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AN APPROACH TO THE FORMULA $L = \lambda V$ VIA THE THEORY OF STATIONARY POINT PROCESSES ON A SPACE OF COMPACT SUBSETS OF R^k

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1. Introduction

Much attention has been drawn to the queueing formula $L = \lambda V$; see [1],[2],[3],[5],[10],[11],[12]. In this context L means "average queue", λ means "intensity of arrivals" and V means "average delay in queue". In this note we point out that the formula $L = \lambda V$ is valid for stationary processes of random compact sets $\underline{\mu}$ in R^k. Informally speaking it is a collection of random compact sets strewed over R^k in a stationary manner. A formal definition will be given in Section 4. In this context L means "average number of sets covering zero", λ means intensity of $\underline{\mu}$, and V means "average volume of a typical set". Clearly concepts of the intensity and average volume of a typical set need explanations and they are studied in Sections 5 and 6. The formula $L = \lambda V$ is studied in Section 6. The proof of $L = \lambda V$ is based on the behaviour of sample path and applies Nguyen-Zessin's ergodic theorem for point processes; see [8]. Finaly in Section 7 we deal with the number of sets from a stationary process of random compact sets $\underline{\mu}$ overlapping a compact set K. The special case of such a problem was considered in [6], and recently, in this setting, was independently studied by Stoyan [12].

2. Preliminaries

Throughout the paper we denote the k-dimensional Euclidean space by R^k and the Borel σ-field of subsets of R^k by $\mathcal{B}^k$. The set of all non-negative real numbers we denote by R_+. For a topological spece E we denote by $\mathcal{B}E$ the Borel σ-field of subsets of E. The open ball in a metric space E with the center x and the radius r we denote by $B(\mathbf{E},x,r)$. The complement of a set A is A^c. The indicator function of a set A is $1_A(x)$. The following notations and convention are used:

$$(0,1)^k = (0,1) \times \dots \times (0,1) \in \mathcal{B}^k ,$$

$$|B| = \int_B dx , \qquad B \in \mathcal{B}^k ,$$

$$\sum_i = \sum_{i=0}^{\infty} .$$

3. Random elements associated with point process

We follow Kallenberg [4] for the definition of random measures or point processes. Let E be a locally compact second countability Hausdorff topological space (LCS). Let $M(E)$ be the class of measures μ such that $\mu(V) < \infty$, for any bounded V and $C(E)$ be the class of continuous functions f with compact support. We can make $M(E)$ into a Polish space with convergence $\mu_n \to \mu$ iff $\int f d\mu_n \to \int f d\mu$ for all $f \in C(E)$. The subset $M(E)$ of integer valued measures in $M(E)$ is Borel.

Let $(\Omega, F, \mathbb{P})$ be a probability space.

Definition 3.1:

A measurable mapping

$$\underline{\mu}: (\Omega, F) \to (M(E), BM(E))$$

is said to be a random measure on E. If $\Omega = M(E)$, $F = BM(E)$ and $\underline{\mu}(\omega) = \omega$ then $\underline{\mu}$ is called the canonical random measure on E. The canonical random measure identifies with the probability space $(M(E), BM(E), \mathbb{P})$.

Definition 3.2:

A measurable mapping

$$\underline{\nu}: (\Omega, F) \to (N(E), BN(E))$$

is said to be a point process on E. If $\Omega = N(E)$, $F = BN(E)$ $\underline{\nu}(\omega) = \omega$ then $\underline{\nu}$ is called the canonical point process on E. The canonical point process identifies with the probability space $(N(E), BN(E), \mathbb{P})$.

Consider a Polish space A and a locally compact secondcountability group E. There is given on A a family ot automorphisms $\{\sigma_x\}$; $x \in E$ fulfilling

(a) $\sigma_x \circ \sigma_y = \sigma_{x+y}$, $x, y \in E$ (group property),

(b) the mapping $A \times E \ni (\alpha, x) \to \sigma_x \alpha \in M$ is measurable.

On $M(E)$ we define a family of automorphisms $\{\tau_x\}$, $x \in E$ by

$$\tau_x \mu(B) = \mu(B+x), \qquad x \in E , \quad B \in BE .$$

Denote for $(\alpha, \mu) \in A \times M(E)$

$$T_x(\alpha, \mu) = (\sigma_x \alpha, \tau_x \mu) , \qquad x \in E .$$

Let $\underline{\alpha}$ be a random element on a probability space $(\Omega, F, \mathbb{P})$ assuming values at A. Let $\underline{\mu}$ be a random measure on E defined on the probability space $(\Omega, F, \mathbb{P})$.

