AN APPROACH TO THE FORMULA \( L = \lambda V \) VIA THE THEORY OF STATIONARY POINT PROCESSES ON A SPACE OF COMPACT SUBSETS OF \( \mathbb{R}^k \)

Tomasz Rolski

University of Wrocław

1. Introduction

Much attention has been drawn to the queueing formula \( L = \lambda V \); see [11,12,13,14]. In this context \( L \) means "average queue", \( \lambda \) means "intensity of arrivals" and \( V \) means "average delay in queue". In this note we point out that the formula \( L = \lambda V \) is valid for stationary processes of random compact sets \( \mathcal{Y} \) in \( \mathbb{R}^k \). Informally speaking it is a collection of random compact sets strewed over \( \mathbb{R}^k \) in a stationary manner. A formal definition will be given in Section 4. In this context \( L \) means "average number of sets covering zero", \( \lambda \) means intensity of \( \mathcal{Y} \), and \( V \) means "average volume of a typical set". Clearly concepts of the intensity and average volume of a typical set need explanations and they are studied in Sections 5 and 6. The formula \( L = \lambda V \) is studied in Section 6. The proof of \( L = \lambda V \) is based on the behaviour of sample path and applies Nguyen-Zessin's ergodic theorem for point processes; see [8]. Finally in Section 7 we deal with the number of sets from a stationary process of random compact sets \( \mathcal{Y} \) overlapping a compact set \( X \). The special case of such a problem was considered in [6], and recently, in this setting, was independently studied by Stoyan [12].

2. Preliminaries

Throughout the paper we denote the \( k \)-dimensional Euclidean space by \( \mathbb{R}^k \) and the Borel \( \sigma \)-field of subsets of \( \mathbb{R}^k \) by \( \mathcal{B}^k \). The set of all non-negative real numbers we denote by \( \mathbb{R}_+ \). For a topological space \( E \) we denote by \( \mathcal{E} \) the Borel \( \sigma \)-field of subsets of \( E \). The open ball in a metric space \( E \) with the center \( x \) and the radius \( r \) we denote by \( B(E, x, r) \). The complement of a set \( A \) is \( A^c \). The indicator function of a set \( A \) is \( 1_A(x) \). The following notations and convention are used:
Consider a probability measure on a measurable set $E$. Let $f$ be such that $f(x) = g(x)$ for bounded $f$. Then $(\mathcal{E}, f) = (\mathcal{E}, g)$.

Next, define a measure $\mathcal{E}$ on $(\mathcal{E}, f)$. For any function $h$ measurable with respect to $(\mathcal{E}, f)$, we define $\mathcal{E}[h] = \int h \, d\mathcal{E}$.

For a measurable mapping $\gamma$, we define $\mathcal{E}[\gamma] = \int \gamma \, d\mathcal{E}$.

If $\mathcal{E}$ is a probability measure on $(\mathcal{E}, f)$, then for any measurable function $h$ on $\mathcal{E}$, we define $\mathcal{E}[h] = \int h \, d\mathcal{E}$.

For a measurable space $(X, \mathcal{X})$, we define $\mathcal{E}[X] = \int \mathcal{X} \, d\mathcal{E}$.

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If $\mathcal{E}$ is a probability measure on $(\mathcal{E}, f)$, then for any measurable function $h$ on $\mathcal{E}$, we define $\mathcal{E}[h] = \int h \, d\mathcal{E}$.
The mean of the point process \( \{ (T_x) \}_{x \in \mathbb{R}} \) is defined as the random variable \( \mu(T_x) = \int_{\mathbb{R}} \rho(x) \delta(x-T_x) dx \), where \( \rho(x) \) is the intensity function.

**Theorem 1.1** (Mecke's Theorem) Let \( \Lambda \) be a point process in \( \mathbb{R}^d \), and let \( \rho(x) \) be its intensity function. Then the random variable \( \mu(T_x) \) is a Poisson random measure with intensity \( \rho(x) \).

**Corollary** If \( \rho(x) \) is a locally finite measure, then \( \mu(T_x) \) is a Poisson point process in \( \mathbb{R}^d \).

**Definition** A point process \( \Lambda \) is said to be a point process in a set \( \mathcal{A} \) without multiple points if there exists a function \( \chi : \mathcal{A} \to \{0,1\} \) such that for any \( a \in \mathcal{A} \),

\[
\chi(a) = \begin{cases} 0 & \text{if } a \in \Lambda \setminus \{a\} \\ 1 & \text{if } a \notin \Lambda \setminus \{a\} \end{cases}
\]

This means that for each point \( a \in \mathcal{A} \), there are no other points of \( \Lambda \) in the set \( \mathcal{A} \) and \( a \) itself.

