Point Processes
Article for EoAS

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August 5, 2003

The theory of point processes originates from problems of statistical demography, insurance, and mortality tables, although the connection may seem remote at the first sight. More recent are problems of risk theory, where one encounters the problem how to describe a stream of arriving claims in a mathematical formal way.

The simplest case of a point process is the POISSON PROCESS*. In general, we understand by a point process a method of randomly allocating points to intervals or rectangles, etc. Formalizing into language of mathematics, a (stochastic) point process is a random denumerable collection of points in a state space $\mathbb{E}$ with some small condition not allowing clustering. It turns out useful sometimes, for a mathematical convenience, to define point processes as random variables assuming values at a space of measures.

In the following, the basic probability space is $(\Omega, \mathcal{F}, \mathbf{P})$.

Point processes on the real line

To determine a point process we have to specify how are the points allocated. There are many ways to do it and we show a few possible approaches in due course. We first consider point processes on the line, that is, on $\mathbb{E} = \mathbb{R}$ (the *state space*). By $\mathcal{B}$ we denote the family of bounded Borel subsets of the state space. A point process on $\mathbb{R}$ is a collection of random variables such that $\ldots \leq \tau_{-1} \leq \tau_0 < 0 \leq \tau_1 \leq \tau_2 \leq \ldots$ and such that in each finite interval there is a finite number of points a.s. (that is, we exclude possibility that a sequence τ_n has points of accumulation). We say that at τ_n there is the n-th point and that the random variables $T_n = \tau_n - \tau_{n-1}$ are the *inter-point* *distances*. Notice that $\ldots, T_{-1}, T_0, \tau_0, \tau_1, T_2, T_3, \ldots$ under some conditions determines the point process respectively and therefore such a collection can also be used as a definition; however, we have to remember the condition of no accumulation of points. For a point process $\{\tau_n\}$ define the number of points $N(B) = \sum_n \mathbb{I}_B(\tau_n)$ in set $B \in \mathcal{B}$. Clearly $N(B)$ is finite for all bounded sets B and therefore the collection of random variables $\{N(B),\ B \in \mathcal{B}\}$ is well defined. Note also that if we know the cardinality $N(B)$ of each set $B \in \mathcal{B}$, then we also know positions of points $\{\tau_n\}$. Actually it suffices to know $N(I)$ for all open intervals $I = (a, b)$. Therefore from now on we say that a point process N is given, which means that we are able to count points in each bounded Borel B, or equivalently we know position of points $\{\tau_n\}$. In general to give a distribution of a point process N it suffices to know the family of finite dimensional *fidi* distributions. Actually we need to know

$$\mu_{B_1,\ldots,B_k}(n_1, \ldots, n_k) = \mathbf{P}(N(B_1) = n_1, \ldots, N(B_k) = n_k), \quad n_1, \ldots, n_k \in \mathbb{Z}$$

for disjoint bounded Borel sets $B_1, \ldots, B_k$ $(k = 1, 2, \ldots)$. We say that two point processes are *identically distributed* if their families of fidi distributions are the same.

For point processes on the real line there are two important notion of stationarity: translation invariance and point to point invariance. We say that a point process $\{\tau_n\}$ is time stationary if the random sequences $\{\tau_n\}$ and $\{\tau_n + t\}$ have the same distribution for all t. Equivalently

$$\begin{aligned} &\mathbf{P}(N(B_1 + t) = n_1, \ldots, N(B_k + t) = n_k) = \\ &\quad = \mathbf{P}(N(B_1) = n_1, \ldots, N(B_k) = n_k), \quad n_1, \ldots, n_k \in \mathbb{Z} \end{aligned}$$

for disjoint bounded Borel sets $B_1, \ldots, B_k$ $(k = 1, 2, \ldots)$. We say that a point process is *Palm* or *event stationary* if $\tau_1 = 0$ (that is there is a point at zero) and $\{T_n\}$ is a stationary sequence.