Definition 3.3:

The pair $(\underline{\alpha}, \underline{\mu})$ is called the random element associated with random measure. The $(\underline{\alpha}, \underline{\mu})$ is said to be stationary if

$$\mathbb{P} \circ (\underline{\alpha}, \underline{\mu})^{-1} \circ T_x = \mathbb{P} \circ (\underline{\alpha}, \underline{\mu})^{-1}$$

or

$$P \circ T_x = P , \qquad x \in E , \tag{3.1}$$

where $P = \mathbb{P} \circ (\underline{\alpha}, \underline{\mu})^{-1}$ is the distribution of $(\underline{\alpha}, \underline{\mu})$.

In the case when $\underline{\mu}$ is a point process we adopt the abreviation REAPP for $(\underline{\alpha}, \underline{\mu})$.

We aim now to define the Palm distribution of P. It is a simply modification of the definition from [7], where the Palm distribution was defined not for the random elements associated with random measure but for the random measure. From now on P is stationary that is (3.1) holds. Define a measure $\lambda(\cdot)$ on (E, BE) by

$$\lambda(B) = E\underline{\mu}(B) , \qquad B \in BE . \tag{3.2}$$

We assume that $\lambda(\cdot)$ is non-zero and $\lambda(B) < \infty$ for bounded B. Then $\lambda(\cdot)$ is a Haar measure on (E, BE). Let $g: E \to R_+$ be such that

$$\int_E g(x) \lambda(dx) = 1$$

and set for $F \in B(A \times M(E))$

$$P^o(F) = \int_{A \times M(E)} \int_E g(x) \, 1_F \circ T_x(\alpha, \mu) \mu(dx) P(d\alpha \times d\mu) .$$

Clearly P^o is a probability measure. We shall call it as the Palm distribution of P.

<u>Theorem</u> 3.1 (Mecke [7]):

The Palm distribution P^o of P is the only probability on $A \times M(E)$ such that for all nonnegative $\mathcal{B}(A \times M(E) \times E)$ -measurable functions $z((\alpha,\mu),x)$

$$(3.3) \qquad \int_{A\times M(E)} \int_E z((\alpha,\mu),x)\mu(dx)P(d\alpha \times d\mu) = \int_{A\times M(E)} \int_E z(T_{-x}(\alpha,\mu),x)\lambda(dx)P^o(d\alpha \times d\mu) .$$

Let $(\underline{\alpha},\underline{\nu})$ be a stationary REAPP and suppose that $\underline{\nu}$ is a point process on R^k. In such a case $E = R^k$, $\lambda(dx) = \lambda dx$, where λ is the intensity of $\underline{\nu}$ and dx is the Lebesgue measure on R^k. Denote

$$A \times N^o(R^k) = \{(\alpha,\mu) \in A \times N(R^k) : \mu(\{0\}) \geq 1\} .$$

We have the following corollary.

<u>Corollary</u>:

If P^o is the Palm distribution of a REAPP $(\underline{\alpha},\underline{\nu})$, where $\underline{\nu}$ is a point process on R^k then

$$P^o(A \times N^o(R^k)) = 1 .$$

<u>Proof</u>. Set in (3.3)

$$z((\alpha,\mu),x) = \begin{cases} 1, & \mu(\{x\}) \geq 1, \quad x \in (0,1)^k, \\ 0, & \text{otherwise.} \end{cases}$$

Recall also the following definition.

<u>Definition</u> 3.4:

It is said that a point process $\underline{\nu}$ on E is without multiple points if

$$P(\{\omega: \underline{\nu}(\omega)(\{x\}) \leq 1, \ x \in E\}) = 1 .$$

4. <u>Processes of random sets</u>.

Let K be the space of all compact sets in R^k endowed with the Hausdorff metric given by

$$h(K_1,K_2) = \max(\sup_{x\in K_2} \kappa(x,K_1), \sup_{x\in K_1} \kappa(x,K_2)) .$$

Here $\kappa(x,K)$ denotes the distance from x to K in the Euclidean metric. It is known (see e.g. [6]) that (K,h) is LCS.

