A NOTE ON SPEED OF CONVERGENCE TO THE QUASI-STATIONARY DISTRIBUTION

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Abstract. In this note we show that for Z being a birth and death process on $\mathbb{Z}$ or Brownian motion with drift and $\mathcal{E}^{'} = (0, \infty)$, the speed of convergence to the quasi-stationary distribution is of order $1/t$. The corresponding version that X is the number of calls in M/M/1 queue or the reflected Brownian motion is also considered. The result is obtained by asymptotic expansions of some transition functions. For this we use some new asymptotic expansion of the Bessel function.

1. Introduction

In this note we will work out the speed of convergence to the quasi-stationary distribution for two special cases: the stationary number of calls in the M/M/1 queue and the reflected Brownian motion with drift. To be specific, suppose that $Z(t)$ is a continuous time Markov process with state space $\mathcal{E}$, $\mathcal{E}^{'} \subset \mathcal{E}$ and let $\tau = \inf\{t > 0 : Z(t) \notin \mathcal{E}^{'}\}$. The law of process starting from Z_0 with distribution π will be denoted by $\mathbb{P}_\pi$ and when $Z_0 = x$ a.s. then by $\mathbb{P}_x$. If $x = 0$ we skip the subscript. By the quasi-stationary distribution π^{RQS} we mean

$$w - \lim_{t\to\infty} \mathbb{P}_x(Z(t) \in dy | \tau > t) = \pi^{\mathrm{RQS}}(dy), \qquad x \in \mathcal{E}^{'}, dy \subset \mathcal{E}^{'}.$$

The convergence above is in a weak sense. Sometimes π^{RQS} is said to be a Yaglom limit. Following [7] we can also work with a modified version of this problem. Suppose now that $X(t)$ is an ergodic Markov process for which

$$\mathbb{P}_x(X(t) \in dy) = \mathbb{P}_x(Z(t) \in dy), \qquad x \in \mathcal{E}^{'}, dy \subset \mathcal{E}^{'}$$

and π be its stationary distribution. Let $T = \inf\{t > 0 : X(t) \notin \mathcal{E}^{'}\}$. It is said that a distribution π^{QS} on $\mathcal{E}^{'} \times \mathcal{E}^{'}$ is quasi-stationary if

$$w - \lim_{t\to\infty} \mathbb{P}_\pi(X(0) \in dx, X(t) \in dy | T > t) = \pi^{\mathrm{QS}}(dx, dy), \qquad dx \subset \mathcal{E}^{'}, dy \subset \mathcal{E}^{'}.$$

In our specific examples Z is a birth and death process on $\mathbb{Z}$ or the Brownian motion with drift on $\mathbb{R}$ and X is the stationary M/M/1 queue on $\mathbb{Z}_+$ or stationary reflected Brownian motion with drift on $\mathbb{R}_+$ and $E^{'} = (0, \infty)$ respectively. It is interesting to note that in both cases the speed of convergence is very slow of order $1/t$. We conjecture that this property should hold for a larger class of Levy processes considered by Kyprianou and Palmowski [5] and Mandjes et al. [7].

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In the Appendix we show some asymptotic expansions of the Bessel function $I_n(t)$. We follow here the idea from Asmussen [2, III.8e.], wherein there is given an expansion with first two terms and the error in the form $n^4 o(t^{-3/2})$ (however we show here the correct error is $n^6 o(t^{-3/2})$). The novelty in [2] is that $o(t^{-3/2})$ is not depending on n. Nevertheless, in this paper we need the expansion of this type with one more term. We also conjecture a general formula with an arbitrary number of terms.

2. M/M/1

In this section $X(t)$ is the number of calls in the M/M/1 queue at which calls arrive according to a Poisson arrivals with rate $\lambda > 0$ and service times are i.i.d. with parameter $\mu > 0$; see eg. Asmussen [2, III.8.]. Let $Z(t) = \Pi_A(t) - \Pi_D(t)$, where Π_A and Π_D are independent Poisson processes with intensity λ and μ, respectively. Under $\mathbb{P}_k = \mathbb{P}_k^{\lambda,\mu}$, process $Z(t)$ is a birth and death process on $\mathbf{Z}$ with the birth and death rates λ, μ respectively and $Z(0) = k$ a.s. Let π be the stationary distribution of $X(t)$; $\pi_i = (1-\rho)\rho^i$, where $\rho = \lambda/\mu$. From now on we assume $\rho < 1$. Under $\mathbb{P}_\pi$, process $(X(t))_{t\geq 0}$ is stationary. In this section $\mathcal{E}' = \{1, 2, \ldots\}$, that is

$$\tau = \inf\{t \geq 0 : Z(t) = 0\}, \qquad T = \inf\{t \geq 0 : X(t) = 0\}.$$

Let

$$p^*_{kl}(t) \;=\; \mathbb{P}_k(X(t) = l, T > t) = \mathbb{P}_k^{\lambda,\mu}(Z(t) = l, \tau > t)$$

