Chapter III

The concept of reserve; one life and one risk

1 The concept of prospective and retrospective net premium reserves.

Consider a policy with termination date n. For a whole life insurance we may set $n = \infty$ and consider in interval $[0, \infty)$, otherwise we $[0, n]$. We make an abreviation for the current value at time t - CV_t.

We define *prospective* and *retrospective loss* respectively by

$$
\begin{aligned}
{}_tL^{\mathrm{pro}} &= \mathrm{CV}_t \text{ of benefit outgo in } [t, n] \\
&\quad - \mathrm{CV}_t \text{ of premium income in } [t, n]
\end{aligned}
$$

and

$$
\begin{aligned}
{}_tL^{\mathrm{retro}} &= \mathrm{CV}_t \text{ of benefit outgo in } [0, t) \\
&\quad - \mathrm{CV}_t \text{ of premium income in } [0, t).
\end{aligned}
$$

Notice that ${}_0L^{\mathrm{pro}} = L$ is the loss defined in Chapter **[????]** and unless it said otheriwse we consider net models with net premiums only, that is $\mathrm{E}\, L = 0$. We now define prospective and retrospective reserve by

$$
{}_t\bar{\mathcal{V}}^{\mathrm{pro}} = \mathrm{E}\,[{}_tL^{\mathrm{pro}} | T_x > t] \qquad \text{and} \qquad {}_t\bar{\mathcal{V}}^{\mathrm{retro}} = \mathrm{E}\,[{}_tL^{\mathrm{retro}} | T_x > t].
$$

In the sequel we assume that

(*) ${}_tL^{\text{pro}} = {}_tL^{\text{retro}} = 0$ for $t > T_x$, that is after death books are closed.

Notice that

$$L = v^t({}_tL^{\text{pro}} + {}_tL^{\text{retro}}),$$

from which we obtain

$$\mathrm{E}\,[{}_tL^{\text{pro}}] = \mathrm{E}\,[-{}_tL^{\text{retro}}].$$

In view of the above equation we can consider only prospective loss and reserve and therefore we will denote them from now on by ${}_tL$ and ${}_t\bar{\mathcal{V}}$, $t \in [0, n]$ for continuous polices and ${}_kL$, ${}_k\mathcal{V}$, $k = 0, \ldots, n$ for discrete ones.

Proposition 1.1 *Under assumption* $(*)$

$$ {}_t\bar{\mathcal{V}} = {}_t\bar{\mathcal{V}}^{\text{retro}}.$$

Proof We have

$$\begin{aligned} {}_t\bar{\mathcal{V}}^{\text{retro}} &= \mathrm{E}\,[-{}_tL^{\text{retro}}|T_x > t] = \frac{-{}_tL^{\text{retro}}}{{}_tp_x} \\ &= \frac{-{}_tL}{{}_tp_x} = \mathrm{E}\,[-{}_tL|T_x > t] = {}_t\bar{\mathcal{V}}. \end{aligned}$$

□

We may rewrite ${}_t\bar{\mathcal{V}}^{\text{retro}}$ as follows. Define ${}_tG^{\text{retro}}$ as APV of premium income in $[0, t)$ minus APV of benefits outgo in $[0, t)$. Then

$$ {}_t\bar{\mathcal{V}}^{\text{retro}} = \frac{{}_tG^{\text{retro}}}{{}_tE_x},$$

where ${}_tE_x = v^t\,{}_tp_x$ is the actuarial discounting function.

Remark Let us see the mechanism of actuarial accumulation. Suppose we consider a cohort of size l_x of lives (x). Each pay 1 on a bank account with interest rate i. After time t they have $l_x(1 + i)^t$ and for one survived there is $l_x(1 + i)^t/l_{[x]+t}$. However because ${}_tp_x = l_{[x]+t}/l_x$ they accumulated $1/(v^t\,{}_tp_x)l_x = 1/{}_tE_xl_x$. On the other hand we see that ${}_tE_x$ is an actuarial discounting function.

Survay of formulas for net premium and reserve function; continuous contracts

Whole life insurance

net premium: $\bar{P}(\bar{A}_x) = \bar{A}_x/\bar{a}_x$,

reserve: ${}_t\bar{V}(\bar{A}_x) = \bar{A}_{[x]+t} - \bar{P}(\bar{A}_x)\bar{a}_{[x]+t}$.