Let u(K) and $\rho(K)$ denote the center and the radius of the ball circumscribing a compact set $K \subset R^k$. The ball circumscribing a compact set K is a ball of the smallest volume containing K. The following argument shows existence and uniqueness. Define the function r(x) = $= \min\{t: B(R^k,x,t) \supset K\}$, which is clearly continuous. Thus there exists x_o such that $r(x_o) = \min_{x\in K} r(x)$. If there exists another ball $B(R^k,y_0,r(x_0)) \supset K$, $x_0 \neq y_0$ then also $B(R^k,(x_0-y_0)/2,(r^2(x_0) - (\|x_0-y_0\|/2/^2)^{\frac{1}{2}}) \supset K$. However in such a case $(r^2(x_0) - (\|x_0-y_0\|/2)^2)^{\frac{1}{2}} < r(x_0)$ which is impossible.

The mappings $u: K \to R^k$ and $\rho: K \to R_+$ are continuous and

$$(4.1) \qquad u(K+x) = u(K) + x ,$$

$$(4.2) \qquad \rho(K+x) = \rho(K), \quad x \in R^k, \quad K \in K.$$

Consider a canonical point process $\underline{\mu}$ on K, that is a probability space $(N(K),\mathcal{B}N(K),\mathbb{P})$. We assume that $\underline{\mu}$ is stationary. It was shown in [4], Lemma 2.3, that

$$(4.3) \qquad \underline{\mu} = \sum_i 1_{\{\underline{K}_i\}} ,$$

where $\underline{K}_i$, i=0,1,... is a sequence of r.e.'s assuming values at K.

<u>Definition</u> 4.1:

The point process $\underline{\mu}$ is said to be a process of random sets.

Using representation (4.3) define random measures

$$(4.4) \qquad \underline{\mu}_\rho = \sum_i \rho(\underline{K}_i) \, 1_{\{u(\underline{K}_i)\}} ,$$

$$(4.5) \qquad \underline{\mu}_1 = \sum_i |\underline{K}_i| \, 1_{\{u(\underline{K}_i)\}}$$

and the point process

$$(4.6) \qquad \underline{\mu}^* = \sum_i 1_{\{u(\underline{K}_i)\}} .$$

The measurability of mappings (4.4)-(4.6) it follows from the measurability of $\rho(K)$, $|K|$, $u(K)$. It is easy to show that $\underline{\mu}_\rho$, $\underline{\mu}_1$, $\underline{\mu}^*$ are sationary.

5. Intensity of $\underline{\mu}$.

Consider a stationary ergodic process of random sets $\underline{\mu}$. Let

$$(5.1) \qquad \lambda = E\, \underline{\mu}^*((0,1)^k)$$

be the intensity of $\underline{\mu}^*$. We assume $\lambda < \infty$, and μ^* is without multiple points.

Definition 5.1.:

The value λ we call the intensity of the process of random sets $\underline{\mu}$.

The following definition was proposed in [8],[9].

Definition 5.2.:

A system of open, bounded, convex sets $\{Q_R\}$, $R > 0$ is said to be regular if

(a) $Q_R \subset B(R^k,0,R)$, $R > 0$,

(b) there exists $k > 0$ and $R_0 > 0$ such that $|Q_R| > k|B(R^k,0,R)|$, $R > R_0$.

In the sequel $\{Q_R\}$ always denotes a regular system of sets. The following proposition justifies Definition 5.1. Recall representation $\underline{\mu} = \sum_i 1_{\underline{K}_i}$.

Proposition 5.1:

The limit

$$(5.2) \qquad \lim_{R\to\infty} \frac{\#\{i:\ \underline{K}_i \subset Q_R\}}{|Q_R|} = \lambda , \qquad \text{a.s. } \mathbb{P}.$$

Moreover

$$(5.3) \qquad \lim_{R\to\infty} \frac{\#\{i:\ \underline{K}_i \cap Q_R \neq \emptyset\}}{|Q_R|} = \lambda , \qquad \text{a.s. } \mathbb{P}.$$

