Remark Sometimes the force of decrement is called *crude force of mortality*. In survival analysis there are used another notations than used in these notes. Namely functions like q_x are preceded by prefix a from *all – at present of all causes*. In particular, for live (x)

- ${}_t(aq)_x = {}_tq_x^{(\tau)}$ – probability of death within t years,
- ${}_t(aq)_x^{(j)} = {}_tq_x^{(j)}$ – probability of death within t years due to decrement j,
- $(al)_x = l_x^{(\tau)}$ – the number of survived age x from the cohort of of $(al)_0$ lives,
- ${}_t(ad)_x^{(j)} = {}_td_x^{(j)}$,
- $(a\mu)_{[x]+t} = -\frac{1}{(al)_x} \cdot \frac{\mathrm{d}(al)_x}{\mathrm{d}x}$,
- $(a\mu)_{[x]+t}^{(j)} = -\frac{1}{(al)_x} \cdot \frac{\mathrm{d}(al)_x^{(j)}}{\mathrm{d}x}$.

For the associated models we have

- q_x^j lub q_{jx} – probability of death in year $[x, x+1)$ in the associated model for decrement j (in these notes $q_x^{'(j)}$),
- l_x^j lub l_{jx} - the number of survived age x from the cohort of $(al)_0$ live in the associated model for decrement j (in these notes $l_x^{'(j)}$.

Exercises; on line lecture 3

1. Prove formulas (1.1), (1.2), (1.4), (1.3).

2 Model of competing risks

Vector (T_x, J_x) can be represented by the so called *model of competing risks*. Remark however that the presented molde is not a unique representation, unless associated decrements are independent, which will be assumed in this subsection.

Consider the joint density $f_x(t, j)$ of vector (T_x, J_x) for a live (x) and force of decrements $\mu_x^{(j)}(t)$, $(j = 1, \ldots, m)$. Define an *associated model for* *decrement* j. It is a single decrement model with with lifetime $T_x^{'(j)}$ having distribution given by

$$\Pr(T_x^{'(j)} > t) = e^{-\int_0^t \mu_x^{(j)}(t)\,\mathrm{d}s} \qquad t \geq 0. \tag{2.13}$$

For each **separate** decrement j, $T^{'(j)}$ is defined by (2.13) and we assume that $T^{'(j)}$ $(j = 1, \ldots, m)$ are independent. By $j_0 = \operatorname{minarg}_j(a_j)$ we denote the index j_0 in sequence (a_j) such that $a_{j_0} \leq a_j$ for all j.

Define (T_x, J_x) by

$$T_x = \min_{j=1,\ldots,m} T_x^{'(j)}, \tag{2.14}$$

$$J_x = \operatorname*{minarg}_{j=1,\ldots,m} T_x^{'(j)}. \tag{2.15}$$

We will show that if $T_x^{'(j)}$ are defined as in (2.13) and $T_x^{'(1)}, \ldots, T_x^{'(m)}$ are independent, then (T_x, J_x) has the joint density $f_x(t, j)$. Such the representation of (T_x, J_x) is said to be given by competing risks.

Remark It turns out that without the assumption on the independence of $T_x^{'(j)}$'s, it is also possible to find joint distribution of $T_x^{'(1)}, \ldots, T_x^{'(m)}$ such that the joint density of (T_x, J_x) defined by (2.14) and (2.15) is $f_x(t, j)$. However we do not pursue this way further on.

Distribution of $T_x^{'(j)}$ and its survival function have traditional actuarial notation:

$$_tp_x^{'(j)} = \Pr(T_x^{'(j)} > t) = 1 - {_tq_x^{'(j)}}.$$

Notice that $_tp_x^{'(j)}$ need not converge to zero for $t \to \infty$, which means $T_x^{'(j)}$ may assume ∞ with a positive probability (please give an example). Quantity $_tq_x^{'(j)}$ is calle sometimes [2] *independent rate of decrement due to cause* j (or *net probability of exit from the status caused by decrement* j).

To show that (T_x, J_x) defined by (2.14) and (2.15) has density $f_x(t, j)$ we need the following lemma.

[2] *absolutnym wskanikiem wychodzenia ze statusu z przyczyny (ryzyka) j* lub te *prawdopodobiestwem netto wyjcia ze statusu z przyczyny (ryzyka) j.*

Lemma 2.1 *Let* (T_x, J_x) *be defined with help* (2.14)–(2.15). *Then*

$$\Pr(T_x \le t, J_x = j) = \int_0^t f_x^{'(j)}(s) \prod_{\ell \ne j} {}_s p_x^{'(\ell)} \, \mathrm{d}s,$$

where $f_x^{'(j)}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\, {}_t p_x^{'(j)}$ *is the density function of* $T^{'(j)}$.

