LECTURES ON GAUSSIAN PROCESSES

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Skrypt został opracowany na podstawie notatek do wykładu *Procesy Gaussowskie*, który odbył się w semestrze zimowym 2001/2002. Prezentowany materiał jest w trakcie opracowywania. Tekst może zawierać błędy. Układ skryptu jak i oznaczenia mogą ulegać zmianom. Autor będzie wdzięczny za wszelkie uwagi oraz listy błędów, ktore można przesłać e-mailem na adres: rolski@math.uni.wroc.pl.

Wersja drukowana w dniu

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Chapter I

Gaussian variables, vectors and processes

1 Gaussian random variables

Let

$$\phi(x) = \phi_{0,1}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad x \in \mathbb{R}$$

and for $a \in \mathbb{R}$ and $\sigma > 0$

$$\phi_{a,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}}, \qquad x \in \mathbb{R}. \tag{1.1}$$

Clearly we have

$$\begin{aligned} \int_{-\infty}^{\infty} x\phi_{a,\sigma^2}(x)\ \mathrm{d}x &= a, \\ \int_{-\infty}^{\infty} (x-a)^2\phi_{a,\sigma^2}(x)\ \mathrm{d}x &= \sigma^2. \end{aligned}$$

Definition 1.1 We say that a random variable X is *Gaussian* or *normal* $\mathcal{N}(a, \sigma^2)$ with mean a and variance σ^2 if X has density wrt the Lebesgue measure equal to $\phi_{a,\sigma^2}(x)$. If $a = 0$ and $\sigma^2 = 1$, then we say that X is a *standard Gaussian variable.* If $a = 0$, then we say that X is *centered.*

We denote the standard Gaussian distribution function by

$$\Phi(x) = \int_{-\infty}^{x} \phi(y)\ \mathrm{d}y$$

and its tail function by

$$\Psi(x) = \int_x^\infty \phi(y)\ \mathrm{d}y, \qquad x \in \mathbb{R}$$

The moment generating function of Gaussian random variables are computed in the next Proposition.

Proposition 1.2 *For all $z \in \mathbb{C}$*

$$\int_{-\infty}^{\infty} e^{zx} \phi_{a,\sigma^2}(x)\ \mathrm{d}x = \exp\left(az + \frac{\sigma^2 z^2}{2}\right).$$

In particular, the characteristic function of X is

$$\widehat{\phi}(t) = \mathbb{E}\, e^{itX} = e^{iat - \frac{\sigma^2 t^2}{2}}, \quad t \in \mathbb{R}.$$

Proof

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_\infty^\infty e^{zx} e^{-\frac{(x-a)^2}{2}}\ \mathrm{d}x &= \frac{1}{\sqrt{2\pi}} e^{az} \int_\infty^\infty e^{zx} e^{-\frac{x^2}{2}}\ \mathrm{d}x \\
\frac{1}{\sqrt{2\pi}} e^{az} \int_\infty^\infty e^{zx} e^{-\frac{x^2}{2}}\ \mathrm{d}x &= \frac{1}{\sqrt{2\pi}} e^{az} \int_\infty^\infty e^{-\frac{x^2 - 2zx + z^2}{2} + \frac{z^2}{2}}\ \mathrm{d}x \\
e^{az} e^{\frac{z^2}{2}} \int_{-\infty}^\infty \exp\left(-\frac{(x-z)^2}{2}\right)\ \mathrm{d}x &= e^{az + \frac{z^2}{2}}
\end{aligned}$$

because for all $z \in \mathbb{C}$ [1]

$$\int_{-\infty}^\infty \exp(-\frac{(x-z)^2}{2})\ \mathrm{d}x = \sqrt{2\pi}. \tag{1.2}$$

□

We have the following estimations for the tail function $\Psi(x)$.

Proposition 1.3 *For all $x > 0$*

$$\frac{\exp(-\frac{x^2}{2})}{(2\pi)^{1/2}} \left(x^{-1} - x^{-3}\right) \le \Psi(x) \le \frac{\exp(-\frac{x^2}{2})}{(2\pi)^{1/2}} x^{-1} \qquad x > 0.$$

Proof Using substitution $s = x + x^{-1}t$ we have

$$\begin{aligned}
\Psi(x) &= \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{s^2}{2}\right)\ \mathrm{d}s = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{x^2 + 2t + x^{-2}t^2}{2} x^{-1}\right) dt \\
&= \frac{\exp(-\frac{x^2}{2})}{(2\pi)^{1/2}} x^{-1} \int_0^\infty \exp\left(-\frac{x^{-2}t^2}{2}\right) \exp\left(-t\right)\ \mathrm{d}t.
\end{aligned}$$

[1] Dac referencje

Since for all $x > 0$, $t \geq 0$

$$1 - \frac{x^{-2}t^2}{2} \leq \exp\left(-\frac{x^{-2}t^2}{2}\right) \leq 1$$

we have

$$1 \geq \int_0^\infty \exp\left(-\frac{x^{-2}t^2}{2}\right) \exp(-t)\ \mathrm{d}t \geq \int_0^\infty (1 - \frac{x^{-2}t^2}{2}) = 1 - x^{-2}$$

from which the result follows. □

Exercises

1.1 Show that for all $z \in \mathbb{C}$

$$\int_{-\infty}^\infty \exp\left(-\frac{(x-z)^2}{2}\right)\ \mathrm{d}x = \sqrt{2\pi}.$$

2 Gaussian random vectors

Let $\boldsymbol{X}$ be a random vector, that is a measurable mapping $\boldsymbol{X} : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^d$ or a equivalently sequence of random variables $\boldsymbol{X} = (X_1, \ldots, X_d)^T$.

Definition 2.1 We say that a random vector $\boldsymbol{X}$ is *Gaussian* if for all $\boldsymbol{t} = (t_1, \ldots, t_d)^T$ random variable $\sum_{j=1}^d t_j X_j = \boldsymbol{X}^T \boldsymbol{t}$ is Gaussian.

To determine the exact distribution of $\sum_{j=1}^d t_j X_j$ we must know its mean $a(\boldsymbol{t})$ and variance $\sigma^2(\boldsymbol{t})$. However

$$a(\boldsymbol{t}) = \mathbb{E} \sum_{j=1}^d t_j X_j = \sum_{j=1}^d t_j \mathbb{E}\, X_j = \boldsymbol{t}^T \boldsymbol{a},$$

where $\boldsymbol{a} = (a_1, \ldots, a_n)^T = (\mathbb{E}\, X_1, \ldots, \mathbb{E}\, X_n)^T$. Furthermore we have (for simplicity we assume $\boldsymbol{a} = \boldsymbol{0}$

$$\begin{aligned} \sigma^2(\boldsymbol{t}) = \mathbb{E}\, (\sum_{j=1}^d t_j X_j)^2 &= \mathbb{E} \sum_{j=1}^d \sum_{k=1}^d t_j X_j t_k X_k \\ = \sum_{j=1}^d \sum_{k=1}^d t_j \sigma_{jk} t_k &= \boldsymbol{t}^T \boldsymbol{\Sigma} \boldsymbol{t}, \end{aligned} \tag{2.3}$$

where $\mathbf{\Sigma} = (\sigma_{jk})_{j,k=1,\dots,d}$ is the *covariance matrix* of $\boldsymbol{X}$ with entries:

$$\sigma_{jk} = \mathrm{Cov}\,(X_j, X_k)\ .$$

Since the variance is nonnegative, and from (2.3), we have that for all $\boldsymbol{t} \in \mathbb{R}^d$

$$\boldsymbol{t}^T \mathbf{\Sigma} \boldsymbol{t} \geq 0$$

which means that $\mathbf{\Sigma}$ the matrix is *nonnegative definite*. It is also symmetric. Thus we see that any vector $\boldsymbol{a} \in \mathbb{R}^d$ and symmetric and nonnegative matrix $\mathbf{\Sigma}$ determines uniquely one Gaussian distribution (sometimes called *multivariate Gaussian (normal) distribution*), which we denote by $\mathcal{N}_d(\boldsymbol{a}, \mathbf{\Sigma})$.

From the above we also see that each covariance matrix is symmetric and nonnegative definite.

Proposition 2.2 *If $\boldsymbol{X} \sim \mathcal{N}_d(\boldsymbol{a}, \mathbf{\Sigma})$, then its characteristic function is*

$$\widehat{\phi}(\boldsymbol{t}) = \mathbb{E}\, e^{i\boldsymbol{X}^T\boldsymbol{t}} = \exp\left(i\boldsymbol{a}^T\boldsymbol{t} - \frac{\boldsymbol{t}^T\mathbf{\Sigma}\boldsymbol{t}}{2} \right).$$

Proof The random variable $\boldsymbol{X}^T\boldsymbol{t} \sim \mathcal{N}_d(\boldsymbol{a}^T\boldsymbol{t}, \boldsymbol{t}^T\mathbf{\Sigma}\boldsymbol{t})$ and next use Proposition 1.2. □

Let $\boldsymbol{X}$ be the *standard Gaussian vector* $\mathcal{N}_d(\mathbf{0}, \boldsymbol{I})$, where $\mathbf{0} = (0, \dots, 0)^T \in \mathbb{R}^d$ and $\boldsymbol{I} = \mathrm{diag}(1, \dots, 1)^T$ is the unit $d \times d$–matrix. Note that it has the density function

$$\phi_{\mathbf{0},\boldsymbol{I}}(\boldsymbol{x}) = (2\pi)^{-d/2} e^{-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{x}}\ .$$

We will utilize matrix notations. For example $\mathbb{E}\,\boldsymbol{X}\boldsymbol{X}^T$ is a covariance matrix of $\boldsymbol{X}$.

For a $d \times d$ matrix $\boldsymbol{L}$ and a matrix $\boldsymbol{a}$ define $\boldsymbol{Y}$ by

$$\boldsymbol{Y} = \boldsymbol{L}^T\boldsymbol{X} + \boldsymbol{a}\ .$$

Proposition 2.3 *For $\boldsymbol{X} \sim \mathcal{N}_d(\mathbf{0}, \boldsymbol{I})$,*

$$\boldsymbol{Y} \sim \mathcal{N}_d(\boldsymbol{a}, \mathbf{\Sigma})$$

where $\mathbf{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T$.

Proof Since $\boldsymbol{Y}^T\boldsymbol{t} = (\boldsymbol{l}\boldsymbol{X})^T\boldsymbol{t} + \boldsymbol{a}^T\boldsymbol{t} = \boldsymbol{X}^T(\boldsymbol{L}\boldsymbol{t}) + \boldsymbol{a}^T\boldsymbol{t}$, we have that $\boldsymbol{Y}$ is Gaussian. Clearly

$$\mathbb{E}\,\boldsymbol{Y} = \mathbb{E}\,(\boldsymbol{L}\boldsymbol{X} + \boldsymbol{a}) = \boldsymbol{L}\mathbb{E}\,\boldsymbol{L}\boldsymbol{Z} + \boldsymbol{a} = \boldsymbol{L}\mathbb{E}\,\boldsymbol{X} + \boldsymbol{a} = \boldsymbol{a}.$$

For the covariance we compute

$$\begin{aligned}\boldsymbol{\Sigma} = \mathbb{E}\,(\boldsymbol{Y}-\boldsymbol{a})(\boldsymbol{Y}-\boldsymbol{a})^{=}\mathbb{E}\,\boldsymbol{L}\boldsymbol{X}(\boldsymbol{L}\boldsymbol{X})^T &= \boldsymbol{L}\mathbb{E}\,\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{L} \\ &= \boldsymbol{L}\boldsymbol{I}\boldsymbol{L}^T = \boldsymbol{L}\boldsymbol{L}^T.\end{aligned}$$

□

Proposition 2.4 *Let $\boldsymbol{X} \sim \mathcal{N}_d(\boldsymbol{a}, \boldsymbol{\Sigma})$ and assume the covariance matrix $\boldsymbol{\Sigma}$ is nonsingular. Then there exists a density with respect λ_d of form*

$$\phi_{\boldsymbol{a},\boldsymbol{\Sigma}}(\boldsymbol{x}) = (2\pi)^{-d/2}|\boldsymbol{\Sigma}|^{-1/2}e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{a})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{a})}\,.$$

Proof

□

In the following proposition we give an algebraic proof of an inverse theorem to Proposition (2.4). The presented result is a special case of a more general theorem on inverse Fourier transform of integrable characteristic functions.

Proposition 2.5 *Let $\boldsymbol{\Sigma}$ be nonsingular. Then*

$$\phi_{\boldsymbol{a},\boldsymbol{\Sigma}}(\boldsymbol{x}) = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} e^{-i\boldsymbol{t}^T\boldsymbol{x}}e^{i\boldsymbol{a}^T\boldsymbol{t}-\frac{\boldsymbol{t}^T\boldsymbol{\Sigma}\boldsymbol{t}}{2}}\,\mathrm{d}\boldsymbol{t},$$

for all $\boldsymbol{x} \in \mathbb{R}^d$.

Proof For simplicity we assume $\boldsymbol{a} = \boldsymbol{0}$. Let $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T$. We have for $\boldsymbol{t}, \boldsymbol{x} \in \mathbb{R}$

$$(\boldsymbol{t}+i\boldsymbol{L}^{-1}\boldsymbol{x})^T(\boldsymbol{t}+i\boldsymbol{x}\boldsymbol{L}^{-1}) = \boldsymbol{t}^T\boldsymbol{t} + i(\boldsymbol{t}^T\boldsymbol{L}^{-1}\boldsymbol{x} + \boldsymbol{x}^T(\boldsymbol{L}^{-1})^T\boldsymbol{t} - \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}.$$

Hence

$$\begin{aligned}&\exp\left(-\frac{\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}}{2}\right)\int_{\mathbf{R}^d}\exp\left(-\frac{1}{2}(\boldsymbol{t}+i\boldsymbol{L}^{-1}\boldsymbol{x})^T(\boldsymbol{t}+i\boldsymbol{x}\boldsymbol{L}^{-1})\right)\,\mathrm{d}\boldsymbol{t} \\ =&\int_{\mathbf{R}^d}\exp\left(-\frac{1}{2}(\boldsymbol{t}^T\boldsymbol{t}+i(\boldsymbol{t}^T\boldsymbol{L}^{-1}\boldsymbol{x}+\boldsymbol{x}^T(\boldsymbol{L}^{-1})^T\boldsymbol{t}))\right)\,\mathrm{d}\boldsymbol{t} \\ =&\ |\boldsymbol{\Sigma}|^{1/2}\int_{\mathbf{R}^d}\exp\left(-\frac{1}{2}(\boldsymbol{s}\boldsymbol{\Sigma}\boldsymbol{s}^T+2i\boldsymbol{s}^T\boldsymbol{x})\right)\,\mathrm{d}\boldsymbol{s},\end{aligned}$$

where in the last equation we used substitution $\boldsymbol{t} = \boldsymbol{L}^T\boldsymbol{s}$. Now

$$\int_{\mathbf{R}^d}\exp\left(-\frac{1}{2}(\boldsymbol{t}+i\boldsymbol{L}^{-1}\boldsymbol{x})^T(\boldsymbol{t}+i\boldsymbol{x}\boldsymbol{L}^{-1})\right)\,\mathrm{d}\boldsymbol{t} = \int_{\mathbf{R}^d}\exp\left(-\frac{\boldsymbol{t}^T\boldsymbol{t}}{2}\right)\,\mathrm{d}\boldsymbol{t} = (2\pi)^d/2.$$

Hence the result follows. □

2.1 Simulation of Gaussian vectors

The aim is to express Gaussian vectors through a sequence of independent random variables $U_1, \ldots, U_d$ having the same uniform distribution $\mathcal{U}[0,1]$, called *random numbers.* In principle such uniformly distributed random variables can be generated on computers, however on computers we can generate only pseudorandom numbers behaving similarly to random numbers. In the simplest case of a sequence of i.i.d standard Gaussian random variables we can use $\Phi^{-1}(U_1), \ldots, \Phi^{-1}(U_d)$, where $\Phi^{-1}(x)$ is the inverse function to $\Phi(x)$ but the method requires the explicit knowledge of the inverse function.

2.2 Box-Muller method

The following method, known as Box-Muller method, allows to generate a pair (X, X') of independent standard Gaussian random variables. We express them in terms of polar coordinates $X = R\cos\Theta$, $X' = R\sin\Theta$.

Proposition 2.6 *Random variables R and Θ are independent, Θ uniformly distributed $\mathcal{U}[0, 2\pi)$ and R having the Raleigh distribution with density $f(r) = r\exp\left[-r^2/2\right] \mathit{1}(r > 0)$.*

Proposition 2.7 [Box-Muller] *Let U, U' are independent and uniformly $\mathcal{U}[0,1]$ distributed. Random variables*

$$X = (-2\log U)^{1/2}\cos 2\pi U', \quad X' = (-2\log U)^{1/2}\sin 2\pi U'$$

are independent standard Gaussian variables.

2.3 Recursive and Cholevsky method

We now discuss how to simulate a sequence of $n+1$ Gaussian variables $X_0, \ldots, X_n$ or a Gaussian vector $\boldsymbol{X}_n = (X_0, \ldots, X_n)^T \sim \mathcal{N}_{n+1}(\mathbf{0}, \mathbf{\Sigma}_n)$. We will consider centered variables only. Let $\boldsymbol{X}_{n+1} = (X_1, X_2, \ldots, X_{n+1})$ be a Gaussian vector with distribution $\mathcal{N}_{n+2}(\mathbf{0}, \mathbf{\Sigma}_{n+1})$, where

$$\mathbf{\Sigma}_{n+1} = \begin{pmatrix} \mathbf{\Sigma}_n & \boldsymbol{\sigma}_{n+1}^T \\ \boldsymbol{\sigma}_{n+1} & \sigma_{n+1\ n+1} \end{pmatrix}, \tag{2.4}$$

where $\boldsymbol{\sigma}_{n+1} = (\sigma_{n+1\ 0}, \ldots, \sigma_{n+1\ n})$. It turns out that knowing a vector $\boldsymbol{X}_n \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_n)$ we cn generate $\boldsymbol{X}_{n+1}$ in view of the following proposition.

Proposition 2.8

$$(X_{n+1}|X_0 = x_0, \ldots, X_n = x_n) \sim \mathcal{N}(a(\boldsymbol{x}_n), s_n^2),$$

where

$$a_n(\boldsymbol{x}_n) = (\boldsymbol{\sigma}_{n+1})^T \boldsymbol{\Sigma}_n^{-1} \boldsymbol{x}, \qquad \boldsymbol{x}_n = (x_0, \ldots, x_n)$$

and

$$s_n^2 = \sigma_{n+1\ n+1} - (\boldsymbol{\sigma}_{n+1})^T \boldsymbol{\Sigma} \boldsymbol{\sigma}_{n+1}.$$

Thus we can simulate $X_0, \ldots$ recursively using recurrence from Proposition 2.8. The drawback of this method is that it is difficult to organize the calculations economically.

A remedy is the following simulation method, which we call *Cholevsky method*, although in principle the Cholevsky method is an algorithm for the following factorization of the matrix $\boldsymbol{\Sigma}_n$:

$$\boldsymbol{\Sigma}_n = L_n L_n^T,$$

where $L_n = (l_{jk})_{j,k=1,\ldots,n}$ is a *lower triangular matrix* (that is $l_{jk} = 0$ for all $j < k$). Notice that if L_n is known, and $Y_0, \ldots, Y_n$ is a sequence of i.i.d. standard Gaussian r.v.s, then $\boldsymbol{X} = \boldsymbol{L}\boldsymbol{Y}$ has the required distribution $\mathcal{N}_{n+1}(\boldsymbol{0}, \boldsymbol{\Sigma}_n)$. The Cholevsky factorization is given in the next proposition.

Proposition 2.9

$$l_{00}^2 = \sigma_{00} \tag{2.5}$$

$$\ldots = \ldots \tag{2.6}$$

$$l_{i+1\ j} = \frac{1}{l_{jj}(\sigma_{i+1\ j} - \sum_{k=0}^{j-1} l_{i+1\ k} l_{j\ k})}, \tag{2.7}$$

$$l_{i+1\ i+1} = \sigma_{i+1\ i+1} - \sum_{k=0}^{i} l_{i+1\ k}^2 \quad 0 \leq j \leq i \tag{2.8}$$

It is important to note that to compute factorization of $\boldsymbol{\Sigma}_{n+1}$ fulfilling (2.4), we have

$$\boldsymbol{L}_{n+1} = \begin{pmatrix} \boldsymbol{L}_n & \boldsymbol{l}_{n+1}^T \\ \boldsymbol{l}_{n+1} & l_{n+1\ n+1} \end{pmatrix}, \tag{2.9}$$

where $\boldsymbol{l}_{n+1} = (l_{n+1\ 0}, \ldots, l_{n+1\ n})^T$ and $l_{n+1\ n+1}$ are computed by (2.8).

In conclusion, simulation via Cholevsky factorization is exact (no approximation is involved) and one does not need to set the time horizon in advance. Moreover no matrix inversion is needed. The drawback of the method is that becomes slow and storage demanding as n becomes large.[2].

Exercises

[2]Cited from Asmussen (1999), p. 101

2.1 Let $\mathbf{\Sigma}$ be a covariance matrix. Show that

$$\mathbb{R}^d \ni \boldsymbol{t} \to \exp\left(-\frac{1}{2}\boldsymbol{t}^T\mathbf{\Sigma}\boldsymbol{t}\right)$$

is integrable if and only if $\det \mathbf{\Sigma} \neq 0$.