Proof. First we note that

$$\#\{i:\ \underline{K}_i \subset Q_R\} ,$$

$$\#\{i:\ u(\underline{K}_i) \in Q_R\}$$

are random variables. It follows from the fact that

$$C = \{B \in K:\ B \subset Q_R\} \in \mathcal{B}K \qquad \text{(see [6])},$$

$$\#\{i:\ \underline{K}_i(\mu) \subset Q_R\} = \mu(C) ,$$

and that the mapping $\mu \to \mu(C)$ is measurable. Also $\#\{i:\ u(\underline{K}_i) \in Q_R\}$ is a random variable because $\underline{\mu}^*$ is a point process and

$$(5.4) \qquad \#\{i:\ u(\underline{K}_i)(\mu) \in Q_R\} = \underline{\mu}^*(\mu)(Q_R) .$$

To prove (5.2) it suffices to show

$$(5.5) \qquad \lim_{R\to\infty} \frac{\#\{i:\ u(\underline{K}_i) \in Q_R\}}{|Q_R|} = \lambda , \qquad \text{a.s. } \mathbb{P}$$

and

$$(5.6) \qquad \lim_{R\to\infty} \left(\frac{\#\{i:\ u(\underline{K}_i) \in Q_R\}}{|Q_R|} - \frac{\#\{i:\ \underline{K}_i \subset Q_R\}}{|Q_R|} \right) = 0 , \qquad \text{a.s. } \mathbb{P}.$$

The existence of the limit in (5.5) is the immediate consequence of (5.4) and the Corollary 1 after Theorem 1 in [7].

To prove (5.6) define a subset of R^k

$$J^-(Q,r) = \{x \in Q:\ B(R^k,x,r) \subset Q\} ,$$

where $Q \in \mathcal{B}^k$. Notice that $J^-(Q,r_1) \subset J^-(Q,r_2)$ whenever $r_1 \geq r_2$. We have for each $r > 0$

$$\{i:\ u(\underline{K}_i) \in Q_R\} - \{i:\ \underline{K}_i \subset Q_R\} \subset$$

$$\subset \{i:\ u(\underline{K}_i) \in Q_R - J^-(Q_R,\rho(\underline{K}_i))\} =$$

$$= \{i:\ u(K_i) \in Q_R - J^-(Q_R,\rho(\underline{K}_i)),\ 0 \leq \rho(\underline{K}_i) \leq r\} \cup$$

$$\cup \{i:\ u(\underline{K}_i) \in Q_R - J^-(Q_R,\rho(\underline{K}_i)), \rho(\underline{K}_i) > r\} \subset$$

$$\subset \{i:\ u(\underline{K}_i) \in Q_R - J^-(Q_R,r)\} \cup \{i:\ u(\underline{K}_i) \in Q_R, \rho(\underline{K}_i) > r\} .$$

From Lemma 2 in [8] we have

$$\lim_{R\to\infty} \frac{|Q_R - J^-(Q_R,r)|}{|Q_R|} = 0 .$$

Hence by the Corollary 1 after Theorem 1 from [8]

$$\lim_{R\to\infty} \frac{\#\{i:\ u(\underline{K}_i) \in Q_R - J^-(Q_R, r)\}}{|Q_R|} = 0 , \qquad \text{a.s. } \mathbb{P}.$$

The same corollary from [8] also provides

$$\lim_{R\to\infty} \frac{\#\{i:\ u(\underline{K}_i) \in Q_R, \rho(\underline{K}_i) > r\}}{|Q_R|}$$

$$= E(\#\{i:\ u(\underline{K}_i) \in (0,1)^k, \rho(\underline{K}_i) > r\}), \qquad \text{a.s. } \mathbb{P}.$$

Thus for completing the proof of (5.6) we remark that

$$\lim_{R\to\infty} E(\#\{i:\ u(\underline{K}_i) \in (0,1)^k, \rho(\underline{K}_i) > r\}) = 0 , \qquad \text{a.s. } \mathbb{P}$$

which follows from that

$$\#\{i:\ u(\underline{K}_i) \in (0,1)^k, \rho(\underline{K}_i) > r_1\} \le$$

$$\le \#\{i:\ u(\underline{K}_i) \in (0,1)^k, \rho(\underline{K}_i) > r_2\} \qquad \text{if } r_1 \ge r_2$$

and

$$\lim_{r\to\infty} E\#\{i:\ u(\underline{K}_i) \in (0,1)^k, \rho(\underline{K}_i) > r\}$$

$$= \lim_{r\to\infty} \lambda P^o_\rho(\{\mu:\ \mu(\{0\}) > r\}) = 0$$

where P^o denotes the Palm distribution of the distribution of $\underline{\mu}$. This completes the proof of (5.2).

