# 3 Processes defined by Markov chains

## 3.1 Discrete time

To begin with consider a DTMC $(\mathcal{J}_k)$ with states of life $\mathcal{E}$. We define two sequences: for $i \in \mathcal{E}$

$$I_k^i = \begin{cases} 1 & \text{if at instant k state of life is } i \\ 0 & \text{othewise} \end{cases}$$

and $i \neq j$

$$I_k^{ij} = \begin{cases} 1 & \text{if at instant k there is transition from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

$k = 0, 1, \ldots$ Using indicator notations we can write precisely $I_k^i = 1(\mathcal{J}_k = i)$ oraz $I_k^{ij} = 1(\mathcal{J}_{k-1} = i, \mathcal{J}_k = j)$.

**Lemma 3.1** *Suppose $\mathcal{J}_0 = i_0$ and $i \neq j$. If $\sum_{k=0}^{\infty} |c_k| < \infty$, then*

$$\mathrm{E}\,[\sum_{k=0}^{\infty} c_k I_k^i] = \sum_{k=0}^{\infty} c_k P^{i_0 i}(0, k)\ ,$$

*and*

$$\mathrm{E}\,[\sum_{k=1}^{\infty} c_k I_k^{ij}] = \sum_{k=1}^{\infty} c_k P^{i_0 i}(0, k-1) P^{ij}(k-1, k).$$

One defines a counting sequence of transitions from $i$ to $j$ $(i \neq j)$ up to time $k$ by:

$$N_k^{ij} = \sum_{l=1}^{k} I_l^{ij}.$$

## 3.2 Continuous time

We restrict ourselves to time interval $[0, n]$. Consider now a CTMC $\mathcal{J}(t)$ and for $i \neq j$ we denote by $0 \leq \tau_l^{ij} \leq n$ $(l = 1, 2, \ldots)$ the consecutive moments of transitions from $i$ to $j$. The counting process of transitions is

$$N^{ij}(t) = \sum_{l=1}^{\infty} 1(\tau_l^{ij} \leq t), \qquad 0 \leq t \leq n.$$

**Lemma 3.2** *Let $c(t)$ and $\mu^{ij}(t)$ $0 \le t \le n$ are piecewise continuous (and hence bounded). If $\mathcal{J}(0) = i_0$, then*

$$\mathrm{E}\,[\int_0^n c(t)\mathit{1}(\mathcal{J}(t) = i)\ \mathrm{d}t] = \int_0^n c(t) P_{i_0 i}(0,t)\ \mathrm{d}t,$$

*and*

$$\mathrm{E}\,[\int_0^n c(t)\mathit{1}(\mathcal{J}(t) = i)\ \mathrm{d}N^{ij}(t)] = \int_0^n c(t) P_{i_0 i}(0,t)\mu^{ij}(t)\ \mathrm{d}t.$$

*Proof* Suppose for a moment that functions $c(t)$ and $\mu^{ij}(t)$ are continuous. We define for $\ell = 1, 2, \dots$ a DTMC $\mathcal{J}^\ell_k = \mathcal{J}(\frac{k}{\ell})$, $k = 0, 1, \dots$ with transition matrices $(P^{ij}_\ell)^m_{i,j=0}$. The starting point is an observation that

$$\int_0^n c(t)1(\mathcal{J}(t) = i)\ \mathrm{d}N^{ij}(t) = \lim_{\ell\to\infty} \sum_{0\le k\le n\ell} c(k/\ell)1(\mathcal{J}(k/\ell) = i, \mathcal{J}((k{+}1)/\ell) = j). \tag{3.39}$$

This is true since $c(t)$ is continuous and realizations of $\mathcal{J}(t)$ has with probability one a finite number of separate jumps and because $N^{ij}(t)$ has a finite number of separate jumps of size 1. Hence, using boundness of $c$ and that $N^{ij}(n)$ is a proper random variable we can prove that

$$\begin{aligned}
&\mathrm{E}\,[\int_0^n c(t)1(\mathcal{J}(t) = i)\,\mathrm{d}N^{ij}(t)] \\
&\quad = \lim_{\ell\to\infty} \mathrm{E}\,[\sum_{0\le k\le n\ell} c(k/\ell)1(\mathcal{J}(k/\ell) = i, \mathcal{J}((k+1)/\ell) = j)] \\
&\quad = \lim_{\ell\to\infty} (\sum_{0\le k\le nl} c(k/\ell)P^{i_0 i}(0, k/\ell)P^{ij}(k/\ell, (k+1)/\ell).
\end{aligned}$$