2.2 Let $X_1, \ldots, X_d$ has joint normal distribution such that there exists only one index k for which $\sigma_j^2 < \sigma_k^2$, for all $j \neq k$. Then for $t \to \infty$

$$\mathbb{P}(\max_j X_j > t) \sim \mathbb{P}(X_k > t) \sim \Psi\left(\frac{t - a_k}{\sigma_k}\right). \tag{2.10}$$

Hence

$$\lim_{t\to\infty} \frac{-1}{t} \log \mathbb{P}(\max_{1\le j\le n} X_j > t) = \frac{1}{2\sigma_k^2}.$$

Hint. $\mathbb{P}(X_k > t) \le \mathbb{P}(\max_j X_j > t) \le \sum_{j=1}^d \mathbb{P}(X_j > t)$, for all $t \ge 0$.[3]

2.3 Let $U_1, \ldots, U_n$ are independent uniformly distributed $\mathcal{U}[0,1]$ random variables. Show that $\Phi^{-1}(U_1), \ldots, \Phi^{-1}(U_n)$ are independent standard Gaussian random variables.

2.4 Complex variables

Let $(X, X') \sim \mathcal{N}_2(\boldsymbol{a}, \mathbf{\Sigma})$. Random variable of form

$$Z = X + iX'$$

is called *complex Gaussian*. The expected value or mean of Z is

$$\mathbb{E}\, Z = \mathbb{E}\, X + i\mathbb{E}\, X' = a + ia' = m.$$

Notice that the knowledge of $\boldsymbol{a}$ and

$$\mathbb{E}\,(Z - m)(\overline{Z - m}) = \mathbb{E}\,(X - a)^2 + \mathbb{E}\,(X' - a')^2 = \sigma_1^2 + \sigma_2^2$$

is not sufficient to determine the distribution of Z.

Let now $X_1, \ldots, X_n, X_1', \ldots, X_n'$ be Gaussian and set $Z_j = X_j + iX_j'$. Then $(Z_1, \ldots, Z_n)$ is a complex Gaussian random vector.

Exercises

2.1 Let X and X' be independent and set $Z = X + iX'$. Show that $\mathbb{E}\, Z\overline{Z}$ and $\mathbb{E}\, ZZ$ determine the distribution of Z.

[3] A co sie dzieje gdy maksimum jest osiagniete dla 2,3, itd sigm?

3 Gaussian processes

Let $\{X(t),\ t \in T\}$ be a real valued stochastic process (random function) with a general *space of parameters* T. In particular we are interested in $T = \mathbb{R}, \mathbb{R}_+, [0,1]$ or $T = \mathbb{Z}, \mathbb{Z}_+$, however we will need in some places the space T to be a family of subsets. For example in the next chapter T can be a sigma field of subsests. By a *finite dimensional distribution* (fi–di) [4] of $\{X(t)\}$ we mean

$$\mu_{(t_1,\ldots,t_n)}(B) = \mathbb{P}((X(t_1),\ldots,(t_n)) \in B), \qquad B \in \mathcal{B}(\mathbb{R}^n).$$

Let $\mathcal{L}$ be the family of all finite sequences of nonrepeating elements from T; that is $\mathcal{L} = \bigcup_{j=1}^{\infty} V_j(T)$, where $V_j(T)$ denotes the set if one-to-one functions from $1,\ldots,n$ to T. The family of finite dimensional (fi–di) distributions of the stochastic process $\{X(t)\}$ is $\{\mu_L,\ L \in \mathcal{L}\}$. Let L be a sequence (not necessarily finite) and $L^{'}$ a subsequence of L. By $\pi_{L,L'} : \mathbb{R}^L \to \mathbb{R}^{L^{'}}$ we denote the restricting a function in $\mathbb{R}^L$ to the domain L'. The family of fi-di distributions of $\{X(t)\}$ has the following consistency property:

$$\mu_{L'} = \mu_L \circ (\pi_{L,L'})^{-1}. \tag{3.11}$$

In the following theorem we consider an inverse problem. Instead from starting from a stochastic process $\{X(t)\}$, which defines the corresponding family of fi-di distributions, we begin now with a family of distributions $\{\mu_L,\ L \in \mathcal{L}\}$, where μ_L is a distribution on $\mathbb{R}^L$ and fulfilling (3.11). We say then that there is given a consistent family of fi-di distributions.

Theorem 3.1 [Daniell-Kolmogorov] *For a given consistent family of fi-di distributions $\{\mu_L\}$, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stochastic process $\{X(t),\ t \in T\}$ having $\{\mu_L\}$ as its family of fi-di distributions.*

Typically nothing can be said about regular properties of the process $\{X(t)\}$ from Theorem 3.1. In fact, processes from the proof of Daniell–Kolmogorov theorem have not required properties of their realizations. One of approaches to define processes with required properties of realizations (like for example with continuous realizations) is proving the existence of a good modification. Thus we say that a stochastic process $\{X^*(t),\ t \in T\}$ is a *modification* of $\{X(t),\ t \in T\}$ if $\mathbb{P}(X^*(t) = X(t)) = 1$ for each $t \in T$. Moreover, for a topological space of parameters T, we say that the modification is *continuous* if for almost all (with respect to $\mathbb{P}$) $\omega \in \Omega$, the function $T \ni t \to X^*_\omega(t)$ is continuous.

[4] ???

Theorem 3.2 [Kolmogorov criterion] *Let $T = \mathbb{R}^d$ or $[0,1]^d$. If for some $\alpha, C > 0$*

$$\mathbb{E}\,|X(t) - X(s)|^\alpha \le C|t-s|^{d+\epsilon}, \qquad s,t \in T$$

then there exists its continuous modification.

Definition 3.3 We say that a process $\{X(t),\ t \in T\}$ is *Gaussian* if its all fi-di distribution μ_L are normal. Functions

$$\begin{aligned} a(t) &= \mathbb{E}\,X(t), \qquad t \in T \\ R(s,t) &= \mathrm{Cov}\,(X(s), X(t)), \qquad s,t \in T \end{aligned}$$

are said to be the *mean function* and *covariance function* respectively.

It is clear that for $L = (t_1, \dots, t_n)$, the distribution μ_L of $(X(t_1), \dots, X(t_n))$ is jointly normal with mean $(a(t_1), \dots, a(t_n))$ and covariance matrix $(R(t_j, t_k)_{j,k=1,\dots,n}$. In Section **[????]** we prove that each covariance function is a positive definite function. In contrast to covariance functions any function (not even measurable) can be the mean function. By Daniell–Kolmogorov theorem functions $a(\cdot), R(\cdot,\cdot)$ uniquely determine the family of fi-di distributions of a Gaussian process $\{X(t),\ t \in T\}$. Therefore we will sometimes say that $\{X(t)\}$ is $a(\cdot), R(\cdot,\cdot)$–Gaussian process and denote it by $\{X(t)\} \sim \mathcal{N}(a(\cdot), R(\cdot,\cdot))$.

Comments. Johnson and Kotz (1972), Lifshits (1995), Tong (1990)

Chapter II

Order relations between Gaussian vectors

Let $\mathbb{F}$ be a class of functions $f : \mathbb{R}^d \to \mathbb{R}$. We will consider only integral relations between two distributions on $\mathbb{R}^d$ defined as follows. We say that distributions μ and μ' are related with respect to $\le_{\mathbf{F}}$ and write $\mu \le_{\mathbf{F}} \mu'$ if

$$\int_{\mathbb{R}^d} f(x)\,\mu(dx) \le \int_{\mathbb{R}^d} f(x)\,\mu'(dx)$$

for all $f \in \mathbb{F}$ for which the integrals exist and are finite. We also write for two random vectors $\boldsymbol{X}$ and $\boldsymbol{X}'$ with distributions μ, μ' respectively that $\boldsymbol{X} \le_{\mathbf{F}} \boldsymbol{X}'$.

1 Two basic lemmas

In the following lemma $f : \mathbb{R}^d \to \mathbb{R}$ is bounded with bounded and continuous second derivatives.[1]

Lemma 1.1 *Let $\boldsymbol{X}$ and $\boldsymbol{X}'$ be Gaussian vectors with mean and covariance matrices $\boldsymbol{a}$ and $\boldsymbol{a}'$ and $\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}'$ respectively and moreover we assume that the covariance matrices are nonsingular. For $0 \le v \le 1$ we define a Gaussian vector X_v with density $\phi_v(\boldsymbol{x})$ having the mean and covariance matrix $v\boldsymbol{a} + (1-v)\boldsymbol{a}'$ and $v\boldsymbol{\Sigma} + (1-v)\boldsymbol{\Sigma}'$ respectively, where $0 \le v \le 1$. Furthermore we assume that $f : \mathbb{R}^d \to \mathbb{R}$ is bounded with bounded and continuous second*

[1] Tozsamosc pojawie sie Mueller and Scarsini; info od R. Kulik

derivatives. Then

$$
\begin{aligned}
\mathbb{E}\, f(\boldsymbol{X}') - \mathbb{E}\, f(\boldsymbol{X}) &= \\
&= \int_0^1 [\sum_{j=1}^{d} (a_j' - a_j) \int_{\mathbf{R}^d} \phi_v(\boldsymbol{x}) \frac{\partial}{\partial x_j} f(\boldsymbol{x}) \ \mathrm{d}\boldsymbol{x} \\
&+ \sum_{j,k=1}^{d} \frac{(\sigma_{jk}' - \sigma_{jk})}{2} \int_{\mathbf{R}^d} \phi_v(\boldsymbol{x}) \frac{\partial^2}{\partial x_j \partial x_k} f(\boldsymbol{x})\, d\boldsymbol{x}] \ \mathrm{d}v \qquad (1.1)
\end{aligned}
$$

Proof Let

$$g(v) = \int_{\mathbf{R}} f(\boldsymbol{x}) \phi_v(\boldsymbol{x})\, d\boldsymbol{x}.$$

Since

$$\int_0^1 \frac{\partial}{\partial v} g(v) \ \mathrm{d}v = g(1) - g(0) = \mathbb{E}\, f(\boldsymbol{X}) - \mathbb{E}\, f(\boldsymbol{X}'),$$

we need to compute

$$\frac{\partial}{\partial v} g(\lambda) = \int_{\mathbf{R}^n} f(\boldsymbol{x}) \frac{\partial}{\partial v} \phi_v(\boldsymbol{x}) \ \mathrm{d}\boldsymbol{x}.$$

We have from Proposition I.2.2

$$\widehat{\phi}_v(\boldsymbol{t}) = e^{\boldsymbol{t}^T (v\boldsymbol{a} - (1-v)\boldsymbol{a}') - \frac{\boldsymbol{t}^T (\boldsymbol{\Sigma} - \boldsymbol{\Sigma}')\boldsymbol{t}}{2}}.$$

Hence

$$
\begin{aligned}
\frac{\partial}{\partial v} \phi_v(\boldsymbol{t}) &= \frac{1}{2\pi} \int_{\mathbf{R}^d} (i\boldsymbol{t}^T (\boldsymbol{a} - \boldsymbol{a}') - \frac{\boldsymbol{t}^T (\boldsymbol{\Sigma} - \boldsymbol{\Sigma}')\boldsymbol{t}}{2}) \widehat{\phi}_v(\boldsymbol{t}) \ \mathrm{d}\boldsymbol{t} \\
&= i \frac{1}{2\pi} \int_{\mathbf{R}^d} (\boldsymbol{t}^T (\boldsymbol{a} - \boldsymbol{a}') \widehat{\phi}_v(\boldsymbol{t}) \ \mathrm{d}\boldsymbol{t} - \frac{1}{2\pi} \int_{\mathbf{R}^d} \frac{\boldsymbol{t}^T (\boldsymbol{\Sigma} - \boldsymbol{\Sigma}')\boldsymbol{t}}{2}) \widehat{\phi}_v(\boldsymbol{t}) \ \mathrm{d}\boldsymbol{t}.
\end{aligned}
$$

We now have

$$
\begin{aligned}
\frac{i}{(2\pi)^d} \int_{\mathbf{R}^d} t_j e^{-i\boldsymbol{t}^T \boldsymbol{x}} \widehat{\phi}_v(\boldsymbol{t}) \ \mathrm{d}\boldsymbol{t} &= \frac{\partial}{\partial x_j} \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-i\boldsymbol{t}^T \boldsymbol{x}} \widehat{\phi}_v(\boldsymbol{t}) \ \mathrm{d}\boldsymbol{t}, \\
&= i \frac{\partial}{\partial x_j} \phi_v(\boldsymbol{x}), \qquad (1.2) \\
\frac{-1}{(2\pi)^d} \int_{\mathbf{R}^d} t_j t_k e^{-i\boldsymbol{t}^T \boldsymbol{x}} \widehat{\phi}_v(\boldsymbol{t}) \ \mathrm{d}\boldsymbol{t} &= \frac{\partial}{\partial x_j \partial x_k} \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-i\boldsymbol{t}^T \boldsymbol{x}} \widehat{\phi}_v(\boldsymbol{t}) \ \mathrm{d}\boldsymbol{t} \\
&= \frac{\partial}{\partial x_j \partial x_k} \phi_v(\boldsymbol{x}). \qquad (1.3)
\end{aligned}
$$

Therefore

$$\frac{\partial}{\partial v}\phi_v(\boldsymbol{x}) = \sum_{j=1}^{d}(a_j - a_j^{'})\frac{\partial}{\partial x_j}\phi_v(\boldsymbol{x}) + \frac{1}{2}\sum_{j,k}^{d}(\sigma_{jk}^{'} - \sigma_{jk})\frac{\partial^2}{\partial x_j \partial x_k}\phi_v(\boldsymbol{x}). \quad (1.4)$$

Integrating by parts

$$\int_{\mathbf{R}^d} f(\boldsymbol{x})\frac{\partial \phi_v(\boldsymbol{x})}{\partial x_j}\,\mathrm{d}\boldsymbol{x} = -\int_{\mathbf{R}^d} \phi_v(\boldsymbol{x})\frac{\partial f(\boldsymbol{x})}{\partial x_j}\,\mathrm{d}\boldsymbol{x}, \quad (1.5)$$

$$\int_{\mathbf{R}^d} f(\boldsymbol{x})\frac{\partial^2 \phi_v(\boldsymbol{x})}{\partial x_j \partial x_j}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbf{R}^d} \phi_v(\boldsymbol{x})\frac{\partial^2 f(\boldsymbol{x})}{\partial x_j \partial x_k}\,\mathrm{d}\boldsymbol{x} \quad (1.6)$$

Hence by (1.4), (1.5) and (1.6)

$$\begin{aligned}\frac{\partial g(v)}{\partial v} = \int_{\boldsymbol{R}^d} f(\boldsymbol{x})\frac{\partial \phi_v(\boldsymbol{x})}{\partial v}\,\mathrm{d}\boldsymbol{x} &= \sum_{j=1}^{d}(a_j' - a_j)\int_{\mathbf{R}^d}\phi_v(\boldsymbol{x})\frac{\partial f(\boldsymbol{x})}{\partial x_j}\,\mathrm{d}\boldsymbol{x} + \\ &+ \sum_{j,k=1}^{d}\frac{\sigma_{jk}^{'} - \sigma_{jk}}{2}\int_{\mathbf{R}^d}\phi_v(\boldsymbol{x})\frac{\partial^2 f(\boldsymbol{x})}{\partial x_j \partial x_k}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

□

Remark The result of Lemma 1.1 can be written using the following notations. For the *gradient* we write

$$\nabla f(\boldsymbol{x}) = \left(\frac{\partial}{\partial x_1}f(\boldsymbol{x}), \ldots, \frac{\partial}{\partial x_d}f(\boldsymbol{x})\right)^T,$$

and for the*Hesse matrix*

$$(\boldsymbol{H}f)(\boldsymbol{x}) = \left(\frac{\partial^2}{\partial x_j \partial x_k}f(\boldsymbol{x})\right)_{i,j=1,\ldots,d}.$$

Furthermore for a $d \times d$–matrix

$$\mathrm{tr}(\boldsymbol{A}) = \sum_{j=1}^{d} a_{jj};$$

for two $d \times d$–matrices $\boldsymbol{A}, \boldsymbol{B}$ we have

$$\mathrm{tr}(\boldsymbol{AB}) = \sum_{j,k=1}^{d} a_{jk}b_{jk}.$$

Then

$$\begin{aligned}
\mathbb{E} f(\boldsymbol{X}') - \mathbb{E} f(\boldsymbol{X}) &= \\
&= \int_0^1 \int_{\mathbf{R}^d} [(\boldsymbol{a}' - \boldsymbol{a})^T (\nabla f)(\boldsymbol{x}) + \frac{1}{2}\mathrm{tr}[(\boldsymbol{\Sigma}' - \boldsymbol{\Sigma})(\boldsymbol{H} f)(\boldsymbol{x})] \phi_v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbf{R}^d} [(\boldsymbol{a}' - \boldsymbol{a})^T (\nabla f)(\boldsymbol{x}) + \frac{1}{2}\mathrm{tr}[(\boldsymbol{\Sigma}' - \boldsymbol{\Sigma})(\boldsymbol{H} f)(\boldsymbol{x})] \, \pi(\mathrm{d}\boldsymbol{x}), \qquad (1.7)
\end{aligned}$$

where $\pi(\mathrm{d}\boldsymbol{x}) = \int_0^1 \phi_v(\boldsymbol{x}) \, \mathrm{d}v \, \mathrm{d}\boldsymbol{x}$.

Let $\boldsymbol{X}$ and $\boldsymbol{X}'$ are two independent copies of the standard Gaussian vectors in $\mathbb{R}^d$ and

$$Y_v = v\boldsymbol{X} + \sqrt{1 - v^2}\boldsymbol{X}', \quad 0le \leq 1.$$

note that

$$\begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y}_v \end{pmatrix} \sim \mathcal{N}_{2d}\left(\boldsymbol{0}, \begin{pmatrix} \boldsymbol{I} & v\boldsymbol{I} \\ v\boldsymbol{I} & \boldsymbol{I} \end{pmatrix}\right),$$

where $\boldsymbol{I}$ is $d \times d$ identity matrix.

Lemma 1.2 *Let $f, g \in \mathcal{C}^1(\boldsymbol{R}^d, \boldsymbol{R})$ such that the functions with first partial derivatives are bounded by $A\exp(B|x|)$. Then*

$$\mathrm{Cov}\,(f(\boldsymbol{X}), g(\boldsymbol{X})) = \int_0^1 \mathbb{E}\,(\nabla f(\boldsymbol{X})^T (\nabla g(\boldsymbol{Y}_v) \, \mathrm{d}v.$$

Proof

□

Corollary 1.3 *Let $f, g \in \mathcal{C}^1(\boldsymbol{R}^d, \boldsymbol{R})$ such that the functions with first partal derivatives are bounded by $A\exp(B|x|)$. Then for $X \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma})$*

$$\mathrm{Cov}\,(f(\boldsymbol{X}), g(\boldsymbol{X})) = \int_0^1 \mathbb{E}\,(\nabla f(\boldsymbol{X})^T \boldsymbol{\Sigma} (\nabla g(\boldsymbol{Y}_v) \, \mathrm{d}v.$$

Remark Denote now μ_v the joint distribution of $\boldsymbol{X}, \boldsymbol{Y}_v$ (that is a $2d$-dimensional Gaussian distribution) and by $\pi(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{y}) = \int_0^1 \mu(\mathrm{d}\boldsymbol{x}, d\boldsymbol{y}) \, \mathrm{d}v$. Then from Lemma 1.2

$$\mathrm{Cov}\,(f(\boldsymbol{X}), g(\boldsymbol{X})) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} < (\nabla f)(\boldsymbol{x}), (\nabla g)(\boldsymbol{y}) > \pi(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{y}).$$

Exercises

1.1 Let $\boldsymbol{X}$ and $\boldsymbol{X}'$ be two independent Gaussian vectors with distribution $\mathcal{N}_d(\mathbf{0},\boldsymbol{\Sigma})$ and $\mathcal{N}_d(\mathbf{0},\boldsymbol{\Sigma}')$ respectively. Let $0 \le v \le 1$ Show that $v^{1/2}\boldsymbol{X} + (1-v)^{1/2}\boldsymbol{X}'$ has the distribution $\mathcal{N}_d(\mathbf{0}, v\boldsymbol{\Sigma} + (1-v)\boldsymbol{\Sigma}')$.

Comments. Mueller and Stoyan (2002), Bobkov et al (2001)

2 Stochastic order

For two vectors $\boldsymbol{X}$ and $\boldsymbol{X}'$ with distributions μ and μ' respectively, we say that they are *stochastically ordered* and write $\boldsymbol{X} \le_{\rm st} \boldsymbol{X}'$ or $\mu \le_{\rm st} \mu'$ if

$$\int_{\mathbf{R}^d} f(x)\,\mu(dx) \le \int_{\mathbf{R}^d} f(x)\,\mu'(dx) \tag{2.8}$$

for all nondecreasing functions such that the integrals exists. In the following result we need a notion of an *upper set* A, that is a set from $\mathcal{B}(\mathbb{R}^d)$ such that if $\boldsymbol{x} \in A$, then for all $\boldsymbol{y} \ge \boldsymbol{x}$ we have $\boldsymbol{y} \in A$.