be the transition probability function of Markov process $X(t)$ killed at a the exit to 0. Following Asmussen [2, page 99], we have

$$p^*_{kl}(t) = e^{-(\lambda+\mu)t}\rho^{\frac{l-k}{2}}(I_{k-l}(2\sqrt{\lambda\mu}t) - I_{k+l}(2\sqrt{\lambda\mu}t)), \tag{2.1}$$

where $I_n(t)$ is the n-modified Bessel function. In the appendix we will prove in Proposition 1 the following universal estimation:

$$I_n(t) = e^t\mathbb{P}(Z(t) = n) = e^t\left[\frac{1}{\sqrt{2\pi t}}\left(1 - \frac{4n^2-1}{8t}\right) + n^6 o(t^{-3/2})\right], \tag{2.2}$$

for each $n \in \mathbb{Z}$ and $t \to \infty$.
Denote

$$\pi_l^{\mathrm{RQS}} = \frac{(1-\rho^{\frac{1}{2}})^2}{\rho^{\frac{1}{2}}} l\rho^{\frac{l}{2}}, \qquad \pi_{k,l}^{\mathrm{QS}} = \left(\frac{(1-\rho^{\frac{1}{2}})^2}{\rho^{\frac{1}{2}}}\right)^2 kl\rho^{\frac{k+l}{2}},$$

for $k, l = 1, 2, \ldots$ and

$$P_{i|t}(j) = \mathbb{P}_i(X(t) = j | T > t), \qquad P_{\pi|t}(j,k) = \mathbb{P}_\pi(X(0) = j, X(t) = k | T > t).$$

Our aim is to prove the following theorem.

Theorem 1. *We have*

(i)

$$||P_{i|t}(\cdot) - \pi^{\mathrm{RQS}}|| = \frac{1}{2}\sum_{j=1}^{\infty}\left|P_{i|t}(j) - \pi_j^{\mathrm{RQS}}\right| = \frac{1}{t}(C_1 + o(1)),$$

where C_1 is given by (2.3). Furthermore,

(ii)

$$||P_{\pi|t}(\cdot) - \pi^{\mathrm{QS}}|| = \frac{1}{2}\sum_{i,j=1}^{\infty}\left|\pi_i P_{i|t}(j) - \pi_{ij}^{\mathrm{QS}}\right| = \frac{1}{t}(C_2 + o(1)),$$

where C_2 is given by (2.4).

The proof will follow from Lemma 1.

Lemma 1. *For each $k = 1, 2, \dots$*

$$p_{kl}^*(t) \;=\; Akl\rho^{\frac{l-k}{2}}t^{-\frac{3}{2}}e^{-\gamma t} - P_1(k,l)\rho^{\frac{l-k}{2}}t^{-\frac{5}{2}}e^{-\gamma t} + P_2(k,l)\rho^{\frac{l-k}{2}}t^{-\frac{5}{2}}e^{-\gamma t}o(1),$$

where $o(1)$ is universal with respect to k, l and

$$A = \left(2\sqrt{\pi}(\lambda\mu)^{\frac{3}{4}}\right)^{-1},\ \ \gamma = (\lambda^{1/2} - \mu^{1/2})^2\ ,$$

and P_1, P_2 are polynomials of variables k,l. Furthermore

$$\begin{aligned}\mathbb{P}_k(\tau > t) &= \sum_{l\geq 1} p_{kl}^*(t)\\ &= A\frac{\rho^{1/2}}{(1-\rho^{1/2})^2}k\rho^{-\frac{k}{2}}e^{-\gamma t}t^{-\frac{3}{2}}\\ &+ D_1(k)\rho^{-\frac{k}{2}}e^{-\gamma t}t^{-\frac{5}{2}} + D_2(k)\rho^{-\frac{k}{2}}e^{-\gamma t}t^{-\frac{5}{2}}o(1),\end{aligned}$$

where D_1, D_2 are polynomials of variable k.

Proof. Applying (3.11) from the Appendix to (2.1) we obtain

$$\begin{aligned}p_{kl}^*(t) &= Akl\rho^{\frac{l-k}{2}}t^{-\frac{3}{2}}e^{-\gamma t} - \frac{A}{2(\lambda\mu)^{1/2}}\frac{k^3l + \frac{1}{2}kl + kl^3}{2}\rho^{\frac{l-k}{2}}t^{-\frac{5}{2}}e^{-\gamma t}\\ &+ \left((k+l)^{12} - (k-l)^{12}\right)\rho^{\frac{l-k}{2}}t^{-\frac{5}{2}}e^{-\gamma t}o(1)\\ &= Akl\rho^{\frac{l-k}{2}}t^{-\frac{3}{2}}e^{-\gamma t} - P_1(k,l)\rho^{\frac{l-k}{2}}t^{-\frac{5}{2}}e^{-\gamma t} + P_2(k,l)t^{-\frac{5}{2}}\rho^{\frac{l-k}{2}}e^{-\gamma t}o(1).\end{aligned}$$