We have $P^{ij}(k/\ell, (k+1)/\ell) = \frac{1}{\ell}\mu^{ij}(k/\ell) + o(1/\ell)$. Hence

$$\begin{aligned}
&\lim_{\ell\to\infty} (\sum_{0\le k\le nl} c(k/\ell)P^{i_0 i}(0, k/\ell)P^{ij}(k/\ell, (k+1)/\ell) \\
&\qquad = \lim_{\ell\to\infty} \frac{1}{\ell} \sum_{0\le k\le nl} c(k/\ell)P^{i_0,i}(0, \frac{k}{\ell})\mu^{ij}(k/\ell).
\end{aligned}$$

Now from the definition of Riemann integral we have

$$\lim_{\ell\to\infty} \frac{1}{\ell} \sum_{0\le k\le nl} c(k/\ell)P^{i_0\, i}(0, \frac{k}{\ell})\ \mu^{ij}(\frac{k}{\ell}) = \int_0^n c(t)P^{i_0,i}(0,t)\mu^{ij}(t)\, dt.$$

and the proof is completed. □

# 4 Markovian theory of multi-state insurances

In the theory of mullti-state insurances we will use an extension of hpothesis HHP and HA. Thus we say that hypotheisis HA-M holds if $\mathbf{P}_x(k) = \mathbf{P}(x+k)$, where $\mathbf{P} = \mathbf{P}_0$. Analogously we say that HHP-M holds if $\mathbf{P}_x(t,t') = \mathbf{P}(x+t, x+t')$ where $\mathbf{P} = \mathbf{P}_0$ or $\boldsymbol{\mu}_x(t) = \boldsymbol{\mu}(x+t)$, where $\boldsymbol{\mu} = \boldsymbol{\mu}_0$. We begin with actuarial notations, which are used in this section. Thus by ${}_tp_x^{ij}$ we denote the probability abd life $(x)$ being in state $i$ after times $t$ is in state $j$. In notattion of the last section ${}_tp_x^{ij} = P^{ij}(x, x+t)$. In particular $p_x^{ij}$ denotes the probability, that life $(x)$ being in state $i$ after one year is in state $j$. In this section we suppose all needed hypothesis like HHP-M or HA-M are valid.

## 4.1 Discrete models; Thiele recursion for reserves.

Consider first a discrete model for a life $(x)$ described by a DTMC $\mathcal{J}_{[x]+k}$. We suppose that hypothesis HA-M holds, that is $p_{[x]+t}^{ij} = p_{x+t}^{ij}$. Thus we have states $\mathcal{J}_{x+k}$ $(k = 0, 1, \ldots)$. We consider a general insurance model for life $(x)$ with states of life $\mathcal{E}$ (together with one absorbing state 0). Thus we have

- states $0, 1, \ldots, m$, where 0 means death,
- termination time $n$,
- if there is a transition at moment $k+1$ (that is $\mathcal{J}_{x+k} = i \neq \mathcal{J}_{x+k+1} = j$, then at time $k+1$ there is paid benefit $b_{k+1}^{ij}$; we set $b_k^{ii} = 0$,
- if $\mathcal{J}_{x+k} = i$, then it is paid annuity $c_k^i > 0$,
- if $\mathcal{J}_{x+k} = i$, then it is paid premium $\Pi_k^i > 0$,
- if at the end $\mathcal{J}_{x+n} = i$, then it is paid endowment $b_n^i$,
- discount factor is $v$.

We also set

$$H_k^i = \begin{cases} c_k^i - \Pi_k^i & 0 \leq k < n \\ b_n^i & n = k \end{cases}$$

Let us make some comments. An index $k$ (e.g. in $c_k^i$), means that the payment is at $k$. Since at the same moment $k$ one can register an annuity and premium (annuities are often paid at the end of fiscal year but premium always at the beginning), therefore the resulted quantity $H^i_k$ has sense. Notice however that $\Pi^i_n = 0$. We now write down the future loss:

$$
\begin{aligned}
{}_kL = \sum_{j'\neq j}\sum_{\ell=0}^{n-k-1} v^{\ell+1} b^{jj'}_{k+\ell+1} 1(\mathcal{J}_{[x]+k+\ell} = j, \mathcal{J}_{[x]+k+\ell+1} = j') \\
+ \sum_{j\in\mathcal{E}}\sum_{\ell=0}^{n-k} v^{\ell} H^j_{k+\ell} 1(\mathcal{J}_{[x]+k+\ell} = j).
\end{aligned}
$$