**Lemma 1.2** If \( \Lambda \) is a point process in \( \mathcal{A} \) without multiple points, then \( \chi(a) \) is a random variable with distribution \( \rho(a) \).

**Proof** (Continued)
are random variables. It follows from the fact that
\[ \mathbb{E}(\mathbf{X}_t | \mathbf{X}_0) = \mathbf{T}_t \mathbf{X}_0. \]

**Proof.** First we note that
\[ \mathbb{E}(\mathbf{X}_t | \mathbf{X}_0) = \mathbf{T}_t \mathbf{X}_0. \]

To prove (5.9), denote a subset of \( \mathbb{R}^d \)
\[ \mathcal{F}_t = \{ \mathbf{X}_t | \mathbf{X}_0 \in \mathcal{F}_0 \}. \]
and the Corollary 4 after Theorem 5.1 is the immediate consequence of
the existence of the limit in (5.9). The first three cases do not apply because a regular system of sets, the
regularity of open, bounded, convex sets \( \mathcal{F}_0 \), \( R > 0 \) is said to be.

The value \( y \) is called the intensity of the process of random sets \( \mathbb{P}. \)

Consider a stochastically ergodic processes of random sets \( \mathbb{P}. \) Let

\[ \mathbb{P} \text{ is intensity of } \mathbb{P}. \]
6. Formula 1 = \( A^2 \).

\[ L_1^* \]

\[ \text{Let } L \text{ be a canonical process of random sets in } R, \text{ in other words,} \]

\[ \text{need not to be strongly uniform, } H, \Delta, \gamma, \text{ and } K. \]

\[ \text{The proof of (9.6) is similar to the proof of (9.5). Note after that} \]

\[ a. \]

\[ 0 = (\frac{L_{1,0}^*}{L_{1,0}}) \]

\[ \text{and} \]

\[ (\frac{L_{1,0}^*}{L_{1,0}^*}) \text{ for } \gamma \neq L_{1,0}^* \]

\[ \text{It follows from the inequalities} \]

\[ a. \]

\[ 0 = (\frac{L_{1,0}^*}{L_{1,0}}) \]

\[ b. \]

\[ (\frac{L_{1,0}^*}{L_{1,0}}) \text{ for } \gamma \neq L_{1,0}^* \]

\[ \text{It was shown in the proof of (9.5) that} \]

\[ (\frac{L_{1,0}^*}{L_{1,0}}) = (\frac{L_{1,0}^*}{L_{1,0}}) \]

\[ \text{where} \]

\[ (\frac{L_{1,0}^*}{L_{1,0}}) \text{ for } \gamma \neq L_{1,0}^* \]

\[ \text{Now we prove (9.7), in view of (9.5) we need only to show} \]

\[ \text{complete the proof of (9.5).} \]

\[ \text{Where do assume the p.m. distribution of the distribution of } \]

\[ 0 = (\frac{L_{1,0}^*}{L_{1,0}}) \]

\[ (\frac{L_{1,0}^*}{L_{1,0}}) \text{ for } \gamma \neq L_{1,0}^* \]

\[ \text{which follows from that} \]

\[ 0 = (\frac{L_{1,0}^*}{L_{1,0}}) \]

\[ (\frac{L_{1,0}^*}{L_{1,0}}) \text{ for } \gamma \neq L_{1,0}^* \]

\[ \text{Thus for completing the proof of (9.5) we remark that} \]

\[ 0 = (\frac{L_{1,0}^*}{L_{1,0}}) \]

\[ (\frac{L_{1,0}^*}{L_{1,0}}) \text{ for } \gamma \neq L_{1,0}^* \]

\[ \text{The same logically from another proof:} \]

\[ 0 = (\frac{L_{1,0}^*}{L_{1,0}}) \]

\[ (\frac{L_{1,0}^*}{L_{1,0}}) \text{ for } \gamma \neq L_{1,0}^* \]
and from the corollary we have theorem 1 in [8].

\[ (\mathcal{L}(\rho))_{\mathbf{0}} = \mathcal{L}(\rho) = \mathcal{I}(\rho) \]

Let \( \mathcal{H}(\rho) \) be the measure. Denote

\[ \mathcal{G}(\rho, (0))_{\mathbf{0}} = \mathcal{I}(\rho) \]

where

\[ (\mathcal{I}(\rho))_{\mathbf{0}} \neq 0 \]

