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T. ROLSKI (Wrocław)

SOME INEQUALITIES FOR $GI/M/n$ QUEUES

1. Introduction. The paper deals with queueing systems $GI/M/n$ under incomplete information concerning the inter-arrival time distribution. Only the first two moments will be assumed to be known. There will be given sharp bounds for some characteristics of systems satysfying the above assumption. For this purpose the author applies known sharp bounds for the Laplace transform of a distribution function for which only the first and the second moment is given.

The author has encountered some publications dealing with similar problems. In [5] Rogozin has shown that if only the first moment of inter-arrival time is known and the service time distribution is given, the shortest possible average waiting time is reached for the deterministic distribution of inter-arrival time. He has also shown that if inter-arrival times have a fixed distribution and only the first moment of service time is given, then the shortest average waiting time is reached for the service distribution concentrated in one point. In [4] Marshall has given some estimates of the characteristics of the system $GI/G/1$ by means of the moments of inter-arrival and service time distributions. In [3], which had been published a bit earlier, Kingman had given approximative estimations of the average waiting time for service in systems $GI/G/1$.

2. Bounds for Laplace transform. Let K_2 be the set of all distribution functions having the support $[0, \infty)$ and the first and the second moment equal to m_1 and m_2, respectively $(m_2 \geq m_1^2)$:

$$K_2 = \left\{F(x):\ F(x) \text{ is d.f., } F(-0) = 0,\ \int_0^\infty x\,dF(x) = m_1,\ \int_0^\infty x^2\,dF(x) = m_2\right\}.$$

Vasilyev and Kozlov [7] have shown that

$$\inf_{G\in K_2} \int_0^\infty e^{-st}\,dG(t) = e^{-sm_1}, \tag{1}$$

$$\max_{G\in K_2} \int_0^\infty e^{-st}\,dG(t) = 1-\frac{m_1^2}{m_2}+\frac{m_1^2}{m_2}e^{-(m_2/m_1)s}, \qquad s \geq 0, \tag{2}$$

where maximum is obtained for

$$G(t) = G_{\max}(t) = \begin{cases} 0, & t < 0, \\ 1 - \dfrac{m_1^2}{m_2}, & 0 \leq t < \dfrac{m_2}{m_1}, \\ 1, & \dfrac{m_2}{m_1} \leq t. \end{cases}$$

The somewhat tedious computational proof of this fact can be immediately obtained using the Tchebycheff systems method (see [2]) if one notices that $(1, t, t^2)$, $(1, t, t^2, e^{-st})$ form Tchebycheff systems on $[0, \infty)$.

3. Bounds for $GI/M/1$ queues. In Takács' book [6] it is proved that the size η of the queue at the moment immediately before the arrival of a customer to a $GI/M/1$ system in stationary conditions has the geometrical distribution

$$\mathrm{P}(\eta = k) = (1-\delta_G)\,\delta_G^k, \qquad k = 0, 1, \ldots$$

The parameter δ_G fulfills the equation

$$x = \Phi_G(\mu(1-x)), \tag{3}$$

where $\Phi_G(t)$ denotes the Laplace transform of the distribution function $G(t)$ of inter-arrival time and μ is the service rate. It is assumed here that $r = m_1\mu > 1$. Now we shall prove the following

THEOREM. *We have*

$$\inf_{G \in K_2} \delta_G = l, \tag{4}$$

$$\max_{G \in K_2} \delta_G = \delta_{G_{\max}} = 1 + \frac{m_1^2}{m_2}(l-1), \tag{5}$$

where l is the root of the equation

$$x = \exp(-r + rx). \tag{6}$$

Proof. Notice that the function $\varphi_G(x) = \Phi_G(\mu(1-x)) - x$ is convex and

$$\varphi_G(0) > 0, \qquad \varphi_G(1) = 0. \tag{7}$$

From (2) it follows that

$$\Phi_G(\mu(1-x)) - x \leq \Phi_{G_{\max}}(\mu(1-x)) - x. \tag{8}$$

First we shall show that

$$\max_{G \in K_2} \delta_G = \delta_{G_{\max}}.$$

Let us assume, to the contrary, that there exist $G_1 \in K_2$ and $\delta_{G_1} > \delta_{G_{\max}}$. Formula (8) implies $0 \leq \varphi_{G_{\max}}(\delta_{G_1})$. Convexity and (7) implies $\varphi_{G_{\max}}(\delta_{G_1}) \leq 0$. Hence $\varphi_{G_{\max}}(\delta_{G_1}) = 0$, which is impossible because $\delta_{G_{\max}}$ is the only root of this equation (see Takács [6]).

From the above considerations it follows that the greatest root we search for is the root of equation

$$z = p + q \exp\left[(z-1)\frac{r}{q}\right], \qquad \text{where } p = 1 - \frac{m_1^2}{m_2},\ q = \frac{m_1^2}{m_2}.$$

Let l be the root of equation (4), i.e. $l = \exp[(l-1)r]$. Hence

$$l = \exp\left[(lq - q)\frac{r}{q}\right].$$

By a suitable transformation of this equality we achieve

$$1 - \frac{m_1^2}{m_2} + \frac{m_1^2}{m_2} l = 1 - \frac{m_1^2}{m_2} + \frac{m_1^2}{m_2} \exp\left[\left(l \frac{m_1^2}{m_2} - \frac{m_1^2}{m_2}\right)\frac{r}{q}\right].$$

Hence we can see that

$$1 + \frac{m_1^2}{m_2}(l-1)$$

is the root of the equation $\Phi_{G_{\max}}(\mu(1-x)) = x$. The lower bound (4) can be proved in a similar way.