By the *intensity measure* of a point process N we mean the measure μ on the state space $\mathbb{E}$ (here the real line) such that

$$\mathbf{E}N(B) = \mu(B), \quad \text{for all } B \in \mathcal{B}.$$

For a stationary real line point process N we have $\mu(B) = \lambda|B|$ where $|\cdot|$ is Lebesgue measure (say $|B| = b - a$ if B is an interval with endpoints $a < b$) and $\lambda = \mathbf{E}N([0, 1])$ is said to be the *intensity* of the point process N.

We say that a point process N is *simple* if $\mathbf{P}(N(\{\tau_n\}) > 1) = 0$ for all $n \in \mathbb{Z}$. It means that there are no *multiple points*, or $\tau_n < \tau_{n+1}$ for all n a.s.

Two basic operations on simple point processes are:

- *independent thinning.* The n-th point of N at $\tau_n = t$ is deleted with probability $1-p(t)$ independently of everything else. In the case when the *retention* probability $p(t) = p$ is constant, we talk about i.i.d. thinning.

- *i.i.d. shifting.* The point τ_n is shifted to $\tau_n + X_n$ for all n, where $\{X_n\}$ are independent and identically distributed random variables.

Examples

- *Renewal point process* on $\mathbb{R}_+$ or $\mathbb{R}$. A point process $\{\tau_n\}$ on $\mathbb{R}_+$ is a renewal point process if $T_1 = \tau_1, T_2, T_3, \ldots$ are independent and identically distributed or is a *delayed renewal point process* if $T_2, T_3, \ldots$ are identically distributed with common distribution F. If $\mathbf{E}T = \int_0^\infty x\, dF(x) < \infty$ we define the integrated tail distribution

$$F^{\mathrm{i}}(x) = \begin{cases} \frac{\int_0^x (1-F(y))\, dy}{\mathbf{E}T} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

  To define a stationary renewal point process on $\mathbb{R}_+$, for a given inter-point distribution F of a non-negative generic inter-renewal time T, we must suppose that τ_1 is distributed according to $F^{\mathrm{i}}(x)$. Frequently it is useful to define the stationary renewal point process on $\mathbb{R}$, for which $\ldots, T_{-1}, T_0, (\tau_0, \tau_1), T_2, T_3, \ldots$ are independent, $\ldots, T_{-1}, T_2, T_3, \ldots$ identically distributed and the joint distribution of (τ_0, τ_1) is determined by $\mathbf{P}(\tau_0 > x, \tau_1 > y) = 1 - F^{\mathrm{i}}(x, y)$ for $x, y \geq 0$. See further RENEWAL THEORY*.

- *Markov renewal point process (semi-Markovian point process).* Let $(F_{ij}(x))_{i,j=1,\ldots,k}$ be an array of subdistribution functions, that is non-negative functions such that $F_{ij}(0) = 0$ $(i, j = 1, \ldots, k)$ and $\sum_{j=1}^k F_{ij}(x) = 1$ for all $x \geq 0$ and $i = 1, \ldots, k$. Then a point process $\{\tau_n\}$ on $\mathbb{R}_+$ is said to be a Markov renewal process if the sequence $\{\tau_n\}$ jointly with a background sequence $\{I_n\}$ is a Markov chain with transition kernel of form

$$\mathbf{P}(\tau_{n+1} \leq y, I_{n+1} = j | \tau_n = x, I_n = i) = F_{ij}(y - x), \\ x, y \geq 0, \quad i, j = 1, \ldots, k$$

  (see MARKOV CHAINS AND PROCESSES*).

