- How do these results compare to the values in discrete times. Use the following mortality rate:

$$\mu(t) = \exp(-7.85785 + 0.01538x + 5.77355 \times 10^{-4}x^2).$$

8. Show the following formula:

$$\begin{aligned} &{}_t\bar{V}(\bar{A}^{\,1}_{x:\overline{n}|}) &(2.13)\\ &= \frac{1}{{}_tE_x}\int_0^t\int_t^n v^{s+w}\,{}_sp_x\,{}_wp_x(\mu_{[x]+w}-\mu_{[x]+s})\,\mathrm{d}w\,\mathrm{d}s \quad 0\le t\le n\,. \end{aligned}$$

(Hint: Use the retropective formula for the reserve and

$$\bar{P}(\bar{A}^{\,1}_{x:\overline{n}|}) - \mu_{[x]+s} = \frac{1}{\bar{a}_{x:\overline{n}|}}\int_0^n v^w\,{}_wp_x(\mu_{[x]+w}-\mu_{[x]+s})\,\mathrm{d}w\,.$$

Then notice that double integral $\int_0^t \mathrm{d}s\int_0^t\dots\ \mathrm{d}w = 0$.)
Conclude that if mortality rate $\mu_{[x]+s}$ is a nondecreasing function s, then the expression on the RHS of (2.13) is nonnegative.

9. Show that if ${}_sp_{[x]+t}$ as a function of s, then ${}_t\bar{V}(\bar{A}_{x:\overline{t}|})$ is nondecreasing. Hint. Convert formula

$$ {}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{[x]+t:\overline{n-t}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{[x]+t:\overline{n-t}|}\,, \tag{2.14}$$

into

$$ {}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = 1 - (\delta + \bar{P}(\bar{A}_{x:\overline{n}|}))\bar{a}_{[x]+t:\overline{n-t}|}\,. \tag{2.15}$$

Then notice that ${}_t\bar{V}(\bar{A}_{x:\overline{n}|})$ is nodecreasing provided $\bar{a}_{[x]+t:\overline{n-t}|}$ decrases for $t \to n$. Now use

$$\bar{a}_{[x]+t:\overline{n-t}|} = \int_0^{n-t} v^s\,{}_sp_{[x]+t}\,\mathrm{d}s.$$

2.3 A deterministic approach

We now show anothr approach to the theory of prospective and retrospecitve reserves In this subsection we assume hyptothesis HA.

Consider a cohort $\{l_{x+n}\}_{n=0}^{\infty}$ of lifes (x). Recall the relationship ${}_hp_{x+k} = l_{x+k+h}/l_{x+k})$, and that $d_{x+k} = l_{x+k} - l_{x+k+1}$ is the number of deaths in year k.

Remark Notice that it is not important the cohort is of size l_x. We can suppose that at the beginning the cohort of lifes (x) counts N. Then after the first year it counts Nl_{x+1}/l_x, after two years Nl_{x+2}/l_x, etc.

The portfolio consists of l_x lifes. Each of the life (x) is insured under our general policy from Section 2.2. We now analyse the cash flows in this portfolio. At the beginning $(k=0)$, the insurer obtains $l_x\Pi_0$. After the first year $(k=1)$, it pays b_1 to the family of each from d_x deaths and obtains yearly premium Π_1 from l_{x+1} survived, etc. Thus for those dying in period $[k-1,k)$ $(k\geq 1)$ it pays total benefits $d_{x+k-1}b_k$ and obtains premiums $l_{x+k}\Pi_k$.

Now looking prospectively at $k\geq 1$ for future benefits (at $k+1,k+2,\ldots$) and premiums (paid at $k,k+1,\ldots$) we have the amount

$$\sum_{h=0}^{n-k-1} b_{k+h+1}v^{h+1}d_{x+k+h}+b_n^* l_{x+n}v^{n-k}-\sum_{h=0}^{n-k-1}\Pi_{k+h}v^h l_{x+k+h}.$$

This gives per capita of one **alive** at k

$${}_k\mathcal{V}=\frac{1}{l_{[x]+k}}\left(\sum_{h=0}^{n-k-1} b_{k+h+1}v^{h+1}d_{x+k+h}+b_n^* v^{n-k}l_{x+n}-\sum_{h=0}^{n-k-1}\Pi_{k+h}v^h l_{x+k+h}\right). \tag{2.16}$$

Upon a reflection remembering that $d_{x+k+h}/l_{x+k}=\Pr(K_{x+k}=h)={}_hp_{x+k}\,q_{x+k+h}$ and $l_{x+k+h}/l_{x+k}=\Pr(K_{x+k}\geq h)={}_hp_{x+k}$ the above is ${}_k\mathcal{V}$.

