 10^{-7}$	-99.996
5	$1.05 \cdot 10^{-2}$	$1.49 \cdot 10^{-7}$	-99.999
10	$1.24 \cdot 10^{-4}$	$3.47 \cdot 10^{-10}$	-100
20	$1.75 \cdot 10^{-8}$	$5.40 \cdot 10^{-13}$	-99.997
30	$2.50 \cdot 10^{-12}$	$1.10 \cdot 10^{-14}$	-99.56
40	$1.60 \cdot 10^{-15}$	$6.71 \cdot 10^{-16}$	-58.17
50	$1.21 \cdot 10^{-16}$	$7.56 \cdot 10^{-17}$	-37.69

Table 9.2: Approximation to ruin probabilities for subexponential claims

for instance the initial risk reserve $u = 20$. Then the ruin probability $\psi(u)$ is $1.75 \cdot 10^{-8}$ which is so small that it is not interesting for practical purposes. However, the approximation error is still almost 100%. Thus, in the case of heavy-tailed claim sizes, the approximation (9.11) can be poor, even for large values of u.

9.4 Ordering of Ruin Functions

We compare the ruin functions $\psi(u)$ and $\psi'(u)$ of two compound Poisson models with arrival rates λ and λ', premium rates β and β', and claim size distributions F_U and $F_{U'}$ respectively.

Theorem 9.4 *If $\lambda \le \lambda'$ and $\beta \ge \beta'$ and if $U \le_{\mathrm{sl}} U'$, then $\psi(u) \le \psi'(u)$ for all $u \ge 0$.*

Corollary 9.1 *If F_U is* NBUE, *then for all $u \ge 0$*

$$\psi(u) \le \frac{\lambda \mu_U}{\beta} \, \mathrm{e}^{-(\mu_U^{-1} - \lambda\beta^{-1})u} . \tag{9.12}$$

Similarly, the reversed inequality in (9.12) *is true if F_U is* NWUE.

Exercises

9.1 Let $\beta = \lambda = 1$ and $F_U(x) = 1 - \frac{1}{3}(\mathrm{e}^{-x} + \mathrm{e}^{-2x} + \mathrm{e}^{-3x})$. Show that the mean value of claim sizes is $\mu = 0.611111$ and that the Cramér–Lundberg approximation (9.8) to $\psi(u)$ is $\psi_{\mathrm{app}}(u) = 0.550790\mathrm{e}^{-0.485131u}$. Compare this approximation to the exact formula obtained in Exercise 8.9.

9.2 Let the claim sizes U_n have distribution $\mathrm{Erl}(2, \delta)$. Determine the adjustment coefficient γ and the Cramér-Lundberg approximation (9.8). Compare this result to the exact formula for the ruin probability $\psi(u)$ obtained in Exercise 8.5.

9.3 Find the asymptotic behaviour of the ruin function $\psi(u)$ as $u \to \infty$ if the claim sizes U_n have a Pareto-type distribution. Show in particular that the ruin function $\psi(u)$ has the asymptotics
(a) $\psi(u) \sim \frac{3}{2}u^{-1}$ as $u \to \infty$ if the claim sizes are PME(2)-distributed and $\rho = 0.75$,
(b) $\psi(u) \sim 9(1+u)^{-10}$ as $u \to \infty$ if $\beta = 1$, $\lambda = 9$ and $F_U(x) = 1 - (1+x)^{-11}$,
(c) $\psi(u) \sim \rho(\alpha(1-\rho))^{-1}(u/c)^{-(\alpha-1)}$ as $u \to \infty$ if the claim sizes are $\mathrm{Par}(\alpha, c)$-distributed with $\alpha > 1$.

9.4 Let $\eta = \beta(\lambda\mu)^{-1} - 1$ be the *relative safety loading*. Show that the risk reserve process can be expressed as

$$R(t) = u + (1+\eta)\lambda t - \sum_{j=1}^{N(t)} U_j \ .$$

Think about an interpretation of η.