Now

$$\begin{aligned}\mathbb{P}_k(\tau > t) &= \sum_{l\geq 1} p_{kl}^*(t)\\ &= A\frac{\rho^{1/2}}{(1-\rho^{1/2})^2}k\rho^{-\frac{k}{2}}e^{-\gamma t}t^{-\frac{3}{2}}\\ &+ \frac{A}{2(\lambda\mu)^{1/2}}\frac{\rho^{1/2}}{(1-\rho^{1/2})^2}\left(\frac{k^3}{2} + \frac{k}{4} + k\frac{\rho + 4\rho^{1/2} + 1}{(1-\rho^{1/2})^2}\right)\rho^{-\frac{k}{2}}e^{-\gamma t}t^{-\frac{5}{2}}\\ &+ D_2(k)\rho^{-\frac{k}{2}}e^{-\gamma t}t^{-\frac{5}{2}}o(1)\\ &= A\frac{\rho^{1/2}}{(1-\rho^{1/2})^2}k\rho^{-\frac{k}{2}}e^{-\gamma t}t^{-\frac{3}{2}}\\ &+ D_1(k)\rho^{-\frac{k}{2}}e^{-\gamma t}t^{-\frac{5}{2}} + D_2(k)\rho^{-\frac{k}{2}}e^{-\gamma t}t^{-\frac{5}{2}}o(1)\end{aligned}$$

and the proof is completed. □

Proof. **of Theorem 1**. Using Lemma 1 we have

$$\begin{aligned}
&\sum_{l=1}^{\infty}\left|\mathbb{P}_k(X(t)=l \mid \tau>t)-\pi_l^{\mathrm{RQS}}\right| \\
&= \sum_{l=1}^{\infty}\left|\frac{p^*_{kl}(t)}{\mathbb{P}_k(\tau>t)}-\frac{(1-\rho^{\frac{1}{2}})^2}{\rho^{\frac{1}{2}}}l\rho^{\frac{l}{2}}\right| \\
&= \sum_{l=1}^{\infty}\left|\frac{Ae^{-\gamma t}t^{-3/2}\rho^{\frac{l-k}{2}}\left(kl+t^{-1}P_1(k,l)+t^{-1}P_2(k,l)o(1)\right)}{\frac{A\rho^{\frac{1}{2}}}{(1-\rho^{\frac{1}{2}})^2}e^{-\gamma t}t^{-3/2}\rho^{-\frac{k}{2}}\left(k+t^{-1}D_1(k)+t^{-1}D_2(k)o(1)\right)}-\frac{1-\rho^{\frac{l}{2}}}{\rho^{\frac{l}{2}}}l\rho^{\frac{l}{2}}\right| \\
&= t^{-1}\frac{(1-\rho^{\frac{1}{2}})^2}{\rho^{\frac{1}{2}}}\sum_{l=1}^{\infty}l\rho^{\frac{l}{2}}\left|\frac{W_1(k,l)+W_2(k,l)o(1)}{kl+t^{-1}lE_1(k)+t^{-1}lE_2(k)o(1)}\right|,
\end{aligned}$$

where A was defined in Lemma 1, P_1, P_2, W_1, W_2 are polynomials of variables k, l and D_1, D_2, E_1, E_2 are polynomials of variable k. Since all the polynomials are of fixed orders respectively,

$$\sup_{t\geq 0}\left|\frac{W_1(k,l)+W_2(k,l)o(1)}{kl+t^{-1}lE_1(k)+t^{-1}lE_2(k)o(1)}\right| = V_1(k,l)<\infty$$

and it is, for each k, a rational function of l. Hence by the dominated convergence theorem

$$\sum_{l=1}^{\infty}l\rho^{\frac{l}{2}}\left|\frac{W_1(k,l)+W_2(k,l)o(1)}{kl+t^{-1}lE_1(k)+t^{-1}lE_2(k)o(1)}\right|\to C_1=\frac{1}{k}\sum_{l=1}^{\infty}\rho^{\frac{l}{2}}\left|W_1(k,l)\right|. \tag{2.3}$$

This completes the proof of the part (i). The proof of the second part uses similar arguments:

$$\begin{aligned}
\mathbb{P}_\pi(\tau>t) &= \sum_{k\geq 1}(1-\rho)\rho^k\sum_{l\geq 1}p^*_{kl}(t) \\
&= (1-\rho)\sum_{k\geq 1}k\rho^k A\frac{\sqrt{\rho}}{(1-\sqrt{\rho})^2}e^{-\gamma t}t^{-\frac{3}{2}}\left(1+t^{-1}D_1(k)+t^{-1}D_2(k)o(1)\right) \\
&= (1-\rho)A\left(\frac{\sqrt{\rho}}{(1-\sqrt{\rho})^2}\right)^2 e^{-\gamma t}t^{-\frac{3}{2}}\left(1+t^{-1}F_1(\rho)+t^{-1}F_2(\rho)o(1)\right),
\end{aligned}$$