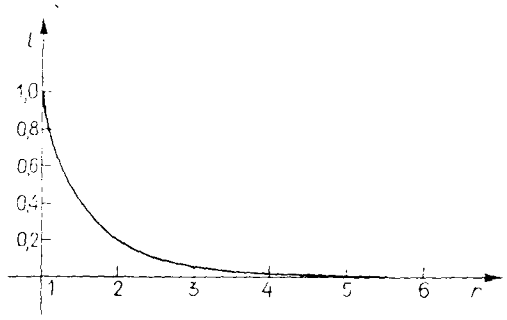

Fig. 1

Equation (6) has a unique solution for any $r > 1$. Values of the root l for some values of r are given in Table 1. In Fig. 1 there is shown the dependence of l on r.

TABLE 1

r	1.01	1.02	1.05	1.10	1.50	2.50	3.50	4.50	5.00
l	.9802	.9610	.9063	.8239	.4172	.1074	.0340	.0117	.0069

Once the estimations of δ are found, we can give estimations of

$$\mathrm{E}\eta = \frac{\delta}{1-\delta}, \qquad \mathrm{Var}\,\eta = \frac{\delta}{(1-\delta)^2}$$

in the class of systems with an inter-arrival time distribution belonging to K_2:

$$1/(1-\ell) < \mathrm{E}\eta \leqslant \frac{p+q\ell}{q(1-\ell)}, \qquad \frac{\ell}{1-\ell} < \mathrm{Var}\,\eta \leqslant \frac{p+q\ell}{q^2(1-\ell)^2}.$$

It has been shown in [1] that the distribution of the waiting time W is of the form

$$\mathrm{P}(W < t) = \begin{cases} 0, & t < 0, \\ 1-\delta e^{-(1-\delta)\mu t}, & t \geqslant 0, \end{cases}$$

and its expected value is equal to

$$\mathrm{E}W = \frac{\delta}{\mu(1-\delta)}.$$

4. Bounds for many-server queues. There are known formulas of the system $GI/M/n$ in terms of the Laplace transform of the inter-arrival time distribution function. Applying bounds for the Laplace transform of the distribution in class K_2 it may be possible to estimate those characteristics.

Thus e.g. the r-th binomial moment B_r of the steady-state distribution of queue size in $GI/M/\infty$ systems fulfills the inequalities

$$B_r^{\min} = \frac{\exp\left[-\frac{r}{2}l(l+1)\right]}{\prod_{i=1}^{r}(1-\exp[-ri])} < B_r \leqslant \prod_{i=1}^{r} \frac{1-\frac{m_1^2}{m_2}+\frac{m_1^2}{m_2}\exp\left[-\frac{m_2}{m_1}i\mu\right]}{\frac{m_1^2}{m_2}-\frac{m_1^2}{m_2}\exp\left[-\frac{m_2}{m_1}i\mu\right]} = B_r^{\max}$$

and the expected value $\mathrm{E}S$ of the average distance between consecutive lost calls in telephone traffic processes the inequalities

$$\frac{1}{\mu}\sum_{j=0}^{m}\binom{m}{j}\prod_{r=0}^{j} B_r^{\min} < S \leqslant \frac{1}{\mu}\binom{m}{j}\prod_{r=0}^{j} B_r^{\max}.$$

These inequalities can be found, applying (1) and (2) to the formulae

$$B_r = \prod_{i=1}^{r}\frac{\varphi(i\mu)}{1-\varphi(i\mu)}, \qquad \mathrm{E}S = \frac{1}{\mu}\sum_{i=0}^{m}\binom{m}{j}\prod_{r=0}^{i} B_r$$

(see Takács [6]).

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MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW

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T. ROLSKI (Wrocław)

PEWNE NIERÓWNOŚCI DLA SYSTEMÓW MASOWEJ OBSŁUGI $GI/M/n$

STRESZCZENIE

Niech $F(x)$ będzie dystrybuantą o supporcie $[0, \infty)$,

$$K_2 = \left\{F: \int_0^\infty x dF(x) = m_1, \int_0^\infty x^2 dF(x) = m_2\right\}$$

i δ_F pierwiastkiem równania

$$x = \Phi_F(\mu(1-x)),$$

gdzie $\Phi_F(s)$ jest transformatą Laplace'a dystrybuanty F.

W pracy zostaje znaleziony kres dolny i górny δ_F po wszystkich dystrybuantach F należących do K_2. Wynik ten zostaje wykorzystany wraz ze znanym już oszacowaniem dla tansformat Laplace'a do podania dwustronnych nierówności dla pewnych charakterystyk systemów $GI/M/n$ $(1 \leqslant n \leqslant \infty)$.