(Prospective) reserve at $k$ we define as

$$
{}_k\mathcal{V}^i = \mathrm{E}\left[{}_kL|\mathcal{J}_x(k) = i\right], \qquad k = 0, 1, \ldots, n.
$$

**Theorem 4.1**

$$
\begin{aligned}
{}_k\mathcal{V}^i = \sum_{j\neq j'}\sum_{\ell=0}^{n-k-1} v^{\ell+1} b^{jj'}_{k+\ell+1}\, {}_\ell p^{ij}_{x+k}\ p^{jj'}_{x+k+\ell} \\
+ \sum_{j\in\mathcal{E}}\sum_{\ell=0}^{n-k} v^{\ell} H^j_{k+\ell}\ {}_\ell p^{ij}_{x+k}, \quad k = 0, 1, \ldots, n.
\end{aligned}
\tag{4.40}
$$

*Furthermore we have Thiele recursion*

$$
{}_k\mathcal{V}^i - H^i_k = v\sum_{j\in\mathcal{E}} p^{ij}_{x+k}({}_{k+1}\mathcal{V}^j + b^{ij}_{k+1}), \tag{4.41}
$$

*for* $0 \leq k < n$ *and* ${}_n\mathcal{V}^j = b^j_n$ *for* $j \in \mathcal{E}$.

We show a simple but tedious proof at the end of this subsection.

**Example 4.2** [One life, one risk model revisited] Consider a general endowment model from Section III.2.2 for life $(x)$ with termination at $n$. Benefit function is $b_k$, premium $\Pi_k$, $k = 0, 1, \ldots$ and endowment $b^*_n$ which is pait if at

$n$, the insured is still alive. wypacan jeli ubezpieczony przeyje $n$ lat. We suppose that hypothesis HA holds. In the framework of multi-state insuracnes, we have two states, $\mathcal{E} = \{0, 1\}$. Then

$$\begin{aligned}
{}_k\mathcal{V}^1 + \Pi^1_k &= v(p^{11}_{x+k}({}_{k+1}\mathcal{V}^1 + b^{11}_{k+1}) + p^{10}_{x+k}({}_{k+1}\mathcal{V}^0 + b^{10}_{k+1})) \\
{}_k\mathcal{V}^0 + \Pi^0_k &= v(p^{00}_{x+k}({}_{k+1}\mathcal{V}^0 + b^{00}_{k+1}) + p^{01}_{x+k}({}_{k+1}\mathcal{V}^1 + b^{01}_{k+1})).
\end{aligned}$$

Clearly we have $p^{11}_{x+k} = p_{x+k}$, $p^{10}_{x+k} = q_{x+k}$, $b^{11}_k = 0$, $b^{10}_k = b_k$, $p^{00}_{x+k} = 1$, $p^{01}_{x+k} = 0$, $\Pi^1_k = \Pi_k$, $\Pi^0_k = 0$ and with boundary konditions: ${}_n\mathcal{V}^1 = b^*_n$, ${}_n\mathcal{V}^0 = 0$. Then the above recurrence is reduced to

$$\begin{aligned}
{}_k\mathcal{V} + \Pi^1_k &= v(p_{x+k}\, {}_{k+1}\mathcal{V}^1 + q_{x+k}({}_{k+1}\mathcal{V}^0 + b_{k+1}) \\
{}_k\mathcal{V}^0 &= v\ {}_{k+1}\mathcal{V}^0
\end{aligned}$$

Using the boundary condition we obtain ${}_k\mathcal{V}^0 = 0$, for $k = 0, \ldots, n$ and then

$${}_{k+1}\mathcal{V} + \Pi_k = v(p_{x+k}\, {}_{k+1}\mathcal{V} + q_{x+k} b_{k+1}),$$

where ${}_{k+1}\mathcal{V} = {}_{k+1}\mathcal{V}^1$ and the boundary condition is ${}_n\mathcal{V} = b^*_n$. This exactly Thiele recurrence (III.2.5) derived earlier in Chapter III.