where F_1 and F_2 are constants depending on ρ. Hence:

$$\begin{aligned}
&\sum_{k,l\geq 1}\left|\frac{\pi_k p^*_{kl}(t)}{\mathbb{P}_\pi(\tau>t)}-\pi^{\mathrm{QS}}_{k,l}\right| \\
&= t^{-1}\left(\frac{(1-\rho^{\frac{1}{2}})^2}{\rho^{\frac{1}{2}}}\right)^2\sum_{k,l\geq 1}kl\rho^{\frac{k+l}{2}}\left|\frac{Z_1(k,l)+Z_2(k,l)o(1)}{kl+t^{-1}klF_1(\rho)+t^{-1}klF_2(\rho)o(1)}\right|.
\end{aligned}$$

Since all polynomials are of fixed orders, we have:

$$\sup_{t\geq 0}\left|\frac{Z_1(k,l)+Z_2(k,l)o(1)}{kl+t^{-1}klF_1(\rho)+t^{-1}klF_2(\rho)o(1)}\right| = V_2(k,l)<\infty.$$

Thus by the dominated convergence theorem we have:

$$\sum_{l=1}^{\infty} kl\rho^{\frac{k+l}{2}} \left| \frac{Z_1(k,l) + Z_2(k,l)o(1)}{kl + t^{-1}klF_1(\rho) + t^{-1}klF_2(\rho)o(1)} \right| \to C_2 = \sum_{l=1}^{\infty} \rho^{\frac{k+l}{2}} \left| Z_1(k,l) \right|. \tag{2.4}$$

□

3. Brownian motion with drift

In this section we will study the reflected Brownian motion with drift $X(t)$. The governing process is $Z(t) = B(t) - ct$; $c > 0$. In this case the stationary distribution π of $X(t)$ is $\pi(dx) = 2ce^{-2cx}\,dx$. Suppose that $\tau = \inf\{t > 0 : Z(t) = 0\}$. The distribution of $\mathbb{P}_x(Z(t) \in \cdot | \tau > t)$ is absolutely continuous with pdf denoted by $f_{x|t}(y)$. It is known the quasi-stationary pdf

$$f^{RQS}(y)dy = \lim_{t\to\infty} \mathbb{P}_x(Z(t) \in dy | \tau > t) = c^2 y e^{-cy}\,dy,$$

see [8]. Consider now the reflected process X and $T = \inf\{t > 0 : X(t) = 0\}$. The distribution of $\mathbb{P}_\pi(X(0) \in dx, X(t) \in dy | T > t)$ is absolutely continuous with pdf denoted by $f_{\pi|t}(x,y)$. The question is about the quasi-stationary distribution $\pi^{QS}(dx \times dy) = f^{\mathrm{QS}}(x,y)\,dx\,dy$, where

$$\lim_{t\to\infty} \mathbb{P}_\pi(X(0) \in dx, X(t) \in dy | T > t) = \pi^{QS}(dx \times dy).$$

Our aim is to study the speed of convergence to the quasi-stationary distribution. Let $f^{\mathrm{QS}}(x,y) = c^4 xy e^{-c(x+y)}$.

Theorem 2. *We have*

(i)

$$\int_0^\infty \left| f_{x|t}(y) - f^{\mathrm{RQS}}(y) \right| \, dy = \frac{1}{t}(C_3 + o(1)),$$

where C_3 is given by (3.3)*. Furthermore*

(ii)

$$\int_0^\infty \left| f_{\pi|t}(x,y) - f^{\mathrm{QS}}(x,y) \right| \, dy = \frac{1}{t}(C_4 + o(1)),$$

where C_4 is given by (3.5)*.*

Proof. We start the proof by writing pdf $f_{x|t}(y)$ in the form

$$f_{x|t}(y)dy = \frac{\mathbb{P}_x(Z(t) \in dy, \tau > t)}{\mathbb{P}_x(\tau > t)},$$

and using formula (1.2.8) from Borodin and Salminen [4, page 252] we have

$$\begin{aligned}
&\mathbb{P}_x(Z(t) \in dy, \tau > t) \\
&= \mathbb{P}_x(Z(t) \in dy, \inf_{0\le s\le t} Z(s) > 0) \\
&= \frac{1}{\sqrt{2\pi t}} \left[\exp\left(-\frac{(y-x+ct)^2}{2t} \right) - \exp\left(-\frac{(x+y)^2}{2t} - c(y-x) - \frac{c^2 t}{2} \right) \right] dy \\
&= \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{-\frac{c^2}{2}t} e^{-cy} e^{cx} \left[\exp\left(-\frac{(x-y)^2}{2t} \right) - \exp\left(-\frac{(x+y)^2}{2t} \right) \right] dy.
\end{aligned}$$

