ZADANIA DO WYKŁADU Z STOCHASTIC GEOMETRY

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1 Lista 1

1.1 Let $\mathbb{Q}$ be a probability measure on $\mathbb{Y}$, N a point process on $\mathbb{X}$ and Φ an independent Q marking of N. Prove that for every measurable $f : \mathbb{X} \times \mathbb{Y} \to [0, \infty)$

$$L_\Phi(f) = L_N(f^*),$$

where

$$f^*(x) = -\log \int e^{-f(x,y)}\, Q(dy), \qquad x \in \mathbb{X}\,.$$

1.2 Let K be a stochastic kernel from $\mathbb{X}$ to $\mathbb{Y}$, N a point process on $\mathbb{Y}$ and Φ a K-marking of N. Let $f : \mathbb{X} \times \mathbb{Y} \to [0, \infty)$ be measurable and define

$$f^*(x) = -\log \int e^{-f(x,y)}\, K(x, dy), \qquad x \in \mathbb{X}\,.$$

Prove that

$$L_\Phi(f) = L_N(f^*).$$

1.3 (M/G/∞ queueing system). Assume that customers arrive at a queueing system with infinite many servers at the instants $T_1, T_2, \dots$ of a homogenous Poisson process $N = \sum_{n=1}^{\infty} \delta_{T_n}$ of intensity $\gamma > 0$. Assume that the n-th customer requires service Y_n, where the Y_n are i.i.d. with distribution G and independent of N. Show that the departure process

$$D = \sum_{n=1}^{\infty} \delta_{(T_n + Y_n)}$$

is a Poisson process and compute its intensity measure $\tilde{\Lambda}$. Show that $\tilde{\Lambda}$ is locally finite and that $\tilde{\Lambda}(dt) = \tilde{\lambda}(t)\, dt$. Compute $\lim_{t\to\infty} \tilde{\lambda}(t)$.

1.4 Let N be a simple point process on $\mathbb{X}$, identified with its support. Define

$$S_0(B) = \mathbb{P}(N \cap B \neq \emptyset), \qquad B \in \mathcal{X},$$

and then inductively, for $k \in \mathbb{N}$

$$S_k(B_0; B_1, \dots, B_k) = \\ S_{k-1}(B_0; B_1, \dots, B_{k-1}) - S_{k-1}(B_0 \cup B_k; B_1, \dots, B_{k-1}), \qquad B_0, \dots, B_k \in \mathcal{X}.$$

(i) Show that for $k \geq 0$

$$S_k(B_0; B_1, \ldots, B_k) = \mathbb{P}(N \cap B_0 \neq \emptyset, N \cap B_1 \neq \emptyset, \ldots, N \cap B_n \neq \emptyset) \,.$$

(ii) Show that for $k \geq 1$

$$S_k(B_0; B_1, \ldots, B_k) = \sum_{r=0}^{k} (-1)^r \sum_{1 \leq i_1 < \ldots < i_r \leq k} S_0(B_0 \cup B_{i_1} \cup \ldots \cup B_{i_r}),$$

where the inner sum equal $1 - S_0(B_0)$, for $r = 0$.

1.5 Let N be a simple point process on a finite set $\mathbb{X}$ and define

$$\begin{aligned} p(F) &= \mathbb{P}(N = F), \qquad F \subset \mathbb{X}, \\ c(F) &= \mathbb{N}(N \subset F),, \qquad F \subset \mathbb{X} \,. \end{aligned}$$

Then

$$c(F) := \sum_{K \subset F} p(K) \,.$$

Show the Möbius inversion formula

$$p(F) := \sum_{K \subset F} (-1)^{\mathrm{card}(F \setminus K)} c(K), \qquad F \subset \mathbb{X} \,.$$

1.6 Find a stationary point process, which is not isotropic.

1.7 Let Z be a random closed set and U a random rotation SO_d - valued random variable) independent of Z.
(i) Show that $U \cdot Z$ is stationary random cloised set if Z is stationary.
(ii) Assume that U is distributed according to the unique rotation invariant probability measure ν on SO_d (that is $\int f(vv')\nu(dv) = \int f(v'v)\nu(dv) = \int f(v)\nu(dv), \;\; v' \in \mathrm{SO}_d$). Prove that UZ is isotropic.

1.8 Consider a Boolean model Z in $\mathbb{R}^2$ with typical grain $Z_0 = B(0, R)$, where $R \geq 0$ is a random variable satisfying $\mathrm{E}\, R^2 < \infty$. Let

$$C_0(x) := \mathrm{E}\,[V_2(Z_0 \cap (Z_0 - x))].$$

(i) Show that

$$C_0(x) = \int_{\frac{||x||}{2}}^{\infty} (2r^2 \arccos \frac{||x||}{2} - \frac{||x||}{2} \sqrt{4v^2 - ||x||^2})\, F(dv),$$

where F is the distribution of R.
(ii) Show that the covariance function $C(x)$ of Z satisfies

$$C(x) \sim (2p-1) + (1-p)\exp\left[-\gamma\pi \int_0^{\frac{||x||}{2}} v^2\, F(dv)\right],$$

as $||x|| \to \infty$.
[Hint. Let B_1, B_2 be two ontersecting discs of equal radius r, whose centers are at distance t. Probe that $V_2(B_1 \cap B_2) = r^2 \arccos \frac{t}{r} - t\sqrt{r^2 - t^2}$.]