Proposition 2.1 *The following sentences are equivalent.*
(i) $\boldsymbol{X} \le_{\rm st} \boldsymbol{X}'$.
(ii) *For all bounded nondecreasing functions f relations (2.8) holds.*
(iii) *For all bounded and continuous nondecreasing functions f relations (2.8) holds.*
(iv) *For all upper sets U*

$$\mathbb{P}(\boldsymbol{X} \in U) \le \mathbb{P}(\boldsymbol{X}' \in U).$$

Proof (i)→(ii) obvious.
(ii)→(iii) obvious.
(iii)→(iv). Let $d(x,A) = \inf_{x\in A}|x-A|$ be the distance from x to A. For each A it is a continuous function of x[2] For an upper set U define

$$f_n(x) = (1 - nd(x,U))_+.$$

This function is bounded (obvious), nondecreasing (because U is an upper set) and continuous and moreover $\lim_{n\to\infty} f_n(x) \downarrow 1(x \in U)$ for all $x \in \mathbb{R}^d$. Since $\int f_n \,\mathrm{d}\mu \le \int f_n \,\mathrm{d}\mu'$, passing with $n \to \infty$ we obtain (iv).
(iv)→(ii) Let f be bounded and nondecreasing and $\gamma = \inf f(x)$. Then

$$U_\alpha = \{x \in \mathbb{R}^d : f(x) \ge \alpha\}$$

[2] REf?

is an upper set. Let

$$f_n(x) = \gamma + \sum_{k=0}^{\infty} \frac{1}{n} 1(x \in U_{\gamma+\frac{k}{n}}).$$

Since

$$f(x) - \frac{1}{n} \le f_n(x) \le f(x) + \frac{1}{n},$$

we have

$$\int_{\mathbf{R}^d} f \, d\mu - \frac{1}{n} \le \int_{\mathbf{R}^d} f_n \, \mathrm{d}\mu \le \int_{\mathbf{R}^d} f_n \, d\mu' \le \int_{\mathbf{R}^d} f \, \mathrm{d}\mu + \frac{1}{n}.$$

Hence (ii) follows.
(ii)$\rightarrow$(i). Let f be nondecreasing such that $\int f \, \mathrm{d}\mu$ and $\int f \, \mathrm{d}\mu'$ are finite. Let for $M < N$

$$f_M^N(x) = \begin{cases} M & \text{for } x < M \\ f(x) & \text{for} M \le x \le N \\ N & \text{for} x > N \end{cases}$$

Since f_M^N is bounded and nondecreasing $\int f_M^N \, \mathrm{d}\mu \le \int f_M^N \, \mathrm{d}\mu'$. Now passing first with $N \to \infty$ and with $M \to -\infty$ we obtain (i). □

Lemma 2.2 *Let f be a nondecreasing and bounded continuous function. Then there exists a sequence f_n of functions from $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ with all derivatives bounded.*

Proof For a nondecreasing bounded continuous function define

$$f_n(\boldsymbol{x}) = \int_{-\infty}^{\infty} f(\boldsymbol{x} - \boldsymbol{y}) \phi_{(\mathbf{0}, \frac{1}{n}\boldsymbol{I})}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \tag{2.9}$$

$$= \int_{-\infty}^{\infty} \phi_{(\mathbf{0}, \frac{1}{n}\boldsymbol{I})}(\boldsymbol{y}) f(\boldsymbol{x} - \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \tag{2.10}$$

From (2.9) we conclude that f_n is nondecreasing and from (2.10) we conclude that f_n is $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ with all derivatives bounded. Moreover $f_n(x) \to f(x)$ for all $x \in \mathbb{R}^d$. □

Corollary 2.3 $\boldsymbol{X} \le_{\mathrm{st}} \boldsymbol{X}'$ *is equivalent to*

$$\mathbb{E} f(\boldsymbol{X}) \le \mathbb{E} f(\boldsymbol{X}')$$

for all nondecreasing and bounded functions from $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ with bounded derivatives.

Proof We have to use Proposition 2.1 (iii) and Lemma 2.2. □

In the next lemma we characterize the stochastic ordering of Gaussian vectors.

Theorem 2.4 *Let* $\boldsymbol{X} \sim \mathcal{N}_d(\boldsymbol{a}, \boldsymbol{\Sigma})$ *and* $\boldsymbol{X}' \sim \mathcal{N}_d(\boldsymbol{a}', \boldsymbol{\Sigma}')$. *Then* $\boldsymbol{X} \leq_{\mathrm{st}} \boldsymbol{X}'$ *if and only if* $\boldsymbol{a} \leq \boldsymbol{a}'$ *and* $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}'$.

Proof Müller and Stoyan p.96. The sufficient condition follows from Lemma 1.1. Conversely □

Exercises

2.1 Show the equivalence of the following sentences for random variables X and X':
(i) $X \leq_{\mathrm{st}} X'$.
(ii) $\mathbb{P}(X > t) \leq \mathbb{P}(X' > t)$ for all $t \in \mathbb{R}$.
(iii) $\mathbb{P}(X \leq t) \geq \mathbb{P}(X' \leq t)$ for all $t \in \mathbb{R}$.

Comments. Mueller and Stoyan (2002)

3 Upper and lower orthant orders

Conditions (ii) and (iii) from Exercise 2.1 can be expressed for random vectors as follows.
(ii$_n$) $\mathbb{P}(\boldsymbol{X} > \boldsymbol{t}) \leq \mathbb{P}(\boldsymbol{X}' > \boldsymbol{t})$ for all $\boldsymbol{t} \in \mathbb{R}^d$. (iii$_n$) $\mathbb{P}(\boldsymbol{X} \leq \boldsymbol{t}) \geq \mathbb{P}(\boldsymbol{X}' \leq \boldsymbol{t})$ for all $\boldsymbol{t} \in \mathbb{R}^d$.

Unfortunately for vectors of dimension greater or equal 2 conditions (ii$_n$) (iii$_n$) are not equivalent. Therefore we introduce the following definitions of the *upper orthant order* that is $\boldsymbol{X} \leq_{\mathrm{uo}} \boldsymbol{X}'$ if condition (ii$_n$) hold. Similarly we define the *lower orthant* that is $\boldsymbol{X} \leq_{\mathrm{lo}} \boldsymbol{X}'$ if condition (iii$_n$) hold. Recall that $\boldsymbol{X} = (X_1, \ldots, X_d)$ and $\boldsymbol{X}' = (X_1^{'}, \ldots, X_d^{'})$

Proposition 3.1 *The following conditions are equivalent.*
(i) $\boldsymbol{X} \leq_{\mathrm{uo}} \boldsymbol{X}'$.
(ii) *We have*

$$\mathbb{E} \prod_{j=1}^{d} f_j(X_j) \leq \mathbb{E} \prod_{j=1}^{d} f_j(X_j^{'}) \tag{3.11}$$

for all nonnegative nondecreasing and bounded function $f_j : \mathbb{R} \to \mathbb{R}$.
(iii) *We have (3.11) for all nonnegative nondecreasing, bounded and continuous function* $f_j : \mathbb{R} \to \mathbb{R}$.

Proof (i)→(ii). Let f_j $(j = 1, \dots, d)$ fulfill condition of (ii) and define upper set

$$U^j_\alpha = \{x \in \mathbb{R} : f_j(x) \geq \alpha\}$$

and $\gamma_j = \inf f_j(x)$. Define

$$f_{jm}(x_j) = \gamma_j + \sum_{k=1}^{\infty} \frac{1}{m} 1(x_j \in U^k_{\gamma_j + \frac{k}{m}}).$$

From (i)

$$\mathbb{E} \prod_{j=1}^{d} f_{jm}(X_j) \leq \mathbb{E} \prod_{j=1}^{d} f_{jm}(X_j^{'}).$$

Since

$$\prod_{j=1}^{d} f_j(x_j) \leq \prod_{j=1}^{d} f_{jm}(x_j) \leq \prod_{j=1}^{d} f_j(x_j) + \frac{\text{const}}{m}$$

hence (ii) follows.
(ii)→(i) Obvious.
The implication (ii)→(iii) is obvious. The converse can be proved similarly like in the proof of (iii)→(iv) in Proposition 2.1. □

Corollary 3.2 *$\boldsymbol{X} \leq_{\text{uo}} \boldsymbol{X}'$ is equivalent to*

$$\mathbb{E} \prod_{j=1}^{d} f_j(X_j) \leq \mathbb{E} \prod_{j=1}^{d} f_j(X_j^{'})$$

for all nonnegative nondecreasing function $f_j \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. with bounded derivaties.

Proof Similar to the proof of the Corollary 2.3.

Parallel results exist for the lower orthant $\leq_{\text{lo}}$ with

$$\mathbb{E} \prod_{j=1}^{d} f_j(X_j) \geq \mathbb{E} \prod_{j=1}^{d} f_j(X_j^{'}) \tag{3.12}$$

for all nonnegative, nonincreasing and bounded fuctions $f_j : \mathbb{R} \to \mathbb{R}$.

In the next result we compare two Gaussian vectors.

Theorem 3.3 *Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{a}, \boldsymbol{\Sigma})$ and $\boldsymbol{X}' \sim \mathcal{N}(\boldsymbol{a}', \boldsymbol{\Sigma}')$.*
(i) *We have $\boldsymbol{X} \leq_{\text{uo}} \boldsymbol{X}'$ if and only if $a_j \leq a_j^{'}$, $\sigma_{jj} = \sigma_{jj}^{'}$ $j = 1, \dots, d$ and $\sigma_{jk} \leq \sigma_{jk}^{'}$ for $j \neq k$.*
(ii) *We have $\boldsymbol{X} \geq_{\text{lo}} \boldsymbol{X}'$ if and only if $a_j \geq a_j^{'}$, $\sigma_{jj} = \sigma_{jj}^{'}$ $j = 1, \dots, d$ and $\sigma_{jk} \leq \sigma_{jk}^{'}$ for $j \neq k$.*

The next corollary is a version of Slepian theorem, an important tool for studying supremums of Gaussian processes.

Corollary 3.4
(i) *If* $a_j \le a_j^{'}$, $\sigma_{jj} = \sigma_{jj}^{'}$ $j = 1, \ldots, d$ *and* $\sigma_{jk} \le \sigma_{jk}^{'}$ *for* $j \ne k$, *then*

$$\min_{1\le j\le d} X_j \le_{\mathrm{st}} \min_{1\le j\le d} X_j^{'}.$$

(ii) *If* $a_j \ge a_j^{'}$, $\sigma_{jj} = \sigma_{jj}^{'}$ $j = 1, \ldots, d$ *and* $\sigma_{jk} \le \sigma_{jk}^{'}$ *for* $j \ne k$, *then*

$$\max_{1\le j\le d} X_j \le_{\mathrm{st}} \max_{1\le j\le d} X_j^{'}.$$

Proof $\mathbb{P}(\boldsymbol{X} > \boldsymbol{t}) \le \mathbb{P}(\boldsymbol{X}^{'} > \boldsymbol{t})$ for all $\boldsymbol{t} \in \mathbb{R}^d$ yields $\mathbb{P}(\min_{1\le j\le d} X_j > t) \le \mathbb{P}(\min_{1\le j\le d} X_j^{'} > t)$ for all $t \in \mathbb{R}$. Similarly $\mathbb{P}(\boldsymbol{X} \le \boldsymbol{t}) \le \mathbb{P}(\boldsymbol{X}^{'} \le \boldsymbol{t})$ for all $\boldsymbol{t} \in \mathbb{R}^d$ yields $\mathbb{P}(\min_{1\le j\le d} X_j \le t) \le \mathbb{P}(\min_{1\le j\le d} X_j^{'} > t)$ for all $t \in \mathbb{R}$. □

For a multivariate distribution we denote by $F_j(x_j) = F(\infty, \ldots, x_j, \ldots, \infty)$ the j–marginal distribution. Let

$$F^{+}(x_1, \ldots, x_d) = \min_{j=1,\ldots,d} F_j(x_j)$$

be the upper Fréchet distribution. If $F_1 = \ldots = F_j$ and $X \sim F_1$, then $(X, X, \ldots, X)$ has distribution F^{+}.

Proposition 3.5

$$F \ge_{\mathrm{lo}} F^{+}.$$

Comments. Mueller and Stoyan (2002), Slepian (1962), Tong (1990)

4 Convex, supermodular and directional convex orderings

Let

$$\Delta_j^v f(\boldsymbol{x}) = f(x_1, \ldots, x_j + v, \ldots, x_d) - f(x_1, \ldots, x_j, \ldots, x_d).$$

Definition 4.1 (i) We say that $f : \mathbb{R}^d \to \mathbb{R}$ is *supermodular* (sm) if for $v_j, v_k > 0$

$$\Delta_j^{v_j} \Delta_k^{v_k} f(\boldsymbol{x}) \ge 0 \quad j \ne k.$$

(ii) We say that $f : \mathbb{R}^d \to \mathbb{R}$ is *directionally convex* (dcx) if for $v_j, v_k > 0$

$$\Delta_j^{v_j} \Delta_k^{v_k} f(\boldsymbol{x}) \geq 0 \quad 1 \leq j, k \leq d.$$

If moreover an sm function is nondecreasing we write then ism, and if a dcx functions is nondecreasing we write idcx.

Definition 4.2
(i) We say that $\boldsymbol{X} \leq_{cx} \boldsymbol{X}'$ if

$$\mathbb{E}\, f(\boldsymbol{X}) \leq \mathbb{E}\, f(\boldsymbol{X}')$$

for all convex functions provided the integrals exist. We call such the ordering by *convex order* or *cx order*
(ii) We say that $\boldsymbol{X} \leq_{icx} \boldsymbol{X}'$ if

$$\mathbb{E}\, f(\boldsymbol{X}) \leq \mathbb{E}\, f(\boldsymbol{X}')$$

for all increasing and convex functions provided the integrals exist. We call such the ordering by *inceasing convex order* or *icx order*.
(iii) We say that $\boldsymbol{X} \leq_{sm} \boldsymbol{X}'$ if

$$\mathbb{E}\, f(\boldsymbol{X}) \leq \mathbb{E}\, f(\boldsymbol{X}')$$

for all sm functions provided the integrals exist. We call such the ordering by *supermodular order* or *sm order*.
(iv) We say that $\boldsymbol{X} \leq_{ism} \boldsymbol{X}'$ if

$$\mathbb{E}\, f(\boldsymbol{X}) \leq \mathbb{E}\, f(\boldsymbol{X}')$$

for all ism functions provided the integrals exist. We call such the ordering by *inceasing supermodular order* or *ism order*.
(v) We say that $\boldsymbol{X} \leq_{dcx} \boldsymbol{X}'$ if

$$\mathbb{E}\, f(\boldsymbol{X}) \leq \mathbb{E}\, f(\boldsymbol{X}')$$

for all dcx functions provided the integrals exist. We call such the ordering by *directionally convex order* or *dcx order*.
(vi) We say that $\boldsymbol{X} \leq_{idcx} \boldsymbol{X}'$ if

$$\mathbb{E}\, f(\boldsymbol{X}) \leq \mathbb{E}\, f(\boldsymbol{X}')$$

for all idcx functions provided the integrals exist. We call such the ordering by *increasing directionally convex order* or *idcx order*.

Proposition 4.3 *The following sentences are equivalent.*
(i) $\boldsymbol{X} \leq_{\mathrm{sm}} \boldsymbol{X}'$.
(ii) For all bounded and continuous sm functions $f : \mathbb{R}^d \to \mathbb{R}$

$$\mathbb{E}\, f(\boldsymbol{X}) \leq \mathbb{E}\, f(\boldsymbol{X}').$$

(ii) For all bounded sm functions $f \in \mathcal{C}^\infty(\mathbb{R}^d \to \mathbb{R})$

$$\mathbb{E}\, f(\boldsymbol{X}) \leq \mathbb{E}\, f(\boldsymbol{X}').$$

In the next result we compare two Gaussian vectors.

Theorem 4.4 *Let* $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{a}, \boldsymbol{\Sigma})$ *and* $\boldsymbol{X}' \sim \mathcal{N}(\boldsymbol{a}', \boldsymbol{\Sigma}')$.
(i) *We have* $\boldsymbol{X} \leq_{\mathrm{sm}} \boldsymbol{X}'$ *if and only if* $a_j = a_j^{'}$, $\sigma_{jj} = \sigma_{jj}^{'}$ $j = 1, \ldots, d$ *and* $\sigma_{jk} \leq \sigma_{jk}^{'}$ *for* $j \neq k$.
(ii) *We have* $\boldsymbol{X} \geq_{\mathrm{dcx}} \boldsymbol{X}'$ *if and only if* $a_j = a_j^{'}$, $\sigma_{jj} \leq \sigma_{jj}^{'}$ $j = 1, \ldots, d$ *and* $\sigma_{jk} \leq \sigma_{jk}^{'}$ *for* $j \neq k$.
(iii) *If* $a_j \leq a_j^{'}$, $\sigma_{jj} \leq \sigma_{jj}^{'}$ $j = 1, \ldots, d$ *and* $\sigma_{jk} \leq \sigma_{jk}^{'}$ *for* $j \neq k$, *then* $\boldsymbol{X} \leq_{\mathrm{icdx}} \boldsymbol{X}^{'}$. *Conversely* $\boldsymbol{X} \leq_{\mathrm{icdx}} \boldsymbol{X}^{'}$ *yields* $a_j \leq a_j^{'}$, *and* $\sigma_{jk} \leq \sigma_{jk}^{'}$ *for* $j \neq k$

Proof (i) $\rightarrow$. Immediate from the definition and simple properties of the sm ordering that marginals have to be equal and that covariances are ordered. ???. Immediate from formula (1.1).
(ii) $\rightarrow$. $\boldsymbol{X} \leq_{\mathrm{dcx}} \boldsymbol{X}'$ implies $\sigma_{jk} \leq \sigma_{jk}^{'}$ and because $X_j \leq_{\mathrm{cx}} X_j^{'}$ we have $a_j = a_j^{'}$ and $\sigma_{jj} \leq \sigma_{jj}^{'}$
???. Immediate from formula (1.1). □

Exercises

4.1 (a) If $f \in C^2(\mathbb{R}^d, \mathbb{R})$, and

$$\frac{\partial^2 f(\boldsymbol{x})}{\partial x_j \partial x_k} \geq 0, \quad j \neq k,$$

then f is sm.
(b) If $f \in C^2(\mathbb{R}^d, \mathbb{R})$, and

$$\frac{\partial^2 f(\boldsymbol{x})}{\partial x_j \partial x_k} \geq 0, \quad 1 \leq j, k \leq d,$$

then f is dc.

4.2 Show that $\min_{1 \leq j \leq d} x_j$ is ism.

Comments. Mueller and Stoyan (2002)

5 Associated Gaussian vectors

We say that a sequence of random variables $X_1, \ldots, X_d$ is *associated* if

$$\mathrm{Cov}\, f(X_1, \ldots, X_d), g(X_, \ldots, X_d)$$

for all nondecreasing functions $f, g : \mathbb{R}^d \to \mathbb{R}$ such that the covariance exists.

Theorem 5.1 [Pitt] *Let $\boldsymbol{X} \sim \mathcal{N}_d(\boldsymbol{a}, \boldsymbol{\Sigma})$ be a Gaussian vector. Then $X_1, \ldots, X_d$ is associated if and only if*

$$\sigma_{jk} \geq 0, \quad 1 \leq j, k \leq d.$$

Proof The sufficient part follows immediately from Corollary 1.3. For he necesary part use $f(\boldsymbol{x}) = x_j$ and $g(\boldsymbol{x}) = x_k$.

Proposition 5.2 *Let $\boldsymbol{X} \sim \mathcal{N}_d(\boldsymbol{a}, \boldsymbol{\Sigma})$ be associated and $\boldsymbol{X}^* = (X_1^*, \ldots, X_d)^T \sim \mathcal{N}_d(\boldsymbol{a}, \mathrm{diag}\{\sigma_{11}, \ldots, \sigma_{dd}\})$ (that is $\boldsymbol{X}^*$ consists of independent Gaussian variables and $X_j =_{\mathrm{d}} X_j^*$ for $j = 1, \ldots, d$). Then*

$$\max_{1 \leq j \leq d} X_j \leq_{\mathrm{st}} \max_{1 \leq j \leq d} X_j^*.$$

Proof Since $\boldsymbol{X}$ is associated, then $\boldsymbol{\Sigma}$ has all entries nonnegative. Let $\boldsymbol{\Sigma}^* = (\sigma_{jk}^*)$ be the covariance matrix of $\boldsymbol{X}^*$. Clearly, $\sigma_{jk}^* \leq \sigma_{jk}$, hence by Proposition 3.4 the result follows. □

Exercises

5.1 Let $\boldsymbol{X} \sim \mathcal{N}_d(\boldsymbol{a}, \boldsymbol{\Sigma})$ and $\det \boldsymbol{\Sigma} \neq 0$. If $\boldsymbol{\Sigma}^{-1}$ has all off-diagonal entries nonpositive, then $X_1, \ldots, X_d$ is associated.

Comments. Mueller and Stoyan (2002), Pitt (1982), Gutmann, S. (1978) Correlations of functions of normal variables. *J. Multivariate Anal.* **8**, 573–578. Tong (1990)

Chapter III

Covariance theory of Gaussian processes

1 Stationarity

Let $L = (t_1, \ldots, t_n)$ and $L + t = (t_1 + t, \ldots, t_n + t)$, where $T = \mathbb{R}, \mathbb{R}_+$ or $\mathbb{Z}, \mathbb{Z}_+$.

Definition 1.1 We say that $\{X(t),\ t \in T\}$ is *stationary* if $\mu_L = \mu_{L+t}$, for all $L \in T^n$ such that $L + t \in T^n$.

Proposition 1.2 $(a(\cdot), R(\cdot,\cdot))$*–Gaussian process* $\{X(t),\ t \in \mathbb{R}\}$ *is stationary if there exists a function* $r : \mathbb{R} \to \mathbb{R}$ *such that* $R(s,t) = r(t-s)$ *and* $a(t) = \text{const}$.

Proof Since μ_t is the same for all $t \in \mathbb{R}$ we have $a(t) = \mathbb{E}\, X(t) = \text{const}$. We have $\mu_{(s,t)} = \mu_{(0,t-s)}$. Without loss of generality we can assume $s \leq t$. Hence $\operatorname{Cov}(X(s), X(t)) = \operatorname{Cov}(X(0), X(t-s))$. Denote $r(t) = \operatorname{Cov}(X(0), X(t))$. Thus $R(s,t) = r(t-s)$. On the other hand, since $R(s,t) = R(t,s)$ we have $R(s,t) = r(|t-s|)$ for all $s, t \in \mathbb{R}$. □

Similarly for $\{X(n),\ n \in \mathbb{Z}\}$ we have $R(n,m) = r_{|n-m|}$.