Term insurance for n years

net premium: $\bar{P}(\bar{A}^{1}_{x:\overline{n}|}) = \bar{A}^{1}_{x:\overline{n}|}/\bar{a}_{x:\overline{n}|}$,

reserve: ${}_t\bar{V}(\bar{A}^{1}_{x:\overline{n}|}) = \bar{A}^{1}_{[x]+t:\overline{n-t}|} - \bar{P}(\bar{A}^{1}_{x:\overline{n}|})\bar{a}_{[x]+t:\overline{n-t}|}$.

umowa: **Pure endowment for n years**

net premium: $\bar{P}(A_{x:\overline{n}|}^{\;\;1}) = \bar{A}_{x:\overline{n}|}^{\;\;1}/\bar{a}_{x:\overline{n}|}$,

reserve: ${}_t\bar{V}(A_{x:\overline{n}|}^{\;\;1}) = \bar{A}_{[x]+t:\overline{n-t}|}^{\;\;1} - \bar{P}(A_{x:\overline{n}|}^{\;\;1})\bar{a}_{[x]+t:\overline{n-t}|}$.

umowa: **Endowment for n years**

net premium: $\bar{P}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x:\overline{n}|}/\bar{a}_{x:\overline{n}|}$,

reserve: ${}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{[x]+t:\overline{n-t}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{[x]+t:\overline{n-t}|}$.

Whole life insurance, premium paid for $h < n$ years

net premium: ${}_h\bar{P}(\bar{A}_x) = \bar{A}_x/\bar{a}_{x:\overline{h}|}$,

reserve: $${}^{h}_{t}\bar{V}(\bar{A}_x) = \begin{cases} \bar{A}_{[x]+t} - {}_h\bar{P}(\bar{A}_x)\bar{a}_{[x]+t:\overline{h-t}|} & \text{dla } t \le h \\ \bar{A}_{[x]+t} & \text{dla } t > h. \end{cases}$$

Endowment for n years, premium paid for z$h < n$ years)

net premium: ${}_h\bar{P}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x:\overline{n}|}/\bar{a}_{x:\overline{h}|}$,

reserve: $${}^{h}_{t}\bar{V}(\bar{A}_{x:\overline{n}|}) = \begin{cases} \bar{A}_{[x]+t:\overline{n-t}|} - {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{[x]+t:\overline{h-t}|} & \text{dla } t \le h, \\ \bar{A}_{[x]+t:\overline{n-t}|} & \text{dla } h < t \le n. \end{cases}$$

Life annuity deffered on m years, with level premium paid for h years ($h \le m$)

net premium: ${}_h\bar{P}({}_{m|}\bar{a}_x) = {}_{m|}\bar{a}_x/\bar{a}_{x:\overline{h}|}$,

$$\textit{reserve:}^1\ {}^h_t\bar{V}({}_{m|}\bar{a}_x) = \begin{cases} {}_{m-t|}\bar{a}_{[x]+t} - {}_h\bar{P}({}_{m|}\bar{a}_x)\bar{a}_{[x]+t:\overline{h-t|}} & \text{dla } t \le h, \\ {}_{m-t|}\bar{a}_{[x]+t} & \text{dla } h < t \le m, \\ \bar{a}_{[x]+t} & \text{dla } t > m. \end{cases}$$

We now demonstrate a formula for net premium reserve for the whole life insurance.

Example 1.2 Consider the whole life insurance from Section II.0.2 In this case we can write loss ${}_tL$ at t by

$${}_tL = v^{T-t}1(T_x > t) - \bar{P}(\bar{A}_x)\bar{a}_{\overline{T-t|}}1(T_x > t). \tag{1.1}$$

Since $(T_x - t|T_x > t)$ ma taki sam rozkad co $T_{[x]+t}$ wic

$$\begin{aligned} \mathrm{E}\,[{}_tL|T_x > t] &= \mathrm{E}\,[v^{T_{[x]+t}} - \bar{P}(\bar{A}_x)\bar{a}_{\overline{T_{[x]+t}|}}] \\ &= \bar{A}_{[x]+t} - \bar{P}(\bar{A}_x)\bar{a}_{[x]+t}. \end{aligned} \tag{1.2}$$

Net premium reserve for the whole life insurance is denoted by ${}_t\bar{V}(\bar{A}_x)$.