Since then

\[ I \text{ is a stationary process of random sets with the finite inter-} \]

Theorem 6.1.5

\[ (\mathcal{I}(\rho, (0)), (0))_{\mathbf{0}} = (\mathcal{I}(\rho, (0)), (0))_{\mathbf{0}} \]

Then

\[ (\mathcal{I}(\rho))_{\mathbf{0}} \neq 0 \]

Proceed, substitute to (6.1.5) the function

\[ \mathcal{H}(\rho)_{\mathbf{0}} = \mathcal{H}(\rho)_{\mathbf{0}} \]

where

\[ (\mathcal{I}(\rho))_{\mathbf{0}} \neq 0 \]

...
From (6) it follows that $b < 8$, we ask for the expected value

$$d = \frac{1}{\mathbb{P}[\mathcal{X}]} \mathbb{E}[X]$$

K.K. define

where $\mathcal{X} = \{x\}$, $x^* = \mathbb{E}[X]$. Let $\mathcal{X}$ be a stationary countable process of random sets. Recall that $\mathcal{X}$ is a special process of random sets was derived (inequality (3.1)).

To finish the proof considering to the Mathematics' textbook (6),

4. The number of sets overlapping a set $X$

$$\sum_{X \in \mathcal{X}} \mathbb{I}(X) = \mathbb{E}[X] = \mathbb{E}[X]$$

Hence $\mathbb{E}[X] = \mathbb{E}[X]$ and

$$\mathbb{E}[X] = \mathbb{E}[X]$$

and

$$\mathbb{E}[X] = \mathbb{E}[X]$$

Lemma 7 and the Corollary 1 after Theorem 1 in (8) (2)

To prove (6) (9) it is easy to find a similar argument as in the proof of Proposion 8.

From (1) that for each $X$

$$\lim_{T \to \infty} \frac{X}{\mathbb{E}[X]} = \lim_{T \to \infty} \frac{X}{\mathbb{E}[X]}$$

then from the ergodic theorem

$$\lim_{T \to \infty} \frac{X}{\mathbb{E}[X]} = \lim_{T \to \infty} \frac{X}{\mathbb{E}[X]}$$

we have $X = X$. Moreover, if $x_0 \in X$ and $\lim X = 0$

$$\mathbb{I}(x_0) = \frac{\mathbb{I}(x_0)}{\mathbb{I}(x_0)} = \frac{\mathbb{I}(x_0)}{\mathbb{I}(x_0)}$$

$$\mathbb{E}[X] = \mathbb{E}[X]$$

$$\mathbb{E}[X] = \mathbb{E}[X]$$

Thus it suffices to show that

$$\lim_{T \to \infty} \frac{X}{\mathbb{E}[X]} = \lim_{T \to \infty} \frac{X}{\mathbb{E}[X]}$$

The proof of the theorem is based on the hypothesis

$$\mathbb{P}[\mathcal{X} = \mathbb{E}[X]]$$

From Lemma 6.1 we have

$$\mathbb{P}[\mathcal{X} = \mathbb{E}[X]]$$

where $\mathcal{X}$ denotes the collection of invariant sets with respect to (7).
Note that if $X(0)$ then $7.2$ reduces to (6.2).

\[ N^X(r) = (\mathbb{P}^0)^X \cdot \mathbb{E}^{X(0)} \cdot N \]

Thus we arrived at the relation where $O$ is the distribution of random variable $X$ on $(N^X, (\mathbb{P}^0)^X)$. Hence by (3.2):

\[ N^X(r) = (\mathbb{P}^0)^X \cdot \mathbb{E}^{X(0)} \cdot N \]

Then $\mathbb{E}^{X(0)}$ is the distribution of random variable $X$ on $(N^X, (\mathbb{P}^0)^X)$. Therefore, the distribution of $X$ on $(N^X, (\mathbb{P}^0)^X)$ can be obtained by

\[ N^X(r) = (\mathbb{P}^0)^X \cdot \mathbb{E}^{X(0)} \cdot N \]

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\[ N^X(r) = (\mathbb{P}^0)^X \cdot \mathbb{E}^{X(0)} \cdot N \]
Let \( \{ x_i \} \) be a complete orthonormal system defined on \( \mathbb{N} \), such that

\[
\sum_{i=1}^{\infty} x_i = 0, 1, 2, \ldots
\]

Then, we have that

\[
\sum_{i=1}^{\infty} \frac{x_i}{x_i^2} = \frac{1}{x} \quad \text{for all } x \neq 0, 1, 2, \ldots
\]

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