Wstęp do matematyki finansowej i aktuarialnej

Lista 1

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1. (A box calculus representation) The purpose of this exercise is to generalize and unify the calculus we made for functions of Brownian motion with drift and geometric Brownian motion. It provides a proof for validity of the box calculus for processes that are functions of Brownian motions and time.

   (a) Let $X_t = f(t, W_t)$. Use Ito's formula to calculte dX_t. Next use the chain rule and Ito's formula to calculte dY_t, where $Y_t = g(t, X_t) = g(t, f(t, W_t))$.

   (b) Calculate $dX_t \cdot dX_t$ by the box calculus.

2. Suppose that X satisfies the SDE
$$dX_t = \alpha X_t \, dt + \sigma X_t \, dW_t,$$
and Y satisfies
$$dY_t = \gamma Y_t \, dt + \delta Y_t \, dV_t,$$
where V is the Wiener process which is independent of W. Define $Z = X/Y$ and drive the SDE for Z computing dZ and substituting Z for X/Y in the RHS of dZ. (If X is nominal income and Y describes the inflation, then Z decribes the real income.

3. Suppose that X satisfies the SDE
$$dX_t = \alpha X_t \, dt + \sigma X_t \, dW_t,$$
and Y satisfies
$$dY_t = \gamma Y_t \, dt + \delta Y_t \, dV_t,$$
where V is the Wiener process which is independent of W. Define $Z = X \cdot Y$ and derive an SDE for Z. If X describes the price process and Y is the currence rate, then Z describes the dynamics expressed in the new currency.

4. The object of this exercise is to give an argument for the formal identity
$$dW_1 \cdot dW_2 = 0,$$
where W_1, W_2 are two indpendent Wiener processes. Let us therefor fix a time t, and divide the interval $[0, t]$ into equidistant points $0 = t_0 < t_1 < \ldots < t_n = t$, where $t_i = (it)/n$. We use the notation
$$\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1}), \qquad i = 1, 2.$$
Define Q_n by
$$Q_n = \sum_{k=1}^{n} \Delta W_1(t_k) \cdot \Delta W_2(t_k).$$
Show that $Q_n \to 0$ in L^2, i.e. show that $E[Q_n] = 0$, $Var[Q_n] \to 0$.