2 Thiele differential equation

We will deal with random cash flows. In this case the present value of such cash flow is a random variable. Therefore we need a characteristic for such the random variable and hence we will use the notion of the *expected cash flow* (EPV). Sometimes it is called the '*actuarial value.*

2.1 Continuous time modelling

We consider a general model for a life insurance[2] of life (x), for a period n, with a benefit function $b(t)$, and endowment b^*_n at n, paid by a premium with rate $\bar{\Pi}(u)$. Define $\mathrm{Pr}_t(\cdot) = \mathrm{Pr}(\cdot|T_x > t)$. One of the basic notions are reserves. To define it we first consider the *future loss* after $t \in [0, n]$. Thus under Pr_t we can define

$${}_tL = \begin{cases} 0 & \text{if } T_x \le t, \\ b(T_x)v^{T_x-t}1(T_x \le n) + b^*_n v^{n-t}1(T_x > n) & \\ \qquad - \int_t^{T_x \wedge n} \bar{\Pi}(u)v^{u-t}\,\mathrm{d}u & \text{if } T_x > t. \end{cases} \tag{2.3}$$

[1] Cos tu nie gra. Czy teraz ok.TR

[2] assurance?

We will call ${}_tL$ by the *prospective loss* of the insurer. We now define *prospective reserve* (or if is clear reserve) by

$$ {}_t\bar{\mathcal{V}} = \mathrm{E}_t[\,{}_tL\,]. $$

In the proof here and later we will use the following. Denote by $T_{[x]+t}$ the conditional future lifetime of life (x), under condition $T_x > t$, that is $T_{[x]+t} =_{\mathrm{d}} (T_x - t | T_x > t)$. Notice that $\Pr(T_{[x]+t} > u) = {}_up_{[x]+t}$. Its denity function we denote by $f_{[x]+t}(s)$. Under HHP we have ${}_up_{[x]+t} = {}_up_{x+t}$.

Proposition 2.1 *If* $\bar{\Pi}(t)$ *is a premium rate, then the mathematical prospective reserve* ${}_t\bar{\mathcal{V}}$ *fulfills*

$$ {}_t\bar{\mathcal{V}} = \int_t^n v^{u-t}\ {}_{u-t}p_{[x]+t}\Big(b(u)\mu_{[x]+u} - \bar{\Pi}(u)\Big)\ \mathrm{d}u + b_n^* v^{n-t}\ {}_{n-t}p_{[x]+t}. \tag{2.4} $$

for $0 \le t \le n$.

Proof For clarity of the considerations we make an assumption that $\Pi(t) = b(t) = 0$ for $t > n$. From the definition

$$
\begin{aligned}
&\mathrm{E}\,[{}_tL|T_x > t] = \\
&= \mathrm{E}\left[b(T_x)v^{T_x - t}1(T_x \le n) - \int_t^{T_x\wedge n} \bar{\Pi}(u)v^{u-t}\ \mathrm{d}u\middle| T_x > t\right] \\
&\qquad + b_n^* v^{n-t}\Pr(T_x > n|T_x > t) \\
&= \mathrm{E}\left[b(T_x - t + t)v^{T_x - t}1(T_x \le n) - \int_t^{(T_x - t + t)\wedge n} \bar{\Pi}(u)v^{u-t}\ \mathrm{d}u\middle| T_x > t\right] \\
&\qquad + b_n^* v^{n-t}\ {}_{n-t}p_{[x]+t} \\
&= \mathrm{E}\left[b(T_{[x]+t} + t)v^{T_{[x]+t}}1(T_{[x]+t} \le n - t) - \int_t^{(T_{[x]+t} + t)\wedge n} \bar{\Pi}(u)v^{u-t}\ \mathrm{d}u\right] \\
&\qquad + b_n^* v^{n-t}\ {}_{n-t}p_{[x]+t} \\
&= \int_0^\infty \left(v^s b(s+t) - \int_t^{s+t} \bar{\Pi}(u)v^{u-t}\ \mathrm{d}u\right) f_{[x]+t}(s)\ \mathrm{d}s \\
&\qquad + b_n^* v^{n-t}\ {}_{n-t}p_{[x]+t}
\end{aligned}
$$

and continuouing

$$\int_0^\infty \left(v^s b(s+t) - \int_t^{s+t} \bar{\Pi}(u) v^{u-t} \, \mathrm{d}u \right) f_{[x]+t}(s) \, \mathrm{d}s$$
$$= \int_0^\infty v^s b(s+t) \; f_{[x]+t}(s) \, \mathrm{d}s$$
$$- \int_0^\infty \int_0^\infty 1(t \le u \le s+t) \bar{\Pi}(u) v^{u-t} \, \mathrm{d}u \; f_{[x]+t}(s) \, \mathrm{d}s = (\Delta) - (\Delta\Delta).$$