The proof will be demonstrated for case $m = 2, 3$ later on.

Proposition 2.2 *Random vector* (T_x, J_x) *defined by* (2.14)–(2.15) *has joint density function* $f_x(t, j)) = \frac{\mathrm{d}}{\mathrm{d}t} \Pr(T_x \le t, J_x = j)$

Proof From Lemma 2.1 and (2.13) we have

$$\begin{aligned} \Pr(T_x \le t, J_x = j) &= \int_0^t \mu_{[x]+s}^{(j)} e^{-\int_0^s \mu_{[x]+v}^{(j)} \, \mathrm{d}v} e^{-\int_0^s \sum_{\ell \ne j} \mu_{[x]+v}^{(\ell)} \, \mathrm{d}v} \, \mathrm{d}s \\ &= \int_0^t \mu_x^{(j)}(s) \, {}_s p_x^{(\tau)} \, \mathrm{d}s \, . \end{aligned}$$

Now using (1.3) we see that the joint density

$$\frac{\mathrm{d}}{\mathrm{d}t} \Pr(T_x \le t, J_x = j)$$

is $f_x(t, j)$. □

From the above facts we see that variables $T_x^{(j)}$ $(j = 1, \ldots, n)$ compete. The winning determines the exit time from the status and the decrement.

We have the following simple property.

Proposition 2.3

$$\begin{aligned} {}_t p_x^{(\tau)} &= \prod_{j=1}^m {}_t p_x^{'(j)}, &\quad (2.16) \\ {}_t q_x^{'(j)} &\ge {}_t q_x^{(j)}. &\quad (2.17) \end{aligned}$$

Proof We prove only (2.17) leaving (2.16) to the reader. Thus

$${}_t p_x^{'(j)} \ge {}_t p_x^{(\tau)}, \qquad t \ge 0$$

and hence

$$ {}_tq_x^{\prime(j)} = \int_0^t {}_sp_x^{\prime(j)} \mu_{[x]+s}^{(j)}\,\mathrm{d}s \geq \int_0^t {}_sp_x^{(\tau)} \mu_{[x]+s}^{(j)}\,\mathrm{d}s = {}_tq_x^{(j)} \ . $$

□

For associated model for decrement j we can formulate hypothesis HHP and HA, and they are denoted by $\mathrm{HHP}^{\prime(j)}$ i $\mathrm{HA}^{\prime(j)}$ and if they are true for all $j = 1, \ldots, n$, then we write $\mathrm{HHP}^{\prime}$ i $\mathrm{HA}^{\prime}$ respectively. Similarly we can define interpolation hypothesis $\mathrm{HU}^{\prime(j)}$ and $\mathrm{HCFM}^{\prime(j)}$ and if they are true for all $j = 1, \ldots, m$, then we write $\mathrm{HU}^{\prime}$ and $\mathrm{HCFM}^{\prime}$ respectively.

2.1 Case $m = 2, 3$.

Consider now the case with two and three risks, for which we derive useful formulas. To avoid working with non-proper distribtutions we assume that

$$ \int_0^\infty \mu_x^{(j)}(s)\,\mathrm{d}s = \infty, \qquad j = 1, \ldots, m, \tag{2.18} $$

holds.

For $m = 2$, suppose that we have two independent associated models: $T_x^{\prime(j)}$ is associated future lifetime with intensity $\mu_x(t)^{(j)}$, $(j = 1, 2)$. We postulate that our exit time from the status "alive" is

$$ \begin{aligned} T_x &= \min(T_x^{\prime(1)}, T_x^{\prime(2)}), &(2.19)\\ J_x &= \begin{cases} 1 & \text{jeli } T_x^{\prime(1)} < T_x^{\prime(2)}, \\ 2 & \text{jeli } T_x^{\prime(2)} \leq T_x^{\prime(1)}. \end{cases} &(2.20) \end{aligned} $$

Hence we have ${}_tp_x^{\prime(j)} = \exp(-\int_0^t \mu_x(s)^{(j)}\,\mathrm{d}s)$ dla $j = 1, 2$.