Definition 1.3 We say that a stochastic process $\{X(t), t \in T\}$ has *stationary increments* (si) if

$$
\begin{aligned}
&(X(t_2) - X(t_1), \ldots, X(t_n) - X(t_1)) =_{\mathrm{d}} \\
&\quad =_{\mathrm{d}} \quad (X(t_2 + h) - X(t_1 + h), \ldots, X(t_n + h) - X(t_1 + h))
\end{aligned}
$$

for all $t_1, \ldots, t_n, t_1 + h, \ldots, t_n + h \in T$ and $n = 1, \ldots$.

Note that if $\{X(t)\}$ is si, if and only if $\{X(t)-X(0)\}$ is si.

Definition 1.4 We say that a real valued stochastic process $X(t),\ t\in T$, where $T=\mathbb{R},\mathbb{R}_+$ or $\mathbb{R}^d,\mathbb{R}^d_+$ is *self-similar* (ss) with *self-similarity index* H (H-ss) if for all $c>0$ we have

$$\{c^{-H}X(ct),\ t\in T\} =_{\mathrm{d}} \{X(t), t\in T\}.$$

Proposition 1.5 *Let $\{X(t),\ t\in\mathbb{R}\}$ be $(a(\cdot),R(\cdot,\cdot))$–Gaussian process. The following sentences are equivalent.*
(i) *The processes with stationary increments.*
(ii) *The mean and covariance function fulfill*

$$a(s+t) = a(s)+a(t), \qquad s,t\in\mathbb{R}, \tag{1.1}$$

$$R(s,t) = \frac{1}{2}(\sigma^2(t)+\sigma^2(s)-\sigma^2(|t-s|)), \tag{1.2}$$

where $\sigma^2(t)=R(t,t)=\mathrm{Var}\,X(t)$.
(iii) *There exist a constant $a\in\mathbb{R}$ and a function $\sigma^2:\mathbb{R}_+\to\mathbb{R}$ such that*

$$\mathbb{E}\,(X(t)-X(s)) = a(t-s), \tag{1.3}$$

$$\mathrm{Var}\,(X(t)-X(s)) = \sigma^2(|t-s|). \tag{1.4}$$

Example 1.6 Let $\{Z(t),\ t\in\mathbb{R}\}$ be a stationary centered Gaussian process with covariance function $\{r(t-s)\}$. We define the *integrated Gaussian* process (IG) by

$$X(t) = \begin{cases} \int_0^t Z(s)\ \mathrm{d}s & \text{for} t\geq 0 \\ \int_t^0 Z(s)\ \mathrm{d}s & \text{for} t\leq 0. \end{cases}$$

Actually we need measurability of $\{Z(t)\}$, however in our applications the process Z has continuous realizations and therefore $\int_0^t Z(s)\ \mathrm{d}v$ is a random variable.[1] This IG process is with stationary increments because for$s<t$

$$\mathbb{E}\,(X(t)-X(s))^2 = \mathbb{E}\,(\int_s^t Z(v)\ \mathrm{d}v)^2\mathbb{E}\,(\int_0^{t-s} Z(v)\ \mathrm{d}v)^2 = \sigma^2(t-s).$$

We now express $\sigma^2(t)$ in terms of the covariance function of $\{Z(t)\}$. Thus

$$\sigma^2(t) = \mathbb{E}\int_0^t Z(v)\ \mathrm{d}v\int_0^t Z(w)\ \mathrm{d}w$$

$$= \int_0^t\int_0^t \mathbb{E}\,Z(v)Z(w)\,\mathrm{d}v\,\mathrm{d}w = 2\int_0^t\int_v^t r(w-v)\ \mathrm{d}v\,\mathrm{d}w = 2\int_0^t\int_0^s r(v)\,\mathrm{d}v\,\mathrm{d}s.$$

[1] ??

1.1 Complex Gaussian processes

Exercises

1.1 Show that the following sentences are equivalent.
(i) Stochastic process $\{X(t),\ t \in \mathbb{R}\}$ (not necessarily Gaussian) has stationary increments.
(ii) $\{X(t+h) - X(h), t \in \mathbb{R}\} =_{\mathrm{d}} \{X(t) - X(0),\ t \in \mathbb{R}\}$.
(iii) For $t_1 \leq \ldots \leq t_n$, $n = 1, \ldots$ and $h \in \mathbb{R}$

$$\begin{aligned}&(X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})) =_{\mathrm{d}} \\ &\quad =_{\mathrm{d}} (X(t_2+h) - X(t_1+h), \ldots, X(t_n+h) - X(t_{n-1}+h)).\end{aligned}$$

1.2 Prove that if $\{X(t), t \geq 0\}$ is H-ss, then $\{Y(t),\ t \in \mathbb{R}\}$ defined by $Y(t) = e^{-tH} X(e^t)$ is stationary. Conversely if $Y(t),\ t \geq 0$ is stationary, then $X(t) = t^H Y(\log t)$ is H-ss.

1.3 Let $\{X(t)\}$ be a centered stationary Gaussian process with covariance function $R(t)$. Show that the process is continuous in probability if and only if $R(t)$ is continuous.

2 Covariance functions

In this section we study further properties of covariance functions. For a real valued process $\{X(t),\ s, t \in T\}$ such that $\mathbb{E}\,(X(t))^2 < \infty$ we define its covariance function by

$$R(s,t) = \mathrm{Cov}\,(X(s), X(t)), \qquad t \in T\ .$$

In case of complex valued stochastic processes $\{X(t),\ t \in T\}$, where $X(t) = X^{\mathrm{r}}(t) + iX^{\mathrm{i}}(t)$ and $X^{\mathrm{r}}(t)$, $X^{\mathrm{i}}(t)$ are two real valued stochastic processes corresponding to the real and imaginary part of $X(t)$ respectively. For such the process one define the mean and covariance function by

$$\mathbb{E}\,X(t) = a(t) = \mathbb{E}\,X^{\mathrm{r}}(t) + i\mathbb{E}\,X^{\mathrm{i}}(t),$$

and

$$\mathrm{Cov}\,(X(s), X(t)) = R(s,t) = \mathbb{E}\,(X(s) - \mathbb{E}\,X(s))(\overline{X(t) - \mathbb{E}\,X(t)}), \qquad s, t \in T\ .$$

Recall that a complex valued functions $\{g(s,t),\ s,t \in T\}$ is *Hermitian* if $g(s,t) = \overline{g(t,s)}$ for all $s, t \in T$. For a real valued functions this property means that it is symmetric.

Definition 2.1 We say that a real or complex valued function $g : T^2 \to \mathbb{R}$ is positive definite if for all $t_1, \ldots, t_n$ and $z_1, \ldots, z_n \in \mathbb{C}$

$$\sum_{j,k=1}^{n} g(t_j, t_k) z_j \overline{z}_k \geq 0. \tag{2.5}$$

In case of real valued function $g(s,t)$ in definition (2.5) it suffices to consider $z_1, \ldots, z_n \in \mathbb{R}$.

Proposition 2.2 *Covariance function is a Hermitian positive definite function.*

For the case of processes with stationary increments we have the following characterization of covariance functions.

Let

$$f(s,t) = \int_{-\infty}^{\infty} (e^{isu} - 1)\overline{(e^{itu} - 1)}\, \nu(du), \tag{2.6}$$

where ν is a measure such that

$$\int_{-\infty}^{\infty} \min(u^2, 1)\, \nu(du) < \infty.$$

It is straightforward to prove that $f(s,t)$ is Hermitian and positive definite.

Proposition 2.3 *Let $f(s,t)$ be defined by (2.6). We have that*

1. *$f(s,t)$ is well defined,*
2. *$f(s,t)$ is Hermitian and positive definite,*
3. *if ν is symmetric, then $f(s,t)$ is real (and this will be assumed from now on),*
4. *for $s, t \in \mathbb{R}$*

$$f(s,t) = \frac{1}{2}(f(t,t) + f(s,s) - f(t-s, t-s)). \tag{2.7}$$

Later in Theorem 6.11 we show that any positive definite function $f(s,t)$ of has to be of form (2.7). From the above proposition it follows the following result.

Proposition 2.4 *A centered real valued Gaussian process with covariance function $R(s,t)$ of form (2.6) has stationary increments.*

Example 2.5 (i) [Wiener process] Let $R : \mathbb{R}_+^2 \to \mathbb{R}$ be defined by $R(s,t) = s \wedge t$. The following argument shows that $R(s,t)$ is positive definite (that is symmetric is obvious). We have

$$s \wedge t = \int_0^\infty 1(v \le s) 1(v \le t)\, dv$$

and hence for all $t_1, \ldots, t_n \in \mathbb{R}_+$ and $z_1, \ldots, z_n \in \mathbb{R}$

$$\begin{aligned} \sum_{j,k=1}^n t_j \wedge t_k \; z_j z_k &= \sum_{j,k=1}^n \int_0^\infty 1(v \le t_j) 1(v \le t_k)\, dv z_j z_k \\ &= \int_0^\infty \left(\sum_{j=1}^n 1(v \le t_j) z_j \right)^2 dv \ge 0. \end{aligned}$$

Centered Gaussian process $\{W(t),\ t \ge 0\}$ with covariance function $s \wedge t$ is called *Wiener process*. Later on we shall also require that $W(t)$ has continuous trajectories. The Wiener process is 1/2-ss.

(ii) [Brownian bridge] Let $R : [0,1]^2 \to \mathbb{R}$ be defined by $R(s,t) = s \wedge t - st$. The following argument shows that $R(s,t)$ is positive definite (that is symmetric is obvious). We have for all $t_1, \ldots, t_n \in \mathbb{R}_+$ and $z_1, \ldots, z_n \in \mathbb{R}$

$$\begin{aligned} &\sum_{j,k=1}^n (t_j \wedge t_k - t_j t_k) \; z_j z_k = \\ &= \sum_{j,k=1}^n \Big(\int_0^1 1(v \le t_j) 1(v \le t_k)\, dv - \int_0^1 1(v \le t_j)\, dv \int_0^1 1(v \le t_k)\, dv \Big) z_j z_k \\ &= \int_0^1 \left(\sum_{j=1}^n (1(v \le t_j) z_j) \right)^2 dv - \left(\int_0^1 \sum_{j=1}^n (1(v \le t_j) z_j)\, dv \right)^2 \ge 0. \end{aligned}$$

Centered Gaussian process $\{\overset{\circ}{W}(t),\ 0 \le t \le 1\}$ with covariance function $s \wedge t - st$ is called *Brownian bridge process.*

(iii) [Wiener–Chentsov random field] Let for $\boldsymbol{s} = (s_1, \ldots, s_d) \in \mathbb{R}_+^d$ and $\boldsymbol{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}_+^d$

$$R(\boldsymbol{s}, \boldsymbol{t}) = \prod_{j=1}^d (s_j \wedge t_j).$$

We have

$$R(\boldsymbol{s}, \boldsymbol{t}) = \int_{\mathbb{R}_+^d} 1(\boldsymbol{v} \le \boldsymbol{s}) 1(\boldsymbol{v} \le \boldsymbol{t})\, d\boldsymbol{v}, \tag{2.8}$$

from which we immediately show that it is positive definite. Symmetry is obvious. Centered Gaussian process $\{W(\boldsymbol{t}),\ \boldsymbol{t} \in \mathbb{R}_+^d\}$ with covariance functions $R(s,t)$ is called *Wiener-Chentsov* random field. The Wiener-Chentsov process is $d/2$–ss.

Exercises

2.1 Show that function $R(s,t)$ defined by (2.8) is symmetric positive definite.

2.2 Show that the Wiener process is 1/2-ss, Wiener-Chentsov random field is $d/2$-ss.

2.3 Show that $\cos(t-s) - \cos t - \cos s + 1$ is positive definite.

2.4 Let A, Θ be two independent random variables — A having *Raleigh distribution* with density function $g(x) = x \exp(-x^2/2) 1(x > 0)$ and Θ being uniformly distributed over $[0, 2\pi]$. Show that

$$X(t) = A\cos(\Theta - t),\ \ t \in \mathbb{R}$$

is a centered stationary Gaussian process and that

$$X(t) = A\sin(\Theta - t),\ \ t \in \mathbb{R}$$

is its independent copy. Find the covariance function.

2.1 Fractional Brownian motion

The main result proved in this subsection is the following theorem.

Theorem 2.6 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by*

$$f(s,t) = \frac{1}{2}(|s|^\alpha + |t|^\alpha - |t-s|^\alpha).$$

If $0 < \alpha \le 2$, *then* f *is positive definite.*

The proof for $\alpha = 2$ is straightforward, because $f(s,t) = st$ and we can immediately check condition (2.5). To prove the theorem for $0 < \alpha < 2$ we need the following facts.

Lemma 2.7 *For* $0 < \alpha < 2$ *and* $s > 0$

$$\int_0^\infty t^{-\alpha} \sin ts \ \mathrm{d}t = \frac{\pi}{2\Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right)} s^{\alpha-1}.$$

Proof Fichtenholz II, 539.3

Lemma 2.8 *For* $a > 0$

$$\Gamma(a+1) = a\Gamma(a).$$

Proof Fichtenholz II, 531.

Proof of theorem We show that $f(s,t)$ can be written as (2.6), in particular we use (2.7) and that

$$f(s,s) = \int_{-\infty}^{\infty} |e^{isu} - 1|^2 \, \nu(\, du) = |s|^\alpha,$$

holds for

$$\nu(\, du) = (2\pi)^{-1}\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)|u|^{-1-\alpha}\, du. \tag{2.9}$$

Since

$$|e^{isu} - 1|^2 = 2(1-\cos su)$$

we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |e^{iut} - 1|^2 \, \nu(\, du) &= 2(2\pi)^{-1}\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)\int_{-\infty}^{\infty}(1-\cos tu)|u|^{-1-\alpha}\, du \\
&= 4(2\pi)^{-1}\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)\int_0^{\infty}\int_0^t \frac{\sin su}{u^\alpha}\, ds\, du \\
&= 4(2\pi)^{-1}\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)\int_0^{t}\int_0^\infty \frac{\sin su}{u^\alpha}\, du\, ds = t^\alpha,
\end{aligned}$$

where in the last equation we used the results of Lemma 2.7 and 2.8 □

Definition 2.9 Centered Gaussian process $\{B_H(t),\ t \geq 0\}$ with covariance function

$$R(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

is called *fractional Brownian motion* with *Hurst index* H. We denote it by H–fBm.

Clearly $0 < H \leq 1$. We will use rather $0 < H \leq 1$, then $0 < \alpha \leq 2$ remembering that $\alpha = 2H$. Formula (2.9) defines *spectral density* $h_H(u)$ of the fBm:

$$h_H(u) = C_2^{-2}(H)|u|^{-2H-1},$$

where

$$C_2(H) = \sqrt{\frac{2\pi}{\Gamma(2H+1)\sin(\pi H)}}. \tag{2.10}$$

We will study spectral representations for fBm's in Section 6.4.

Remark Since $\mathbb{E}\, B_H(0) = 0$ and $\mathrm{Var}\, B_H(0) = R(0,0) = 0$ we have $B_H(0) = 0$ a.s.. Note that H-fBm is H–ss. For $H = 1/2$, $\{B_{1/2}(t)\}$ is the Wiener process. If $W \sim \mathcal{N}(0,1)$ and $X(t) = Wt$, then $X = B_1$, that is an 1–fBm.

Remark The H–fBm for $H > 1/2$ serves as an example of a Gaussian process with *long range dependence property.* Thus consider the sequence $X_n,\ n = 0,1,\ldots$, defined by $X_n = B_H(n+1) - B_H(n)$. Its covariance function is for $n = 0,1,\ldots,$ and $m = 1,\ldots$

$$R(n, n+m) = \mathbb{E}\, X_n X_{n+m} = \frac{1}{2}(|m-1|^{2H} - 2m^{2H} + |m+1|^{2H}) \quad (2.11)$$

and so the sequence is stationary with covariance function $r_m = R(n, n+m)$ and $r_0 = 1$. Gaussian sequence with the covariance function like in (2.11) is called a *fractional Brownian noise* (H–fBn)

Exercises

2.1 Prove that fractional Brownian motion is sample continuous. Hint. Prove that $\mathbb{E}\,(X(t) - X(s))^{2n} = \mathbb{E}\, W^{2n}|t-s|^{2Hn}$, where $W\mathcal{N}(0,1)$.

3 Positive and negative definite functions

Besides of the concept of positive definite functions we have also the following one: a function $g : T^2 \to \mathbb{R}$ is said to be *negative definite* if for $t_1, \ldots, t_n \in T$ and $w_1, \ldots, w_n \in \mathbb{R}$ such that $w_1 + \ldots + w_n = 0$

$$\sum_{j,k=1}^{n} g(t_j, t_k) w_j w_k \le 0.$$

We will not consider complex functions but mention that in such the case if $g : T^2 \to \mathbb{C}$, then g is positive definite if for $t_1, \ldots, t_n \in T$ and $w_1, \ldots, w_n \in \mathbb{C}$ such that $w_1 + \ldots + w_n = 0$

$$\sum_{j,k=1}^{n} f(t_j, t_k) w_j \bar{w}_k \le 0.$$

We assume that $g(\mathbf{0}) = 0$ which yields

$$g(\boldsymbol{s}, \mathbf{0}) = g(\mathbf{0}, \boldsymbol{s}) = 0.$$

Moreover $g(\boldsymbol{t} - \boldsymbol{s}) = g(\boldsymbol{s} - \boldsymbol{t})$, which yields that f is symmetric.

In the sequel $T = \mathbb{R}^d$ and g is real.

Lemma 3.1 *Let*

$$f(\boldsymbol{s},\boldsymbol{t}) = \frac{1}{2}(g(\boldsymbol{t}) + g(\boldsymbol{s}) - g(\boldsymbol{t}-\boldsymbol{s})).$$

$f(\boldsymbol{s},\boldsymbol{t})$ *is positive definite if and only if* $g(\boldsymbol{t})$ *is negative definite.*

Proof Let f be positive define and $w_1,\ldots,w_n \in \mathbb{R}$ such that $\sum_{j=1}^n w_j = 0$. We have

$$\begin{aligned} 0 &\geq \sum_{j,k=1}^n f(\boldsymbol{t}_j,\boldsymbol{t}_k)w_jw_k \\ &= \frac{1}{2}\left(\sum_{j,k=1}^n g(\boldsymbol{t}_j)w_jw_k + \sum_{j,k=1}^n g(\boldsymbol{t}_k)w_jw_k - \sum_{j,k}^n g(\boldsymbol{t}_j-\boldsymbol{t}_k)w_jw_k\right) \\ &= -\frac{1}{2}\sum_{j,k=1}^n g(\boldsymbol{t}_j-\boldsymbol{t}_k)w_jw_k, \end{aligned} \tag{3.12}$$

and so $g(\boldsymbol{t})$ is negative definite. Conversely suppose $g(\boldsymbol{t})$ is negative definite and let $\boldsymbol{t}_1,\ldots,\boldsymbol{t}_n \in T$,$w_1,\ldots,w_n \in \mathbb{R}$ (we do not assume that $\sum_{j=1}^n w_j = 0$). Let $\boldsymbol{t}_0 = \boldsymbol{0}$ and $w_0 = -\sum_{j=1}^n w_j$. We have

$$0 \geq \sum_{j,k=1}^n g(\boldsymbol{t}_j-\boldsymbol{t}_k) = -\sum_{j,k=0}^n f(\boldsymbol{t}_j,\boldsymbol{t}_k)w_jw_k = -\sum_{j,k=1}^n f(\boldsymbol{t}_j,\boldsymbol{t}_k)w_jw_k.$$

□

Theorem 3.2 [I. Schur] (i) *If* f_1 *and* f_2 *are positive definite, then* f_1f_2 *is also positive definite.*
(ii) *If* f *is positive definite, then* e^f *is positive definite too.*

Proof (i) To each positive definite function f_j we can associate a Gaussian process $\{X_j(\boldsymbol{t}),\ \boldsymbol{t} \in T\}$ defined on a probability space $(\Omega_j, \mathcal{F}_j, \mathbb{P}_j)$, $(j = 1,2)$. Let $\{Y(\boldsymbol{t}),\ \boldsymbol{t} \in T\}$, where $Y(\boldsymbol{t}) = X_1(\boldsymbol{t})X_2(\boldsymbol{t})$ be defined on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$. Clearly the covariance function of Y is f_1f_2 and the covariance function must be positive definite.
(ii) It follows from

$$e^f = \sum_{j=0}^{\infty} \frac{f^j}{j!}.$$

□

Theorem 3.3 [Schoenberg] *For all $c > 0$ function $\mathbb{R}^d \ni \boldsymbol{t} \to \exp(-cg(\boldsymbol{t}))$ is positive definite, if and only if $g(\boldsymbol{t})$ is negative definite.*

Proof We have

$$cg(\boldsymbol{t}) = 1 - e^{-cg(\boldsymbol{t})} + o(c, \boldsymbol{t}),$$

where $\lim_{c \to 0} o(c, \boldsymbol{t})/c = 0$ for all $\boldsymbol{t}$. Hence for $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n \in \mathbb{R}^d$ and $w_j \in \mathbb{R}$ such that $\sum_j w_j = 0$

$$c \sum_{j,k}^{n} g(\boldsymbol{t}_j - \boldsymbol{t}_k) w_j w_k + \sum_{j,k=1}^{n} o(c, \boldsymbol{t}_j - \boldsymbol{t}_k) w_j w_k = - \sum_{j,k}^{n} e^{-cg(\boldsymbol{t}_j - \boldsymbol{t}_k)} w_j w_k.$$

If we choose $c > 0$ sufficiently small, then the LHS is negative. The converse part follows from Schur theorem 3.2. □

Proposition 3.4 *Function $\mathbb{R}^d \ni \boldsymbol{t} \to g(\boldsymbol{t})$ is negative definite if and only if there exists a symmetric measure ν $(\nu(A) = \nu(-A))$ on $\mathbb{R}^d$ such that*

$$\int_{\mathbf{R}^d} \min(|\boldsymbol{u}|^2, 1)\, \nu(\,\mathrm{d}\boldsymbol{u}) < \infty$$

and

$$\begin{aligned} g(\boldsymbol{t}) &= \int_{\mathbf{R}^d} |e^{i\boldsymbol{t}\boldsymbol{u}^T} - 1|^2\, \nu(\,\mathrm{d}\boldsymbol{u}) \\ &= 2 \int_{\mathbf{R}^d} (1 - \cos \boldsymbol{t}\boldsymbol{u}^T)\, \nu(\,\mathrm{d}\boldsymbol{u}). \end{aligned}$$

Proof $\boldsymbol{t} \to f(\boldsymbol{t}) = \exp(-g(\boldsymbol{t}))$ is a characteristic function (recall that $g(\mathbf{0}) = 0$) of an infinitly divisible distribution. Therefore, taking under account that $g(\boldsymbol{t})$ is real, we have by the Lévy-Khinchin representation that there exists a symmetric measure ν'

$$g(\boldsymbol{t}) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} (1 - \cos \boldsymbol{t}\boldsymbol{u}) \nu'(\,\mathrm{d}\boldsymbol{u})$$

and

$$\int_{\mathbf{R}^d} \min(|\boldsymbol{u}|^2, 1)\, \nu'(\,\mathrm{d}\boldsymbol{u}) < \infty$$

To complete the proof note that $|e^{i\boldsymbol{t}\boldsymbol{u}^T} - 1|^2 = 2(1 - \cos \boldsymbol{t}\boldsymbol{u}^T)$ and so $\nu = \nu'/2$. □

Comments. Bozejko (1987)

3.1 Lévy fractional Brownian field

We show now that $e^{-c|\boldsymbol{t}|^{2H}}$ is a characteristic function for $0 < H \leq 1$, or equivalently $|\boldsymbol{t}|^{2H}$ is negative definite. Hence it follows from Lemma 3.1 that

$$R(\boldsymbol{s},\boldsymbol{t}) = \frac{1}{2}(|\boldsymbol{t}|^{2H} + |\boldsymbol{s}|^{2H} - |\boldsymbol{s}-\boldsymbol{t}|^{2H}) \tag{3.13}$$

is a covariance function. A centered Gaussian process $\{\boldsymbol{X}(\boldsymbol{t}),\ \boldsymbol{t} \in \mathbb{R}^d\}$ with a covariance function given by (3.13) (and $X(\boldsymbol{0}) = 0$) is called *Lévy fractional Brownian field* (H–LfBf). A special case of interest is when $H = 1/2$. Then the process is called is called *Lévy Brownian field* or *Lévy Brownian motion.*

That function $g(\boldsymbol{t}) = |\boldsymbol{t}|^2$ is negative definite can be easily seen directly from definition. We now show that g^H is negative definite too. It follows from a more general result presented below.