- *(time–homogeneous) Poisson point process.* This is the stationary renewal process with the generic inter-point distribution $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$. The equivalent definition of the Poisson point process is given in terms of $N(B)$ defined for bounded Borel sets: $N(B_1), \ldots, N(B_k)$ are independent for mutually disjoint $B_1, \ldots, B_k$ and all $k = 2, \ldots$ and
$$\mathbf{P}(N(B) = n) = \frac{\lambda |B|^n}{n!} e^{-\lambda |B|}, \quad n = 0, 1, \ldots.$$
We have that $\mathbf{E}N([0,t]) = \lambda t$ and so the parameter λ is the intensity. We have $\mathbb{D}^2 N([0,t]) = \mathbf{E}N([0,t])$. Independent thinning of a Poisson process with intensity λ with retention probability p is again a Poisson process with intensity $p\lambda$. Similarly an i.i.d. shift of a Poisson process with intensity λ is again the Poisson process with the same intensity. See also POISSON PROCESS*.

- *Mixed Poisson point process.* It is the process defined by a doubly stochastic argument: the intensity λ is chosen according to a mixing distribution $G(x)$ with support in $\mathbb{R}_+$ and next the Poisson process with this intensity is defined. Formally a mixed point process is defined by the family of fidi distributions
$$\mathbf{P}(N(B_1) = n_1, \ldots, N(B_k) = n_k) =$$
$$= \int_0^\infty \frac{\lambda |B_1|^{n_1}}{n_1!} e^{-\lambda |B_1|} \cdots \frac{\lambda |B_1|^{n_k}}{n_k!} e^{-\lambda |B_1|} \, dG(\lambda).$$
We have $\mathbb{D}^2 N(B) \geq \mathbf{E}N(B)$.

- *Inhomogenous Poisson process* with *intensity function* $\{\lambda(t)\}$ is defined by the following conditions: $N(B_1), \ldots, N(B_k)$ are independent for mutually disjoint $B_1, \ldots, B_k$ and all $k = 2, \ldots$ and
$$\mathbf{P}(N(B) = n) = \frac{(\int_B \lambda(t)\, dt)^n}{n!} e^{-\int_B \lambda(t)\, dt}, \quad k = 0, 1, \ldots$$
It can be obtained from the homogeneous Poisson process by the time change $t \to \Lambda(t)$, where
$$\Lambda(t) = \begin{cases} \int_0^t \lambda(s)\, ds, & \text{for } t \geq 0 \\ -\int_t^0 \lambda(s)\, ds, & \text{for } t < 0. \end{cases}$$
If N is an inhomogeneous Poisson point process with intensity $\lambda(t)$ then after thinning we obtain again an inhomogeneous Poisson point process with intensity $p(t)\lambda(t)$.

- *Cox process.* Here there is given a stochastic process $\{\lambda(t)\}$ such that for all ω, the deterministic function $\lambda(t,\omega)$ is non-negative and locally integrable. We first choose a realization and then we define an inhomogeneous Poisson point process with intensity function $\lambda(t,\omega)$. Cox point processes serve as a model for bursty arrivals because $\mathbb{D}^2 N(t) \geq \mathbf{E}N(t)$, while for the homogeneous Poisson process $\mathbb{D}^2 N(t) = \mathbf{E}N(t)$. The following special cases are of interest in risk theory:

  the *Markov modulated Poisson point process (MMPPP).* Here there is given a *background* continuous time Markov chain $J(t)$ (see MARKOV CHAINS AND PROCESSES*) with finite state space $\{1,\ldots,k\}$ and with intensity matrix $\boldsymbol{Q}$. We can always suppose that $J(t)$ has right-continuous piecewise constant realizations. The Markov modulated Poisson point process is a Cox process with intensity process $\lambda(t) = \beta_{J(t)}$, where $\beta_1,\ldots,\beta_k > 0$. Thus it is determined by $\beta_1,\ldots,\beta_k$ and intensity matrix $\boldsymbol{Q}$ of the Markov process.