We now find the asymptotic expansion of $\mathbb{P}_x(Z(t) \in dy, \tau > t)$ as $t \to \infty$. Using Lemma 2 from the Appendix

$$\begin{aligned} &\exp\left(-\frac{(x-y)^2}{2t}\right) - \exp\left(-\frac{(x+y)^2}{2t}\right) \\ &= \; 2xyt^{-1} - (x^2+y^2)xyt^{-2}\left(1+o_t(1)\right), \end{aligned}$$

where $o_t(1)$ is a function of t tending to 0 as $t \to \infty$. Thus we have

$$\begin{aligned} &\mathbb{P}_x(Z(t) \in dy, \tau > t) \\ &= \; \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}t} e^{-cy} e^{cx} \left(2xyt^{-1} - (x^2+y^2)xyt^{-2}\left(1+o_t(1)\right)\right) dy. \end{aligned}$$

Moreover,

$$\mathbb{P}_x(\tau > t) \;=\; \mathbb{P}_x(\inf_{0 \le s \le t} B(t) - ct > 0) = \mathbb{P}(\tau_x > t),$$

where τ_x is inverse Gaussian process. Hence following Kyprianou [6, page 9]

$$\mathbb{P}_x(\tau > t) \;=\; \int_t^\infty \frac{x}{\sqrt{2\pi s^3}} \exp\left(cx - \frac{c^2}{2}s - \frac{x^2}{2s}\right) ds.$$

Again using Lemma 2

$$\begin{aligned} \mathbb{P}_x(\tau > t) &= \frac{x}{\sqrt{2\pi}} e^{cx} \int_t^\infty s^{-\frac{3}{2}} \exp\left(-\frac{c^2}{2}s\right)\left(1 - \frac{x^2}{2s}(1+o_s(1))\right) ds \\ &= \frac{x}{\sqrt{2\pi}} e^{cx} \int_t^\infty s^{-\frac{3}{2}} \exp\left(-\frac{c^2}{2}s\right) ds - \frac{x^3}{2\sqrt{2\pi}} e^{cx} \int_t^\infty s^{-\frac{5}{2}} \exp\left(-\frac{c^2}{2}s\right)(1+o_s(1))\, ds. \end{aligned}$$

Notice that the last integral can be expressed uaing incomplete gamma function notation:

$$\begin{aligned} \int_t^\infty s^{-\frac{3}{2}} \exp\left(-\frac{c^2}{2}s\right) ds &= \left(\frac{c^2}{2}\right)^{\frac{1}{2}} \int_{\frac{c^2}{2}t}^\infty u^{-\frac{3}{2}} e^{-u} du \\ &= \frac{c}{\sqrt{2}} \Gamma\left(-\frac{1}{2}, \frac{c^2}{2}t\right). \end{aligned} \tag{3.1}$$

Thus

$$\mathbb{P}_x(\tau > t) \;=\; \frac{x}{\sqrt{2\pi}} e^{cx} \left[\frac{c}{\sqrt{2}} \Gamma(-\frac{1}{2}, \frac{c^2}{2}t) - \left(\frac{c}{\sqrt{2}}\right)^3 \frac{x^2}{2} \Gamma\left(-\frac{3}{2}, \frac{c^2}{2}t\right)(1+o_t(1))\right].$$

We now need an expansion given in Abramovitz and Stegun [1, page263]:

$$\Gamma(a,z) \sim z^{a-1} e^{-z}\left(1 + \frac{a-1}{z} + o(z^{-1})\right). \tag{3.2}$$

Using formula (3.2) we can write

$$\begin{aligned} &\mathbb{P}_x(\tau > t) = \\ &= \frac{1}{\sqrt{2\pi}} e^{cx} e^{-\frac{c^2}{2}t} 2c^{-2} \left[xt^{-\frac{3}{2}} - \left(\frac{x^3}{2} - 3c^{-2}x\right) t^{-\frac{5}{2}} + x^3 t^{-\frac{5}{2}} o_t(1)\right]. \end{aligned}$$

Hence we have

$$\begin{aligned} f_{x|t}(y)dy &= \mathbb{P}_x(X_t \in dy|\tau > t) \\ &= \frac{2xyt^{-\frac{3}{2}} - (x^2+y^2)xyt^{-\frac{5}{2}}\,(1+o_t(1))}{2c^{-2}\left(t^{-\frac{3}{2}}x - t^{-\frac{5}{2}}(\frac{x^3}{2} - 3c^{-2}x) + t^{-\frac{5}{2}}x^3 o(1)\right)} e^{-cy}dy. \end{aligned}$$

