Chapter IV

One life, few decrements

[1] We now consider the theory with one life but few risks. Another name is the theory with multi decrements and we will stick to such a terminology. In this theory we are allowed to distinguish more causes of death. For example we can have: 1 – *natural death*, 2 – *death from an accident*. In this case we have two decrements. We can add for example the third one like 3 – *disability*. Another natural example with two decrements is 1 – *natural death*, 2 – *withdrawal*. In this chapter we consider only models for one life (x), for which at an instant of death, there can happen one event from the list of decrements. In the classical life insurance we have only two states or in other words statuses "ALIVE" and "DEAD". So we have the event at the moment of death. Thus the moment of deaths is the moment of exit from status "ALIVE". One can say the exit happened due to specific decrement. Using multistate diagrams we can illustrate our situation as on Figure IV.

1 Basic probabilistic notions

1.1 Time and cause of exit from status

We denote by T_x the future lifetime of (x) (other interpretations are also possible, in general this is the exit time from the status), and by J_x the number of decrement, which caused the exit. Notice that we allowed to have m different decrements, which are numbered. Thus we have a vector (T_x, J_x), where $T_x \geq 0$, and J_x assumes values in $\{1, \ldots, m\}$. We suppose that there

[1] 03.2020

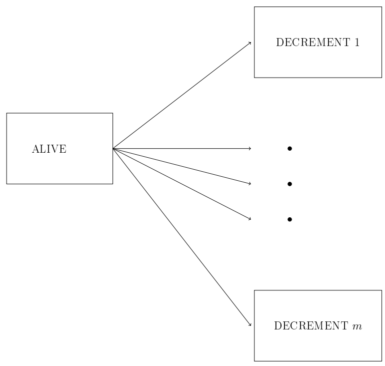

Figure 0.1: A general multiple decrement model.

exists a joint density function in the form $f_x(t, j)$ $t \geq 0$, $j = 1, \ldots, m$. For example the probability that the death time is in interval (a, b) and was caused by decrement j is

$$\Pr(\, a < T_x \leq b, J_x = j \,) = \int_a^b f_x(t, j)\ \mathrm{d}t.$$

Hence

$$\Pr(\, J_x = j \,) = \int_0^\infty f_x(t, j)\ \mathrm{d}t,$$

and the distribution function of T_x is

$$\Pr(\, T_x \leq t \,) = F_x(t) = \sum_{j=1}^{m} \int_0^t f_x(v, j)\ \mathrm{d}v,$$

and its density functions

$$f_x(t) = \sum_{j=1}^{m} f_x(t,j).$$

In actuarial notations we write

- the probability of exit from the status for life (x) before time t:

$$ {}_tq_x^{(\tau)} = \Pr(T_x \le t),$$

- the probability of not exiting before t is

$$ {}_tp_x^{(\tau)} = 1 - {}_tq_x^{(\tau)},$$

- the probability of exit for life (x) before time t from the decrement j

$$ {}_tq_x^{(j)} = \Pr(T_x \le t, J_x = j),$$

- the conditional probability of the exit caused by decrement j of life (x) before $t+s$ under the condition that within time s was in the status

$$\begin{aligned} {}_tq_{[x]+s}^{(\tau)} &= \Pr(T_x \le s+t \,|\, T_x > s)\,, \\ {}_tq_{[x]+s}^{(j)} &= \Pr(T_x \le s+t, J_x = j \,|\, T_x > s). \end{aligned}$$

Notice that for multidecrement models we do not have ${}_tp_x^{(j)}$ (why?; see Exercise 3.2).

We define *force of decrement due to decrement* j $\mu_x^{(j)}(t)$ as

- $$\mu_x^{(j)}(t) = \frac{f_x(t,j)}{S_x(t)}.$$

Heuristics behind this concept are as follows.