*Proof of Theorem 4.1.* Using that

$$\begin{aligned}
\mathrm{E}\,[1(\mathcal{J}_x(k+\ell) = j, \mathcal{J}_x(k+\ell+1) = j')|\mathcal{J}_x(k) = i] &= {}_\ell p^{ij}_{x+k}\ p^{jj'}_{x+k+\ell} \\
\mathrm{E}\,[1(\mathcal{J}_x(k+\ell) = j)|\mathcal{J}_x(k) = i] &= {}_\ell p^{ij}_{x+k}
\end{aligned}$$

we have

$$\begin{aligned}
{}_k\mathcal{V}^i &= \mathrm{E}\,[\sum_{j\neq j'} \sum_{\ell=0}^{n-k-1} v^{\ell+1} b^{jj'}_{k+\ell+1} 1(J_x(k+\ell) = j, J_x(k+\ell+1) = j')|J_x(k) = i] \\
&+ \mathrm{E}\,[\sum_{j\in\mathcal{E}} \sum_{\ell=0}^{n-k} v^\ell H^j_{k+\ell} 1(J_x(k+\ell) = j)||J_x(k) = i] \\
&= \sum_{j\neq j'} \sum_{\ell=0}^{n-k-1} v^{\ell+1} b^{jj'}_{k+\ell+1}\, {}_\ell p^{ij}_{x+k}\ p^{jj'}_{x+k+\ell} + \sum_{j\in\mathcal{E}} \sum_{\ell=0}^{n-k} v^\ell H^j_{k+\ell}\ {}_\ell p^{ij}_{x+k}
\end{aligned}$$

To demonstrate the recurrence we write

$$
\begin{aligned}
{}_k\mathcal{V}^i = & \sum_{j,j'\in\mathcal{E}:j'\neq j} \sum_{\ell=0}^{n-k-1} v^{\ell+1} b^{jj'}_{k+\ell+1}\, {}_\ell p^{ij}_{x+k}\ p^{jj'}_{x+k+\ell} \\
& + \sum_{j\in\mathcal{E}} \sum_{\ell=0}^{n-k} v^\ell H^j_{k+\ell}\ {}_\ell p^{ij}_{x+k} \\
= \ v & \sum_{j,j'\in\mathcal{E}:j'\neq j} (b^{jj'}_{k+1} 1(j=i)\, p^{jj'}_{x+k} + \sum_{\ell=1}^{n-k-1} v^\ell b^{jj'}_{k+\ell+1}\, {}_\ell p^{ij}_{x+k}\, p^{jj'}_{x+k+\ell}) \\
& + \sum_{j\in\mathcal{E}} (H^j_k 1(j=i) + v \sum_{\ell=1}^{n-k} v^{\ell-1} H^j_{k+\ell}\ {}_\ell p^{ij}_{x+k}).
\end{aligned}
$$

Next we use Chapmana-Kolmogorowa equation for $\ell = 1, 2, \ldots$

$$
{}_\ell p^{ij}_{x+k} = \sum_{i'\in\mathcal{E}} p^{ii'}_{x+k}\ {}_{l-1}p^{i'j}_{x+k+1},
$$

and substitution $\ell' = \ell - 1$. Hence we have

$$
\begin{aligned}
v \sum_{i'\in\mathcal{E}} b^{ii'}_{k+1}\, p^{ii'}_{x+k} \quad + \quad & v \sum_{i'\in\mathcal{E}} p^{ii'}_{x+k} \sum_{j,j'\in\mathcal{E}:j'\neq j} \sum_{\ell'=0}^{n-k-2} v^{\ell'+1} b^{jj'}_{k+1+\ell'+1}\ {}_{\ell'+1}p^{i'j}_{x+k+1}\ p^{jj'}_{x+k+1+\ell'} \\
& + H^i_k + v \sum_{i'\in\mathcal{E}} p_{x+k} ii' \sum_{j\in\mathcal{E}} \sum_{\ell'=0}^{n-k-1} v^{\ell'} H^j_{k+1+\ell'}\ {}_{l'}p^{i'j}_{x+k+1} \\
= \qquad & H^i_k + v \sum_{i'\in\mathcal{E}} p^{ii'}_{x+k} (b^{ii'}_{k+1} + {}_{k+1}\mathcal{V}^{i'}).
\end{aligned}
$$

□

## 4.2 Continuous model

Suppose that for new born person states of life are described by a CTMC $\mathcal{J}(t)$ with matrix transition intensity function $\mu(t)$, the assuming HHP-M, we have that for life $(x)$ state of lifes are described by a CTMC, $\mathcal{J}(x+t)$ with with matrix transition intensity function $\mu(x+t)$. Recall that

$$
{}_h p^{ij}_{x+t} - 1(i=j) = h\mu^{ij}_{x+t} + o(h),
$$

provided $t$ is a continuity point of $\mu^{ij}(x+t)$. We consider now a general endowment for life $(x)$ for time $n$ such that

- states are $0, 1, \ldots, m$, where 0 (death) is an absorbing state,
- if there is a transition at $t$ from $i$ to $j$, then benefit $b^{ij}(t)$ is paid; we assume $b^{ii}(t) = 0$,
- if $\mathcal{J}(x+t) = i$, then the annuity is paid with rate $c^i(t)$,
- if $\mathcal{J}(x+t) = i$, then the premium is paid with rate $\Pi^i(t)$,
- at $n$ we have $\mathcal{J}(x+t) = i$, then endowment is paid $b^i(n)$ (clearly $b^0(n) = 0$),
- force of interest is $\delta$.