Now we prove (5.3). In view of (5.2) we need only to show

$$\text{(5.7)} \qquad \lim_{R\to\infty} \left(\frac{\#\{i:\ \underline{K}_i \cap Q_R \neq \emptyset\}}{|Q_R|} - \frac{\#\{i:\ \underline{K}_i \subset Q_R\}}{|Q_R|}\right) = 0 , \qquad \text{a.s. } \mathbb{P}.$$

However

$$\{i:\ \underline{K}_i \cap Q_R \neq \emptyset\} - \{i:\ \underline{K}_i \subset Q_R\} \subset$$

$$\subset \{i:\ u(\underline{K}_i) \in Q_R, \underline{K}_i \cap Q_R^c \neq \emptyset\} \cup \{i:\ u(\underline{K}_i) \notin Q_R, \underline{K}_i \cap Q_R \neq \emptyset\} \subset$$

$$\subset \{i:\ u(\underline{K}_i) \in Q_R - J^-(Q_R, \rho(\underline{K}_i))\} \cup$$

$$\cup \{i:\ u(\underline{K}_i) \in J^+(Q_R, \rho(\underline{K}_i)) - Q_R\} ,$$

where

$$J^+(Q,r) = \{x:\ B(R^k, x, r) \cap Q \neq \emptyset\} .$$

It was shown in the proof of (5.2) that

$$\lim_{R\to\infty} \frac{\#\{i:\ u(\underline{K}_i) \in Q_R - J^-(Q_R, \rho(\underline{K}_i))\}}{|Q_R|} = 0 , \qquad \text{a.s. } \mathbb{P}.$$

In the similar way we can show that

$$\lim_{R\to\infty} \frac{\#\{i:\ u(\underline{K}_i) \in J^+(Q_R, \rho(\underline{K}_i)) - Q_R\}}{|Q_R|} = 0 , \qquad \text{a.s. } \mathbb{P}.$$

which completes the proof of (5.3).

<u>Remark</u>:

We related with each K_i the point $u(K_i)$ from $B(R^k, u(K_i), \rho(K_i))$. Sometimes it is more convenient to choose not $u(K_i)$ but another point $u'(K_i) \in B(R^k, u(K_i), \rho(K_i))$. This is just the case in queueing theory where K_i are intervals in R^1 and $u'(K_i)$ is the left end-point of K_i. We assume that the function u' is measurable. Then $\underline{\mu}' = \sum_i 1_{\{u'(\underline{K}_i)\}}$ is a point process and

$$\text{(5.8)} \qquad \lim_{R\to\infty} \frac{\underline{\mu}'(Q_R)}{|Q_R|} = \lambda , \qquad \text{a.s. } P.$$

It follows from the inequalities

$$\#\{i:\ \underline{K}_i \subset Q_R\} \le \underline{\mu}'(Q_R) \le \#\{i:\ B(R^k, u(\underline{K}_i), \rho(\underline{K}_i)) \cap Q_R \neq \emptyset\}$$

and

$$\text{(5.9)} \qquad \lim_{R\to\infty}\left(\frac{\#\{i:\ B(R^k, u(\underline{K}_i), \rho(\underline{K}_i)) \cap Q_R \neq \emptyset\}}{|Q_R|} - \frac{\#\{i:\ \underline{K}_i \subset Q_R\}}{|Q_R|}\right) = 0$$

a.s. $\mathbb{P}$.

The proof of (5.9) is similar to the proof of (5.7). Note also that $\underline{\mu}$ need not to be stationary unless $u'(K) + x = u'(K+x)$, $x \in R^k$, $K \in \mathcal{K}$.

6. <u>Formula</u> $L = \lambda V$.

Let $\underline{\mu}$ be a canonical process of random sets in R^k. In other words $\underline{\mu}$ is a point process on $\mathcal{K}$ and has representation $\underline{\mu} = \sum_i 1_{\{\underline{K}_i\}}$.