Consider now

$$\begin{aligned} (\Delta\Delta) &= \int_0^\infty \int_0^\infty 1(t \le u \le s+t) \bar{\Pi}(u) v^{u-t} \, \mathrm{d}u \; f_{[x]+t}(s) \, \mathrm{d}s \\ &= \int_0^\infty \int_t^\infty 1(0 \le u-t \le s) \bar{\Pi}(u) v^{u-t} \, \mathrm{d}u \; f_{[x]+t}(s) \, \mathrm{d}s \\ &= \int_t^n \bar{\Pi}(u) v^{u-t} \; {}_{u-t}p_{[x]+t} \, \mathrm{d}u. \end{aligned}$$

Now

$$\begin{aligned} (\Delta) &= \int_0^\infty v^s b(s+t) \; f_{[x]+t}(s) \, \mathrm{d}s \\ &= \int_t^n v^{u-t} b(u) f_{[x]+t}(u-t) \, \mathrm{d}\, u \\ & \int_t^n v^{u-t} b(u) \mu_{[x]+u} \, {}_{u-t}p_{[x]+t} \, \mathrm{d}u. \end{aligned}$$

The proof is completed. □

Corollary 2.2 *(Thiele differential equation) Suppose functions $b(t)$, $\bar{\Pi}(t)$, $\mu_{[x]+t}$ are continuous. Then*

$$\frac{\mathrm{d}_t\bar{\mathcal{V}}}{\mathrm{d}t} = \bar{\Pi}(t) + \delta \, {}_t\bar{\mathcal{V}} + ({}_t\bar{\mathcal{V}} - b(t)) \mu_{[x]+t} \, . \tag{2.5}$$

Furthermore ${}_t\bar{\mathcal{V}}$ given by formula (2.4) is the unique solution such that ${}_n\bar{\mathcal{V}} = b^*_n$.

Proof From formula (2.4), using the identity ${}_{u-t}p_{[x]+t} = {}_u p_x / {}_t p_x$ for $0 \le t \le u$ we obtain

$$ {}_t\bar{\mathcal{V}} = \frac{1}{v^t \ {}_t p_x} \left(\int_t^\infty \Big(b(u)\mu_{[x]+u} - \bar{\Pi}(u) \Big) v^u \ {}_u p_x \ \mathrm{d}u + v^n b_n^* {}_n p_x \right) . $$

Differentiating with respect t

$$ \frac{\mathrm{d}_t\bar{\mathcal{V}}}{\mathrm{d}t} = $$

$$ = -\Big(b(t)\mu_{[x]+t} - \bar{\Pi}(t) \Big) + \frac{\mu_{[x]+t} + \delta}{v^t \ {}_t p_x} (\int_t^\infty \Big(b(u)\mu_{[x]+u} - \bar{\Pi}(u) \Big) v^u \ {}_u p_x \ \mathrm{d}u, $$

from which we obtain (2.5). Notice that passing in (2.4) with $t \uparrow n$ we obtain ${}_n\bar{\mathcal{V}} = b_n^*$. Since this is a linear differential equation, it has with a given boundary condition the unique solution and it has to be (2.4). □

For many contracts continuity assumption in Corollary 2.2 is not fulfilled. However in most cases these functions are piecewise continuous with finite number of discontinuities.[3] Then we can proceed recursively. Start at n with ${}_n\bar{\mathcal{V}} = b_n^*$ and solve to backward to the last jump t_0 before n. Next continue with the new boundary ${}_{t_0}\bar{\mathcal{V}}$ up to the next jump before t_0.

Remark Equation (2.5) has the following intuitive meaning. The increment of reserves $\frac{\mathrm{d}_t\bar{\mathcal{V}}}{\mathrm{d}t}$ is a result of inflows of premiums at rate $\bar{\Pi}(t)$ dt minus the outflow of benefits $({}_t\bar{\mathcal{V}} - b(t))$ paid with probability $\mu_{[x]+t}$.

2.2 Discrete time policies

Consider prospective reserve ${}_kL$ at $k = 0, \ldots$. We define prospective (net premium) reserve

$$ {}_k\mathcal{V} = \mathrm{E}[{}_kL | K_x \ge k]. $$

The general concept will be illustrated first by the special case of endowment.

Example 2.3 Consider the following endowment policy for life (x), with maturity at n. The insurance sum is $C = 1$, and the net level premium is

[3] In these notes by piecewise continuous we mmean always that the number of jumps is finite and that segments are right continuous.

computed at discount factor v. In this policy the loss is

$$L = v^{K_x+1}1(K_x < n) + v^n 1(K_x \geq n) - \Pi \sum_{l=0}^{K_x \wedge (n-1)} v^l.$$

The future loss (or prospective loss) after year $k = 0, \ldots, n-1$ is

$${}_kL = v^{K_x+1-k}1(K_x < n) + v^{n-k} 1(K_x \geq n) - \Pi \sum_{l=k}^{K_x \wedge (n-1)} v^{l-k}.$$