In the sequel we suppose that functions $c^i(t)$, $\Pi^i(t)$, $b^{ij}(t)$ and $\boldsymbol{\mu}(x+t)$ are piecewises continuous. Similarly as in the discrete case we set $H^i(t) = c^i(t) - \Pi^i(t)$.

As usual we first write the future loss

$$
\begin{aligned}
{}_tL = \sum_{j \neq j'} \int_t^n & v^{s-t} b^{jj'}(s)\, 1(\mathcal{J}_x(s) = j)\ \mathrm{d}N^{jj'}(s) \\
&+ \sum_{j \in \mathcal{E}} v^{n-t} b^j(n) 1(\mathcal{J}_x(n) = j) \\
&+ \sum_{j \in \mathcal{E}} \int_t^n v^s H^j(s) 1(\mathcal{J}_x(s) = j). \qquad (4.42)
\end{aligned}
$$

Then (prospective) reserve at $k$ is

$$
{}_t\bar{\mathcal{V}}^i = \mathrm{E}\left[{}_tL | \mathcal{J}(x+t) = i\right], \qquad 0 \leq t \leq n.
$$

We now show the reserves ${}_t\bar{\mathcal{V}}^i$ fulfill the system of differential equation (called Thiele's system of d.e.)

**Theorem 4.3** *For $0 \le t \le n$ and $i = 0, \dots, m$*

$$
\begin{aligned}
{}_t\bar{\mathcal{V}}^i = & \sum_{j,j' \in \mathcal{E}} \int_t^n v^{s-t} \; {}_{s-t}p_{x+t}^{ij} b^{jj'}(s) \mu^{jj'}(x+s) \, \mathrm{d}s \\
& + \sum_{j \in \mathcal{E}} \int_t^n v^{s-t} \; {}_{s-t}p_{x+t}^{ij} H^j(s) \, \mathrm{d}s \\
& + \sum_{j \in \mathcal{E}} v^{n-t} \; {}_{n-t}p_{x+t}^{ij} b^j(n) \, . \qquad (4.43)
\end{aligned}
$$

*Furthermore reserves ${}_k\bar{\mathcal{V}}^i$, $i \in \mathcal{E}$, $0 \le t \le n$ fulfill*

$$
\frac{\mathrm{d}}{\mathrm{d}t} {}_t\bar{\mathcal{V}}^i = -H^i(t) + \delta \, {}_t\bar{\mathcal{V}}^i - \sum_{i' \ne i} \mu^{ii'}(x+t)({}_t\bar{\mathcal{V}}^{i'} - {}_t\bar{\mathcal{V}}^i + b^{ii'}(t)) \qquad (4.44)
$$

*for $0 \le t \le n$ and ${}_n\bar{\mathcal{V}}^j = b^j(n)$, for $j = 0, 1 \dots, m$.*

We can rewrite the above Thiele system of differential equations in the form

$$
-H^i(t) = \frac{\mathrm{d}}{\mathrm{d}t} {}_t\bar{\mathcal{V}}^i - \delta \, {}_t\bar{\mathcal{V}}^i + \sum_{i' \ne i} \mu^{ii'}(x+t)({}_t\bar{\mathcal{V}}^{i'} - {}_t\bar{\mathcal{V}}^i + b^{ii'}(t)).
$$

The we have

- $-H^i(t)$ is premium/annuity,
- $\frac{\mathrm{d}\, {}_t\bar{\mathcal{V}}^i}{\mathrm{d}t} - \delta \, {}_t\bar{\mathcal{V}}^i$ is *saving premium*,
- $\sum_{i' \ne i} \mu^{ii'}(x+t)(b^{ii'}(t) + {}_t\bar{\mathcal{V}}^{i'} - {}_t\bar{\mathcal{V}}^i)$ is *risk premium* (with is compenstate risks related to changes of states).