  the *Björk–Grandell point process* is a Cox process with the intensity process defined as follows. Let the random vectors $\{(\Lambda_i, I_i),\ i = 1,2,\ldots\}$ are independent and identically distributed for $i = 2,3,\ldots$ and

$$\lambda(t) = \sum_{i=1}^{\infty} \Lambda_i \mathbb{I}(\sum_{k=1}^{i-1} I_i \leq t < \sum_{k=1}^{i} I_i).$$

  If we choose the distribution of (Λ_1, I_1) such that

$$\mathbf{P}(\Lambda_1 \in B, I_1 \in B') = \frac{1}{\mathbf{E}I_2}\int_{B'} \mathbf{P}(\Lambda_2 \in B, I_2 > x)\, dx$$

  then the process $\lambda(t)$ is stationary and ergodic. If $I_i = 1$ for $i = 2,3,\ldots$, then the Björk–Grandell point process is called the *Ammeter process.*

- *Markov arrival point process (MArPP).* This is a generalization of the MMPPP. The definition is as for the MMPPP with intensity matrix $\boldsymbol{Q}$ and rate vector $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_k)$ except that some additional points may occurwhen $J(t)$ changes state: with probability a_{ij} a point occurs at a jump from i to $j \neq i$. Therefore the MArPP is defined by $\boldsymbol{\beta}$, $\boldsymbol{Q}$ and $\boldsymbol{A}$, where $\boldsymbol{A} = (a_{ij})$ and $a_{ii} = 0$. Note that if all $\beta_j = 0$ and $a_{k-1,k} = 1$ and otherwise $a_{ij} = 0$, then one obtains a renewal process with a inter-point distribution which is PHASE TYPE*. It is known that the class of MArPP's is dense in the class of all point processes on $\mathbb{R}_+$; see [1]

Point processes on a subset of $\mathbb{R}^d$

Similarly as for case $\mathbb{E} = \mathbb{R}$, by a point process on $\mathbb{E}$ we mean a collection of points $\{\tau_n,\ n \in \mathbb{Z}\}$, where $\tau_n \in \mathbb{E} \subset \mathbb{R}^d$. We denote by $N(B)$ the number of points in B that is $N(B) = \sum_{n\in\mathbb{Z}} \mathbb{I}_B(\tau_n)$.

Poisson point process

We say that N is a *Poisson point process* on $\mathbb{E} \subset \mathbb{R}^d$ with intensity measure μ if for mutually disjoint sets $B_1, \ldots, B_n$ $(n = 1, 2\ldots)$, the number of points in these sets $N(B_1), \ldots, N(B_n)$ are independent Poisson random variables,

$$\mathbf{P}(N(B) = k) = \frac{(\mu(B))^k}{k!} e^{-\mu(B)}, \quad k = 0, 1, \ldots$$

We now show how to construct a Poisson process with intensity measure μ, provided that μ is finite, that is $\mu(\mathbb{R}^d) < \infty$. Let $Z_1, Z_2, \ldots$ be a sequence of independent random elements from $\mathbb{E}$ with the common distribution $F(dx) = \mu(dx)/\mu(\mathbb{E})$ and Y be an independent from the sequence random variable having the Poisson distribution with mean $\mu(\mathbb{E})$. Then $\tau_1 = Z_1, \ldots, \tau_Y = Z_Y$ is the Poisson process. If $\mu(\mathbb{E}) = \infty$, then we have to split $\mathbb{E}$ into disjoint sets $\mathbb{E}_j$ such that $\mu(\mathbb{E}_j) < \infty$ and $\mathbb{E} = \bigcup_i \mathbb{E}_i$. Then on each subspace $\mathbb{E}_i$ $(i = 1, \ldots)$ we define independent Poisson processes to define the Poisson process with intensity measure μ.