Compute

$$\Pr(T_x \le t+h, J_x = j | T_x > t) = \frac{\int_t^{t+h} f_x(s,j)\,\mathrm{d}s}{S_x(t)}.$$

Now if t is a continuity point of $f_x(t,j)$, then

$$\frac{\int_t^{t+h} f_x(s,j)\ \mathrm{d}s}{1-F_X(t)} = \mu_x^{(j)}(t)h + o(h).$$

On the other hand we define the *total force of decrement*

$$\mu_x^{(\tau)}(t) = \frac{f_x(t)}{1-F_x(t)}.$$

One can easily prove

$$\mu_x^{(\tau)}(t) = \sum_{j=1}^{m} \mu_x^{(j)}(t)\,, \tag{1.1}$$

$$ {}_tp_x^{(\tau)} = e^{-\int_0^t \mu_x^{(\tau)}(s)\ \mathrm{d}s}\,. \tag{1.2}$$

$$\frac{\mathrm{d}\ {}_tq_x^{(j)}}{\mathrm{d}t} = f_x(t,j) = {}_tp_x^{(\tau)}\ \mu_x^{(j)}(t)\,, \tag{1.3}$$

$$ {}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)}\ \mu_x^{(j)}(s)\ \mathrm{d}s\,. \tag{1.4}$$

From these formulas we see that the knowledge of all forces of decrement $\mu_x^{(j)}(t)$, $j=1,\ldots,m$ gives a full information about the distribution of (T_x, J_x) and allows to determine all characteristics as ${}_tp_x^{(\tau)}$, and ${}_tq_x^{(j)}$. We leave the proof to the reader in exercise 3.3.

Example 1.1 Consider

$$\mu_x^{(1)}(t) = 0.01, \quad \mu_x^{(2)}(t)^{(2)} = 0.02\,,$$

for all $t \geq 0$. To determine $f_x(t,j)$ we first compute

$$\mu_x^{(\tau)}(t) = \mu_x(t)^{(1)} + \mu_x^{(2)}(t) = 0.03,$$

and hence using (1.2)

$$ {}_tp_x^{(\tau)} = e^{-0.03t}.$$

Now (1.3) yields for $t \geq 0$

$$f_x(t,j) = \begin{cases} 0.01\, e^{-0.03t}, & \text{for } j=1, \\ 0.02\, e^{-0.03t}, & \text{for } j=2. \end{cases} \tag{1.5}$$

We will say that *hypothesis of homogeneous population* ($\text{HHP}^{(\tau)}$) holds for the multi decrement model, if if

$$\Pr(\,T_x \le t, J_x = j\,) = \Pr(\,T_0 - x \le t, J_0 = j \,|\, T_0 > x\,),$$

for all $j = 1, \ldots, m$, $x, t \ge 0$. Then the standard hyphotesis of homogeneous population (HHP) holds because

$$\begin{aligned}
\Pr(T_x > t) &= 1 - \sum_{j=1}^{m} \Pr(T_x \le t, J_x = j) \\
&= 1 - \sum_{j=1}^{m} \Pr(T_0 - x \le t, J_0 = j | T_0 > x) \\
&= \Pr(T_0 - x > t | T_0 > x).
\end{aligned}$$

Under $\text{HHP}^{(\tau)}$ we have

$$\mu_x^{(j)}(t) = \mu_x^{(j)}(t)\,, \tag{1.6}$$

$${}_tq_{[x]+s}^{(j)} = {}_tq_{x+s}^{(j)}\,. \tag{1.7}$$

Curtate future lifetime is $K_x = \lfloor T_x \rfloor$. In actuarial convention we write

- ${}_{k|}q_x^{(j)} = \Pr(K_x = k, J_x = j)$ – probability of death from the risk j of the life with complete number of years lived is k $(k = 0, \ldots)$.