Define $\underline{L}(t): \mathcal{N}(\mathcal{K}) \to R_+$ by

$$\underline{L}(t) = \sum_i 1_{\{\underline{K}_i\}}(t) , \qquad t \in R^k ,$$

i.e. $\underline{L}(t)$ is the number of sets from $\{\underline{K}_i\}$ covering the point t. Let $\{\sigma_s\}$, $s \in R^k$ be a group of automorphisms on the class of functions $f: R^k \to R^1$ defined by

$$\sigma_s f(t) = f(t-s) , \qquad t,s \in R^k .$$

<u>Lemma</u> 6.1:

$\{L(t)\}$, $t \in R^k$ is a random process on $(\mathcal{N}(\mathcal{K}), \mathcal{BN}(\mathcal{K}), \mathbb{P})$. Moreover if $\mathbb{P}$ is stationary then $\{\underline{L}(t)\}$ is.

<u>Proof</u>. Measurability:

$$\{\mu: \underline{L}(s)(\mu) = k\} = \{\mu: \mu(\mathcal{K}_{\{s\}}) = k\} ,$$

where as it was shown in [6]

$$\mathcal{K}_{\{s\}} = \{K \in \mathcal{K}: K \cap \{s\} \neq \emptyset\} \in \mathcal{BK} .$$

For the stationarity it suffices to point out that

$$\sigma_t \underline{L}(s)(\mu) = \underline{L}(s)(\tau_t \mu) , \qquad t,s \in R^k, \quad \mu \in \mathcal{N}(\mathcal{K}) .$$

Consider a REAPP $(\underline{\mu}_1, \underline{\mu}^*)$, where $\underline{\mu}_1$ and $\underline{\mu}^*$ are defined in (4.5) and (4.6). Let $\mathbb{P}_1$ be the distribution $(\underline{\mu}_1, \underline{\mu}^*)$ and $\mathbb{P}_1^o$ its Palm distribution. Denote $P_1^o(A) = \mathbb{P}_1^o(A \times \mathcal{N}(R^k))$, $A \in R^k$. The measure $\lambda(\cdot)$ defined in (3.2) is λdx, where λ is the intensity of $\underline{\mu}^*$ and dx is the Lebesgue measure in R^k. Then for a non-negative $\mathcal{B}(\mathcal{M}(R^k) \times R^k)$ - measurable function $z(\mu,x)$ we have by (3.3)

$$(6.1) \qquad \int_{\mathcal{M}(R^k)\times\mathcal{N}(R^k)} \int_{R^k} z(\mu,x)\nu(dx)\mathbb{P}_1(d\mu \times d\nu)$$
$$= \lambda \int_{\mathcal{M}(R^k)\times R^k} \int_{R^k} z(\tau_{-x}\mu,x)dx\, \mathbb{P}_1^o(d\mu \times d\nu) .$$

<u>Lemma</u> 6.2:

$$E\, \underline{\mu}_1((0,1)^k) = \lambda E_{P_1^o} \underline{\mu}_1(\{0\}) ,$$

where

$$E_{P_1^o} \underline{\mu}_1(\{0\}) = \int_{\mathcal{M}(R^k)} \mu(\{0\}) P_1^o(d\mu) .$$

Proof. Substitute to (6.1) the function

$$z(\mu,x) = \begin{cases} \mu(\{x\}) , & x \in (0,1)^k , \\ 0 , & \text{otherwise} , \end{cases}$$

Then

$$z(\tau_{-x}\mu,x) = \begin{cases} \mu(\{0\}) , & x \in (0,1)^k , \\ 0 , & \text{otherwise.} \end{cases}$$

<u>Theorem</u> 6.1:

If $\underline{\mu}$ is a stationary process of random sets with the finite intensity λ then

$$(6.2) \qquad E\underline{L}(t) = \lambda E_{\mathbb{P}_1^o} |\underline{K}| ,$$

where

$$E_{\mathbb{P}_1^o} |\underline{K}| = \int_{\mathcal{M}(R^k)} \mu(\{0\}) P_1^o(d\mu) .$$

<u>Proof</u>. Denote

$$a(R) = \{i: u(\underline{K}_i) \in Q_R\} , \qquad b(R) = \{i: \underline{K}_i \subset Q_R\} ,$$

$$c(R) = \{i: \underline{K}_i \cap Q_R \neq \emptyset\} .$$

We have

$$\sum_{i \in a(R)} |\underline{K}_i| = \underline{\mu}_1(Q_R)$$

and from the Corollary 1 after Theorem 1 in [8]

$$\lim_{R\to\infty} \frac{\underline{\mu}_1(Q_R)}{|Q_R|} = E\, \underline{\mu}_1((0,1)^k) | J) , \qquad \text{a.s. } \mathbb{P},$$