Π is the level net premium if $\mathrm{E}L = 0$. Then

$$A_{x:\overline{n}|} - \Pi \ddot{a}_{x:\overline{n}|} = 0$$

and hence $\Pi = P_{x:\overline{n}|} = A_{x:\overline{n}|}/\ddot{a}_{x:\overline{n}|}$

Furthermore

$$\begin{aligned} {}_k\mathcal{V} &= \mathrm{E}[{}_kL | K_x \geq k] \\ &= \mathrm{E}[v^{K_x+1-k}1(K_x < n) | K_x \geq k] && (2.6) \\ &+ \mathrm{E}[v^{n-k}1(K_x \geq k) | K_x \geq k] && (2.7) \\ &- \mathrm{E}[\Pi \sum_{l=k}^{K_x \wedge (n-1)} v^{l-k} | K_x \geq k]. && (2.8) \end{aligned}$$

Recalling $(K_x - k | K_x \geq k) =_{\mathrm{d}} K_{[x]+k}$ we have

$$\begin{aligned} (2.6) &= \mathrm{E}[v^{K_{x+k}+1}1(K_{x+k} < n-k)], \\ (2.7) &= v^{n-k}\Pr(K_x \geq n-k), \\ (2.8) &= -\Pi \ddot{a}_{x:\overline{n-k}|}. \end{aligned}$$

Hence remembering that

$$\mathrm{E}[v^{K_{[x]+k}+1}1(K_{[x]+k} < n-k)] + v^{n-k}\Pr(K_x \geq n-k) = A_{x:\overline{n-k}|}$$

we have

$${}_k\mathcal{V} = A_{x:\overline{n-k}|} - \Pi \ddot{a}_{x:\overline{n-k}|}.$$

Small manipulations yield the formula for prospective reserve written in actuarial notations:

$${}_kV_{x:\overline{n}|} = A_{[x]+k:\overline{n-k}|} - P_{x:\overline{n}|}\, \ddot{a}_{[x]+k:\overline{n-k}|}. \tag{2.9}$$

We now introduce a general discrete model. We denote by b_k - benefit paid at $k = 0, \ldots, n$, b_n^* – endowment paid at n, Π_j – premium paid at $j = 0, \ldots, n-1$ (beginning of the j-th year of the policy). In this model the prospective reserve is for $k \geq K_x$

$$_kL = b_{K_x+1-k} v^{K_x+1-k} 1(K_x < n) + b_n^* v^{n-k} 1(K_x \geq n) - \sum_{j=k}^{K_x \wedge (n-1)} \Pi_j v^{j-k}.$$

Proposition 2.4 *We have*

$$\begin{aligned} {}_k\mathcal{V} = & \sum_{j=0}^{n-k-1} b_{k+j+1} v^{j+1} \ {}_{j|}q_{[x]+k} + v^{n-k} b^*_n \, {}_{n-k}p_{[x]+k} \\ & - \sum_{j=0}^{n-k-1} \Pi_{k+j} v^j \ {}_jp_{[x]+k} \end{aligned}$$

Proof Denote by $K_{[x]+k}$ the conditional curtate future lifetine of (x) after k years under condition $K_x \geq k$; that is $\Pr(K_{[x]+k} \geq n) = {}_np_{[x]+k}$. Recall also the convention that $\Pi_j = 0$ for $j \geq n-1$ and $b_j = 0$ for $j > n$. Beginning from the definition we have

$$\begin{aligned} & \mathrm{E}\left[{}_kL | K_x \geq k\right] \\ = & \ \mathrm{E}\left[b_{K_x+1} v^{K_x+1-k} 1(K_x < n) + b_n^* v^{n-k} 1(K_x \geq n) - \sum_{j=k}^{K_x} \Pi_j v^{j-k} \middle| K_x \geq k \right] \\ = & \ \mathrm{E}\left[b_{K_x-k+1+k} v^{K_x-k+1} + b_n^* v^{n-k} 1(K_x \geq n) - \sum_{j=0}^{K_x-k} \Pi_{k+j} \, v^j \middle| K_x \geq k \right] \\ = & \ \mathrm{E}\left[b_{K_{[x]+k}+k+1} v^{K_{[x]+k}+1} - \sum_{i=0}^{K_{[x]+k}} \Pi_{k+i} v^i \right] + b_n^* v^{n-k} \ {}_{n-k}p_{[x]+k} \\ = & \ \sum_{j=0}^{\infty} \left(v^{j+1} b_{k+j+1} - \sum_{i=0}^{j} v^i \Pi_{k+i} \right) {}_jp_{[x]+k} \ q_{[x]+k+j} + b_n^* v^{n-k} \ {}_{n-k}p_{[x]+k} \\ = & \ \sum_{j=0}^{\infty} b_{k+j+1} v^{j+1} \ {}_jp_{[x]+k} \ q_{[x]+k+j} + b_n^* v^{n-k} \ {}_{n-k}p_{[x]+k} - \sum_{j=0}^{\infty} \Pi_{k+j} v^j \ {}_jp_{[x]+k} \, , \end{aligned}$$