We now look retrospectively how much money we collecte up to $k-1$:

$$\sum_{h=0}^{k-1}\Pi_h(1+i)^{k-h}l_{x+h}-\sum_{h=0}^{k-1}b_{h+1}(1+i)^{k-(h+1)}d_{x+h},$$

which gives per survived capita the amount

$$\frac{1}{l_{x+k}}\left(\sum_{h=0}^{k-1}\Pi_h(1+i)^{k-h}l_{x+h}-\sum_{h=0}^{k-1}b_{h+1}(1+i)^{k-(h+1)}d_{x+h}\right). \tag{2.17}$$

Comparing the present values of (2.16) and (2.17) we notice that these two amounts are equal if

$$\frac{1}{l_{x+k}}\left(\sum_{h=0}^{n-k-1} b_{k+h+1}v^{k+h+1}d_{x+k+h}+b_n^* v^n l_{x+n}-\sum_{h=0}^{n-k-1}\Pi_{k+h}v^{k+h}l_{x+k+h}\right)=$$

$$=\frac{1}{l_{x+k}}\left(\sum_{h=0}^{k-1}\Pi_h v^h l_{x+h}-\sum_{h=0}^{k-1}b_{h+1}v^{h+1}d_{x+h}\right),$$

which after small manipulations yields

$$\sum_{h=0}^{n-1} b_{h+1} v^{h+1} d_{x+h} - \sum_{h=0}^{n-1} \Pi_h v^h l_{x+h} = 0. \tag{2.18}$$

This is exactly the condition that (Π_j) is net premium.

Consider now (2.17). We rewrite it in the form

$$\begin{aligned} &\frac{1}{l_{x+k}}\left(\sum_{h=0}^{k-1} \Pi_h (1+i)^{k-h} l_{x+h} - \sum_{h=0}^{k-1} b_{h+1}(1+i)^{k-(h+1)} d_{x+h}\right) \\ &= \frac{1}{v^k\, {}_x p_k}\left(\sum_{h=0}^{k-1} \Pi_h v^h\, {}_x p_h - \sum_{h=0}^{k-1} b_{h+1} v^{h+1}\, {}_{h|}q_x\right) = {}_k\mathcal{V}^{\text{retro}} \end{aligned}$$

that is, this is the retrospective reserve. Thus

$${}_k\mathcal{V} = \frac{1}{{}_kE_x}\, {}_k\mathcal{V}^{\text{retro}}$$

where ${}_kE_x = v^k\, {}_x p_k$ is the actuarial discounting function.

Example 2.7 Consider for a theoretical group of l_{40} lifes $x = 40$-the term insurance for 10 years on the sum 1000, dying according to according TT Tablicy Trwania Zycia TTZ-Pl97m from Błaszczyszyn and Rolski []. Suppose the interest rate is $i = 0.04$ (or $v = 0.96154$). In table 2.1, we present the mathematical reserve computed with net level premium

$$1000 P_{\overset{1}{40}:\overline{10}|} = \frac{1000 A_{\overset{1}{40}:\overline{10}|}}{\ddot{a}_{40:\overline{10}|}} = \frac{\sum_{h=0}^{9} 1000 v^{h+1} d_{40+h}}{\sum_{h=0}^{9} v^h l_{40+h}} = 6.41053.$$

Reserves in column (1) are computed with use of formula (2.16). Considering year $[k, k+1)$ in column (5) we obtain the amount gathered at the end of year, which divided by l_{40+k+1} gives ${}_{k+1}V_{\overset{1}{40}:\overline{10}|}$. Remember that ${}_0V_{\overset{1}{40}:\overline{10}|} = 0$. Thus we have a recurrence

$$(5) = 1000[l_{40+k}({}_kV_{\overset{1}{40}:\overline{10}|} + P_{\overset{1}{40}:\overline{10}|})(1+i) - d_{40+k}].$$

Table 2.1: Term insurance from Example 2.7). Consecutive columns consist:
$(1) = 1000\,{}_kV_{\overset{1}{40}:\overline{10}|} = (5)_{k-1}/l_{40+k}$,
$(2) = 1000\,{}_kV_{\overset{1}{40}:\overline{10}|}\, l_{40+k}(1+i)$,
$(3) = 1000P_{40:\overline{10}|}\, l_{40+k}(1+i)$
$(4) = 1000b_{k+1}d_{40+k}$,
$(5) = (2) + (3) - (4)$.