where J denotes the σ-field of invariant sets with respect to $\{\tau_x\}$ on $(N(K), BN(K), \mathbb{P})$. Denote

$$\underline{\Psi} = E(\underline{\mu}_1((0,1^k))\,|\,J) \ .$$

From Lemma 6.1 we have

$$E\underline{\Psi} = \lambda \int_{M(K)} \mu(\{0\}) P_1^o(d\mu)$$

The proof of the theorem is based on the inequality

$$(6.3) \qquad \sum_{i\in b(R)} |\underline{K}_i| \le \int_{Q_R} \underline{L}(s)ds \le \sum_{i\in c(R)} |\underline{K}_i| \ .$$

Thus it suffices to show that

$$(6.4) \qquad \lim_{R\to\infty} \frac{\sum_{i\in b(R)} |\underline{K}_i|}{|Q_R|} = \underline{\Psi} \ , \qquad \text{a.s. } \mathbb{P},$$

$$(6.5) \qquad \lim_{R\to\infty} \frac{\sum_{i\in c(R)} |\underline{K}_i|}{|Q_R|} = \underline{\Psi} \ , \qquad \text{a.s. } \mathbb{P}.$$

To prove (6.4) we find by a similar argument as in the proof of Proposition 5.1 that for each $r > 0$

$$0 \le \sum_{i\in a(R)} |\underline{K}_i| - \sum_{i\in b(R)} |K_i| \le$$

$$\le \sum_{\{i:u(\underline{K}_i)\in Q_R - J^-(Q_R,r)\}} |\underline{K}_i| + \sum_{\{i:u(\underline{K}_i)\in Q_R,\rho(K_i)>r\}} |\underline{K}_i| \ .$$

Lemma 2 and the Corollary 1 after Theorem 1 in [8] yields

$$\lim_{R\to\infty} \frac{\sum_{\{i:u(\underline{K}_i)\in Q_R - J^-(Q_R,r)\}} |K_i|}{|Q_R|}$$

$$= \lim_{R\to\infty} \frac{\underline{\mu}_1(Q_R - J^-(Q_R,r))}{|Q_R|} = 0 \ , \qquad \text{a.s. } \mathbb{P}.$$

They also provide existence of the limit

$$\lim_{R\to\infty} \frac{\sum_{\{i:u(K_i)\in Q_R,\rho(K_i)>r\}} |K_i|}{|Q_R|} =$$

$$= \lim_{R\to\infty} \frac{\sum_{i\in a(R)} |\underline{K}_i| 1_{(r,\infty)}(\rho(\underline{K}_i))}{|Q_R|} = \underline{\Phi}_r \ , \qquad \text{a.s. } \mathbb{P}.$$

We have $\underline{\Phi}_{r_1} \ge \underline{\Phi}_{r_2}$, whenever $r_1 \le r_2$ and $\lim_{r\to\infty} E\underline{\Phi}_r = 0$ which yields $\lim_{r\to\infty} \underline{\Phi}_r = 0$, a.s. $\mathbb{P}$. The proof of (6.5) is similar. Thus by (6.3)-(6.5),

$$\lim_{R\to\infty} \frac{\int_{Q_R} \underline{L}(s)ds}{|Q_R|} = \underline{\Psi} \ , \qquad \text{a.s. } \mathbb{P}.$$

On the other hand from the stationarity of $\{\underline{L}(s)\}$, by the ergodic theorem

$$\lim_{R\to\infty} \frac{\int_{Q_R} \underline{L}(s)ds}{|Q_R|} = \underline{\mathbf{Y}} \ , \qquad \text{a.s. } \mathbb{P}$$

and

$$E\underline{Y} = E\underline{L}(0) \ .$$

Hence $\underline{Y} = \underline{\Psi}$, a.s. $\mathbb{P}$ and

$$E\underline{Y} = E\underline{L}(t) = E\underline{\Psi} = \lambda E_{\mathbb{P}_1^o} |\underline{K}| \ .$$

7. The number of sets overlapping a set K

We finish the paper contributing to the Matheron's textbook [6], where a special process of random sets was derived (namely stationary Poisonian).