9.5 Consider the following reinsurance scheme, in which reinsurer's payments are functions of individual claims. Suppose that $h(x)$ is a *retention function* that is if claim is x, then $y - h(x)$ is paid by the reinsurer. Premiums and reinsurance premiums are paid continuously with constant rate. It is assumed that the reinsurer uses the same safety loading η. Show that the risk reserve at time t in the classical model is:

$$R_h(t) = u + (1+\eta)\lambda \mathbb{E}\, h(U) - \sum_{j=1}^{N(t)} U_j \ .$$

9.6 (Continuation). In the classical model show that the adjustment coefficient γ_h is the solution to

$$1 + \gamma_h \mathbb{E}\, h(U)(1+\eta) = \mathbb{E}\, e^{\gamma_h h(U)} \ ,$$

and that the probability of ruin $\psi_h(u)$ under this contract with retention function $h(x)$ fulfills

$$\psi_h(u) \le e^{-\gamma_h u}, \qquad u \ge 0 \ .$$

Other exercises

9.7 (Continuation). Consider the *stop-loss contract* (deductible a), when $h_a(x) = \min(x, a)$. Show that γ_{h_a} is a decreasing function of a. Prove also that the stop-loss contract is optimal in the following sense. Among all reinsurance contracts with $\mathbb{E}\, h_a(U) = \mathbb{E}\, h(U)$ we have $\gamma_a \ge \gamma_h$. Using Cramer-Lundberg approximation find a probabilistic interpretation of this result.

10 Lecture: Further Studies

10.1 Finite-Horizon Ruin Probabilities

Seal's Formulae: We now express the survival probability $\overline{\psi}(u;x)$ in terms of the distribution $F_{X(t)}$ of the aggregate claim amount $X(t) = \sum_{i=1}^{N(t)} U_i$. If U has density f_U then the distribution function of $X(t)$ can be expressed as

$$F_{X(t)}(x) = \mathrm{e}^{-\lambda x} + \int_0^x \tilde{f}_{X(t)}(y)\,dy\,, \qquad x \geq 0\,,$$

where $\tilde{f}_{X(t)}(y) = \sum_{n=1}^{\infty}((\lambda t)^n/n!)\mathrm{e}^{-\lambda t} f_U^{*n}(y)$. Formulae (10.1) and (10.2) below are known in actuarial mathematics as *Seal's formulae*;.

Theorem 10.1 *Assume that* $\mathbb{P}(U > 0) = 1$.
(a) *Then, for initial risk reserve* $u = 0$,

$$\overline{\psi}(0;x) = \frac{1}{\beta x}\mathbb{E}\,(R(x)_+) = \frac{1}{\beta x}\int_0^{\beta x} F_{X(x)}(y)\,dy\,. \tag{10.1}$$

(b) *If* $u > 0$ *and* U *has density* $f_U(y)$ *then*

$$\overline{\psi}(u;x) = F_{X(x)}(u+\beta x) - \beta\int_0^u \overline{\psi}(0,x-y)\tilde{f}_{X(y)}(u+\beta y)\,dy\,. \tag{10.2}$$

We now give the ruin function for the case of the exponential service time $U \sim \mathrm{Exp}(\delta)$. Let

$$J(x) = \sum_{n=0}^{\infty}\frac{z^n}{n!n!} = I_0(2\sqrt{x})\,, \tag{10.3}$$

where I_0 is modified Bessel function of order 0. We have

Theorem 10.2 *Assume that* $F_U(x) = 1 - \mathrm{e}^{-\delta x}$ *for all* $x \geq 0$. *Then*

$$\psi(u;x) = 1 - \mathrm{e}^{-\delta u-(1+c)\lambda x} g(\delta u + c\lambda x, \lambda x) \tag{10.4}$$

where $c = \delta\beta/\lambda$ *and*

$$g(z,\theta) = J(\theta z) + \theta J^{(1)}(\theta z) + \int_0^z \mathrm{e}^{z-v}J(\theta v)\,dv - \frac{1}{c}\int_0^{c\theta}\mathrm{e}^{c\theta-v}J(zc^{-1}v)\,dv\,. \tag{10.5}$$

10.2 Random Walks

Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed (not necessarily integer-valued) random variables with distribution F which can take both positive and negative values. The sequence $\{S_n,\ n \in \mathbb{N}\}$ with $S_0 = 0$ and

$S_n = Y_1 + \ldots + Y_n$ for $n = 1, 2, \ldots$ is called a *random walk*. We assume that the first moment $\mathbb{E}\,Y$ exists and that Y is not concentrated at 0, i.e. $\mathbb{P}(Y = 0) < 1$. Look at the first entrance time of the random walk $\{S_n\}$ into the positive half-line $(0, \infty)$:

$$\nu^+ = \min\{n > 0 : \; S_n > 0\} \tag{10.6}$$

setting $\nu^+ = \infty$ if $S_n \le 0$ for all $n \in \mathbb{N}$, and call ν^+ the (first strong) *ascending ladder epoch* of $\{S_n\}$. Suppose that $\mathbb{E}\,Y < 0$. In some cases it is possible to compute G^+ – the defective distribution of S_{ν^+}. This is done by the so called Wiener-Hopf factorization.