*Proof* In the proof (4.43) we use formula (4.42) and results of Lemma 3.2.

The following matrix/vector notations will be convenient in the furhter considerations:

$$\begin{array}{rcl} {}_t\mathbf{p}_x & = & ({}_tp_x^{ij})_{i,j\in\mathcal{E}} \\ \mathbf{B}(s) & = & (b^{jj'}(s)\mu^{jj'}(x+s))_{j,j'\in\mathcal{E}} \\ {}_t\bar{\mathbf{V}} & = & ({}_t\bar{\mathcal{V}}^0,\ldots,{}_t\bar{\mathcal{V}}^m)^T\ , \\ \mathbf{e} & = & (1,\ldots,1)^T\ , \\ \mathbf{H}(t) & = & (H^0(t),\ldots,H^m(t))^T \\ \mathbf{b}(n) & = & (b^0(n),\ldots,b^m(n))^T \end{array}$$

From Chapman-Kolomogorov equation[6]

$$_{s-t}\mathbf{p}_{x+t} = ({}_t\mathbf{p}_x)^{-1}\ {}_s\mathbf{p}_x\ .$$

Then (4.43) can be written as

$$\begin{array}{rcl} {}_t\bar{\mathbf{V}} & = & \displaystyle\int_t^n v^{s-t}\ {}_{s-t}\mathbf{p}_{x+t}\ \mathbf{B}(s)\ \mathrm{d}s\mathbf{e} \\ & & \displaystyle +\int_t^n v^{s-t}{}_{s-t}\mathbf{p}_{x+t}\mathbf{H}(s)\ \mathrm{d}s\mathbf{e} \\ & & +v^{n-t}{}_{n-t}\mathbf{p}_{x+t}\mathbf{b}(n) \\ & = & \displaystyle (v^t\ {}_t\mathbf{p}_x)^{-1}\int_t^n v^s\ {}_s\mathbf{p}_x\ \mathbf{B}(s)\ \mathrm{d}s\,\mathbf{e} \\ & & \displaystyle +(v^t\ {}_t\mathbf{p}_x)^{-1}\int_t^n v^s{}_s\mathbf{p}_x\mathbf{H}(s)\ \mathrm{d}s\mathbf{e} \\ & = & +(v^t\ {}_t\mathbf{p}_x)^{-1}{}_n\mathbf{p}_x\mathbf{b}(n) \end{array}$$

We also write (2.17) in the present notations Kolmogorov prospective equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}_t\mathbf{p}_x = {}_t\mathbf{p}_x\,\boldsymbol{\mu}(x+t)\ .$$

We will need the following facts from matrix calculus.

**Lemma 4.4** *Suppose $\mathbf{A}(t),\mathbf{B}(t)$ are matrices $m\times m$ with differentiable elements. Then*

$$\frac{d}{\mathrm{d}t}(\mathbf{A}(t)\mathbf{B}(t)) = (\frac{d}{\mathrm{d}t}\mathbf{A}(t))\mathbf{B}(t) + \mathbf{A}(t)(\frac{d}{\mathrm{d}t}\mathbf{B}(t)).$$

[6] We assume that ${}_t\mathbf{p}_x$ is non-singular.

*Furthermore assuming non-sigularity of* $\mathbf{A}(t)$, *we have* $\mathbf{A}(t)^{-1}\mathbf{A}(t) = \mathbf{I}$. *Hence*

$$\frac{d}{\mathrm{d}t}\mathbf{A}(t)^{-1} = -\mathbf{A}(t)^{-1}(\frac{d}{\mathrm{d}t}\mathbf{A}(t))\mathbf{A}(t)^{-1}.$$

We are now ready to demonstrate Thiele's system of differential equations 4.3.

*Proof* Differentiating

$${}_t\bar{\mathbf{V}} = (v^t{}_t\mathbf{p}_x)^{-1}(\int_t^n v^s{}_s\mathbf{p}_x\mathbf{B}(s)\ \mathrm{d}s\,\mathbf{e} + \int_t^n v^s\mathbf{H}(s)\ \mathrm{d}s + v^n{}_n\mathbf{p}_x\mathbf{b}(n)),$$