Proposition 3.5 *If $g(\boldsymbol{s},\boldsymbol{t})$ is negative definite and $g(\boldsymbol{s},\boldsymbol{t}) \geq 0$, then g^{2H} is negative definite too, for $0 < H \leq 1$.*

Proof We need the following integral formula

$$t^H = c_H \int_0^\infty (1 - e^{-v^2 t})\, \frac{\mathrm{d}v}{v^{1+H}}, \quad t \geq 0$$

where $0 < H < 1$ and

$$c_H^{-1} = \int_0^\infty (1 - e^{-v^2})\, \frac{\mathrm{d}v}{v^{1+H}}.$$

□

Corollary 3.6 $e^{-c|\boldsymbol{s}-\boldsymbol{t}|^{2H}}$ *is positive definite.*

Exercises

3.1 Show that $|\boldsymbol{t}|^2$ is negative definite.

4 Space $\mathbb{L}^2(\mu)$ and $\mathbb{L}^2_{\mathbb{C}}(\mu)$

Let $(\mathbb{E}, \mathcal{E}, \mu)$ be a measurable space. We define $\mathbb{L}^2(\mu)$ to be a family of all functions $f : \mathbb{E} \to \mathbb{R}$ such that $\int f^2\, d\mu < \infty$. On this space we introduce the scalar product $< f, g >_\mu = \int f g\, d\mu$. The square of the distance between f and g is $||f - g||^2_\mu = \int (f - g)^2\, d\mu$.

We define $\mathbb{L}^2_{\mathbb{C}}(\mu)$ to be a family of all functions $f : \mathbb{E} \to \mathbb{C}$ such that $\int |f|^2 \, d\mu < \infty$. On this space we introduce the scalar product $< f, g >_\mu = \int f\overline{g} \, d\mu$. The square of the distance between f and g is $||f-g||^2_\mu = \int |f-g|^2 \, d\mu$.

Two cases are of interest:

- $\mathbb{E} = \Omega$, $\mathcal{E} = \mathcal{F}$ and $\mu = \mathbb{P}$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is our basic probability space. In this case functions are random variables X such that $\mathbb{E}\, X^2 < \infty$ or in the case of complex random variables $\mathbb{E}\, |Z|^2 < \infty$. In this case the scalar product is defined by $< X, X' >_{\mathbf{P}} = \mathbb{E}\, XX'$ and in case of complex random variables $< Z, Z' >_{\mathbf{P}} = \mathbb{E}\, Z\overline{Z'}$.
- $\mathbb{E} = [-a, a]$, $\mathcal{E} = \mathcal{B}[-a, a]$ and $\mu = \nu$ is a measure. Another case of interest is when $\mathbb{E} = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$.

5 Integration wrt white noise

Our aim is to define $\int_T f \; \mathrm{d}M$ from a deterministic functions on the space of parameters with respect to the so called white noise defined below. We assume that $(T, \mathcal{T}, \nu)$ is a measurable space with a finite measure ν. Define

$$\mathcal{R}(A, A') = \nu(A \cap A'), \qquad A, A' \in \mathcal{T}.$$

Since $\mathcal{R}(A, A') = \int_T 1_A 1_{A'} \, d\nu$ it is easy to prove that $\mathcal{R}$ is positive definite and of course symmetric.

Definition 5.1 Real centered Gaussian process $\{M(A),\ A \in \mathcal{T}\}$ with covariance function $\mathcal{R}(A, A') = \nu(A \cap A')$ is said to be a real *Gaussian white noise* with intensity ν.

Example 5.2 Let ν be the Lebegue measure on $\mathbb{R}_+$. We call then the defined Gaussian white noise by *Wiener white noise*

Proposition 5.3 *If $A_1, \ldots, A_n \in \mathcal{T}$ are disjoint, then $M(A_1), \ldots, M(A_n)$ are independent an*

$$M(\bigcup_{j=1}^n A_j) = \sum_{j=1}^n M(A_j), \quad \text{a.s.}$$

It turns out that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a Gaussian white noise $\{M(A)\}$ defined on it and such that, for almost all $(\omega \in \Omega)$, $A \to M(A)$ is finitely additive. One can construct examples, however, that $A \to M(A)$ is not σ-additive. We now show how to construct $\int f \, dM$ for functions $f \in \mathbb{L}^2(E, \mathcal{E}, \nu)$. We denote the scalar product on this space by $< f, g >_\nu$ and $||f - g||_\nu^2 =< f - g, f - g >_\nu$. We denote by $\mathbb{L}_{\text{step}}(T)$ the family of step functions.

Step 1. For a simple function $f(t) = \sum_{j=1}^n a_j 1(t \in A_j)$, where $a_j \in \mathbb{R}$ and $A_j \in \mathcal{T}$ we define

$$\int_T f(u)\, M(du) = \sum_{j=1}^n a_j M(A_j) \ .$$

Lemma 5.4 *Let $f_1, \dots, f_n$ be simple functions on T.*
(i) *We have*

$$< \int_T f_1\, dM, \int_T f_2\, dM) >_{\mathbf{P}} =< f_1, f_2 >_\nu \ .$$

(ii) *Random vector*

$$(\int_T f_1\, dM, \dots, \int_T f_n\, dM)$$

has joint multivariate normal distribution $\mathcal{N}(\mathbf{0}, (< f_j, f_k >)_{j,k})$.

Corollary 5.5 *Let f, f' be simple functions.*
(i) $\mathbb{E}\,(\int f\, dM - \int f'\, dM)^2 = \int (f - f')^2\, d\nu$.
(ii) *If $f(t) = f'(t)$, then $\mathbb{E}\,(\int f\, dM - \int f'\, dM)^2 = 0$.*

Step 2. Let $f \in \mathbb{L}^2(\nu)$. There exists a sequence of simple functions f_n (they of course belong to $\mathbb{L}^2(T)$) such that $f_n \stackrel{2}{\to} f$. Since

$$\mathbb{E}\,(\int_T f_n\, dM - \int_T f_m\, dM)^2 = \int_T (f_n - f_m)^2\, d\nu$$

the sequence $\{\int_T f_n\, dM\}$ is a Cauchy sequence in $\mathbb{L}^2(\mathbb{P})$. Therefore there exists the limit

$$\lim_{n\to\infty} \int_T f_n\, dM$$

in $\mathbb{L}^2(\mathbb{P})$. This limit we denote by $\int_T f\, dM$. It is an exercise to check that it does not depend on the choice of the sequence $\{f_n\}$.

Lemma 5.6 *Let $f_1, \dots, f_n \in \mathbb{L}^2(\nu)$.* (i) *W have*

$$< \int_T f_1\, dM, \int f_2\, dM) >_{\mathbf{P}} =< f_1, f_2 >_\nu \ .$$

In particular

$$\mathbb{E}\,(\int_T f_1\, dM)^2 = \int_T f\, d\nu.$$

(ii) *Random vector*

$$(\int f_1\,dM,\ldots,\int f_n\,dM)$$

has joint multivariate normal distribution $\mathcal{N}(\mathbf{0},(< f_j,f_k >)_{j.k})$.

Example 5.7 Let ν be an intensity measure on $\mathbb{R}$ absolute continuous wrt Lebesgue measure $\lambda(\cdot)$. Let $h(x)$ be the density of ν. Define

$$M(A)=\int_{-\infty}^{\infty}1(x\in A)h^{1/2}(x)\,W(dx),\quad A\in\mathcal{B}(\mathbb{R}),$$

where $\{W(A)\}$ is the Wiener white noise. Then $\{M(A),\ A\in\mathcal{B}(\mathbb{R})\}$ is a Gaussian white noise with intensity measure ν. Namely using Lemma 5.6 for A,A'

$$\mathbb{E}\,M(A)M(A')=\int_{-\infty}^{\infty}1(x\in A)h^{1/2}(x)1(x\in A')h^{1/2}(x)\,dx=\nu(A\cap A').$$

Therefore $\{M(A)\}$ is a Gaussian white noise with intensity measure ν.

5.1 Integration wrt complex white noise

Although in these notes we study real valued processes (or eventually assuming values in $\mathbb{R}^d$), to get some representations, like some harmonizable representations, we need to use complex white noise.

As before $(T,\mathcal{T},\nu)$ be a measurable space with a finite measure ν. Let

$$Z(A)=M^{\mathrm{r}}(A)+iM^{\mathrm{i}}(A),\qquad A\in\mathcal{T} \tag{5.14}$$

be a complex stochastic process, where $\{M^{\mathrm{r}}(A)\}$ and $\{M^{\mathrm{i}}(A)\}$ are real white noises. A *complex white noise* with an intensity measure ν hs to fulfill the following conditions:

$$\mathbb{E}\,|Z(A)|^2=\nu(A), \tag{5.15}$$
$$\mathbb{E}\,Z(A)\overline{Z(A')}=0. \tag{5.16}$$

for $A,A'\in\mathcal{T}$ disjoint. Conditions (5.15) and (5.16) is equivalent to

$$\mathbb{E}\,[(M^{\mathrm{r}})^2(A)+(M^{\mathrm{i}})^2(A)]=\nu(A), \tag{5.17}$$
$$\mathbb{E}\,M^{\mathrm{r}}(A)M^{\mathrm{r}}(A')+\mathbb{E}\,M^{\mathrm{i}}(A)M^{\mathrm{i}}(A')=0, \tag{5.18}$$
$$\mathbb{E}\,[M^{\mathrm{i}}(A)M^{\mathrm{r}}(A')]-\mathbb{E}\,[M^{\mathrm{r}}(A)M^{\mathrm{i}}(A')]=0 \tag{5.19}$$

for $A,A'\in\mathcal{T}$ disjoint.

Note that conditions (5.15), (5.16) case the intensity measure ν does not determine uniquely the white noise. An example of a complex white noise is as follow.

Example 5.8 (i) Let $\{M^{\mathrm{r}}(A),\ A \in \mathcal{T}\}$ be a real Gaussian white noise with intensity measure $(1/2)\nu$ and $\{M^{\mathrm{i}}(A),\ A \in \mathcal{T}\}$ be an independent copy of $\{M^{\mathrm{r}}(A)\}$. Then $\{Z(A)\}$ defined by (5.14) is a complex white noise that is conditions (5.15),(5.16) hold.

We now show how to construct $\int_T f\, dZ$ for functions $f \in \mathbb{L}^2_{\mathbb{C}}(T, \mathcal{T}, \mu)$, that is the space of functions $f : T \to \mathbb{C}$, such that $\int |f|^2\, d\nu < \infty$. Recall that the scalar product on this space is defined by: $< f, g >_\nu = \int f\overline{g}\, d\nu$. We denote by $\mathbb{L}_{\mathbb{C},\mathrm{s}}(\nu)$ the family of simple functions.
Step 1. For a simple function $f(t) = \sum_{j=1}^n a_j 1(t \in A_j)$, where $a_j \in \mathbb{C}$ and $A_j \in \mathcal{T}$ we define

$$\int_T f(u)\, Z(du) = \sum_{j=1}^n a_j Z(A_j)\ .$$

Lemma 5.9 *Let $f_1, \ldots, f_n \in \mathbb{L}^2_{\mathbb{C},\mathrm{s}}(T)$.*
(i) *We have*

$$< \int_T f_1\, dM, \int_T f_2\, dM) >_{\mathbf{P}} = < f_1, f_2 >_\nu\ .$$

(ii) *Random vector*

$$(\int_T f_1\, dZ, \ldots, \int_T f_n\, dZ)$$

has a joint multivariate normal distribution.

Corollary 5.10 *Let $f, f' \in \mathbb{L}^2_{\mathbb{C},\mathrm{s}}(\nu)$.*
(i) $\mathbb{E}\,(\int_T f\, dZ - \int_T f'\, dZ)^2 = \int_T (f - f')^2\, d\nu$.
(ii) *If $f(t) = f'(t)$, then $\mathbb{E}\,(\int f\, dZ - \int f'\, dZ)^2 = 0$.*

Step 2. Let $f \in \mathbb{L}^2_{\mathbb{C}}(\nu)$. There exists a sequence of simple functions $f_n \in \mathbb{L}^2_{\mathbb{C},\mathrm{s}}(\nu)$ (they of course belong to $\mathbb{L}^2_{\mathbb{C}}(\nu)$) such that $f_n \stackrel{2}{\to} f$. Since

$$\mathbb{E}\,(\int_T f_n\, dZ - \int_T f_m\, dZ)^2 = \int_T (f_n - f_m)^2\, d\nu$$

the sequence $\{\int f_n\, dZ\}$ is a Cauchy sequence in $\mathbb{L}^2(\mathbb{P})$. Therefore, there exists the limit

$$\lim_{n\to\infty} \int_T f_n\, dZ$$

in $\mathbb{L}^2(\mathbb{P})$. This limit we denote by $\int_T f\, dZ$. It is an exercise to check that the limit does not depend on the choice of the sequence $\{f_n\}$.

Proposition 5.11 *Let* $f_1, \dots, f_n \in \mathbb{L}^2_{\mathbb{C}}(\nu)$.
(i) *W have*

$$< \int_T f_1\, dZ, \int_T f_2\, dZ >_{\mathbf{P}} = < f_1, f_2 >_\nu .$$

In particular

$$\mathbb{E} \left| \int_T f_1\, dZ \right|^2 = \int_T |f_1|^2\, d\nu.$$

(ii) *Random vector*

$$\left(\int_T f_1\, dZ, \dots, \int_T f_n\, dZ\right)$$

has a joint multivariate complex normal distribution.

5.2 Tilde complex white noise

The following notion will play an important role in defining real harmonizable representation of Gaussian processes. Let $T = \mathbb{R}$, $\mathcal{T} = \mathcal{B}(\mathbb{R})$, and ν is a finite *symmetric measure* (that is $\nu(A) = \nu(-A)$ for $A \in \mathcal{B}(\mathbb{R})$). We define first the real white noise $\{M^{\mathrm{r}}(A),\ A \in \mathcal{B}(0,\infty)\}$ with intensity measure $\frac{1}{2}\nu$ and independent of $M^{\mathrm{r}}(\{0\}) \sim \mathcal{N}(0, \nu(\{0\}))$. Let $\{M^{\mathrm{i}}(A),\ A \in \mathcal{B}(0,\infty)\}$ be independent copy of $\{M^{\mathrm{r}}(A),\ A \in \mathcal{B}(0,\infty)\}$. Furthermore let $M^{\mathrm{r}}(\{0\}) \sim \mathcal{N}(0, \nu(\{0\}))$ be a random variable independent of $\{M^{\mathrm{r}}(A)\}$ and $\{M^{\mathrm{i}}(A)\}$. We next extend uniquely $\{M^{\mathrm{r}}(A)\}$ and $\{M^{\mathrm{i}}(A)\}$ on the whole line $\mathbb{R}$ to satisfy $M^{\mathrm{r}}(A) = M^{\mathrm{r}}(-A)$ and $M^{\mathrm{i}}(A) = -M^{\mathrm{i}}(-A)$ for all $A \in \mathcal{B}(\mathbb{R})$. This condition yields $M^{\mathrm{i}}(\{0\}) = 0$. We now define the *tilde complex Gaussian white noise* $\{\widetilde{Z}(A),\ A \in \mathcal{B}(\mathbb{R})\}$ by $\widetilde{Z}(A) = M^{\mathrm{r}}(A) + iM^{\mathrm{i}}(A)$. Note that

$$\mathbb{E}\, \widetilde{Z(A)}\overline{\widetilde{Z(A)}} = \nu(A).$$

This definition is valid if instead from ν on $\mathbb{R}$ we consider a symmetric measure ν on a symmetric interval $[-a, a]$.

In the case when ν is the Lebesge measure on $\mathbb{R}$ we call the above defined tilde white noise by *tilde complex Wiener white noise.* Such the process we denote by $\{\widetilde{W}\}$.

Consider now the case when $T = \mathbb{R}$ and $\widetilde{\mathbb{L}}^2(\nu)$ be the class of functions $f(x) = f^{\mathrm{r}}(x) + if^{\mathrm{i}}(x)$ be such that

$$f(x) = \overline{f(-x)}, \qquad \int_{-\infty}^{\infty} |f(x)|^2\, \nu(dx) < \infty. \tag{5.20}$$

For such functions we define the integral $I(f)$. The novelty here is that $\widetilde{Z}$ does not fulfill condition (5.16) on the whole line and therefore the second equation below is by definition.

$$\begin{aligned} I(f) &= \int_{-\infty}^{\infty} f(x)\, \widetilde{Z}(dx) \\ &= \int_{(-\infty,0)} f(x)\, \widetilde{Z}(dx) + f(0)\, M^{\mathrm{r}}(\{0\}) + \int_{(0,\infty)} f(x)\, \widetilde{Z}(dx) \\ &= \int_{(0,\infty)} \overline{f(x)}\, \overline{\widetilde{Z}(dx)} + f(0)\, M^{\mathrm{r}}(\{0\}) + \int_{(0,\infty)} f(x)\, \widetilde{Z}(dx) \\ &= 2\left(\int_{(0,\infty)} f^{\mathrm{r}}(x)\, M^{\mathrm{r}}(dx) - \int_{(0,\infty)} f^{\mathrm{i}}(x)\, M^{\mathrm{i}}(dx)\right) + f^{\mathrm{r}}(0)\, M(\{0\}) \end{aligned}$$

Notice that from (5.21) (in the proof we use that $f^{\mathrm{i}}(0) = 0$)

$$\mathbb{E}\,(I(f))^2 = \int_{-\infty}^{\infty} |f(x)|^2\, \nu(dx)$$

and

$$\mathbb{E}\, I(f)\overline{I(f')} = \int_{-\infty}^{\infty} f(x)\overline{f'(x)}\, \nu(dx)\ .$$

Assuming that ν is absolute continuous wrt the Lebesgue measure, let h be the spectral density of ν. We have for

$$\int_{-\infty}^{\infty} g(u)\, \widetilde{Z}(du), \quad g \in \mathbb{L}^2_{\mathbb{C}}(\nu)$$

is equal in distribution to

$$\int_{-\infty}^{\infty} g(u)\sqrt{h(u)}\, \widetilde{W}(du), \quad g \in \mathbb{L}^2_{\mathbb{C}}(\nu),$$

where $\{\widetilde{W}(A)\}$ is the tilde complex Wiener white noise (that is a complex tilde Gaussian white noise with the Lebesgue intensity measure). Actually we can consider a Gaussian process

$$\left\{\int_{-\infty}^{\infty} f(u)\, \widetilde{Z}(du), \quad f \in \mathbb{L}^2_{\mathbb{C}}(\nu)\right\}$$

and show that its fi-di distributions are the same as in the Gaussian process

$$\left\{\int_{-\infty}^{\infty} f(u)\sqrt{h(u)}\, \widetilde{W}(du), \quad f \in \mathbb{L}^2_{\mathbb{C}}(\nu).\right\}$$

Exercises

5.1 Show that

$$\{\int_{-\infty}^{\infty} f(u)\, \widetilde{Z}(du), \quad f \in \mathbb{L}^2_{\mathbb{C}}(\nu)\}$$

is equal in distribution to

$$\int_{-\infty}^{\infty} f(u)\sqrt{h(u)}\, \widetilde{W}(du), \quad f \in \mathbb{L}^2_{\mathbb{C}}(\nu).$$

6 Spectral representation of Gaussian processes

6.1 Stationary processes

Theorem 6.1 [Bochner] *Let $R : \mathbb{R} \to \mathbb{C}$ be continuous. $\{R(t-s),\ s,t \in \mathbb{R}\}$ is Hermitian and positive definite if and only if there exists a finite measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that*

$$R(t) = \int_{-\infty}^{\infty} e^{itu}\, \nu(du), \qquad t \in \mathbb{R}.$$

The measure ν is called spectral measure.