Let $\underline{\mu}$ be a stationary canonical process of random sets. Recall that $\underline{\mu} = \sum_i 1_{\{\underline{K}_i\}}$, where $\underline{K}_i : N(K) \to K$ are measurable. For a fixed set $K \in K$ define

$$D = \{F \in K : F \cap K \neq \emptyset\} \ .$$

From [6] it follows that $D \in BK$. We ask for the expected value

$$E\#\{i: \underline{K}_i \cap K \neq \emptyset\} = E\underline{\mu}(D) \quad (7.1)$$

Let $\mathbb{P}_*$ denotes the distribution of the REAPP $(\underline{\mu},\underline{\mu}^*)$ and $\mathbb{P}^o_*$ its Palm distribution. The marginal distribution we denote by $P^o_*(A) = \mathbb{P}^o_*(A \times N(R^k))$, $A \in BN(K)$. Assume that $0 < \lambda = E\underline{\mu}^*((0,1)^k) < \infty$ and that $\underline{\mu}^*$ is without multiple points.

To find $E\underline{\mu}(D)$ we use formula (3.3) setting for $\mu = \sum_i 1_{\{K_i\}} \in N(K)$

$$z(\mu,x) = \begin{cases} 1, & K_i \cap K \neq \emptyset, \quad u(K_i) = x, \\ 0, & \text{otherwise.} \end{cases}$$

Denote for $K, F \in K$

$$K \oplus \check{F} = \{x \in R^k: K \cap (F+x) \neq \emptyset\} .$$

From [6] it follows that the mapping $K \in F \to \check{F} \oplus K \in K$ is measurable. Note that

$$\int_{R^k} z(\mu,x)\underline{\mu}^*(\mu)(dx) = \underline{\mu}^*(\mu)(D) .$$

Consider the set

$$N^o(K) = \{\mu: \mu^*(\mu)(\{0\}) > 0\} \in BN(K)$$

Define

$$M_i = \{\mu: N^o(K): u(\underline{K}_i(\mu)) = 0\} , \qquad i=0,1,\ldots$$

and

$$\underline{\ell}(\mu) = \begin{cases} i, & \text{if } \mu \in M_i , \\ 0, & \text{otherwise} . \end{cases}$$

Then M_i, $i=0,1,\ldots$, are measurable and

$$\underline{K}_{(0)}(\mu) = \underline{K}_{\underline{\ell}(\mu)}(\mu)$$

is a random element. We have for $\mu \in N^o(K)$

$$\int_{R^k} z(\tau_{-x},x)dx = |K \oplus K_0| .$$

Hence by (3.3)

$$E\underline{\mu}(D) = \int_{N(K)} \int_{R^k} z(\mu,x)\underline{\mu}^*(\mu)(dx)\mathbb{P}(d\mu) =$$

$$= \lambda \int_{N(K)} \int_{R^k} z(\tau_{-x}\mu,x)dx \; P^o_*(d\mu) = \lambda \int_K |K \oplus F| P^o_{(0)}(dF),$$

where $P^o_{(0)}$ is the distribution of random element $\underline{K}_{(0)}$ on $(N(K), BN(K), P^o_*)$. Thus we arrived at the relation

$$E\#\{i: \underline{K}_i \cap K \neq \emptyset\} = \lambda \int_{N(K)} |K \oplus \check{F}| P^o_{(0)}(dF) . \quad (7.2)$$

Note that if $K = \{0\}$ then (7.2) reduces to (6.2).

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SEQUENTIAL ESTIMATES OF A REGRESSION FUNCTION BY ORTHOGONAL SERIES WITH APPLICATIONS IN DISCRIMINATION

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1. Introduction

Let (X,Y) be a pair of random variables. X takes values in a Borel set A, $A \subset R^p$, whereas Y takes values in R. Let f be the marginal Lebesgue density of X. Based on a sample $(X_1, Y_1),\ldots,(X_n, Y_n)$ of independent observations of (X,Y) we wish to estimate the regression r of Y on X, i.e.

$$r(x) = E\left[Y \mid X = x\right].$$

Let $\{g_k\}$, $k = 0,1,2,\ldots$ be a complete orthonormal system defined on A, such that

$$|g_k(x)| \leq G_k \quad (1)$$

for all $x \in A$, where $\{G_k\}$ is a sequence of numbers. Define

$$h(x) = r(x)f(x).$$

We assume that functions h and f have the representations

$$h(x) \sim \sum_{k=0}^{\infty} a_k g_k(x), \quad (2)$$

Lecture Notes in Statistics

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