and recalling that $\mathrm{d}v^{-t}/\,\mathrm{d}t = \delta v^{-t}$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}{}_t\bar{\mathbf{V}} &= (\delta v^{-t}({}_t\mathbf{p}_x)^{-1} - v^{-t}({}_t\mathbf{p}_x)^{-1}(\frac{\mathrm{d}}{\mathrm{d}t}{}_t\mathbf{p}_x)({}_t\mathbf{p}_x)^{-1}) \\
&\quad \times(\int_t^n v^s{}_s\mathbf{p}_x\mathbf{B}(s)\ \mathrm{d}s\ \mathbf{e} + \int_t^n v^s\mathbf{H}(s)\ \mathrm{d}s + v^n{}_n\mathbf{p}_x\mathbf{b}(n)) \\
&\quad + v^{-t}({}_t\mathbf{p}_x)^{-1}(-v^t{}_t\mathbf{p}_x\mathbf{B}(t)\,\mathbf{e} - v^t{}_t\mathbf{p}_x\mathbf{H}(t)) \\
&= \delta_t\bar{\mathbf{V}} - \boldsymbol{\mu}_{x+t}\ {}_t\bar{\mathbf{V}} - \mathbf{B}(t)\mathbf{e} - \mathbf{H}(t)\ .
\end{aligned}$$

# 5 Examples of disability and sickness insurances

## 5.1 Discrete time

Multi-state theory turns out to be usefull in pricing some forms of disability and sickness insurances. The starting point can be modell AID with transitions depicted on diagram 0.2.

Times are slotted into intervals, let say one year. Here states of life are seen only at instances $0, 1, \ldots$. Suppose we life $(x)$ buys an insurance. We suppose that states $\mathcal{J}_x, \mathcal{J}_{x+1}, \mathcal{J}_{x+2}, \ldots$ form a DTMC with known transition matrices.

**Example 5.1** [Disability insurance] When a person buys an insurance at age $x$, he will, during the lifetime of his policy, that is $[0, n]$, run through different states. Rougly speaking, the policy will be active at inception of the policy, the insured can die or become disabled, may re-enter into active states, etc. As usual death is an absorbing state. Policy pays a benefit at the end of year in which $(x)$ became disabled or it lastsin this state. However the benefit is stopped after $c$ consecutive years. Thus to describe this policy we need the following states: A (active), D (death), $I = 1, 2, \ldots, c$ (number of years being disabled). Let $\Pr(\mathcal{J}_{x+t+1} = j | \mathcal{J}_{x+t} = i) = p^{ij}_{x+t}$ and we assume hypothesis HA-M. The non-zero transition probabilities are: clearly $p^{\mathrm{DD}}_x = 1$ and for A and $i = 1, \ldots, c-1$

$p^{i,\mathrm{A}}_x$ – the probability to "reactivate" from state $i$ (resp. to remain in A in case of $i = A$),

$p^{i,i+1}_x$ – the probability to remain disabled (resp. to become entitled to benefit in case $i = 0$),

$p^{i,\mathrm{D}}_x$ – the death probability when being in state $i$,

and for $i = c$

$p^{c,c}_x$ – the probability to remain disabled (resp. not to die, as we have assumed that in state $c$, no recover will take place.

$p^{c,\mathrm{D}}_x$ – the death probability when being in state $c$.

We can have the following payments. In the $i$-the year, ie. $[i-1, i)$, premiums $\Pi^B_{i-1}$ are paid at the beginning of the year and disability benefit at the end of year – $b^{\mathrm{A},1}_i$ and $b^{l,l+1}_i$, where $l = 1, \ldots, c-1$ and death benefit $b^{l,D}_i$.

**Example 5.2** [ Hospital cash benefits with premium restitution][7] In this example we consider a hospital cash plan for life $(x_0)$ with restitution of premiums in case no claims have occured during a certain period. To price this insurance, we build the following multi-state Markovian model in discrete time. Duration of the policy is $n$ years. Let $\mathcal{J}_{x_0}, \mathcal{J}_{x_0+1}, \mathcal{J}_{x_0+2}, \ldots$ be states in consecutive year of life $(x_0)$ (or age of the policy is $x - x_0$). We have the following states: I,1, $\ldots, r$,D, where D denotes death (an absorbing state), and I is sick (gets cash from the insurer) and $1, \ldots, r$ are the number of consecutive no claim years. Under this policy, if after $r$ consecutive years without claim, the whole premium (without interest) is paid back to life $(x_0)$. We have given: $q_x$ – the probability of death of $(x)$ in the present year,

[7] Hospital cash benefits with premium restitution; A simple application of Markovian multi-state model. by Ernst Huber, Swiss Re.