Marked point process on the real line

Here we consider point processes on $\mathbb{R}$ with marks from a space space $\mathbb{K}$. One frequently considers a point τ_n together with some additional information, which is described by $K_n \in \mathbb{K}$. For example in risk theory a claim arriving at τ_n is of size K_n, so the space of marks is $\mathbb{K} = \mathbb{R}_+$. Therefore a marked point process is a collection of $\{(\tau_n, K_n),\ n \in \mathbb{Z}\}$, where $\{\tau_n,\ n \in \mathbb{Z}\}$ is itself a point process on the real line. For marked point processes N we define the shifted process $\theta_t N$ with points at $\{(\tau_n + t, K_n),\ n \in \mathbb{Z}\}$. We say now that N is *time stationary* if N and $\theta_t N$ has the same distribution for all t. Similarly *Palm stationarity* means that $\tau_1 = 0$ (that is there is a point at zero) and $\{(T_n, K_n)\}$ is a stationary sequence. By the intensity of the stationary marked point process $\{(\tau_n, K_n),\ n \in \mathbb{Z}\}$ we mean the intensity λ of the real line stationary point process $\{\tau_n,\ n \in \mathbb{Z}\}$.

Measure theoretical approach

Let $(\mathbb{E}, \mathcal{E})$ be a measurable space and $x \in \mathbb{E}$. By the *Dirac measure* δ_x we mean a probability measure concentrated at point $x \in \mathbb{E}$:

$$\delta_x(A) = \begin{cases} 1 & \text{for } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

To a collection of points with coordinates $\{\tau_n(\omega)\}$ one relates, for each $\omega \in \Omega$, a measure on $\mathbb{E}$ by:

$$\{\tau_n(\omega)\} \to N(\omega, \cdot) = \sum_n \delta_{\tau_n(\omega)}(\cdot).$$

The measure N is locally finite, that is for bounded sets B, $N(B) < \infty$ a.s. To avoid unnecessary technicalities we assume here that $\mathbb{E}$ is a Borel subset of $\mathbb{R}^d$. We denote by $\mathcal{M}(\mathbb{E})$ the space of all locally finite measures on $\mathbb{E}$. An important subset of $\mathcal{M}(\mathbb{E})$, denoted by $\mathcal{N}(\mathbb{E})$, consists of all counting measures $\nu \in \mathcal{M}(\mathbb{E})$, specified by the requirement.

$$\nu(B) \in \{0, 1 \ldots\}, \quad \text{for all } B \in \mathcal{E}.$$

Each point process assumes values in $\mathcal{N}(\mathbb{E})$.

Two important cases are: real line point processes with values in $\mathcal{N}(\mathbb{R})$ and real line marked point processes with values in $\mathcal{N}(\mathbb{R} \times \mathbb{K})$.

This approach to point process allows to introduce a concept of convergence for them. For example in case of $\mathcal{M}(\mathbb{R})$ we define the *vague* topology which makes a sequence $\mu_n \in \mathcal{M}(\mathbb{R})$ convergent to μ if for each continuous function $f : \mathbb{R} \to \mathbb{R}$ with compact support we have $\int f \, d\mu_n \to \int f \, d\mu$. We now say that a sequence of real line point processes N_n converges in distribution to N if for all continuous and bounded function $\phi : \mathcal{M}(\mathbb{R}) \to \mathbb{R}$ we have $\mathbf{E}\phi(N_n) \to \mathbf{E}\phi(N)$.

Palm theory of marked point processes on the real line

In this section we consider marked point processes on R. Let N be a stationary point process with points given by $\{\tau_n, \ n \in \mathbb{Z}\}$. We assume that the point process $\{\tau_n\}$ is *simple*. We aim to define the conditional distribution, called *Palm distribution* or *Palm probability*, of the point process N, under the condition that there is a point at zero. However the condition $\{N(\{0\}) = 1\}$ has probability zero and so this conditional distribution has to be defined with some care. We assume that the intensity

$$\lambda = \mathbf{E}(N[0, 1])$$

of N is nonzero and finite.