In the sequel we will need a special case of Lemma 2.1. If independent random variables X and Y have density function $f_X(t)$ and survival function S_Y respectively, then by the total probability formula

$$ \begin{aligned} \Pr(X \leq t, X < Y) &= \int_0^\infty \Pr(X \leq t, X < Y | X = x)\, f_X(x)\mathrm{d}x \\ &= \int_0^t \Pr(t < Y | X = x)\, f_X(x)\mathrm{d}x \\ &= \int_0^t (S_Y(x))\, f_X(x)\mathrm{d}x \ . \end{aligned} $$

Using the assumption of independence

$$
\begin{aligned}
{}_tq_x^{(1)} &= \Pr(T_x^{'(1)} \leq t, T_x^{'(1)} < T_x^{'(2)}) \\
&= \int_0^t {}_sp_x^{'(1)} \mu_x^{(1)}(s) \, {}_sp_x^{'(2)} \, \mathrm{d}s \ , \qquad (2.21)
\end{aligned}
$$

$$
\begin{aligned}
{}_tq_x^{(2)} &= \Pr(T_x^{'(2)} \leq t, T_x^{'(2)} \leq T_x^{'(1)}) \\
&= \int_0^t {}_sp_x^{'(2)} \mu_x^{(2)}(s) \, {}_sp_x^{'(1)} \, \mathrm{d}s \ , \qquad (2.22)
\end{aligned}
$$

and from the inclusion-exclusion formula (i.e. formulas (VI.2.7) with $m = 2$)

$$
\begin{aligned}
{}_tq_x^{(\tau)} &= \Pr(T_x^{'(1)} \leq t \text{ or } T_x^{'(2)} \leq t) \\
&= {}_tq_x^{'(1)} + {}_tq_x^{'(2)} - {}_tq_x^{'(1)} \ {}_tq_x^{'(2)} \ . \qquad (2.23)
\end{aligned}
$$

We now suppose that $\mathrm{HU}^{'(\tau)}$ holds, ie.., for $j = 1, 2$ we have $\mathrm{HU}^{'(j)}$

$$
{}_{n+u}q_x^{'(j)} = (1-u) \ {}_nq_x^{'(j)} + u \ {}_{n+1}q_x^{'(j)}, \qquad 0 \leq u < 1 \ . \qquad (2.24)
$$

In the proof of the following proposition we will use, assuming HU, that

$$
{}_tp_x \mu_x(t) = p_x
$$

and that

$$
{}_tq_x = tq_x
$$

applied to the associated model. For the proof see table in Chapter 1, in which we have $\mu_x(t) = \frac{q_x}{1-tq_x}$.

Proposition 2.4 *Assuming* $HU^{'(\tau)}$

$$
\begin{aligned}
q_x^{(1)} &= q_x^{'(1)}(1 - \frac{1}{2} q_x^{'(2)}), \qquad (2.25) \\
q_x^{(2)} &= q_x^{'(2)}(1 - \frac{1}{2} q_x^{'(1)}) \ . \qquad (2.26)
\end{aligned}
$$

Proof Recall that because of $\mathrm{HU}^{'(\tau)}$, we have for $j = 1, 2$ and $0 \leq t < 1$

$$
{}_tp_x^{'(j)} \mu_x^{(j)}(t) = p_x^{'(j)}.
$$

and

$$ {}_tq_x^{'(j)} \;=\; t\; q_x^{'(j)}. $$

Now

$$
\begin{aligned}
q_x^{(1)} &= \int_0^1 {}_sp_x^{'(1)}\mu_x^{(1)}(s)\,{}_sp_x^{'(2)}\,\mathrm{d}s \\
&= \int_0^1 {}_sp_x^{'(1)}\mu_x^{(1)}(s)(1-{}_sq_x^{'(2)})\,\mathrm{d}s \\
&= \int_0^1 q_x^{'(1)}(1-s\;q_x^{'(2)})\mathrm{d}s \\
&= q_x^{'(1)}(1-\frac{1}{2}\,q_x^{'(2)}).
\end{aligned}
$$