10.3 Sparre Andersen Model

Consider the risk reserve process $\{R(t), t \ge 0\}$ in continuous time. We assume herethat the claim counting process $\{N(t)\}$ is a renewal counting process, i.e. $\{N(t)\}$ is governed by a sequence of independent and identically distributed inter-occurrence times $\{T_n\}$ with a common distribution F_T. Furthermore, the sequence $\{U_n\}$ of claim sizes consists of independent and identically distributed random variables with distribution F_U and is independent of $\{T_n\}$. This model is called the *Sparre Andersen model* Let $\rho = (\lambda\mu)/\beta < 1$.

Theorem 10.3 *For all $u \ge 0$,*

$$1 - \psi(u) = (1-p)\sum_{k=0}^{\infty}(G^+)^{*k}(u) = \sum_{k=0}^{\infty}(1-p)p^k G_0^{*k}(u)\,, \tag{10.7}$$

where G^+ is the (defective) distribution of the ladder height of the random walk $\{S_n\}$; $S_n = \sum_{i=1}^{n}(U_i - \beta T_i)$, $p = G^+(\infty)$ and $G_0(u) = G^+(u)/G^+(\infty)$.

10.4 Lundberg Bounds

Consider the equation

$$\hat{m}_Y(s) = \hat{m}_U(s)\hat{l}_T(\beta s) = 1\,. \tag{10.8}$$

Clearly, $\hat{m}_Y(0) = 1$. This equation may have a second root. If such a root $s \ne 0$ exists, then it is unique and strictly positive. The solution to (10.8), if it exists, is called the *adjustment coefficient* and is denoted by γ. Let $x_0 = \sup\{x : F_Y(x) < 1\}$.

Theorem 10.4 *Suppose that there exists a positive solution γ to* (10.8)*. Then*

$$b_-\,\mathrm{e}^{-\gamma u} \le \psi(u) \le b_+\,\mathrm{e}^{-\gamma u} \tag{10.9}$$

for all $u \ge 0$, where

$$b_- = \inf_{x\in[0,x_0)} \frac{\mathrm{e}^{\gamma x}\overline{F}_Y(x)}{\int_x^\infty \mathrm{e}^{\gamma y}\,dF_Y(y)}\,, \qquad b_+ = \sup_{x\in[0,x_0)} \frac{\mathrm{e}^{\gamma x}\overline{F}_Y(x)}{\int_x^\infty \mathrm{e}^{\gamma y}\,dF_Y(y)}\,. \tag{10.10}$$

10.5 The Cramér-Lundberg Approximation

We assume that the distribution F of Y is nonlattice and $\mathbb{E}\,Y < 0$. The reader should prove that then the ladder height distribution G^+ corresponding to F is nonlattice too. Furthermore, we assume that (10.8) has a positive solution γ.

Theorem 10.5 *For the Sparre Andersen model,*

$$\lim_{u\to\infty} \mathrm{e}^{\gamma u}\psi(u) = c \tag{10.11}$$

where the constant $c \geq 0$ is finite and given by

$$c = \frac{1 - G^+(\infty)}{\gamma \int_0^\infty v\mathrm{e}^{\gamma v}\, dG^+(v)} \ . \tag{10.12}$$

10.6 Subexponential Claim Sizes

For the Sparre andersen model we have the following generalization.

Theorem 10.6 *If $F_U^{\mathrm{s}} \in \mathcal{S}$, then*

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{F_U^{\mathrm{s}}}(u)} = \frac{\mathbb{E}\,U}{\beta\mathbb{E}\,T - \mathbb{E}\,U} \ . \tag{10.13}$$

Some Problems in the Theory of Risk

Examination Problems

Name ....

Return by 2000.02.15
to my office H1-33 or to Ms T. Higashi H1-22.
Attempt all problems. Show all work.

1. Show that the tail $\overline{F}(x)$ of the mixture $F = \int_0^\infty F_\theta \, \mathrm{d}G(\theta)$ with $F_\theta = \mathrm{Exp}(\theta)$ and $G = \Gamma(a, \lambda)$ is $\overline{F}(x) = (\lambda^{-1}x + 1)^{-a}$. Compute its hazard rate functions and the coefficient of variation.

2. Let F be the distribution on $\mathbb{R}_+$ defined by
$$\overline{F}(x) = \left(\frac{b}{b + x^r}\right)^\alpha, \qquad x \geq 0\,,$$
where $\alpha, b, r > 0$. Show that $F \in \mathcal{S}$ and, if $\alpha r > 1$, $F^{\mathrm{s}} \in \mathcal{S}$. Consider the compound random variable $X = \sum_{i=1}^N U_i$, where $U_1, U_2, \ldots$ are iid with distribution F^{s} and N has negative binomial distribution NB(2,1/2). Find the asymptotics of $\mathbb{P}(X > x)$, for $x \to \infty$.