We are now ready to compute the speed of convergence

$$\begin{aligned} &\mid f_{x|t}(y) - f^{RQS}(y) \mid \\ &= \left| \frac{2xyt^{-\frac{3}{2}} - (x^2+y^2)xyt^{-\frac{5}{2}}\,(1+o_t(1))}{2c^{-2}xt^{-\frac{3}{2}} - (c^{-2}x^3 - 6c^{-4}x)t^{-\frac{5}{2}} + x^3t^{-\frac{5}{2}}o_t(1)} e^{-cy} - c^2ye^{-cy} \right| \\ &= e^{-cy}t^{-1} \left| \frac{-(6c^{-2}+y^2)xy + x^3y^3 o_t(1)}{2c^{-2}x - (c^{-2}x^3 - 6c^{-4}x)t^{-1} + x^3t^{-1}o_t(1)} \right| \\ &= \left(3y + \frac{c^2}{2}y^3\right) e^{-cy}t^{-1}(1 + x^3y^3 o_t(1)). \end{aligned}$$

Hence

$$\begin{aligned} &\int_0^\infty \mid f_{x|t}(y) - f^{RQS}(y) \mid dy \\ &= \int_0^\infty \left(3y + \frac{c^2}{2}y^3\right) e^{-cy}dyt^{-1}\left(1 + x^3 o_t(1)\right) \\ &= 6c^{-2}t^{-1}\left(1 + x^3 o_t(1)\right). \end{aligned} \tag{3.3}$$

This completes the proof of the first part of Theorem 2. We start the proof of the second part by recalling that:

$$\begin{aligned} &\mathbb{P}_\pi(X_0 \in dx, X_t \in dy, \tau > t) \\ &= \frac{2c}{\sqrt{2\pi}} e^{-\frac{c^2}{2}t} e^{-cy} e^{-cx} \left(2xyt^{-\frac{3}{2}} - (x^2+y^2)xyt^{-\frac{5}{2}}\,(1+o_t(1))\right) dy. \end{aligned}$$

By integrating above equation we derive:

$$\begin{aligned} \mathbb{P}_\pi(\tau > t) &= \int_0^\infty \int_0^\infty \mathbb{P}_\pi(X_0 \in dx, X_t \in dy, \tau > t) \\ &= \frac{2c}{\sqrt{2\pi}} e^{-\frac{c^2}{2}t} \int_0^\infty e^{-cx} \left(2xt^{-\frac{3}{2}} \int_0^\infty ye^{-cy}dy - t^{-\frac{5}{2}} \int_0^\infty (x^2+y^2)xye^{-cy}dy\,(1+o_t(1))\right) \\ &= \frac{2c}{\sqrt{2\pi}} e^{-\frac{c^2}{2}t} \left(\frac{2}{c^4}t^{-\frac{3}{2}} - \frac{12}{c^6}t^{-\frac{5}{2}} + t^{-\frac{5}{2}}o_t(1)\right). \end{aligned}$$

Hence we have

$$f_{\pi|t}(x,y) = \frac{2t^{-\frac{3}{2}} - (x^2+y^2)t^{-\frac{5}{2}}\,(1+o_t(1))}{\frac{2}{c^4}t^{-\frac{3}{2}} - \frac{12}{c^6}t^{-\frac{5}{2}} + t^{-\frac{5}{2}}o_t(1)} xye^{-c(x+y)},$$

which gives:

$$\begin{aligned} \mid f_{\pi|t}(x,y) - f^{\mathrm{QS}}(x,y) \mid &= \left| \frac{2xyt^{-\frac{3}{2}} - (x^2+y^2)xyt^{-\frac{5}{2}}\,(1+o_t(1))}{\frac{2}{c^4}t^{-\frac{3}{2}} - \frac{12}{c^6}t^{-\frac{5}{2}} + t^{-\frac{5}{2}}o_t(1)} - c^4xy \right| e^{-c(x+y)} \\ &= c^2xye^{-c(x+y)} \left| 6 - c^2\frac{x^2+y^2}{2} \right| t^{-1}\left(1+o_t(1)\right), \end{aligned}$$

where the quasi-stationary pdf is

$$f^{\mathrm{QS}}(x,y) = c^4 xye^{-c(x+y)}. \tag{3.4}$$

Thus

$$\begin{aligned} &\int_0^\infty \int_0^\infty \mid f_{\pi|t}(x,y) - f^{\mathrm{QS}}(x,y) \mid dxdy \\ &= c^2 \int_0^\infty \int_0^\infty \left| 6 - c^2 \frac{x^2+y^2}{2} \right| xye^{-c(x+y)} t^{-1} (1 + o_t(1))\, dxdy. \end{aligned} \tag{3.5}$$

□

Appendix

Here we follow an idea from Asmussen [2, III.8e.]. We would like to point out that in [2, Lemma 8.11] is a typo, which is corrected in (3.10) of Proposition 1 below. However for our purposes we need an extension given in (3.11). Recall that the n-th Bessel function is

$$I_n(x) = \begin{cases} \sum_{k=0}^\infty \frac{(x/2)^{2k+n}}{k!(k+n)!} & \text{for } n \in \mathbf{Z}_+ \\ I_{-n}(x) & \text{for } -n \in \mathbf{Z}_+. \end{cases} \tag{3.6}$$