k	l_{40+k}	d_{40+k}	(1)	(2)	(3)	(4)	(5)
0	94 012	421	0.00	0	626 774	421 000	205 774
1	93 591	460	2.20	214 005	623 967	460 000	377 971
2	93 131	500	4.06	393 090	620 900	500 000	513 990
3	92 631	544	5.55	534 550	617 567	544 000	608 116
4	92 087	589	6.60	632 441	613 940	589 000	657 381
5	91 498	639	7.18	683 676	610 013	639 000	654 689
6	90 859	693	7.21	680 876	605 753	693 000	593 629
7	90 166	750	6.58	617 374	601 133	750 000	468 507
8	89 416	811	5.24	487 247	596 132	811 000	272 379
9	88 605	874	3.07	283 275	590 725	874 000	0
10	87 731	0	0.00				

Table 2.2: Endowment assurance from Example 2.2). Consecutive columns consist:
$(1) = 1000\,{}_kV_{40:\overline{10}|} = (5)_{k-1}/l_{40+k}$,
$(2) = 1000\,{}_kV_{40:\overline{10}|}\, l_{40+k}(1+i)$,
$(3) = 1000P_{40:\overline{10}|}\, l_{40+k}(1+i)$,
$(4) = 1000b_{k+1}d_{40+k}$,
$(5) = (2) + (3) - (4)$

k	(1)	(2)	(3)	(4)	(5)
0	0.00	0	8 110 736	421 000	7 689 736
1	82.16	7 997 326	8 074 415	460 000	15 611 741
2	167.63	16 236 211	8 034 729	500 000	23 770 940
3	256.62	24 721 777	7 991 593	544 000	32 169 370
4	349.34	33 456 145	7 944 660	589 000	40 811 805
5	446.04	42 444 277	7 893 845	639 000	49 699 122
6	546.99	51 687 087	7 838 716	693 000	58 832 803
7	652.49	61 186 115	7 778 929	750 000	68 215 044
8	762.90	70 943 646	7 714 224	811 000	77 846 869
9	878.58	80 960 744	7 644 256	88 605 000	87 731 000
10	1000.00				

Example 2.8 Consider as in Example 2.7 a cohort l_{40} of lifes (40) having bought the endowment assurance for 10 years for sum $b = 1000$. In Table 2.2 we present reserves collected if the net premium is paid

$$1000P_{40:\overline{10}|} = \frac{1000A_{40:\overline{10}|}}{\ddot{a}_{40:\overline{10}|}} = \frac{\sum_{h=0}^{9} 1000v^{h+1}d_{40+h} + 1000v^{10}l_{40+10}}{\sum_{h=0}^{9} v^h l_{40+h}} = 82.95521 \ .$$

Exercises; on line lecture 2

1. Derive Thiele recurrence (2.10) from formula (2.16).

2. Analyse term insurance from Example 2.7, however terminated after 3 years. Show entries of Table 2.1 in this case.

2.4 An analysis of a real portfolio

Suppose that at the beginning of the k year there is still N_k active polices. We know this number, although in practice it differs from the theoretical one obtained from life tables. We make the following analysis for period $[k, k+1)$ of polices for lifes (x) bought k years ago. The insurer has at $k+1$ (formally just before $k+1$) reserve ${}_{k+1}\mathcal{V}$ per head to be paid obligations for deaths from $[k, k+1)$. Let D_k denotes the number of deaths in $[k, k+1)$. It is a random variable. Knowing that at the beginning of the interval $[k, k+1)$ is N_k, we can expect the number of deaths $N_k\, q_{[x]+k}$. Thus the difference at the end of analysed year is

$$N_k\, q_{[x]+k} - D_k \ .$$

Since for one alive at k there is prepared reserve ${}_{k+1}\mathcal{V}$, and b_{k+1} is the benefit paid at $k+1$ per head, then we have a difference

$$b_{k+1} - {}_{k+1}\mathcal{V}$$

for one insured alive at k. This amount is called *death strain at risk* (DSAR). In the whole portfolio we have

$$I_{k+1} = (N_k\, q_{[x]+k} - D_k)(b_{k+1} - {}_{k+1}\mathcal{V}) \ . \tag{2.19}$$

The quantity I_{k+1} is said to be a mortality profit.