Note that for R to be real we need to assume that ν is symmetric, that is $\nu(A) = \nu(-A)$ for all A.

If we say that $\{X(t)\}$ is a stationary process with spectral measure ν on $\mathbb{R}$ then tacitly assume that the covariance function $R(t)$ is continuous.

Let $\{\widetilde{Z}(A),\ A \in \mathcal{B}(\mathbb{R})\}$ be a tilde complex white noise from Section 5.2.

Theorem 6.2 *A stationary Gaussian process with spectral measure ν on $\mathbb{R}$ has the following spectral representation:*

$$\begin{aligned} X(t) &= \int_{-\infty}^{\infty} e^{itu}\, \widetilde{Z}(\mathrm{d}u) \\ &= 2\int_0^{\infty} \cos tu\, M^{\mathrm{r}}(\mathrm{d}u) - 2\int_0^{\infty} \sin tu\, M^{\mathrm{i}}(\mathrm{d}u) + M^{\mathrm{r}}(\{0\}),\ t \in \mathbb{R} \end{aligned} \tag{6.1}$$

or

$$\begin{aligned} X'(t) &= \int_{-\infty}^{\infty} e^{itu}\, \overline{\widetilde{Z}(\mathrm{d}u)} \\ &= 2\int_0^{\infty} \cos tu\, M^{\mathrm{r}}(\mathrm{d}u) + 2\int_0^{\infty} \sin tu\, M^{\mathrm{i}}(\mathrm{d}u) + M^{\mathrm{r}}(\{0\}),\ t \in \mathbb{R} \end{aligned} \tag{6.2}$$

Proof Formula (6.21) follows directly from (5.21) applied to $f^{\mathrm{r}}(t) = \cos tu - 1$ and $f^{\mathrm{i}}(t) = \sin tu$. Next note that $X(t)$ and $X'(t)$ have the same covariance function. □

Example 6.3 Let $\nu = \delta_{-1} + \delta_0 + \delta_{-1}$. Then

$$R(t) = \int_{-\infty}^{\infty} e^{itu}\, \nu(du) = 1 + 2\cos t\,.$$

Let $M_1^{\mathrm{r}}, M_1^{\mathrm{i}}, M_0^{\mathrm{r}}$ be independent and $M_1^{\mathrm{r}}, M_1^{\mathrm{i}} \sim \mathcal{N}(0,1/2)$ and $M_0^{\mathrm{r}} \sim \mathcal{N}(0,1)$. The tilde complex white noise $\widetilde{Z}$ is concentrated on points $-1, 0, 1$:

$$\widetilde{Z}(\{1\}) = M_1^{\mathrm{r}} + iM_1^{\mathrm{i}}, \quad \widetilde{Z}(\{0\}) = M_0^{\mathrm{r}}, \quad \widetilde{Z}(\{-1\}) = M_1^{\mathrm{r}} - iM_1^{\mathrm{i}}.$$

Then

$$X(t) = \int_{-\infty}^{\infty} e^{itu}\, \widetilde{Z}(\,\mathrm{d}u) = 2M_1^{\mathrm{r}}\cos t - 2M_1^{\mathrm{i}}\sin t + M_0^{\mathrm{r}}.$$

Clearly

$$\begin{aligned} \mathbb{E}\, X(t)X(s) &= 4\mathbb{E}\,(M_1^{\mathrm{r}})^2 \cos t\cos s + 4\mathbb{E}\,(M_1^{\mathrm{i}})^2 \sin t\sin s + \mathbb{E}\,(M_0^{\mathrm{r}})^2 \\ &= 2(\cos t\cos s + \sin t\sin s) + 1 = 2\cos(t-s) + 1. \end{aligned}$$

Notice that

$$X'(t) = 2M_1^{\mathrm{r}}\cos t + 2M_1^{\mathrm{i}}\sin t + M_0^{\mathrm{r}}.$$

is another spectral representation of $\{X(t)\}$.

Example 6.4 Suppose now that a symmetric discrete measure ν has atoms at λ_j, $j = -N, \ldots, N$; at $\lambda_j > 0$ a mass σ_j^2, $j = 1, \ldots, N$ and at $\lambda_0 = 0$ a mass σ_0^2. Let $\{M_j^{\mathrm{r}},\ M_j^{\mathrm{i}}\}$ are independent random variables, $M_j^{\mathrm{r}}, M_j^{\mathrm{i}} \sim \mathcal{N}(0,1/2)$ and $M_0^{\mathrm{r}} \sim \mathcal{N}(0,1)$ Then a process

$$X(t) = 2\sum_{j=1}^{N}(M_j^{\mathrm{r}}\cos\lambda_j t + M_j^{\mathrm{i}}\sin\lambda_j t) + M_0^{\mathrm{r}}$$

is stationary Gaussian with spectral measure ν. Let $\Phi_j \in [-\pi,\pi]$ be such that

$$\cos\Phi_j = \frac{M_j^{\mathrm{r}}}{\sqrt{(M_j^{\mathrm{r}})^2 + (M_j^{\mathrm{i}})^2}}$$

and let

$$A_j = \sqrt{(M_j^{\mathrm{r}})^2 + (M_j^{\mathrm{i}})^2}$$

for $j = 1, \ldots, N$. We now can write

$$X(t) = 2\sum_{j=1}^{N} A_j \cos(\lambda_j t + \Phi_j) + M_0^{\mathrm{r}}. \tag{6.23}$$

Note that A_j, Φ_j $(j = 1, \ldots N)$ are independent , A_j has Raleigh distribution and Φ_j is uniformly distributed.

Example 6.5 Function $e^{-\alpha|t-s|}$ is symmetric positive definite with spectral measure $h(u)\,\mathrm{d}u$, where

$$h(u) = \frac{\alpha}{\pi(\alpha^2 + u^2)}, \qquad u \in \mathbb{R},$$

see Feller II.[2] Centered Gaussian process with covariance function $e^{-\alpha|t-s|}$ is called a *stationary Orstein-Uhlenbeck process.* Let $W(t) = W[0,t]$, where W is the Wiener white noise, that is a Gaussian white noise with the Lebesgue intensity measure (see Example (5.2)). It is straightforward to prove that

$$X(t) = e^{-\alpha t} W(e^{2\alpha t}), \qquad t \in \mathbb{R},$$

or that

$$X'(t) = \int_{-\infty}^{t} e^{-\alpha(t-u)}\, W(\,\mathrm{d}u), \qquad t \in \mathbb{R}.$$

are stationary Orstein-Uhlenbeck processes.

Example 6.6 Consider an IG process defined in Example 1.6 Since $r(t) = \int \exp(itu)\,\nu'(\,\mathrm{d}u)$ for some symmetric and finite measure ν', we have

$$\begin{aligned}
\sigma^2(t) = \int_0^t\int_0^t r(v-w)\,\mathrm{d}w\,\mathrm{d}v &= \int_0^t\int_0^t\int_{-\infty}^{\infty} e^{itu}\,\nu(\,\mathrm{d}u)\,\mathrm{d}v'\,\mathrm{d}w \\
= \int_{-\infty}^{\infty}\int_0^t\int_0^t e^{itu}\,\mathrm{d}v\,\mathrm{d}w\,\nu'(\,\mathrm{d}u) &= \int_{-\infty}^{\infty} \frac{|e^{itu}-1|^2}{u^2}\,\nu'(\,\mathrm{d}u) \\
&= \int_{-\infty}^{\infty} |e^{itu}-1|^2\,\nu(\,\mathrm{d}u),
\end{aligned}$$

where $\nu(\,\mathrm{d}u) = (1/u^2)\nu'(\,\mathrm{d}u)$.

Exercises

[2] ????

6.1 Show from the definition that for two independent standard Gaussian variables X and X', the process

$$X(t) = X(\cos t - 1) + X' \sin t$$

is with stationary increments.

6.2 Show that

$$X(t) = e^{-\alpha t} W(e^{2\alpha t}), \qquad t \in \mathbb{R},$$

or

$$X'(t) = \int_{-\infty}^{t} e^{-\alpha(t-u)}\, M(\,du), \qquad t \in \mathbb{R}.$$

$$X''(t) = \int_{t}^{\infty} e^{-\alpha(u-t)}\, M(\,du), \qquad t \in \mathbb{R},$$

where M is a Gaussian white noise with the Lebesgue intensity measure, are stationary Orstein-Uhlenbeck processes.

6.3 Show that a stationary Orstein-Uhlenbeck process is Markov.

6.4 Show that if $\{X(t)\}$ is a stationary Gaussian process with continuous covariance function is Markovian, then it is an Orstein-Uhlenbeck process. Hint. For a proof of a reverse statement show that the assumptions yield the exponential form of the covariance function.

6.2 Stationary sequences

In the following theorem function f is defined on $d\mathbb{Z} = \{\ldots, -2d, -d, 0, d, 2d, \ldots\}$, for some $d > 0$.

Theorem 6.7 [Herglotz] $\{f(x-y),\ x, y \in d\mathbb{Z}\}$ *is Hermitian and positive definite if and only if there exists a finite measure on* $[-\pi/d, \pi/d]$, $\mathcal{B}[-\pi/d, \pi/d])$ *such that*

$$f(nd) = \int_{[-\pi/d, \pi/d]} e^{indu}\, \nu(\,du), \qquad t \in \mathbb{R}.$$

The measure ν *is called* spectral.

Proof Loève, M. *Probability Theory*, Van Nostrand, Princton.

Note that for $\{f(nd)\}$ to be real we need to assume that ν is symmetric.

In this section we study real stationary Gaussian sequences with covariance function $r_n(d) = \mathbb{E}\, X_0 X_n\; n \in \mathbb{Z}$, which by Herglotz theorem can be expressed by

$$r_n(d) = \int_{[-\pi/d,\pi/d]} e^{indu}\,\nu(\,\mathrm{d}u).$$

The we say that $\{X_n\}$ is a stationary Gaussian sequence with spectral measure ν on $[-\pi/d, \pi/d]$.

Example 6.8 Let $\{X(t),\ t \in \mathbb{R}\}$ be a stationary process with spectral measure ν and define $\{X_n = X(nd),\ n \in \mathbb{Z}\}$. Clearly $\{X_n\}$ is stationary. Compute now its covariance sequence

$$\begin{aligned} r_n(d) &= \mathbb{E}\, X_0 X_n = R(nd) = \int_{-\infty}^{\infty} e^{indu}\,\nu(\,\mathrm{d}u) \\ &= \sum_{k=-\infty}^{\infty} \int_{((2k-1)\pi/d,(2k+1)\pi/d]} e^{indu}\,\nu(\,\mathrm{d}u). \end{aligned}$$

Since function $u \to e^{indu}$ is periodic with period $2\pi/d$ we receive

$$R(nd) = \int_{(-\pi/d,\pi/d]} e^{indu}\,\nu_d(du), \tag{6.24}$$

where

$$\nu_d(A) = \sum_{k=-\infty}^{\infty} \nu((2k-1)\pi/d + A), \quad A \subset (-\pi/d, \pi/d), \tag{6.25}$$

and

$$\nu_d(\{-\pi/d\}) = \nu_d(\{\pi/d\}) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \nu(\{(2k-1)\pi/h + \pi/d\}) \tag{6.26}$$

Note that ν_d is a finite measure on $[-\pi, \pi]$. If ν has a continuous density $h(u)$, then ν_d has density, called a *spectral density*:

$$h_d(u) = \sum_{k=-\infty}^{\infty} h(u + \frac{2\pi k}{d}), \qquad -\pi/d < u < \pi/d \tag{6.27}$$

We want to give a *spectral representation* of a stationary real Gaussian sequence with a given covariance sequence $\{r_n(d)\}$ having the spectral measure ν on $[-\pi/d, \pi/d]$. Let for example

$$X_n = \int_{[-\pi/d,\pi/d]} e^{inu}\,M(\,\mathrm{d}u), \qquad n \in \mathbb{Z},$$

where M is a Gaussian white noise with intensity measure ν on $[-\pi/d, \pi/d]$. From Proposition (5.11) $\{X_n\}$ is a complex Gaussian sequence and

$$\mathbb{E}\, X_{n+m}\overline{X_n} = \int_{[-\pi/d,\pi/d]} e^{imdu}\nu(\,\mathrm{d}u)$$

which shows that $\{X_n\}$ is stationary with covariance sequence $\{r_n(d)\}$. Unfortunately this sequence is tilde complex and therefore to get a real spectral representation we need to use complex Gaussian white noises. It turns out that in this case we need to apply $\{\widetilde{Z}(A) = M^{\mathrm{r}}(A) + iM^{\mathrm{i}}(A),\ A \in \mathcal{B}[-\pi/d, \pi/d]\}$ from Example 5.8 (iii).

Theorem 6.9 *A stationary Gaussian sequence with symmetric spectral measure ν on $[-\pi/d, \pi/d]$, where $d > 0$, has the following spectral representation:*

$$X_n(d) = \tag{6.28}$$

$$= \int_{[-\pi/d,\pi/d]} e^{indu}\, \widetilde{Z}(\mathrm{d}u)$$

$$= 2\int_{(0,\pi/d]} \cos ndu\, M^{\mathrm{r}}(\,\mathrm{d}u) - 2\int_{(0,\pi/d]} \sin ndu\, M^{\mathrm{i}}(\,\mathrm{d}u) + M^{\mathrm{r}}(\{0\}),\ n \in \mathbb{Z} \tag{6.29}$$

or

$$X_n^{'}(d) = \tag{6.30}$$

$$= 2\int_{(0,\pi/d]} \cos nhu\, M^{\mathrm{r}}(\,\mathrm{d}u) + 2\int_{(0,\pi/d]} \sin ndu\, M^{\mathrm{i}}(\,\mathrm{d}u) + M^{\mathrm{r}}(\{0\}),\ n \in \mathbb{Z} \tag{6.31}$$

Proof One has to adapt formula (5.21) and the proof of Theorem 6.9, changing the domain of integration $\mathbb{R}$ to $[-\pi/d, \pi/d]$

Example 6.10 Let ν be discrete symmetric measure on $[-\pi/d, \pi/d]$ of form

$$\sum_{j=-(N-1)}^{N-1} \sigma_j^2 \delta_{s_j}$$

where $s_j = s_{-j}$ and $s_0 = 0$ and $\sigma_j \geq 0$. It is clear that M^{r} and M^{i} are concentrated on $\{s_k,\ k = -(N-1), \ldots, N-1\}$ and let $M^{\mathrm{r}}(\{s_j\}) = M_j^{\mathrm{r}}$, $M^{\mathrm{i}}(\{s_j\}) = M_j^{\mathrm{i}}$, $j = 1, \ldots, N-1$, M_0^{r} are independent and $M_j^{\mathrm{r}}, M_j^{\mathrm{i}} \sim \mathcal{N}(0, 1/2)$ for $j \neq 0$ and $M^{\mathrm{r}} \sim \mathcal{N}(0,1)$. Then from Theorem 6.9

$$X_n(d) = 2\sum_{j=1}^{N-1} \sigma_j (M_j^{\mathrm{r}} \cos ns_jd - M_j^{\mathrm{i}} \sin ns_jd) + M_0^{\mathrm{r}}. \tag{6.32}$$

6.3 Processes with stationary increments

Theorem 6.11 *Let $f(s,t)$ be continuous of form*

$$f(s,t) = \frac{1}{2}(\sigma^2(s) + \sigma^2(t) - \sigma^2(t-s)). \tag{6.33}$$

for some real even (symmetric) function $\sigma^2(t)$. Function $f(s,t)$ is positive definite if and only if

$$\sigma^2(t) = \int_{-\infty}^{\infty} |e^{itu} - 1|^2 \, \nu(du) \tag{6.34}$$

for some spectral *symmetric measure ν on $\mathbb{R}$ such that*

$$\int_{-\infty}^{\infty} \min(u^2, 1) \, \nu(\mathrm{d}u) < \infty \, .$$

Proof Let $f(s,t)$ be positive definite. Then by Lemma 3.1 function $\sigma^2(t)$ is negative definite and such functions were characterized in Proposition 3.4. Conversely substituting (6.34) to (6.33) we have

$$f(s,t) = \int_{-\infty}^{\infty} (e^{isu} - 1)(e^{-itu} - 1) \, \nu(\mathrm{d}u)$$

and the proof of positive definite property is immediate. □

Let M be a white noise with intensity measure ν. Clearly

$$X(t) = \int_{-\infty}^{\infty} (e^{itu} - 1) \, M(\mathrm{d}u)$$

has covariance function

$$R(s,t) = \int_{-\infty}^{\infty} (e^{isu} - 1)(e^{-itu} - 1) \, \nu(\mathrm{d}u)$$

but unfortunately the process is complex. Let $\widetilde{Z} = M^{\mathrm{r}} + iM^{\mathrm{i}}$ be a tilde complex white noise with intensity measure ν.

Theorem 6.12 *The Gaussian process with stationary increments and spectral measure ν on $\mathbb{R}$ has the following spectral representation:*

$$\begin{aligned} X(t) &= \int_{-\infty}^{\infty} (e^{itu} - 1) \, \widetilde{Z}(du) \\ &= 2\int_0^{\infty} (\cos(tu) - 1) \, M^{\mathrm{r}}(du) \\ &\quad -2\int_0^{\infty} \sin tu \, M^{\mathrm{i}}(du) + M^{\mathrm{r}}(\{0\}), \;\; t \in \mathbb{R}. \end{aligned} \tag{6.35}$$

Let $\widetilde{W}$ be the tilde complex Wiener white noise defined in Section 5.2. Note that $\{\widetilde{Z}(A)\}$ defined by

$$\widetilde{Z}(A) = \int_A (h(t))^{-1/2} 1(h(t) > 0)\, \widetilde{W}(dt)$$

is a complex tilde white noise with spectral density h. Note that this is an even function. Since ν is symmetric we can assume that h is an even function. We can express formula (6.35) as follows:

$$X'(t) = \int_{-\infty}^{\infty} (e^{itu} - 1)(h(t))^{1/2} 1(h(t) > 0)\, \widetilde{W}(\,du) \tag{6.36}$$

$$= 2\int_0^{\infty} (\cos tu - 1) h^{1/2}(u)\, W^{\mathrm{r}}(du) + 2\int_0^{\infty} \sin tu h^{1/2}(u)\, W^{\mathrm{i}}(du), \tag{6.37}$$

where W^{r} and W^{i} are two independent Wiener white noise.

6.4 H–fBm

Consider H-fBm with spectral density function $h_H(u)$ defined by (2.9). Then from (6.36) we have

$$X'(t) = \int_{-\infty}^{\infty} (e^{itu} - 1) C_2^{-1}(H) |u|^{-H-1/2}\, \widetilde{Z}(\,du) \tag{6.38}$$

$$= 2\int_0^{\infty} C_2^{-1}(H) u^{-H-1/2} (\cos tu - 1)\, W^{\mathrm{r}}(\,du) - 2\int_0^{\infty} C_2^{-1}(H) u^{-H-1/2} \sin tu\, W^{\mathrm{i}}(\,du). \tag{6.39}$$

where $C_2(H)$ was defined by (2.10). Therefore (6.39) is a spectral representation of the H–fBm. We now rewrite

$$2\int_0^{\infty} C_2^{-1}(H) u^{-H-1/2} (\cos tu - 1)\, W^{\mathrm{r}}(\,du) - 2\int_0^{\infty} C_2^{-1}(H) u^{-H-1/2} \sin tu\, W^{\mathrm{i}}(\,du)$$

$$= 2\int_0^{\infty} C_2^{-1}(H) u^{-(H-1/2)} \frac{\cos tu - 1}{u}\, W^{\mathrm{r}}(\,du) - 2\int_0^{\infty} C_2^{-1}(H) u^{-(H-1/2)} \frac{\sin tu}{u}\, W^{\mathrm{i}}(\,du).$$

Notice that $\mathbb{R} \ni u \to \frac{\cos tu - 1}{u}$ is odd and $\mathbb{R} \ni u \to \frac{\sin tu}{u}$ even and that the process

$$2\int_0^\infty C_2^{-1}(H)u^{-(H-1/2)}\frac{\cos tu - 1}{u}\, W^{\mathrm{r}}(\,du) \\ -2\int_0^\infty C_2^{-1}(H)u^{-(H-1/2)}\frac{\sin tu}{u}\, W^{\mathrm{i}}(du), \quad t \geq 0$$

has the same distribution as the process

$$2\int_0^\infty C_2^{-1}(H)u^{-(H-1/2)}\frac{\cos tu - 1}{u}\, W^{\mathrm{i}}(\,du) \\ +2\int_0^\infty C_2^{-1}(H)u^{-(H-1/2)}\frac{\sin tu}{u}\, W^{\mathrm{r}}(\,du), \quad t \geq 0$$

and hence by (5.21)

$$\begin{aligned} & 2\int_0^\infty C_2^{-1}(H)u^{-(H-1/2)}\frac{\cos tu - 1}{u}\, W^{\mathrm{i}}(\,du) \\ & \quad -2\int_0^\infty C_2^{-1}(H)u^{-(H-1/2)}\frac{\sin tu}{u}\, W^{\mathrm{r}}(\,du) \\ = & \int_{-\infty}^\infty C_2^{-1}(H)|u|^{-(H-1/2)}\frac{\sin tu}{u}\, W^{\mathrm{r}}(\,du) \\ & \quad +2\int_{-\infty}^\infty C_2^{-1}(H)|u|^{-(H-1/2)}\frac{\cos tu - 1}{u}\, W^{\mathrm{i}}(\,du) \\ = & \int_{-\infty}^\infty \frac{e^{iut}-1}{iu} C_2^{-1}(H)|u|^{-(H-1/2)}\widetilde{Z}(\,du) \end{aligned} \tag{6.40}$$

The formula (6.40) is called *harmonizable representation* of the H-fBm.