In Abramovitz and Stegun [1] one can find the following asymptotic expansion

$$\begin{aligned} &I_n(t) \sim \\ &\sim \quad e^t \frac{1}{\sqrt{2\pi t}} \left[1 - \frac{d-1}{8t} + \frac{(d-1)(d-9)}{2!(8t)^2} - \frac{(d-1)(d-9)(d-25)}{3!(8t)^3} + \dots \right] \end{aligned}$$

for $t \to \infty$, where $d = 4n^2$. Unfortunately nothing is said about the error. Here, following the idea from Asmussen [2, page 106], we give two or three terms with universal errors valid for all n. For this consider $Z(t) = \Pi_\beta(t) - \Pi_\delta(t)$, where Π_β and Π_δ are two independent Poisson processes with $\beta = \delta = 1/2$. In particular $\mathbb{E}Z(1) = 0$, $\mathbb{V}arZ(1) = 1$ and furthermore the cummulants are

$$\chi_r = \begin{cases} 1, & \text{if } r \text{ even}, \\ 0, & \text{if } r \text{ odd}. \end{cases}$$

We now give a uniform with respect to n estimation of $\mathbb{P}(Z(t) = n)$ for $t \to \infty$. We will use them to have universal estimations of the Bessel function $I_n(t) = e^t \mathbb{P}(Z(t) = n)$ for $t \to \infty$. For this we need to clarify the following convention. It is said that a family of functions $a_n(t)$ is $n^s o(t^{-\alpha})$ universally for all n if

$$\lim_{t\to\infty} \sup_n \left| \frac{a_n(t)}{n^s} \right| = o(t^{-\alpha}).$$

In the sequel we will use the following result from Rao-Bhattacharya [3, page 231]:

$$\sup_n \left[1 + \left(\frac{n}{\sqrt{t}} \right)^s \right] \left| \mathbb{P}\left(\frac{Z(t)}{\sqrt{t}} = \frac{n}{\sqrt{t}} \right) - q_{t,s}\left(\frac{n}{\sqrt{t}} \right) \right| = o(t^{-\frac{s-1}{2}}), \tag{3.7}$$

where

$$q_{t,s}(x) = \frac{1}{\sqrt{t}} \sum_{r=0}^{s-2} t^{-\frac{r}{2}} P_r(-\phi; \{\chi_v\})(x),$$

and functions $P_r(-\phi;\{\chi_v\})(x)$ are given by

$$P_r(-\phi;\{\chi_v\})(x) = \sum_{m=1}^{r}\frac{1}{m!}\left\{\sum_{j_1,\dots,j_m}^{*}\frac{(-1)^{j_1}\chi_{j_1+2}}{(j_1+2)!}\cdots\frac{(-1)^{j_m}\chi_{j_m+2}}{(j_m+2)!}\phi^{(2m+r)}(x)\right\}. \tag{3.8}$$

The summation $\sum^*$ is over all m-tuples of positive integers $(j_1,\dots,j_m)$ satisfying $\sum_{k=1}^m j_k = r$, and $\phi^{(n)}$ denotes the n-th derivative of probability density function of the standard Gaussian distribution. We will need the following lemma.

Lemma 2. *We have for $t>0$ and $n\in\mathbb{Z}$*

$$\exp\left(-\frac{n^2}{2t}\right) = \sum_{j=0}^{k}\frac{(-1)^j}{j!}\left(\frac{n^2}{2t}\right)^j + n^{2k}o(t^{-k}),$$

as $t\to\infty$ universally for all n.

Proof. Since

$$\sup_{x>0}\left|\frac{e^{-x}-\sum_{j=0}^{k}(-1)^j\frac{x^j}{j!}}{x^{k+1}}\right| = c<\infty,$$

setting $x=n^2/(2t)$ $(t>0, n\in\mathbb{Z})$

$$n^{2k}\frac{c}{(2t)^{k+1}} \geq \left|\exp\left(-\frac{n^2}{2t}\right)-\sum_{j=0}^{k}\frac{(-1)^j}{j!}\left(\frac{n^2}{2t}\right)^j\right|,$$

which completes the proof. □

In particular we have

$$\exp\left(-\frac{n^2}{2t}\right) = 1-\frac{n^2}{2t}+n^2o(1). \tag{3.9}$$

Using (3.7) for $s=4$, $s=6$ respectively we can prove the following result.

Proposition 1. *For each $n\in\mathbb{Z}$*

$$\mathbb{P}(Z(t)=n) = \frac{1}{\sqrt{2\pi t}}\left(1-\frac{4n^2-1}{8t}\right)+n^6o(t^{-3/2}) \tag{3.10}$$

and

$$\mathbb{P}(Z(t)=n) = \frac{1}{\sqrt{2\pi t}}\left(1-\frac{4n^2-1}{8t}+\frac{(4n^2-1)(4n^2-9)}{2^7t^2}\right)+n^{12}o(t^{-5/2}), \tag{3.11}$$

where the symbol $o(\cdot)$ is a function of t universal for all n.