Similarly we show the second equation. □

For $m=3$ we can make an analogous reasoning with

$$ T_x = \min(T_x^{'(1)}, T_x^{'(2)}, T_x^{'(3)}), $$

and

$$ J_x = \begin{cases} 1 & \text{jeli } T_x^{'(1)} < \min(T_x^{'(2)}, T_x^{'(3)}) \\ 2 & \text{jeli } T_x^{'(2)} \le \min(T_x^{'(1)}, T_x^{'(3)}) \\ 3 & \text{jeli } T_x^{'(3)} \le \min(T_x^{'(1)}, T_x^{'(2)}), \end{cases} $$

where $T_x^{('1)}, T_x^{'(2)}$ i $T_x^{'(3)}$ are **independent** random variables, with mortality rates $\mu_x^{(1)}(t), \mu_x^{(2)}(t), \mu_x^{(3)}(t)$, respectively. Remark that in view of the continuity of distributions of $T_x^{'(1)}, T_x^{'(2)}, T_x^{'(3)}$ in definition J_x we do not have to worry what is going on when $T_x^{'(i)} = T_x^{'(j)}$. We will need a formula corresponding to (2.21): since

$$ \Pr(s < \min(T_x^{'(2)}, T_x^{'(3)})) = {}_sp_x^{'(2)}\,{}_sp_x^{'(3)} $$

we have

$$
\begin{aligned}
{}_tq_x^{(1)} &= \Pr(T_x^{'(1)} \le t, T_x^{'(1)} < \min(T_x^{'(2)}, T_x^{'(3)})) \\
&= \int_0^t {}_sp_x^{'(1)}\mu_x^{(1)}(s)\,{}_sp_x^{'(2)}\,{}_sp_x^{'(3)}\,\mathrm{d}s\;. \qquad (2.27)
\end{aligned}
$$

Proposition 2.5 *Under hypothesis $HU^{'(\tau)}$*

$$q_x^{(1)} = q_x^{'(1)}(1 - \frac{1}{2}(q_x^{'(2)} + q_x^{'(3)}) + \frac{1}{3} q_x^{'(2)} q_x^{'(3)}) . \qquad (2.28)$$

$$q_x^{(2)} = q_x^{'(2)}(1 - \frac{1}{2}(q_x^{'(1)} + q_x^{'(3)}) + \frac{1}{3} q_x^{'(1)} q_x^{'(3)}) \qquad (2.29)$$

$$q_x^{(3)} = q_x^{'(3)}(1 - \frac{1}{2}(q_x^{'(1)} + q_x^{'(2)}) + \frac{1}{3} q_x^{'(1)} q_x^{'(2)}) . \qquad (2.30)$$

Proof Using formula (2.27) we have

$$\begin{aligned} q_x^{(1)} &= \int_0^t {}_sp_x^{'(1)} \mu_x^{(1)}(s) \, {}_sp_x^{'(2)} \, {}_sp_x^{'(3)} \, \mathrm{d}s \\ &= \int_0^1 q_x^{'(1)}(1 - s\, q_x^{'(2)})(1 - s\, q_x^{'(3)}) \, \mathrm{d}s \\ &= q_x^{'(1)}(1 - \frac{1}{2}(q_x^{'(2)}). + q_x^{'(3)}) + \frac{1}{3} q_x^{'(2)} q_x^{'(3)}) . \end{aligned}$$

Analogously we show the second and third equation. □

Remark One can solve system of equations (2.25),(2.28) lub (2.28)–(2.30) and get probabilities $q_x^{'(j)}$. Similarly for higher m's. However this could be more interest in medical statisctics than in actuarila applications. For instance, one considers a hypothetical population with respect de one specific disease. In 1875 William Farr asked: what could be an effect on the expected life length if one could eliminate this specific disease. In our setting this would be We caan eliminatiion of one risk. However in calculations of assurances we need simple all probabilities $q_x^{(j)}$.

We now demonstrate the application of the theory to construct multi-decrement life tables.

Example 2.6 We will construct a two decremental life table with two decrements: natural death (1) and accidental death (2). We suppose that due to decrement (1) the cohort dies out according to TTZ-PL97m. A fragment is attached in Table 2.1 i $\mu_{20+t}^{(2)} = 0.009$, dla $0 \le t \le 10$. This means that $q_x^{'(2)} = 1 - e^{-0.009} = 0.0089596$. Next we calculate $q_x^{(j)}$ using equations (2.25),(2.26). We propose the reader to complete table 2.1.