Consider the H-fBn $\{X_n,\ n \in \mathbb{Z}\}$ defined in Section 2.1. Notice that $X(0), X(1), \ldots$ is an H–fBn with the covariance sequence given by (2.11).

Proposition 6.13 *The H–fBn has spectral density*

$$h_H^{\mathrm{a}}(u) = C_2^{-2}(H)|e^{itu}-1|^2 \sum_{k=-\infty}^{\infty} \frac{1}{|u+2\pi k|^{2H+1}} \tag{6.41}$$

$$= \frac{\int_0^\infty \cos(xu)(\sin(x/2))^2 x^{-2H-1}\,dx}{\int_0^\infty (\sin(x/2))^2 x^{-2H-1}\,dx}, \qquad -\pi \leq u \leq \pi \tag{6.42}$$

Proof Samorodnitsky and Taqqu, p. 333.

6.5 Isotropic random fields

Consider now a stationary (sometimes it is said *homogeneous*) Gaussian field $\{X(\boldsymbol{t}),\ \boldsymbol{t} \in \mathbb{R}^d\}$ with covariance function

$$r(\boldsymbol{t}) = \int_{\mathbf{R}^d} e^{i\boldsymbol{t}^T \boldsymbol{x}}\,\nu(\,\mathrm{d}\boldsymbol{x})$$

for some finite measure ν. We say that this field is *isotropic* if $r(\boldsymbol{t}) = \rho(|\boldsymbol{t}|)$ for some function $\rho : \mathbb{R} \to \mathbb{R}$.

Comments. Samorodnitsky and Taqqu (1994), Theorem 6.11 – Doob (1953), p. 552, Lindgren (1999) Loéve, Breiman

7 Integration wrt Gaussian processes

In this section $\{X(t),\ t \in T\}$ is a centered Gaussian process, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $T = \mathbb{R}, \mathbb{R}_+$ or $[0,1]$. Note that $X(t) \in \mathbb{L}^2(\mathbb{P})$ for all $t \in T$. The Gaussian process has the covariance function $R(s,t)$, which is assumed to be continuous. Our aim is to define the notion of $\int_T f(t)\ \mathrm{d}X(t)$. We will deal analogously as it was in the case of integration wrt Gaussian white noise.

Let $\mathbb{H}_X$ be a Hilbert space defined defined as the $\mathbb{L}^2(\mathbb{P})$–closure of the space of all finite linear combinations $\sum_j a_j X(t_j)$. The sapce $\mathbb{H}_X$ is a subspace of $\mathbb{L}^2(\mathbb{P})$. We will show the isometry between a Hilbert space $\mathcal{H}(R)$ defined in a moment and the Hilbert space $\mathbb{H}_X$.

7.1 Space $\mathcal{H}(R)$

As usual we start with the definition of the integral for simple function of form $f(t) = 1(t \in (x, x+w])$;

$$\int_T f(t)\ \mathrm{d}X(t) = X(x+w) - X(w).$$

Note that

$$\int_T f(t)\ \mathrm{d}X(t) \sim \mathcal{N}(0, \sigma^2(w)).$$

Moreover if $f'(t) = 1(t \in (y, y+v])$ is another simple function, then

$$\begin{aligned}
&\mathbb{E}\,(X(x+w) - X(x))(X(y+v) - X(y)) = \\
&= \mathbb{E}\left(\int_0^\infty f(t)\ \mathrm{d}X(t) \int_0^\infty g(t)\ \mathrm{d}X(t)\right) &(7.43)\\
&= R(x+w, y+v) - R(x+w, y) - R(x, y+v) + R(x,y). &(7.44)
\end{aligned}$$

If X is with stationary increments, that is with covariance function

$$R(s,t)=\frac{1}{2}(\sigma^2(s)+\sigma^2(t)-\sigma^2(t-s)),$$

where $\sigma^2(0)=0$, $\sigma^2(t)=\sigma^2(-t)$ we have

$$\begin{aligned}
&\mathbb{E}\,(X(x+w)-X(x))(X(y+v)-X(y)) \\
&= \mathbb{E}\left(\int_0^\infty f(t)\ \mathrm{d}X(t)\int_0^\infty g(t)\ \mathrm{d}X(t)\right) \\
&= -\frac{1}{2}(\sigma^2(x+w-(y+v))-\sigma^2(x+w-y)-\sigma^2(x-(y+v))+\sigma^2(x-y)).
\end{aligned} \tag{7.45}$$

Let $\mathcal{H}_{\mathrm{s}}$ be a space of simple functions of form

$$f(t)=\sum_{j=1}^n a_j 1(t\in A_j]), \tag{7.46}$$

where A_j are bounded Borel subsets of T. For such functions We have to consider two cases:

- $R(s,t)$ is of bounded variation on every finite domain of $T\times T$; we say in short that $R(s,t)$ is of locally bounded variation,
- $R(s,t)$ if of unbounded variation for on a finite domain of $T\times T$; than we say in short that $R(s,t)$ is of locally unbounded variation.

We are going to define an inner product which behaves on indicator function $1(\cdot\in A)$ consistently with (7.43). If R is of locally bounded variation, then we define

$$\ll 1(\cdot\in A),1(\cdot\in A')\gg_R=\int_T\int_T 1(x\in A)\,1(y\in A')\ \mathrm{d}^2R(x,y), \tag{7.47}$$

for every set A,A' which is a finite union of intervals $(x,x+v]$. Unfortunately for $R(s,t)$ of locally unbounded variation the above integral may diverge. For example consider an H–fBm with $H<1/2$. Then

$$\int_0^1\int_0^1\ \mathrm{d}^2R(s,t)=\infty.$$

Thus if R is of locally unbounded variation, then we define

$$\ll 1(\cdot\in A),1(\cdot\in A')\gg_R=\int_T\int_T R(x,y)\ \mathrm{d}1(x\in A)\ \mathrm{d}1(y\in A'), \tag{7.48}$$

for every set A,A' which is a finite union of intervals $(x,x+v]$. In the next lemma we assume that the measre generated by $R(s,t)$ is positive.

Lemma 7.1 (i) *If R is of locally bounded variation positive measure, then $\ll \cdot,\cdot \gg_R$ defined by (7.47) is an inner product on the space $\{1(\cdot \in A),\ A$ bounded Borel subset of $T\}$.*
(ii) (ii)[3] *If R is of locally unbounded variation, then $\ll \cdot,\cdot \gg_R$ defined by (7.48) is an inner product on the space $\{1(\cdot \in A),\ A \in$* {bounded Jordan measurable subsets of T}}.

Proof (i) Let A be bounded Borel. For every ϵ there exists $A' = \sum_{j=1}^n (x_j, x_j + v_]$ such that $R(A \times A???A' \times A') < \epsilon$. Hence we have

$$\int_A \int_A \mathrm{d}^2 R(s,t) \geq 0.$$

Other conditions are obviously fulfilled.
(ii) ????? □It is clear that in these two cases we can extend the

above inner product for simple functions of form $\sum_{j=1}^n a_j 1(\cdot \in A_j)$ where A_j are bounded Borel subsets of T. The space of such the simple functions we denote by $\mathcal{H}_s$. Thus the space $(\mathcal{H}_s, \ll \cdot,\cdot \gg_R)$, where

- if R is a positive measure of locally bounded variation, then

$$\ll f, f' \gg_R = \int_T \int_T f(x) f'(y)\ \mathrm{d}^2 R(x,y),$$

- if R is a positive measure of locally unbounded variation, then

$$\ll f, f' \gg_R = \frac{-1}{2} \int_T \int_T R(x,y)\ \mathrm{d}f(x)\ \mathrm{d}f'(y),$$

is an inner product space but it is not Hilbert.

Note that in the case when R is defined by an si process, then we have

- if R is of bounded variation

$$\ll f, f' \gg_R = \int_T \int_T f(x) f'(y)\ \mathrm{d}^2 \sigma^2(x-y),$$

- otherwise

$$\ll f, f' \gg_R = \frac{-1}{2} \int_T \int_T \sigma^2(x,y)\ \mathrm{d}f(x)\ \mathrm{d}f'(y).$$

Let

$$\|f\|_R = \sqrt{\ll f, f \gg_R}.$$

Two simple functions f, g will be considered identical if $\|f-g\|_R = 0$.

[3] To jest na razie odgadniete. Taka funkcja ma wahanie ograniczne (Lojasiewicz) na dowolnym $[0,t]$. Czy to wahanie jest rowne 0?

7.2 R of bounded variation

For $f, g \in \mathcal{H}_{\mathrm{s}}$ we define formally

$$\ll f, g \gg_R = \int_T \int_T f(s) g(t)\ \mathrm{d}^2 R(s,t). \tag{7.49}$$

Clearly the above definition is consistent with (7.47). In case of processes with stationary increments, we have to integrate, instead from $\mathrm{d}^2 R(s,t)$, wrt $-(1/2)\,\mathrm{d}^2\sigma^2(s-t)$. Note that, under our assumptions, $(s,t) \to \sigma^2(s-t)$ is of bounded variations on every finite domain too.

Lemma 7.2 *$(\mathcal{H}_{\mathrm{s}}, \ll \cdots, \cdot \gg_R)$ is an inner product space and hence for $f_1, f_2 \in \mathcal{H}_{\mathrm{s}}$*

$$(\ll f_1, f_2 \gg_R)^2 \leq \ll f_1, f_1 \gg_R \ll f_2, f_2 \gg_R . \tag{7.50}$$

Proof We have to check whether conditions of the inner product space are fulfilled. All are clear but one condition: $\ll f, f \gg_R \geq 0$ and $\ll f, f \gg_R = 0$ if and only if $f = 0$. However for $f(t) = \sum_{j=1}^n a_j 1(t \in (x_j, x_j + w_j])$

$$\ll f, f \gg_R = \mathrm{Var} \sum_{j=1}^n a_j (X(x_j + w_j) - X(x_j)) \geq 0.$$

The inequality (7.50) holds for inner product spaces; see Sikorski v. II, p. 107. For the second part of the condition note that $\ll f, f \gg_R = 0$ means thar f is equivalent to 0. □

We now define the Hilbert space $\mathcal{H}$ as the completion of $\mathcal{H}_{\mathrm{s}}$, so that it is a Hilbert space with inner product, denoted again by $\ll \cdot, \cdot \gg_R$. A typical element in $\mathcal{H}$ is a Cauchy sequence of simple functions. There is a question about a more detailed describtion of $\mathcal{H}$ and a formula for inner product of two functions from $\mathcal{H}$.

In the case when $R(s,t)$ defines a locally finite **positive** measure $R(\cdot)$ we can proceed as follows. Consider the space of functions

$$\mathcal{H}(R) = \{f : T \to \mathbb{R} : \ll f, f \gg_R < \infty.\}$$

Proposition 7.3 $\mathcal{H} = \mathcal{H}(R)$.

Proof We start from the inner product space $(\mathcal{H}_{\mathrm{s}}, \ll \cdot, \cdot \gg_R)$. Let $\{f_n\}$, where $f_n \in \mathcal{H}_{\mathrm{step}}$, be a Cauchy sequence with respect to norm $\|\cdot\|_R$. We have to show that f_n is also a Cauchy sequence for the convergence with respect to measure R. Then there exists a subsequence $\{f_{n_k}\}$ converging to a function f, R-almost surely. We have to show that $\|f - f_n\|_R \to 0$.[4]

□

Let $|R|(x,y)$ be the of total variation measure defined by R. Let $\mathcal{H}^\circ$ be the class of functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_T \int_T |f(s)f(t)| \ \mathrm{d}^2|R|(s,t) < \infty,$$

$$\int_T \int_T f(s)1(t \in (a,b]) \ \mathrm{d}^2|R|(s,t) < \infty.$$

Under this assumption we have the following result.

Proposition 7.4 *Functions from $\mathcal{H}^\circ$ form a dense subset of $\mathcal{H}_R$. If $f, g \in \mathcal{H}_R$ and $\int_T \int_T |f(s)g(t)| \ \mathrm{d}^2|R|(s,t) < \infty$, then*

$$\ll f, g \gg_R = \int_T \int_T f(s)g(t) \ \mathrm{d}^2 R(s,t).$$

Proof huangcambanis p. 589.

We now define $\int_T f(t) \ \mathrm{d}X(t)$ for simple functions of form $f(t) = \sum_{j=1}^n a_j \int (t \in (x_j, x_j + v_j])$ in the usual way. We denote such the subsepce of simple functions by $\mathcal{H}_{\text{step}}$. Next the map

$$\mathcal{H}_{\text{step}} \ni f \to \int_T f \ \mathrm{d}X \in \mathbb{H}(X)$$

preserves inner product and hence it can be extended to an isomorphism on $\mathcal{H}(R)$ to a closed subspace of $\mathbb{H}(X)$. However $1(\cdot \in [0,t]) \in \mathcal{H}(R)$ and $X(t) = \int 1(s \in [0,t]) \ \mathrm{d}X(s)$, and hence it follows that the isomorphism is onto $\mathbb{H}(X)$. Concluding the map

$$\mathcal{H}(R) \ni f \to I(f) = \int_0^\infty f(t) \ \mathrm{d}X(t) \in \mathbb{H}(X)$$

establishes an isometric embedding.

We now consider special cases of interest. First we study the Wiener process, which is a special case of the fractional Brownian motion.

Example 7.5 Consider $\{W(t), \ t \in \mathbb{R}_+\}$ a Wiener process. In this case $\sigma^2(x-y) = |x-y|$. It is easy to see that for two simple functions f, g

$$\ll f, g \gg_R = \int_0^\infty f(t)g(t) \ \mathrm{d}t. \tag{7.51}$$

[4] Skonczyc. Popatrzec do Sikorskiego v.2, str 16.

In fact

$$\mathcal{H}(R) = \mathcal{L}^2(\lambda)$$

and the inner product is given by (7.51).

Transformation

$$\mathcal{H}(R) \ni f \to I(f) = \int_0^\infty f(t)\ \mathrm{d}W(t) \in \mathbb{H}(W)$$

establishes an isometric embedding.

Example 7.6 Consider $\{X(t),\ t \in \mathbb{R}_+\}$ an IG process, where $X(t) = \int_0^t Z(s)\ \mathrm{d}s$ and Z is stationary with covariance function $\{r(s-t)\}$. In this case $\sigma^2(x-y) = 2\int_0^{|x-y|}\int_0^w r(v)\ \mathrm{d}v\ \mathrm{d}w$. It is easy to see that for two simple functions f, g

$$\ll f, g \gg_R = 2\int_0^\infty f(t)g(s)r(t-s)\ \mathrm{d}t\ \mathrm{d}s. \qquad (7.52)$$

Transformation

$$\mathcal{H}(R) \ni f \to I(f) = \int_0^\infty f(t)\ \mathrm{d}X(t) = \int_0^\infty f(t)Z(t)\ \mathrm{d}t \in \mathbb{H}(W)$$

establishes an isometric embedding.

7.3 Integration wrt fBm; $H > 1/2$

In this subsection we consider $\{X(t),\ t \geq 0\}$, where $X(t) = B_H(t)$. Without loss of generality we suppose that $T = \mathbb{R}_+$. We have $R(s,t) = \frac{1}{2}(\sigma^2(t) + \sigma^2(s) - \sigma^2(t-s))$, where

$$\sigma^2(t) = |t|^{2H}, \quad t \in \mathbb{R}.$$

Remark [5] Note that in this case

$$\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) = \int_0^s \int_0^t H(2H-1)|w-v|^{2H-2}\ \mathrm{d}w\ \mathrm{d}v,$$

and therefore $\mathrm{d}^2 R(s,t) = H(2H-1)|s-t|^{2H-2}\ \mathrm{d}s\ \mathrm{d}t$. Functions $f, g \in \mathcal{L}^2(\lambda)$ belong to $\mathcal{H}(R)$ [6] and

$$\ll f, g \gg_R = H(2H-1)\int_0^\infty \int_0^\infty f(t)g(s)|s-t|^{2H-2}\ \mathrm{d}s\,\mathrm{d}t < \infty.$$

[5] Spawdzic ?

[6] To be proved!!!

For $f \in \mathcal{H}(R)$ we have

$$\int_0^\infty f(t)\ \mathrm{d}B_H(t) \sim \mathcal{N}(0, < f, f >_R).$$

It can be proved (see Ruzmaikina (2000), Statement 4.1) that if $H > 1/2$ and $f \in \mathcal{L}_\lambda^{1/H}$, then $f \in \mathcal{H}(R)$.

Lemma 7.7

$$\int_{-\infty}^{x \wedge y} c_H^2 (x-u)^{H-\frac{3}{2}} (x-u)^{H-\frac{3}{2}}\ \mathrm{d}u = H(2H-1)|x-y|^{2H-2},$$

where

$$c_H = \sqrt{\frac{H(2H-1)\Gamma(3/2-H)}{\Gamma(H-1/2)\Gamma(2-2H)}}.$$

Proof Gripenberg and Norros [JAP, 33,400–410].

Define for $f \in \mathcal{H}(R)$ define (see Oksendal and Zhang, 2001)

$$\Gamma_R f(u) = \int_u^\infty f(x) c_H (x-u)^{H-3/2}\ \mathrm{d}x.$$

Lemma 7.8 Γ_R *defines an isometry from* $\mathcal{H}_R$ *into* $\mathcal{L}^2(\mathbb{R}_+)$.

Lemma 7.9 *If* $\{Y(t),\ t \geq 0\}$ *is a centered Gaussian process with increasing variance function* $\operatorname{Var} Y(t) = \sigma_Y^2(t)$ *and covariance function*

$$\operatorname{Cov}(Y(s), Y(t)) = \min(\sigma^2(s), \sigma^2(t)),$$

then Y *is a process with independent increments.*

Proof Let $0 \leq x < y < z$. Then

$$\mathbb{E}\,(X(y)-X(x))(X(z)-X(y)) = \\ \min(\sigma^2(y),\sigma^2(z)) - \min(\sigma^2(y),\sigma^2(y)) - \min(\sigma^2(x),\sigma^2(z)) + \min(\sigma^2(x),\sigma^2(y)) = 0$$

□

We now define for each $t \geq 0$ function $\gamma(,\cdot)$by

$$\gamma(t,s) = \begin{cases} b_1 s^{1/2-H}(t-s)^{1/2-H}, & 0 < s < t \\ 0 & t \leq s \end{cases},$$

where

$$b_1 = (B(\frac{3}{2}-H, H-\frac{1}{2}))^{-1}.$$

Lemma 7.10 *For $0 < \alpha < 1$, $0 < x < 1$, we have*

$$\int_0^1 v^{-\alpha}(1-v)^{-\alpha}|x-v|^{2\alpha-1}\ \mathrm{d}t = B(\alpha, 1-\alpha).$$

Lemma 7.11

$$\int_0^t \gamma(t,v)|s-v|^{2H-2}\ \mathrm{d}v = 1, \qquad \textit{for}\ \ 0 \le s \le t$$

Proof Norros et al, p. 576. □

Proposition 7.12 *The centered Gaussian process*

$$M(t) = \int_0^t \gamma(t,s)\ \mathrm{d}X(s)$$

has independent increments and variance function $\mathbb{E}\, M^2(t) = Ct^{2-2H}$, where $C =$???.

Proof Let $t \le t'$. We have

$$\begin{aligned}
\mathbb{E}\, M(t)M(t') &= \mathbb{E}\left[\int_0^t \gamma(t,w)\ \mathrm{d}X(w) \int_0^t \gamma(t,v)\ \mathrm{d}X(v)\right] \\
&= \int_0^t \int_0^{t'} \gamma(t',w)\gamma(t',v)H(2H-1)|w-v|^{2H-2}\ \mathrm{d}w\ \mathrm{d}v \\
&= H(2H-1)\int_0^t \gamma(t,w)\ \mathrm{d}w \left(\int_0^{t'} \gamma(t',v)|w-v|^{2H-2}\ \mathrm{d}v\right)
\end{aligned}$$

Proposition 7.13 *Proces $\{M(t)\ t \in \mathbb{R}_+\}$ generates, up to sets of measure zero, the same filtration as $\{X(t),\ t \in \mathbb{R}_+\}$.*

Proof Norros et al, p. 580.

Corollary 7.14 *The process*

$$e^{\theta M(t) - \frac{\theta^2}{2}\mathbb{E}\, M^2(t)}$$

is a mean one martingale.

7.4 R is not of locally bounded variations

For $f, g \in \mathcal{H}_s$ we define the inner product by

$$\ll f, g \gg_R = \int_T \int_T R(t,s)\ \mathrm{d}f(t)\ \mathrm{d}g(s). \qquad (7.53)$$

Clearly the above definition is consistent with (7.48). In case of processes with stationary increments, we have to change the integrand to $\sigma^2(s-t)$. Note that, under our assumptions, $(x,y) \to \sigma^2(x-y)$ is not of bounded variations on some finite domain too. Let

$$||f||_R = \sqrt{\ll f, f \gg_R}.$$

Two functions $f, g \in \mathcal{H}_s$ will be considered identical if $||f-g||_R = 0$. We now define the Hilbert space $\mathcal{H}(R)^*$ as the completion of $\mathcal{H}^*_{\mathrm{step}}$, so that it is a Hilbert space with inner product, denoted again by $\ll \cdot, \cdot \gg_R$. There is a question about a formula for inner product for two functions from $\mathcal{H}(R)$. More precisely we ask for conditions when for $f, g \in \mathcal{H}(R)$ formula (7.53) holds.