$f_x$ – frequence of hospitalisation of life $(x)$. Denote transition probabilities $\mathcal{J}_{x+1} = j|\mathcal{J}_x = i) = p^{ij}(x)$. We then have:

| $*$ | I | 1 | 2 | $\dots$ | $r$ | D |
|---|---|---|---|---|---|---|
| I | $f_x p_x$ | $(1-f_x)p_x$ | 0 | $\dots$ | 0 | $q_x$ |
| 1 | $p_x f_x$ | 0 | $(1-f_x)p_x$ | $\dots$ | 0 | $q_x$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\dots$ | $\vdots$ | $\vdots$ | |
| $r$ | $f_x p_x$ | $(1-f_x)p_x$ | $\dots$ | $\dots$ | 0 | $q_x$ |
| D | 0 | 0 | $\dots$ | $\dots$ | 0 | 1 |

Level premium $\Pi$ is collected at the beginning of fiscal year, benefit and back premiums are paid at the end of the year.

## 5.2 Continuous time

We can show the following important property. If from a state $i$ all transitions are out and there are no into, then

$$P^{ii}(t,t') = P^{\underline{ii}}(t,t') = e^{-\int_t^{t'} \mu^i(v)\,dv}\,.$$

We leave the reader to show this propoery.

We consider now the permanent disability state insurance (AID) from Example 0.2. Recall that we have three states: Healthy – $*$, Disabled – $\Diamond$ and Dead – $\dagger$. Suppose that $\mu^{*\Diamond}(t) = \sigma(t)$, $\mu^{\Diamond *}(t) = \rho(t) = 0$, $\mu^{*\dagger}(t) = \mu(t)$ and $\mu^{\Diamond\dagger}(t) = \nu(t)$; see transition diagram in Fig. 0.2.

In this case we can have explicite formula for $\mathbf{P}(t,t')$. Thus

$$P^{**}(t,t') = e^{-\int_t^{t'} (\mu(s)+\sigma(s))\,\mathrm{d}s} \tag{5.45}$$

$$P^{\Diamond\Diamond}(t,t') = e^{-\int_t^{t'} \nu(s)\,\mathrm{d}s} \tag{5.46}$$

$$P^{*\Diamond}(t,t') = \int_t^{t'} \sigma(s) e^{-\int_t^s (\mu(v)+\sigma(v))\,\mathrm{d}v} e^{-\int_s^{t'} \nu(v)\,\mathrm{d}v}\,\mathrm{d}s \tag{5.47}$$

$$P^{\diamond\dagger}(t,t') = 1 - e^{-\int_t^{t'} \nu(s)\,\mathrm{d}s}. \tag{5.48}$$

We accept the hypothesis HHP-M $\mu^{*\Diamond}(x+t) = \mu(x+t)$ and $\sigma(x+t) =$

$\mu^{*\dagger}(x+t) = \mu^{\Diamond\dagger}(t)$, then

$$
\begin{aligned}
P^{**}(tt') &= \exp\Big(-\int_t^{t'} (\mu(x+s)+\sigma(x+s))\,\mathrm{d}s\Big)\,, \\
P^{*\Diamond}(tt') &= \exp\Big(-\int_t^{t'} \sigma(x+s)\,\mathrm{d}s\Big)\Big(1-\exp\Big(-\int_t^{t'} \mu(x+s)\,\mathrm{d}s\Big)\Big)\,, \\
P^{\Diamond\Diamond}(tt') &= \exp\Big(-\int_t^{t'} \sigma(x+s)\,\mathrm{d}s\Big).
\end{aligned}
$$

We can show these formulas (5.45)– (5.48) in two ways either by substituting to Kolmogorov equations, or in a direct way. Consider now $P^{*\Diamond}(t,t')$. Notice that the exit from $\star$ state, can be modelled as the exit from status $*$ becasue of decrement $\diamond$, while another decrement is $\dagger$. Let $T$ be the exit time and $J$ is the corresponding decrement. Then $\sigma(s)e^{-\int_t^s(\mu(v)+\sigma(v)\,\mathrm{d}v}$, is density function of $T$, when $J=\diamond$. Similarly $e^{-\int_s^{t'}\nu(v)\,\mathrm{d}v}$ is the probability of stay at $\diamond$ in interval $(s,t']$.

**Exercises; on line lecture 7**

1. Write transition matrix for the Markov chain from Example 5.1.
2. Continue Example 2.5, wherein it was proposed how to build the intensity matrix for multi-decrement model. With the use of Theorem 4.3 derive Thiele differential equation for multi-decrement insuracne, that is prove (IV.4.5) in Proposition IV.4.4