3. Consider the risk model in discrete time. Yearly payments of claims $X_1, X_2, \ldots$ are $0, C, 2C$ with probabilities $1/2, 1/4, 1/4$ respectively. Each year the company collects C of premiums. Compute the ruin function $\psi(u)$, where $u = 0, 1, \ldots$. Find the initial capital u such that the ruin happens with probability less than 0.05.

4. Consider the classical ruin model. Put $\rho = \lambda\mu\beta^{-1}$. Let $\rho = 0.75$, $\beta = 1$ and let the claim sizes have the distribution
$$\bar{F}_U(x) = pe^{-a_1 x} + (1 - p)e^{-a_2 x},$$
where $p = 2/3$, $a_1 = 2$ and $a_2 = 1/2$. Find μ and λ and the ruin function $\psi(u)$. Find Lundberg bounds and Cramer-Lundberg approximations. [HINT. For evaluation of roots or to make the partial fraction expansion you may use MATLAB, however you have to report this.

Solutions. 1. We have

$$\begin{aligned}\overline{F}(x) &= \frac{\lambda^a}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-\theta x}\theta^{a-1}\mathrm{e}^{-\lambda\theta}\,\mathrm{d}\theta \\ &= \frac{\lambda^a}{\Gamma(a)}\frac{1}{(x+\lambda)^a}\int_0^\infty t^{a-1}\mathrm{e}{-t}\,\mathrm{d}t = \frac{1}{(\lambda^{-1}+1)^a}\ ,\end{aligned}$$

where the substitution $t = (x+\lambda)\theta$ is used in the second equality. Hence

$$f(x) = -\frac{\mathrm{d}\overline{F}(x)}{\mathrm{d}x} = a\lambda^{-1}(\lambda^{-1}x+1)^{-(a+1)} = \lambda^a a(\lambda+x)^{-(a+1)}.$$

For $a > 1$

$$\mu = \int_0^\infty \overline{F}(x)\,\mathrm{d}x = \frac{\lambda}{a-1},$$

which follows from

$$\int_y^\infty (\lambda^{-1}x+1)^{-a}\,\mathrm{d}x = \frac{\lambda}{a-1}(\lambda^{-1}y+1)^{-a+1}\ .$$

For $a > 2$

$$\mu^{(2)} = 2\int_0^\infty\int_y^\infty \overline{F}(x)\,\mathrm{d}x = \frac{2\lambda^2}{(a-2)(a-1)}\ .$$

Hence

$$\sigma^2 = \frac{\lambda^2 a}{(a-2)(a-1)^2}\ .$$

Thus the hazard rate function is

$$m(x) = \frac{a\lambda^{-1}(\lambda^{-1}x+1)^{-(a+1)}}{(\lambda^{-1}x+1)^{-a}} = a\lambda^{-1}(\lambda^{-1}x+1)^{-1} = \frac{a}{x+\lambda}$$

and

$$\mathrm{cv} = \frac{\lambda\sqrt{a}}{\sqrt{a-2}(a-1)}\frac{a-1}{\lambda} = \sqrt{\frac{a}{a-2}}\ .$$

2. We write

$$\overline{F}(x) = x^{-\alpha r}\left(\frac{b}{bx^{-r}+1}\right)^\alpha = x^{-\alpha r}L(x), \qquad x \ge 0$$

Since $\lim_{x\to\infty} L(x) = b^\alpha$, $L(x)$ is a slowly varying function and so F is Pareta-like (LN p. 12). By Theorem 3.4 we have $F \in \mathcal{S}$. If $\alpha r > 1$, then $\mu < \infty$. We write

$$\overline{F^{\mathrm{s}}}(x) = \frac{1}{\mu}\int_x^\infty \overline{F}(y)\,dy = x^{-\alpha r+1}L_2(x)$$

where $L_2(x)$ is a slowly varying function and the last equation follows from Theorem 4.3. Let $U_i \sim F^{\mathrm{s}}$ and $N \sim \mathrm{B}(2, 1/2)$, $X = \sum_{i=1}^{N} U_i$. By Theorem 3.3

$$\frac{\mathbb{P}(X > x)}{\frac{1}{\mu}\int_x^\infty \overline{F}(y)\,dy} \to \mathbb{E}\,N\ .$$