which completes the proof. □

Moreover we have the *Thiele recursion formula*

Corollary 2.5 *We have* ${}_n\mathcal{V} = b_n^*$ *and for* $0 \le k < n$

$$ {}_k\mathcal{V} = vb_{k+1}\, q_{[x]+k} - \Pi_k + v\ {}_{k+1}\mathcal{V}\, p_{[x]+k}, \tag{2.10}$$

Remark We can rewrite (2.10) as follows

$$ {}_k\mathcal{V} - v\ {}_{k+1}\mathcal{V} + \Pi_k = v(b_{k+1} - {}_{k+1}\mathcal{V})q_{[x]+k}. \tag{2.11}$$

The left hand side can be read as available fund at the beginning of year k and the right hand side is a projection of expenses.

Example 2.6 As in 2.3 we consider the endowment policy with benefit b, paid by net premium (Π_j) with the following specifications. The third one below is the so called Zillmer's net premium. Formulas are stated under hypothesis HA that is $K_{[x]+k} = K_{x+k}$ for all $x, k \in \mathbb{Z}_+$. [4]

(i) Assumption $\Pi = \Pi_0 = \Pi_1 = \ldots = \Pi_{n-1}$ yields

$$\Pi\ddot{a}_{x:\overline{n|}} = b\mathrm{E}\,[v^{K+1\wedge n}] = bA_{x:\overline{n|}}.$$

Hence

$$\Pi = b\,{}_xV_{n:\overline{=|}} \frac{A_{n:\overline{x|}}}{\ddot{a}_{n:\overline{x|}}}$$

is the level premium for this contract.

(ii) Assumption $\Pi_0 > 0$ and $\Pi_1 = \Pi_2 = \ldots = 0$ yields $\Pi_0 = bA_{x:\overline{n|}}$, which is a single net premium for this contract.

(iii) Assumption $\Pi_0 < \Pi_1 = \Pi_2 = \ldots = \Pi_{n-1}$ yields the so called Zillmer's net premium. Denoting $\Pi_0^{\mathrm{Z}} = \Pi_0$, $\Pi_1^{\mathrm{Z}} = \Pi_1$ and $I = \Pi_1 - \Pi_0$ the equivalence principle gives

$$b\mathrm{E}\,[v^{(K+1)\wedge n}] = \Pi_0^{\mathrm{Z}} + \Pi_1^{\mathrm{Z}}\mathrm{E}\,[\sum_{j=1}^{K\wedge(n-1)} v^j]$$

[4]Dr. August Zillmer (1831–1893), German actuary. He was a manager in insureance companys. The author of the first exhaustive text about life insurances: *Die mathematischen Rechnungen bei Lebens und Renternversicherungen*, opublikowanej w Berline w 1861 pierwszego od czasu Tetensa (1736–1807).

or in actuarial notations

$$bA_{x:\overline{n}|} = -I + \Pi_1^{\mathrm{Z}} \ddot{a}_{x:\overline{n}|}$$

which yields

$$\Pi_0^{\mathrm{Z}} = \frac{bA_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} + \frac{I}{\ddot{a}_{x:\overline{n}|}} - I, \qquad \Pi_1^{\mathrm{Z}} = \frac{bA_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} + \frac{I}{\ddot{a}_{x:\overline{n}|}}. \tag{2.12}$$

We now compute formula for reserve, which we denote by ${}_kV^{\mathrm{Z}}_{x:\overline{n}|}$. Thus for $k = 1, 2, \ldots$

$$\begin{aligned}
{}_kV^{\mathrm{Z}}_{x:\overline{n}|} &= bA_{x+k:\overline{n-k}|} - \Pi_1^{\mathrm{Z}} \ddot{a}_{x+k:\overline{n-k}|} \\
&= bA_{x+k:\overline{n-k}|} - \left(\frac{bA_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} + \frac{I}{\ddot{a}_{x:\overline{n}|}}\right) \ddot{a}_{x+k:\overline{n-k}|} \\
&= b\left(A_{x+k:\overline{n-k}|} - \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \ddot{a}_{x+k:\overline{n-k}|}\right) - I \frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}} \\
&= b\ {}_kV_{x:\overline{n}|} - I \frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}} \\
&= b\ {}_kV_{x:\overline{n}|} + I\left(1 - \frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}} - 1\right) \\
&= (b + I)\, {}_kV_{x:\overline{n}|} - I
\end{aligned}$$

where in the last equation we used formula

$${}_kV_{x:\overline{n}|} = 1 - \frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}}.$$

Survay of formulas for net premium and reserve function; discrete policies Recall that now $x, k, h, m \in \mathbb{Z}_+$. For term policies with termination n, reserves are defined for $k < n$.