Proof. **of Proposition 1**. To prove (3.10) we write

$$\mathbb{P}(Z(t)=n) = q_{t,4}\left(\frac{n}{\sqrt{t}}\right)+o(t^{-3/2}) = \frac{1}{\sqrt{t}}\phi\left(\frac{n}{\sqrt{t}}\right)\left(1+\frac{\frac{n^4}{t^2}-6\frac{n^2}{t}+3}{4!t}\right)+o(t^{-3/2}).$$

Now using (3.9)

$$\begin{aligned}\mathbb{P}(Z(t)=n) &= \\ &= \frac{1}{\sqrt{2\pi t}}\left(1-\frac{n^2}{2t}+n^2t^{-1}o(1)\right)\left[1+\frac{1}{8t}+\frac{n^4}{t}\left(\frac{1}{4!t^2}-\frac{1}{4n^2t}\right)\right]+t^{-\frac{3}{2}}o(1) \\ &= \frac{1}{\sqrt{2\pi t}}\left(1-\frac{4n^2-1}{8t}+n^6t^{-1}r(n,t)\right)+t^{-\frac{3}{2}}o(1),\end{aligned}$$

where $r(n,t) = 16n^{-4}t^{-1} + (n^{-4} + n^{-4}t^{-1} + t^{-1})o(1)$ is clearly $o(1)$ as a function of t uniformly with respect to n. Now taking (3.7) with $s = 6$ we have

$$\begin{aligned}&\mathbb{P}(Z(t)=n) \\ &= q_{t,6}\left(\frac{n}{\sqrt{t}}\right)+t^{-\frac{5}{2}}o(1) \\ &= \frac{1}{\sqrt{t}}\left\{\phi\left(\frac{n}{\sqrt{t}}\right)+\frac{1}{t}\frac{\chi_4}{4!}\phi^{(4)}\left(\frac{n}{\sqrt{t}}\right)+\frac{1}{t^2}\left[\frac{\chi_6}{6!}\phi^{(6)}\left(\frac{n}{\sqrt{t}}\right)+\frac{\chi_4^2}{2!4!^2}\phi^{(8)}\left(\frac{n}{\sqrt{t}}\right)\right]\right\}+t^{-\frac{5}{2}}o(1) \\ &= \frac{1}{\sqrt{t}}\phi\left(\frac{n}{\sqrt{t}}\right)\left\{1+\frac{1}{t}\frac{\frac{n^4}{t^2}-6\frac{n^2}{t}+3}{4!}+\frac{1}{t^2}\left(\frac{H_6(\frac{n}{\sqrt{t}})}{6!}+\frac{H_8(\frac{n}{\sqrt{t}})}{2!4!^2}\right)\right\}+t^{-\frac{5}{2}}o(1),\end{aligned}$$

where H_k is the Hermite polynomial of degree k. We have

$$\frac{H_6(\frac{n}{\sqrt{t}})}{6!}+\frac{H_8(\frac{n}{\sqrt{t}})}{2!4!^2} = \frac{1}{2^7}\left(9+n^8o(1)\right),$$

where $o(1)$ is with respect to t uniform for all n. Hence we have

$$\begin{aligned}&\mathbb{P}(Z(t)=n) \\ &= \frac{1}{\sqrt{t}}e^{\frac{-n^2}{2t}}\left[1+\frac{n^4}{4!t^3}-\frac{n^2}{4t^2}+\frac{1}{8t}+\frac{9}{2^7t^2}+\frac{n^8}{t^2}o(1)\right]+t^{-\frac{5}{2}}o(1) \\ &= \frac{1}{\sqrt{2\pi t}}\left[1-\frac{n^2}{2t}+\frac{n^4}{8t^2}+\frac{n^4}{t^2}o(1)\right]\left[1+\frac{n^4}{4!t^3}-\frac{n^2}{4t^2}+\frac{1}{8t}+\frac{9}{2^7t^2}+\frac{n^8}{t^2}o(1)\right]+t^{-\frac{5}{2}}o(1) \\ &= \frac{1}{\sqrt{2\pi t}}\left[1-\frac{4n^2-1}{8t}+\frac{1}{t^2}\left(\frac{n^4}{8}-\frac{5n^2}{16}+\frac{9}{2^7}\right)+r(n,t)\right]+t^{-\frac{5}{2}}o(1),\end{aligned}$$

where

$$r(n,t) = n^{12}t^{-2}o(1),$$

which completes the proof of the proposition. □

In general we can state the following conjecture:

$$\begin{aligned}\mathbb{P}(Z(t)=n) &= \\ &= \frac{1}{\sqrt{2\pi t}}\left[1+\sum_{k=1}^{j}(-1)^k\frac{(d-1)\cdots(d-(2k-1)^2)}{k!(8t)^k}\right]+n^{6j}o(t^{-(j+\frac{1}{2})}),\end{aligned}$$

where $d = 4n^2$ and the symbol $o(\cdot)$ is universal for all n.

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