Exercises

7.1 Show that (7.51) holds for simple functions.

Comments. Huang & Cambanis (1978), Norros *et al* (1999), Gripenberg and Norros (1996), Molcan & Golosov (1969), Ruzmaikina (2000)

8 Simulation

Suppose we want to make a computer simulation of an H–fBm on interval $[0, T]$. Since on computers we can only simulate a finite number of random variables we choose an integer N and $d = d(T, N) > 0$ such that $Nd = T$ and simulate values of $\{X(t)\}$ in points $0, d, 2d, \ldots, Nd$. Therefore in the next subsection we discuss a simulation method of stationary sequences, which bases on spectral representations of stationary sequences.

8.1 Stationary sequence

Formula (6.32) suggest the following procedure of simulation of a stationary Gaussian process $\{X(t),\ t \in \mathbb{R}\}$ with spectral measure ν having a continuous density $h(u)$. For this we have to discretize $\nu(\,\mathrm{d}u) = h(u)\ \mathrm{d}u$:

1. choose the simulated time interval $[0,T]$ and step $d > 0$ such that $Nd = T$, and N

2. set $s_j = (\pi j)/(Nd) = \pi j/N'$, for $j = 0, \ldots, N-1$,

3. approximate

$$h_{\mathrm{appr}}(s_j) \simeq \sum_{j=-J}^{J} h(s_j + \frac{2\pi j}{d}),$$

   where J is taken enough large,

4. compute the mass $\sigma_j^2 = \frac{\pi}{dN'} h_{\mathrm{approx}}(s_j)$, for $j = 0, \ldots, N-1$,

5. generate $U_0, \ldots, U_{2N-1}$ – independent and uniformly distributed random variables and set[7]

$$\begin{aligned} M_j^{\mathrm{r}} &= \cos(2\pi U_j)\sqrt{-\log U_{N+j}} \\ M_j^{\mathrm{i}} &= \sin(2\pi U_j)\sqrt{-\log U_{N+j}} \end{aligned}$$

   for $j = 0, \ldots, N-1$; (see (I.2.6)),

6. compute $X_n(d)$ for $n = 0, \ldots, N-1$ using formula (6.32).

In practice we need to simulate

$$X_n(d), n = 0, \ldots, N-1 \tag{8.54}$$

for large N. Therefore it is recommended to choose $N = 2^m$ for some m and use for computing (8.54) the Fast Fourier Transform (FFT) procedure.

8.2 Simulation of H–fBm

We want to simulate an H–fBm on $[0,T]$ and without loss of generality we assume that $T = 1$. Since fBm's are selfsimilar it suffices for simplicity to study how to simulate, instead from the sequence $B_H(j/N)$, $j = 0, \ldots, N-1$, the sequence $B(0), \ldots, B(N)$. We write $X_n = B_H(n+1) - B_H(n)$ and recall that $X(0), X(1), \ldots$ is an H–fBn with the covariance sequence given by (2.11). Note that from ([**????**]) we can write

$$X_n = 2\int_0^{\pi} \cos nu(h_H^{\mathrm{a}}(u))^{1/2}\, W^{\mathrm{r}}\, \mathrm{d}u) - 2\int_0^{\pi} \sin nu(h_H^{\mathrm{a}}(u))^{1/2}\, W^{\mathrm{i}}\, \mathrm{d}u)$$

[7] Czy 2 pod log wyrzucic?

where h_H^{a} was given in Proposition 6.13. We now discretize the spectral density function as follows: set $s_j = (2\pi j)/(Nd)$ and

$$h_N(u) = \sigma_j, \qquad \text{for} \quad s_j \leq t < s_{j+1},$$

where

$$\sigma_j = \left(\frac{2\pi}{N}\right)^{1/2} h_H^{\mathrm{a}}(s_j).$$

We then obtain

$$X_n \simeq \sqrt{2}\left(\frac{2\pi}{N}\right)^{1/2} \sum_{j=0}^{N-1} W_j^{\mathrm{r}} h_H^{\mathrm{a}}(\frac{2\pi j}{N}) \cos n\frac{2\pi j}{N} - \sqrt{2}\sum_{j=0}^{N-1} W_j^{\mathrm{i}} h_H^{\mathrm{a}}(\frac{2\pi j}{N}) \sin n\frac{2\pi j}{N},$$

for $j = 0, \ldots, N-1$ where $W_j^{\mathrm{r}}, W_j^{\mathrm{i}}$ are independent and identically standard Gaussian random variables.

Remark

8.3 Interpolation error

For a given H–fBm $\{B_H(t)\}$ we define $\{B_H^{(h)}(t),\ t \geq 0\}$ to be the continuous piecewise linear process such that $B_H^{(h)}(kh) = B_H(kh)$, $k = 0, 1, \ldots$. We want to study the difference between $W_H(t)$ and $W_H^{(h)}(t)$ when the time parameter is restricted to $0 \leq t \leq 1$. We define the following measures for the discretization error:

$$\delta_N^{\infty} = \mathbb{E} \sup_{0 \leq t \leq 1} |W_H(t) - W_H^{(1/n)}(t)|,$$

$$\delta_N^{p} = \left(\mathbb{E} \int_0^1 |W_H(t) - W_H^{(1/n)}(t)|^p \, dt\right)^{1/p},$$

where $1 \leq p < \infty$.

To analyze the difference $B_H(t) - B_H^{(d)}(t)$ we brake this process into processes

$$X_j(t) = B_H(t) - B_H^{(d)}(t), \quad (j-1)d \leq t \leq jd,$$

for $j = 1, \ldots, N$ and $Nd = 1$. Rescale these on $[0, 1]$ as follows:

$$X_j^{'}(s) = X_j(d(s + j - 1)), \quad 0 \leq s \leq 1.$$

We have

$$X^{'}(s) = B_H(d(s+j-1)) - B_H((j-1)d) - s(B_H(jd) - B_H((j-1)d)), \quad 0 \leq s \leq 1$$

for $j = 1\ldots, N$ and from the H-ss property of the fBm, these sequence of processes is in distribution equivalent to

$$\{d^H \overset{\circ}{B}_{H,j}(s), \quad 0 \le s \le 1\},$$

for $j = 1, 2, \ldots, N$, where

$$\overset{\circ}{B}_{H,j}(s) = B_H\left((s+j-1)\right) - B_H\left((j-1)\right) - s(B_H\left(j\right) - B_H\left((j-1)\right)).$$

Gaussian process

$$\overset{\circ}{B}_H(s) = B_H(s) - sB_H(1)$$

is the H*–fractional Brownian bridge.*

Proposition 8.1 *We have for* $1 \le p < \infty$

$$N^H \delta_N^p = c$$

where

$$c = \left(\mathbb{E}\int_0^1 |B_H(t) - tB_H(1)|^p\, dt\right)^{1/p}.$$

Proof We have

$$\begin{aligned}
\left(\mathbb{E}\int_0^1 |B_H(t) - B_H^{(1/N)}(t)|^p\, dt\right)^{1/p} &= \left(\mathbb{E}\sum_{j=1}^N \int_{\frac{j-1}{N}\le t<\frac{j}{N}} |B_H(t) - B_H^{(1/N)}(t)|^p\, dt\right)^{1/p} \\
&= \left(\frac{1}{N}\sum_{j=1}^N \mathbb{E}\int_0^1 |X_j^{'}(t)|\ \mathrm{d}t\right)^{1/p} \\
&= N^{-H}\left(\mathbb{E}\int_0^1 |\overset{\circ}{B}_{H,1}(t)|^p\, \mathrm{d}t\right)^{1/p}
\end{aligned}$$

which completes the proof. □

DALSZA CZESC NIE JEST SKONCZONA.

Lemma 8.2

$$\liminf_{n\to\infty} \frac{\mathbb{E}\max_{i=1,\ldots,n}\sup_{0\le t\le 1}\{|\overset{\circ(}{B}_{H,i}\ 1)(t)|}{(\log n)^{1/2}} \quad \ge \quad C_1 > 0 \qquad (8.55)$$

$$\limsup_{n\to\infty} \frac{\mathbb{E}\max_{i=1,\ldots,n}\sup_{0\le t\le 1}|\{\overset{\circ(}{B}_{H,i}\ 1)(t)|}{(\log n)^{1/2}} \quad \le \quad C_2 < \infty. \qquad (8.56)$$

Proof We first show (8.55). Notice that

$$\mathbb{E} \max_{1\le i\le n} \sup_{0\le t\le 1} |\{\overset{\circ}{B}{}^{(1)}_{H,i})(t)| \ge \mathbb{E} \max_{1\le i\le n} \{\overset{\circ}{B}{}^{(1)}_{H,i})(1/2)$$

We have

$$\begin{aligned} r_k &= \mathbb{E}\{\overset{\circ}{B}{}^{(1)}_{H,1})(1/2)\{\overset{\circ}{B}{}^{(1)}_{H,k+1}(1/2) \\ &= \frac{1}{4}[2(k+\frac{1}{2})^{2H} + 2(k-\frac{1}{2})^{2H} + 3k^{2H} - \frac{1}{2}(k+1)^{2H} - \frac{1}{2}(k-1)^{2H}] \sim \frac{1}{2}\binom{2H}{2} k^{2H-2} \end{aligned} \quad (8.57)$$

Therefore conditions of Theorem 4.3.3 from [14] are fulfilled. Remark also that $\{\overset{\circ}{B}{}^{(1)}_{H,i})(1/2)$ is normally distributed with mean 0 and variance $\sigma_H^2 = (\frac{1}{2})^{2H} - 1$. We use part (i) of the Theorem with $u_n = (2\log n)^{1/2}x$. As $n \to \infty$

$$n(1-\Phi(u_n)) \sim \frac{n^{1-x^2}}{\sqrt{2\pi}x(2\log n)^{1/2}} \to \begin{cases} \infty & \text{for } x < 1, \\ 0 & \text{for } x > 1. \end{cases}$$

Thus

$$\frac{\max_{1\le i\le n} B^{(1)}_{H,i})(1/2)}{(2\sigma_H^2 \log n)^{1/2}}$$

converges in probability to 1 as $n \to \infty$. Hence using Fatou lemma

$$\liminf_{n\to\infty} \mathbb{E} \frac{\max_{1\le i\le n} B^{(1)}_{H,i})(1/2)}{(2\sigma_H^2 \log n)^{1/2}} \ge 1,$$

which yields (8.55). □

Proposition 8.3

$$0 < \liminf_{n\to\infty} \frac{\delta_n^\infty}{(\log n)^{1/2}/n^H} \le \limsup_{n\to\infty} \frac{\delta_n^\infty}{(\log n)^{1/2}/n^H} < \infty.$$

Remark The slow convergence with the exponent H is not surprising in the light of the following facts. Following Ciesielski [?] realizations of the H–fBm have the following Hölder property: with probability 1

$$\lim_{d\to 0+} \sup_{0\le s<t\le 1:\ |s-t|\le d} \frac{|B_H(s) - B_H(t)|}{|s-t|^H |\log|s-t||^{1/2}} < \infty.$$

This yields that almost all realizations of $\{B_H(t),\ 0 \le t \le 1\}$ belongs to the class $\mathcal{H}_{H-\epsilon}$ of Hölder functions $(0 < \epsilon < H)$. Recall that a continuous function $f \in \mathcal{C}[0,1]$ is Hölder with exponent α, that is $f \in \mathcal{H}_\alpha$, if

$$L = \sup_{0 \le s < t \le 1} \frac{|f(s) - f(t)|}{|s-t|^\alpha} < \infty \,.$$

On the other hand for a function $f \in \mathcal{H}_\alpha$, the Burkill-Whitney lemma (see e.g. [7, 31]) implies the following result.

Proposition 8.4 *Suppose that* $0 = x_0 \le x_1 \le \ldots \le x_n = 1$, $\Delta = \max |x_i - x_{i-1}|$, $i = 1, \ldots, n$. *If* f^Δ *is a piecewise linear function, that coincides with* $f(x)$ *in* x_i, $i = 0, 1, \ldots, n$, *then*

$$\sup_{x \in [0,1]} |f^\Delta(x) - f(x)| \le L \Delta^\alpha \,.$$

Therefore from Ciesielski theorem we can immediately conclude that, with probability 1, for some random variable L and all $0 < \epsilon < H$, $\sup_{0 \le t \le 1} |B_H(t) - B_H^{(1/n)}(t)| \le L \left(\frac{1}{n}\right)^{H-\epsilon}$.

Comments. Yin (1966), Matheron (1973), Asmussen (1999), Burkill (1952), Whitney (1957), Ciesielski (1961)

Chapter IV

Boundness and continuity

1 Separable modification

There is given a stochastic process $\{X(t),\ t \in T\}$ with a topological space of parameters.

Definition 1.1 We say that $\{X(t),\ t \in T\}$ is *separable* if there exists a denumerable dense subset $T^* \subset T$, called a separant, such that for all open $V \subset T$

$$\sup_{t\in V} X(t) = \sup_{t\in V\cap T^*} X(t) \qquad \inf_{t\in V} X(t) = \inf_{t\in V\cap T^*} X(t) \tag{1.1}$$

From now on we assume that $\mathbb{E}\, X^2(t) < \infty$, which is not a problem when studying Gaussian processes. Let

$$\rho_X(s,t) = \mathbb{E}^{1/2}(X(t) - X(s))^2, \qquad s,t \in T.$$

Proposition 1.2 *ρ_X is a pseudometric that is*

1. $\rho_X(s,t) = \rho_X(t,s)$, $s,t \in T$,
2. $\rho_X(s,u) \le \rho_X(s,t) + \rho_X(t,u)$, $s,t,u \in T$,
3. $\rho_X(t,t) = 0$.

An *intrinsic pseudometric* generates an *intrinsic topology* on T. Note that condition $\rho_X(s,t) = 0$ does not imply always that $s = t$ (show examples).

Proposition 1.3 *If (T, ρ_X) is separable, then for every countable dense set $T^* \subset T$ there exists a separable modification of $\{X(t),\ t \in T\}$ with separant T^*.*

Proof Let $T^* \subset T$ be a countable dense set. Define a ρ_X–measurable mapping

$$\pi_n : T \to T^*$$

such that $\rho_X(\pi_n(t), t) < 2^{-n}$, $n = 1, 2, \ldots$. We now define what will be a separable modification

$$X^*(t) = \frac{1}{2}\{\limsup_{n\to\infty} X(\pi_n(t)) + \liminf_{n\to\infty} X(\pi_n(t))\}.$$

For all $t \in T$

$$\sum_{n=1}^{\infty} \mathbb{P}(|X(t) - X(\pi_n(t))| > \epsilon) \le \epsilon^{-2} \sum_{n=1}^{\infty} \mathbb{E}\,(X(t) - X(\pi_n(t))^2 = \epsilon^{-2} \sum_{n=1}^{\infty} 2^{-2n} < \infty.$$

Hence $X(\pi_n(t)) \to X(t)$ a.s.. Therefore $X^*(t) = X(t)$ a.s. and so $\{X^*(t),\ t \in T\}$ is a modification of $\{X(t),\ t \in T\}$. We now check condition (1.1). Define

$$A = \bigcup_{t \in T^*} \{X^*(t) \neq X(t)\}.$$

Notice that since X^* is a modification of X, we have $\mathbb{P}(A) = 0$. Further on let $\omega \neq A$. Let $t \in V$ for some open $V \subset T$. Then immediately from the definition of $X^*(t)$

$$X^*_\omega(t) \le \limsup_{n\to\infty} X^*_\omega(t)$$

and since

$$\pi_n(t) \in V$$

for enough big n we have

$$\limsup_{n\to\infty} X^*_\omega(t) \le \sup_{t \in V \cap T^*} X * (t).$$

Hence

$$\sup_{t \in V} X^*_\omega(t) \le \sup_{t \in V \cap T^*} X^*(t)$$

which proves the first condition in (1.1). The second condition can be verified similarly. $\square$.

From now on in these notes we assume that **all** processes are separable.

1.1 Metric entropy

For a (pseudo)metric space (T, ρ_X) we define $N(T, \epsilon)$ the smallest number of closed ρ_X–balls of radius ϵ tha cover T. The quantity

$$H(T, \epsilon) = \log N(T, \epsilon)$$

is called *metric entropy* of space T. The quantity

$$\mathcal{D}(T, \epsilon) = \int_0^{\epsilon} H^{1/2}(T, s) \, \mathrm{d}s$$

is called the *Dudley integral.*

Example 1.4 Let $\{W(t),\ t \geq 0\}$ be a Wiener process. Then $\rho_W(s,t) = |t-s|^{1/2}$. In general if $B_H(t),\ t \geq 0\}$, then $\rho_{B_H}(s,t) = |s-t|^H$. We now compute $N(T, \epsilon)$ for $T = [0,1]$. Consider for example the ϵ–ball covering 0. It has the center in $\epsilon^{1/H}$. Therefore it covers the interval $[0, 2\epsilon^{1/H}]$. Therefore

$$N(T, \epsilon) = \lceil 1/(2\epsilon^{1/H}) \rceil$$

and hence $H(T, \epsilon) = \log \lceil 1/(2\epsilon^{1/H}) \rceil$.

Example 1.5 Let $\{X(t), t \geq 0\}$ be a stationary Gaussian process with covariance function $R(t)$ and without loss of generality we may assume that $\operatorname{Var} X(t) = R(0) = 1$. Then

$$\rho_X(s,t) = 2^{1/2}(1 - R(|t-s|)^{1/2}.$$

For a stationary Orstein-Uhlenbeck process with the covariance function $\{\exp(-\alpha|x-y|),\ x, \in \mathbb{R}_+\}$ $\{X(t),\ t \geq 0\}$ we have

$$\rho(s,t) = 2^{1/2}\alpha^{1/2}|t-s|^{1/2} + o(|t-s|^{1/2}).$$

Theorem 1.6 *Let $\{X(t),\ t \in T\}$ be a centered Gaussian process, which is bounded a.s. If*

$$\sigma_T = \sup_{t \in T} \operatorname{Var} X(t),$$

then

$$\mathbb{E} \sup_{t \in T} X(t) \leq 4\sqrt{2}\mathcal{D}(T, \sigma_T/2).$$

Proof Lifshits, p. 179.

We say that a Gaussian process $\{X(t),\ t \in T\}$ has bounded realizations if $\sup_t X(t) < \infty$ and $\inf_t X(t) > -\infty$, a.s. For centered Gaussian process it suffices to consider only one condition $\sup_t X(t) < \infty$ a.s..

Theorem 1.7 [Dudley]. *Let $\{X(t),\ t \in T\}$ be a centered Gaussian process. If $\mathcal{D}(T,\cdot)$ is bounded, then the process is bounded and with uniformly continuous (with respect metric ρ_X).*

Remark Sometimes by Dudley integral one means $\int_0^\infty H^{1/2}(T,s)\ \mathrm{d}s$. Note that $\int_0^\infty H^{1/2}(T,s)\ \mathrm{d}s$ converges if and only if $\mathcal{D}(T,\epsilon)$ does.

Exercises

1.1 Compute the intrinsic metrics for H–fractional Lévy Brownian motion and then the corresponding metric entropy.

1.2 Show that $N(T,\epsilon)$ and $H(T,\epsilon)$ are nonincreasing and so $\mathcal{D}(T,\epsilon)$ is concave. Hence

$$\mathcal{D}(T,\epsilon_1+\epsilon) \leq \mathcal{D}(T,\epsilon_1) + \mathcal{D}(T,\epsilon_2)$$

and

$$\mathcal{D}(T,c\epsilon) \leq c\mathcal{D}(T,\epsilon).$$

1.3 Show that there exists a stationary Gaussian process, variance 1, such that for some $c>0$ and $0<\alpha\leq 2$

$$R(t) = 1 - c|t|^\alpha + o(|t|^\alpha).$$

1.4 Give an estimates for $H([0,1],\epsilon)$ in the case of a stationary Orstein-Uhlenbeck process.

Comments. Lifshits (1995)

Chapter V

Supremum from a Gaussian process; two fundamental results

1 Borell inequality

2 Slepian inequality

Chapter VI

Appendix

1 Analysis

1.1 Special functions

We denote by B the *beta function*

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \, dx,$$

where $\alpha, \beta > 0$.

1.2 Spherical coordinates

We have to recall some notions. Spherical coordinates in $\mathbb{R}^d - \mathbf{0}$ are

$$\begin{aligned}
x_1 &= r\cos\varphi_1, \\
x_2 &= r\sin\varphi_1\cos\varphi_2, \\
x_3 &= r\sin\varphi_1\sin\varphi_2\cos\varphi_3, \\
\ldots &= \ldots \\
x_{n-1} &= r\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\cos\varphi_{n-1} \\
x_n &= r\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\sin\varphi_{n-1}
\end{aligned}$$

Jacobian equals

$$J = \frac{D(x_1, \ldots, x_n)}{D(r, \varphi_1, \ldots, \varphi_{n-1})} = r^{n-1}\sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \ldots \sin\varphi_{n-2},$$

see Fichtenholz, III , p. 335.

1.3 Inner product and Hibert space

Let H be a vector space. *Inner product* is a function

$$H \times H \ni (f, g) \to < f, g > \in \mathbb{R}$$

fulfilling

- $< u, v > = \overline{< v, u >}$,
- $< \alpha u + \beta v, w > = \alpha < u, w > + \beta < v, w >$ for scalars α, β and $u, v, w \in H$.
- $< u, u > \geq 0$ and $< u, u > = 0$ if and only if $u = 0$.

A vector space with an inner product is said to be an *inner product space* H. An complete separable inner product space H is called a Hilbert space.

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