Whole life insurance

net premium: $P_x = A_x / \ddot{a}_x$,

reserve: ${}_kV_x = A_{[x]+k} - P_x\ \ddot{a}_{[x]+k}$.

Term insuracne for n years

net premium : $P_{\overset{1}{x}:\overline{n}|} = A_{\overset{1}{x}:\overline{n}|} / \ddot{a}_{x:\overline{n}|}$,

reserve: ${}_kV_{\overset{1}{x}:\overline{n}|} = A_{\overset{1}{[x]+k}:\overline{n-k}|} - P_{\overset{1}{x}:\overline{n}|}\ \ddot{a}_{[x]+k:\overline{n-k}|}$.

Pure endowment for n years

net premium : $P_{x:\overline{n}|}^{\;\;1} = A_{x:\overline{n}|}^{\;\;1}/\ddot{a}_{x:\overline{n}|}$,

reserve: ${}_kV_{x:\overline{n}|}^{\;\;1} = A_{[x]+k:\overline{n-k}|}^{\;\;1} - P_{x:\overline{n}|}^{\;\;1}\, \ddot{a}_{[x]+k:\overline{n-k}|}$.

Endowment for n years

net premium : $P_{x:\overline{n}|} = A_{x:\overline{n}|}/\ddot{a}_{x:\overline{n}|}$,

reserve: ${}_kV_{x:\overline{n}|} = A_{[x]+k:\overline{n-k}|} - P_{x:\overline{n}|}\, \ddot{a}_{[x]+k:\overline{n-k}|}$.

Whole life insurance, premium paid for $h < n$ years

net premium: ${}_hP_x = A_x/\ddot{a}_{x:\overline{h}|}$,

$$\textit{reserve: } {}_k^hV_x = \begin{cases} A_{[x]+k} - {}_hP_x\, \ddot{a}_{[x]+k:\overline{h-k}|}, & \text{dla } k \le h \\ A_{[x]+k}, & \text{dla } k > h\,. \end{cases}$$

Endowment for n years, premium paid for h years $(h < n)$

net premium: ${}_hP_{x:\overline{n}|} = A_{x:\overline{n}|}/\ddot{a}_{x:\overline{h}|}$,

$$\textit{reserve: } {}_k^hV_{x:\overline{n}|} = \begin{cases} A_{[x]+k:\overline{n-k}|} - {}_hP_{x:\overline{n}|}\, \ddot{a}_{[x]+k:\overline{h-k}|}, & \text{dla } k \le h\,, \\ A_{[x]+k:\overline{n-k}|}, & \text{dla } h < k \le n\,. \end{cases}$$

Whole life annuity due deffered for m years, with premium paid for m years

net premium: $P({}_{m|}\ddot{a}_x) = {}_{m|}\ddot{a}_x/\ddot{a}_{x:\overline{m}|}$,

$$\textit{reserve: } {}_kV({}_{m|}\ddot{a}_x) = \begin{cases} {}_{m-k|}\ddot{a}_{[x]+k} - P({}_{m|}\ddot{a}_x)\, \ddot{a}_{[x]+k:\overline{m-k}|}\,, & \text{dla } k \le m\,, \\ \ddot{a}_{[x]+k}\,, & \text{dla } k > m\,. \end{cases}$$

Exercises; will be shifted later at the end of the section.

1. Define $s_{x:\overline{n}|} = \frac{a_{x:\overline{n}|}}{{}_nE_x}$. Show

$$(1+i)^n \frac{l_[x]}{l_{[x]+n}} \sum_{k=1}^{n} v^k = s_{x:\overline{n}|}\,.$$

Similarly define $\ddot{s}_{x:\overline{n}|} = \frac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x}$ and show

$$(1+i)^n \frac{l_[x]}{l_{[x]+n}} \sum_{k=0}^{n-1} v^k = \ddot{s}_{x:\overline{n}|}\,.$$

2. Show formulas for ${}_t\bar{V}(\bar{A}_x)$:

$$ {}_t\bar{V}(\bar{A}_x) = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x} \ , $$

(ration of annuities)

$$ {}_t\bar{V}(\bar{A}_x) = \frac{\bar{A}_{x+t} - \bar{A}_x}{1 - \bar{A}_x} $$

(ratio of life insurances)

$$ {}_t\bar{V}(\bar{A}_x) = \frac{\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t}) + \delta} $$

(premium ratio formula).