Suppose x is integer. We say that *hypothesis of aggregation for model we few risks* ($\text{HA}^{(\tau)}$) holds if

$$\Pr(\,K_x = k, J_x = j\,) = \Pr(\,K_0 - x = k, J_0 = j \,|\, K_0 \ge x\,),$$

for $k = 0, 1, \ldots$, $j = 1, \ldots, m$. Assumption $\text{HA}^{(\tau)}$ for (K_x, J_x) yields HA for K_x, that is

$$\begin{aligned}
\Pr(K_x = k) &= \sum_{j=1}^{m} \Pr(K_x = k, J_x = j) \\
&= \sum_{j=1}^{m} \Pr(K_0 - x = k, J_0 = j | K_0 \ge x) \\
&= \Pr(K_0 - x = k | K_0 \ge x).
\end{aligned}$$

Under the hypothesis $\text{HA}^{(\tau)}$ we can use the representation

$$\Pr(K_x = k, J_x = j) = {}_kp_x^{(\tau)}\; q_{x+k}^{(j)}, \tag{1.8}$$

$${}_kp_x^{(\tau)} = p_x^{(\tau)}\; p_{x+1}^{(\tau)} \cdots p_{x+k-1}^{(\tau)}\,. \tag{1.9}$$

1.2 Interpolation hypotheses

Suppose x is integer. We say that the distribution of (T_x, J_x) fulfils *hypothesis of uniformity for multi-risk model (we write HU$^{(j)}$)* If HU$^{(j)}$ hold for all $j = 1, \dots, m$, then we write HU$^{\tau}$. if for $n = 0, 1 \dots$

$$ {}_{n+u}q_x^{(j)} = (1-u)\ {}_n q_x^{(j)} + u\ {}_{n+1}q_x^{(j)}, \qquad 0 \le u < 1\,. \tag{1.10} $$

In particular, for $n = 0$ we have the following interpolation formula

$$ {}_u q_x^{(j)} = u\ q_x^{(j)} \quad 0 \le u < 1\,. \tag{1.11} $$

Formula (1.10) implies, summing with respect $j = 1, \dots, m$,

$$ {}_{n+u}q_x^{(\tau)} = (1-u)\ {}_n q_x^{(\tau)} + u\ {}_{n+1}q_x^{(\tau)}, \qquad 0 \le u < 1 $$

which is hypothesis HU for T_x. We recommend the reader to prove the following result.

Proposition 1.2 *Under HU$^{\tau}$ we have*

$$ \mu^{(j)}_{[x]+n+u} = \frac{q^{(j)}_{[x]+n}}{1 - u\ q^{(\tau)}_{[x]+n}} = \frac{q^{(j)}_{[x]+n}}{1 - u + u\ p^{(\tau)}_{[x]+n}}\,. \tag{1.12} $$

Similarly to hypothesis HCFM for distribution of T_x it is said that distribution of (T_x, J_x) fulfils *hypothesis of constant exit rate from status j (we write HCFM$^{(j)}$)*, if

$$ \mu^{(j)}_{[x]+n+u} = \mu^{(j)}_{[x]+n}, \qquad 0 \le u < 1, $$

for $n = 0, 1, \dots$ and $j = 1, \dots, m$.

If HCFM$^{(j)}$ for all $j = 1, \dots, n$, then we write HCFM$^{(\tau)}$ Clearly HCFM$^{\tau}$ for (T_x, J_x) yields HCFM for T_x, and furthermore we have the following relationship.

Proposition 1.3 *If HCFM$^{(\tau)}$ holds, then*

$$ {}_u q_x^{(j)} = \frac{\mu_x^{(j)}}{\mu_x^{(\tau)}}\ {}_u q_x^{(\tau)} \qquad 0 \le u < 1. $$