Table 2.1: Fragment of two decrement table.

x	$q_x^{'(1)}$	$q_x^{'(2)}$	$q_x^{(1)}$	$q_x^{(2)}$	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
20	0.00136	0.0089596	0.0013539	0.0089535	10000	13.539	89.535
21	0.00136	0.0089596			9896.9		
22	0.00136	0.0089596					
23	0.00134	0.0089596					
24	0.00138	0.0089596					
25	0.00144	0.0089596					

Example 2.7 Suppose we consider a population of women in a workplace of age in age interval [50,60). There are possible four causes of exiting: (1) - death, (2) - volontary dismissal, (3) - retirement. We will not take under account the fourth (4) - firing. For decrement (1) we assume mortality like in TTZ-PL97k, due to decrement (2) the exits are with constant intensity $\mu^{(2)} = 0.03$. On the other side the retiremment scheme is as follows. At age 55 there exit 20%, and furthermore $\mu^{(3)}_{55.5} = 0.08$, $\mu^{(3)}_{56.5} = 0.07$, $\mu^{(3)}_{57.5} = 0.03$, $\mu^{(3)}_{58.5} = 0.02$ $\mu^{(3)}_{59.5} = 0.01$. At $x = 60$ all the left workers retire. Suppose that in this work place there are 1000 employees; that is suppose that $l_{50} = 1000$.

We first calculate probabilites for associated models. Assuming the hypothesis HCFM$^{(3)}$, for $x = 55, \ldots, 60$ we have $d_x^{'(1)} = l_{50}^{'(\tau)} q_x$ where q_x is from Table TTZ-PL97k, $d_x^{'(2)} = l_x^{'(\tau)}(1 - \exp(-0.03))$ oraz $d_x^{'(3)} = l_x^{'(\tau)}(1 - \exp(-\mu_{x+0.5}))$. Hence we compute probabilities $q_x^{'(j)}$. We next calculate $q_x^{(j)}$ from formulas (2.28)–(2.30) and sequentially $q_{55}^{(j)}$, $l_{56}^{(\tau)}$, itd. Then for $x = 50, \ldots, 54$ we have $q_x^{'(1)} = q_x$ where q_x is from Table TTZ-PL97k, $q_x^{'(2)} = (1 - \exp(-0.03))$ and $q_x^{'(3)} = e^{-\mu_{x+0.5}}$. We are now ready to work out $q_x^{(j)}$ for ages $x = 50, \ldots, 54$ from formulas (2.25),(2.26) and next sequentially $d_x^{(j)} = l_x^{(\tau)} q_x^{(j)}$. That is we compute $d_{50}^{(1)}$ and $d_{50}^{(2)}$ z $d^{(j)} = l_{50}^{(\tau)} q_{50}^{(j)}$, from which we work out $l_{51}^{\tau} = l_{50}^{(\tau)} - d_{50}^{(1)} - d_{50}^{(2)}$, etc. This gives $l_{55-}^{(\tau)} = l_{50} - \sum_{l=0}^{4}(d_{x+l}^{(1)} + d_{x+l}^{(2)} + d_{x+l}^{(3)})$. Notices that due to our specifications function $l_x^{(\tau)}$ is discontinuous at $x = 55$ because at the beginning of year 55, 20% of women retires ; $l_{55}^{(\tau)} = 0.8 l_{55-}^{(\tau)}$.

Exercise to e-wyklad4

1. In the following fragment of the decremental table for both sexes TSZ-PL99 for $x = 50, \ldots, 54$ fill in the missing columns.

x	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(\tau)}$
50	91708	661	85			
51	90961	656	85			
52	90221	650	84			
53	89487	645	83			
54	88759	640	83			
55	88037	991	80			

Use this table to compute ${}_kp_{50}^{(\tau)}$, for $k = 2, 3, 4$, ${}_2q_{52}^{(1)}$, ${}_{2|}q_{52}^{(1)}$. What is l_{56}^{τ}?

2. Complete table 2.1 from Example 2.6.

3. (*) Consider a life in interval $[x, x + 1)$ with possible exits for status alive by two modes: normal death (decrement 1) and accidental death exactly at time $x+k$ with probability $\nu'^{(2)}$, where $0 < k < 1$ (decrement 2). Write the formula for $q_x^{(j)}$ for $j = 1, 2$. Warning. The theory developed in this chapter cannot be used becasue the second decrement is not absolute continuous.

4. (**; Scott, page 259). Military personel attend a training camp for an intensive course which involve three weeks of continual exercises. Each soldier can be eliminated from the camp because of injury or “failed” by one of the instructors and sent back to his base. For the associated model the weakly rates of decrement are given:

week	hospitalized	failed
1	0.078	0.132
2	0.102	0.092
3	0.058	0.043

Of group of 1000 soldiers who start the course, calculate the number who will successfully complete the course, the number who will be hospitalized and the number who will be failed. Assume the uniform distribution over each week of hospitalization and failure. Anyone sent to hospital or failde leaves the camp immedialelly and does not return. There are no other modes of decrement. Hint. Consider “age” as time (in weeks) since entry the camp. The final result should be 588.