3. Show the retrospective formula for continuous reserve:

$$ {}_t\bar{V}(\bar{A}^{\,1}_{x:\overline{t}|}) = \bar{P}(\bar{A}^{\,1}_{x:\overline{t}|})\bar{s}_{x:\overline{t}|} - \frac{\bar{A}^{\,1}_{x:\overline{t}|}}{{}_tE_x} = \bar{P}(\bar{A}^{\,1}_{x:\overline{t}|})\frac{\bar{a}_{x:\overline{t}|}}{{}_tE_x} - \frac{\bar{A}^{\,1}_{x:\overline{t}|}}{{}_tE_x}. $$

4. Derive the Thiele differential equation for general model if the force of interest is $\delta(t)$. Write the Thiele recurrence if in the k-th year there is discount factor v_k.

5. We consider a term life insurance with death benefit b paid at the instant of death, which is financed by a level premium of $\bar{\Pi}$. Write and next solve the Thiele differential equation for the net premium reserve $\bar{P}(\bar{A}_{x:\overline{n}|})$.

6. We consider a term life insurance with death benefit b paid at the end of death year, which is financed by a level premium of $P_{n:\overline{x}|}$. Write and next solve the Thiele reccurence for the net premium reserve ${}_kV({}_{m|}\ddot{a}_x)$.

7. We consider the endowment policy. The death benefit is 200,000 USD and an endowment is 100,000 USD. We consider a life $x = 30$ and 65 as the age of maturity of the policy.

   - How much is a single premium for this insurance if the technical interest rate is 3.5%?

- How do these results compare to the values in discrete times. Use the following mortality rate:

$$\mu(t) = \exp(-7.85785 + 0.01538x + 5.77355 \times 10^{-4}x^2).$$

8. Show the following formula:

$$\begin{aligned} &{}_t\bar{V}(\bar{A}^{\,1}_{x:\overline{n}|}) &(2.13)\\ &= \frac{1}{{}_tE_x}\int_0^t\int_t^n v^{s+w}\,{}_sp_x\,{}_wp_x(\mu_{[x]+w}-\mu_{[x]+s})\;\mathrm{d}w\,\mathrm{d}s \quad 0\le t\le n\,. \end{aligned}$$

(Hint: Use the retropective formula for the reserve and

$$\bar{P}(\bar{A}^{\,1}_{x:\overline{n}|}) - \mu_{[x]+s} = \frac{1}{\bar{a}_{x:\overline{n}|}}\int_0^n v^w\,{}_wp_x(\mu_{[x]+w}-\mu_{[x]+s})\;\mathrm{d}w\,.$$

Then notice that double integral $\int_0^t \mathrm{d}s\int_0^t \ldots\;\mathrm{d}w = 0$.)
Conclude that if mortality rate $\mu_{[x]+s}$ is a nondecreasing function s, then the expression on the RHS of (2.13) is nonnegative.

9. Show that if ${}_sp_{[x]+t}$ as a function of s, then ${}_t\bar{V}(\bar{A}_{x:\overline{t}|})$ is nondecreasing. Hint. Convert formula

$${}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{[x]+t:\overline{n-t}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{[x]+t:\overline{n-t}|}\,, \qquad (2.14)$$

into

$${}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = 1-(\delta+\bar{P}(\bar{A}_{x:\overline{n}|}))\bar{a}_{[x]+t:\overline{n-t}|}\,. \qquad (2.15)$$

Then notice that ${}_t\bar{V}(\bar{A}_{x:\overline{n}|})$ is nodecreasing provided $\bar{a}_{[x]+t:\overline{n-t}|}$ decrases for $t\to n$. Now use

$$\bar{a}_{[x]+t:\overline{n-t}|} = \int_0^{n-t} v^s\,{}_sp_{[x]+t}\;\mathrm{d}s.$$

2.3 A deterministic approach

In this subsection we assume hyptothesis HA. Consider a cohort $\{l_{x+n}\}_{n=0}^{\infty}$ of lifes (x). Notice the relationship ${}_hp_{x+k} = l_{x+k+h}/l_{x+k})$. Recall that $d_{x+k} = l_{x+k} - l_{x+k+1}$ is the number of deaths in year k.