There are known two approaches. One of Cz. Ryll-Nardzewski proceeds via Radon-Nikodym derivatives but we give another due to K. Matthes, which can be summarized as follows. Let f be a real function on $\mathcal{N}$ (for example $f(\nu) = \nu(B)$ for a counting measure ν). We define a Palm version N° (that is the conditional point process) of the stationary point process N as

$$\mathbf{E} f(N^\circ) = \frac{1}{\lambda}\mathbf{E}\left[\sum_{n:0<\tau_n\leq 1} f(\theta_{\tau_n} N)\right],$$

provided that the expectations are well defined. In particular N° is a real line point process such that

- $\mathbf{P}(N^\circ(\{0\}) > 0) = 1$, that is there is always a point at zero,
- $\theta_{\tau_n} N^\circ$ has the same distribution as N°, and in consequence $\tau_1 = 0$, and the sequence of inter-point distances $\{T_n^\circ,\ n \in \mathbb{Z}\}$ is stationary.

A very useful tool is *Campbell's formula*:

$$\mathbf{E}\left[\sum_n f(N, \tau_n)\right] = \lambda \mathbf{E}\left[\int_{-\infty}^{\infty} f(\theta_{-t} N, t)\right] dt,$$

where f is a function of a point process and time such that all the expectations and integrals are well defined.

We now give the Palm versions N° for the following stationary point processes N:

- *Renewal process.* If N is a stationary renewal point process with generic inter-point distribution F, then N° has the inter-point sequence T_n being an i.i.d. sequence of non-negative random variables with distribution F. Recall that there is a point at zero which means $\tau = 0$ and so $\tau_1 = T_1$.
- *Poisson process.* If N is an inhomogeneousPoisson process with intensity function $\lambda(t)$, then $N^\circ = N + \delta_0$, that is, the Poisson process with intensity λ with a point added at zero.
- *Cox PP.* Here, as for inhomogeneous Poisson processes, N° is a Cox process with some intensity process $\lambda^\circ(t)$ with a point added at zero (for definition of $\lambda^\circ(t)$ see for details Grandell (1976))

A stationary point process N is *ergodic* if in the corresponding Palm version N°, the inter-point distances sequence $\{T_n\}$ is ergodic. This is equivalent to the following: for each $f, g : \mathcal{N} \to \mathbb{R}_+$ (non-negativity is technical)

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{E} f(N) g(\theta_t N)\, dt = \mathbf{E} f(N) \mathbf{E} g(N).$$

Stochastic intensity

Some archetypes can be found in mathematical demography. For a single individual with lifetime T_0 with the survivor function $s(x) = \mathbf{P}(T_0 > t)$ we define the density of the lifetime distribution

$$f(x)\, dx = \mathbf{P}(\text{lifetime terminates between } x, x + dx),$$

while $h(x)$ is the hazard function (or mortality function)

$$h(x) = \mathbf{P}(\text{lifetime terminates between } x, x + dx \quad \text{given it does not terminate before } x) = \frac{f(x)}{s(x)}.$$

In terms of point processes it is a point process with single point at the time of death T_0. Here we present the theory for the half line. We first define the *internal* FILTRATION* $\{\mathcal{F}_t\}$ of a point process N. It is, for each $t \geq 0$, a sub–σ–field of $\mathcal{F}$ generated by the family of random variables $N(B)$, where B runs through all subsets of $[0, t]$.

The main object is a generalization of the hazard function for point processes. Let N be a simple point process with its internal filtration $\mathcal{F}_t,\ t \geq 0\}$ and $\{\lambda(t),\ t \geq 0\}$ be a non-negative process adapted (see FILTRATION**) to $\{\mathcal{F}_t\}$. The process $\{\lambda(t)\}$ is called an $\mathcal{F}_t$–intensity of N if it is locally integrable (that is $\int_B \lambda(s)\, ds < \infty$ for bounded B) and

$$\mathbf{E}[N((a, b])|\mathcal{F}_a] = \mathbf{E}[\int_a^b \lambda(s)\, ds|\mathcal{F}_a],$$

for all $0 \leq a < b < \infty$. For uniqueness it is usually assumed that $\{\lambda(t)\}$ is predictable.