Proof From the hypothesis we have $\mu_{[x]+u} = \mu_x$ for $0 \le u < 1$. Next we have for $0 \le u < 1$

$$\begin{aligned} {}_u q_x^{(j)} &= \int_0^u f_x(t,j)\ \mathrm{d}t \\ &= \int_0^u {}_t p_x^{(\tau)}\ \mu_{[x]+t}^{(j)}\ \mathrm{d}t \\ &= \mu_x^{(j)} \int_0^u {}_t p_x^{(\tau)}\ \mathrm{d}t \\ &= \frac{\mu_x^{(j)}}{\mu_x^{(\tau)}} \int_0^u {}_t p_x^{(\tau)}\ \mu_x^{(\tau)}\ \mathrm{d}t \\ &= \frac{\mu_x^{(j)}}{\mu_x^{(\tau)}}\ {}_u q_x^{(\tau)}. \end{aligned}$$

□

1.3 Multiple decrement tables

By a *multiple decrement table* we call a sequence $\{(l_n^{(\tau)}, d_n^{(1)}, \ldots, d_n^{(m)})\}_{n=0}^{\infty}$, in which the n-th element cosnsits of $m+1$ natural numbers: $l_n^{(\tau)}$ i $d_n^{(j)}$, $j = 1, \ldots, m$. These numbers are: $l_0^{(\tau)}$ is the cardinality of the cohort of newborn lives, $l_n^{(\tau)}$ is the number of survived at exact age n, $d_n^{(i)}$ repesents the number of lives exiting from the cohort in period $[n, n+1)$. Clearly we have $\sum_{j=1}^m d_n^{(j)} = l_n^{(\tau)} - l_{n+1}^{(\tau)}$ and $d_n^\tau = l_x^\tau - l_{x+1}^\tau$. In the section we assume the hypothesis of aggregation HA$^\tau$. Hence formally probability of exiting from the cohort in period $[n, n+k)$ due to decrement j:

$$ {}_k q_n^{(j)} = \frac{\sum_{l=n}^{n+k-1} d_l^{(j)}}{l_n^{(\tau)}} \qquad n, k = 0, 1 \ldots. $$

We can also use notation ${}_k d_n^{(j)} = \sum_{i=n}^{n+k-1} d_i^{(j)}$, and ${}_k d_n^{(\tau)} = \sum_{j=1}^m {}_k d_n^{(j)}$. In particular

$$ d_n^{(j)} = l_n^{(\tau)}\, q_n^{(j)}, \qquad . $$

In the book of Baszczyszyn and Rolski [?] there is attached an illustrative table with two decrements: natural death (1) and accidental death (2).

Remark Sometimes the force of decrement is called *crude force of mortality*. In survival analysis there are used another notations than used in these notes. Namely functions like q_x are preceded by prefix a from *all – at present of all causes*. In particular, for live (x)

- ${}_t(aq)_x = {}_tq_x^{(\tau)}$ – probability of death within t years,
- ${}_t(aq)_x^{(j)} = {}_tq_x^{(j)}$ – probability of death within t years due to decrement j,
- $(al)_x = l_x^{(\tau)}$ – the number of survived age x from the cohort of of $(al)_0$ lives,
- ${}_t(ad)_x^{(j)} = {}_td_x^{(j)}$,
- $(a\mu)_{[x]+t} = -\frac{1}{(al)_x} \cdot \frac{\mathrm{d}(al)_x}{\mathrm{d}x}$,
- $(a\mu)_{[x]+t}^{(j)} = -\frac{1}{(al)_x} \cdot \frac{\mathrm{d}(al)_x^{(j)}}{\mathrm{d}x}$.

For the associated models we have

- q_x^j lub q_{jx} – probability of death in year $[x, x+1)$ in the associated model for decrement j (in these notes $q_x^{'(j)}$),
- l_x^j lub l_{jx} - the number of survived age x from the cohort of $(al)_0$ live in the associated model for decrement j (in these notes $l_x^{'(j)}$.

Exercises; on line lecture 3

1. Prove formulas (1.1), (1.2), (1.4), (1.3).

2 Model of competing risks

Vector (T_x, J_x) can be represented by the so called *model of competing risks*. Remark however that this is not a unique representation, unless associated decrementss are independent and from now on this will be assumed.

The starting point is distribution $f_x(t, j)$ in the model with few decrements considered in Section 1.