We have $\mathbb{E}\,N = 2$

3. Without loss of generality we can assume $C = 1$, because the size of C does not change the ruin function. We have $\mathbb{E}\,X = 3/4$, $\mathbb{E}\,Y = -1/4 < 0$ and

$$\hat{g}_X(s) = \frac{2}{4} + \frac{1}{4}s + \frac{1}{4}s^2$$

and from (7.6)

$$\hat{g}_M(s) = \frac{1-s}{2-3s+s^2} = \frac{1}{2}\frac{1}{1-(s/2)}\ .$$

The g.f. $\hat{g}_M(s)$ corresponds to the geometric distribution Geo(1/2). Thus the ruin function is

$$\psi(u) = \mathbb{P}(M > u) = \left(\frac{1}{2}\right)^{u+1}, \qquad u = 0, 1, \ldots$$

Hence $\psi(u) < 0.05$ for $u + 1 > 4.3219$. Therefore the minimal required capital is $4C$.

4. We have

$$\mu = \int_0^\infty (\frac{2}{3}e^{-2x} + \frac{1}{3}e^{-x/2})\,\mathrm{d}x = 1.$$

From $\rho = \lambda\mu\beta$ we conclude that $\lambda = 0.75$. To compute ruin function $\psi(u)$ we use formula (8.11). We have

$$\hat{l}_U(s) = \frac{4}{3(2+s)} + \frac{1}{3(1+2s)}, \qquad s \geq -1/2.$$

Therefore

$$\hat{L}_\psi(s) = \frac{3(2s+3)}{2s(4s^2+7s+1)} = \frac{0.0486}{s+1.5931} + \frac{0.7014}{s+0.1569}\ .$$

(Note that we write up to 4 digits of accuracy). Inverting we have

$$\psi(u) = 0.7014e^{-0.1569u} + 0.0486e^{-1.593u}, \qquad u \geq 0\ .$$

To compute the adjustment coeficient γ we have to compute $\theta(s) = \lambda(\hat{m}_U(s) - -1) - \beta s$ which is well defined for $u < 1/2$ and

$$\theta(s) = \frac{s(4s^2-7s+1)}{2(s-2)(1-2s)}, \qquad s < 1/2.$$

Note that function $\theta(s)$ is well-defined for $s < 1/2$. The adjustment coefficient $\gamma > 0$ fulfills thus

$$\frac{s(4s^2 - 7s + 1)}{2(s-2)(1-2s)} = 0$$

which formally has three solutions

$$s = 0, \frac{7 - \sqrt{33}}{8} = 0.1569, \frac{7 + \sqrt{33}}{8}.$$

We have to discard the last one because it is greater than $1/2$ and hence $\gamma = 0.1569$. Since by theorem 9.2 we have $\psi(u) \sim \mathrm{const}\exp(-0.1569)$ from the form of the already computed ruin function we immediately conclude that the Cramer-Lundberg approximation is

$$\psi_{\mathrm{app}}(u) = 0.7014 \exp(-0.1569u) \ .$$

To compute a_+ and a_- we have to analyze

$$\begin{aligned} r(x) &= \frac{e^{\gamma x} \int_x^\infty \overline{F}_U(x)\,\mathrm{d}x}{\int_x^\infty e^{\gamma y}\overline{F}_U(x)\,\mathrm{d}x} \\ &= \frac{e^{-2x} + 2e^{-x/2}}{\frac{2}{2-\gamma}e^{-2x} + \frac{2}{1-2\gamma}e^{-x/2}} \\ &= \frac{e^{-2x} + 2e^{-x/2}}{ae^{-2x} + be^{-x/2}} \end{aligned}$$

where $a = 1.0851$ and $b = 2.9146$. We now write

$$\frac{e^{-2x} + 2e^{-x/2}}{ae^{-2x} + be^{-x/2}} = \frac{1}{a}\left(1 - \frac{b - 2a}{ae^{-3x/2} + b}\right) \ .$$

Since $b - 2a > 0$ and

$$\frac{b - 2a}{ae^{-3x/2} + b}$$

is increasing we conclude that $r(x)$ is non-increasing so

$$\sup_{x \geq 0} r(x) = r(0) = \frac{3}{a+b} = 0.7501,$$

$$\inf_{x \geq 0} r(x) = \lim_{x \to \infty} r(x) = \frac{2}{b} = 0.6862 \ .$$

Hence we have the Lundberg bounds:

$$0.6862e^{-0.1569} \leq \psi(u) \leq 0.7501e^{-0.1569}, \qquad u \geq 0 \ .$$