Here are some examples of the $\mathcal{F}_t$-intensity.

- *Poisson process with intensity function* $\lambda(t)$. In view of the independence of the increments, $\{\lambda(t)\}$ is the $\mathcal{F}_t$–intensity of N.

- *Renewal process with inter-point distribution F.* We assume that the hazard function $h(x)$ exists (that is, the lifetime distribution has a density). Then

$$\lambda(t) = \sum_{n-1}^{\infty} h(t - \tau_n)\mathbb{I}(\tau_n < t \leq \tau_{n+1}).$$

- *Transitions in Markov chains.* Let $\{X(t)\}$ be a continuous time, finite Markov chain with state space $\mathbb{E} = \{1, 2, \ldots\}$, see MARKOV CHAINS AND PROCESSES*, with transition intensity matrix $\boldsymbol{Q} = (q_{ij})$. For fixed $i \neq j$, let τ_n be the instant of the n transition from state i to state j. Then the $\mathcal{F}_t$–intensity is given by

$$\lambda(t) = q_{ij}\mathbb{I}(X(t) = i).$$

Ordering of point processes

Important orderings of point processes are defined via the ordering of their fidi distributions. Thus if we have an order $\prec$ for random vectors (as for example stochastic ordering) we say that for two point processes N and N' we have $N \prec N'$ if for all bounded Borel subsets $B_1, \ldots, B_k$ $(k = 1, 2 \ldots)$

$$(N(B_1), \ldots, N(B_n)) \prec (N'(B_1), \ldots, N'(B_n)).$$

We consider only integral orders $\leq_{\mathbb{F}}$ for random vectors defined by a class $\mathbb{F}$ of functions. Here we only mention the classes of: increasing functions (st), supermodular functions (sm) or a subset of increasing supermodular functions called increasing directionally convex functions (idcx). Thus for two random vectors $X, X^{'}$ we have $X \leq_{\mathbb{F}} X^{'}$ if $\mathbf{E}f(X) \leq \mathbf{E}f(X^{'})$ provided the expectations are finite. See also e.g. STOCHASTIC ORDERING*. These classes of functions define the following important orders:

- *st or stochastic ordering* It can be proved for simple point processes that if $N \leq_{\text{st}} N'$, then there exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and two real line point processes N, N' such that for all $\omega \in \Omega$ $N_\omega \subset N'_\omega$, which means that for all ω the set of all of points of N_ω is a subset the set of all of points of N'_ω.

- *sm or supermodular ordering.* If $N \leq_{\text{sm}} N'$, then the marginals $N(B)$ and $N'(B)$ have the same distributions and the order measures the strength of dependence between between $(N(B_1), \ldots, N(B_k))$ and $(N'(B_1), \ldots, N'(B_k))$

- *increasing directionally convex or idcx ordering.* Let N be a real line stationary Poisson point process with intensity λ and N' a Cox process with random intensity process $\lambda'(t)$, which is assumed to be stationary and $\lambda \leq \mathbf{E}\lambda'(0)$. Then $N \leq_{\mathrm{idcx}} N'$.

Recommendations for further reading

The main reference for the foundation, basic notions and examples is the monograph of Daley & Vere–Jones (1988). An easy to read account on the Poisson process is the book [9]. Cox processes are studied in [5] and mixed Poisson processes with emphasis on their applications to risk theory are surveyed in [6]. Stochastic intensity and related problems can be found in [2, 3, 8] and statistical theory based on the notion of the stochastic intensity is given in [7]. More on Palm distribution of point processes can be found in [2, 11, 12] and in the latter there are applications to risk theory. For ordering of point processes we refer to [10, 13].

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