Queues and Communication Networks.

An outline of continuous time theory.

Tomasz Rolski

Mathematical Institute, Wrocaw University

Wrocław 2008

Contents

Chapter I

Introduction

Chapter II

Basic concepts from Markov processes theory

A Markov process is a stochastic process whose dynamics is such that the distribution for its future development after any moment depends only on the present state at this moment and not on how the process arrived in that state.

1 Continuous time Markov chains

In this section we introduce basic notions from the theory of some continuous time Markov chains (CTMC). We consider here time homogeneous CTMCs.

We assume a denumerable state space $\mathbb{E}$. In general considerations, we will sometimes use $\mathbb{E} = \{0, 1, 2, \ldots\}$.

Consider a matrix function $\boldsymbol{P}(t) = (p_{ij}(t))_{i,j\in\mathbf{E}}$ fulfilling

- $\boldsymbol{P}(t)$ is a stochastic matrix, i.e. $p_{ij}(t) \geq 0$, $\sum_{j\in\mathbf{E}} p_{ij}(t) = 1$ for all $i \in \mathbb{E}$,
- $\boldsymbol{P}(0) = \boldsymbol{I}$
- (CP) *Chapmann-Kolmogorov* equation holds, that is $\boldsymbol{P}(t+s) = \boldsymbol{P}(t)\boldsymbol{P}(s)$, for all $s, t \geq 0$

Then $(\boldsymbol{P}(t))_{t\geq 0}$ is said to be *transition probability function*(t.p.f.) [1] We will assume further on that considered transition probability functions are

[1] transition semi-group

continuous, that is

$$\lim_{h\downarrow 0} \boldsymbol{P}(h) = \boldsymbol{I}\ .$$

Definition 1.1 A stochastic process $(X(t))$ assuming values at $\mathbb{E}$, is said to be a continuous time Markov chain (CTMC), with state space $\mathbb{E}$, initial distribution $\boldsymbol{\mu}$ and transition probability function $\boldsymbol{P}(t) = (p_{ij}(t))_{i,j=0,1\ldots}$ if for $0 < t_1 < \ldots < t_n$

$$\mathbb{P}(X(0) = i_0, \ldots, X(t_n) = i_n) \quad = \quad \nu_{i_0} p_{i_0 i_1}(t_1) \ldots p_{i_{n-1} i_n}(t_n - t_{n-1}) \tag{1.1}$$

for all $i_0, \ldots, i_n \in \mathbb{E}$.

We start from the definition of the *intensity matrix* which plays similar role to the probability transition matrix for DTMCs. It is said that $\boldsymbol{Q} = (q_{ij})_{i,j\in\mathbf{E}}$ is an *intensity matrix* if

(i) $q_{ij} \geq 0$ for all $i \neq j$,

(ii) $\sum_{j\in\mathbf{E}} q_{ij} = 0$ for all $i \in \mathbb{E}$.

We set $q_i = -q_{ii}$. Note that $q_i \geq 0$. Now define the matrix $\boldsymbol{P}^\circ = (p^\circ_{ij})_{i,j=0,1,\ldots}$

$$p^\circ_{ij} = \begin{cases} \frac{q_{ij}}{q_i} & \text{for } \ q_i > 0, i \neq j\ , \\ 1 & \text{for } \ q_i = 0, j = i\ , \\ \\ 0 & \text{for } \ q_i = 0, j \neq i\ . \end{cases} \tag{1.2}$$

It is easy to check that $\boldsymbol{P}^\circ$ is a transition probability matrix.

1.1 Constructive definition via the minimal processes

We now define a process $\{X(t), t \geq 0\}$ in a constructive way. This will be so called *minimal CTMC* defined by $\boldsymbol{Q}$. Suppose we start off $X(0) = i_0$. Then the process stays at i_0 for an exponential time with parameter q_i and next jumps to i_1 with probability $p^\circ_{i_0,i_1}$ ($i_1 \in \mathbb{E}$). Next it stays at i_1 for an exponential time with parameter q_{i_1} and after that it jumps to i_2 with probability $p^\circ_{ii_2}$ ($i_2 \in \mathbb{E}$), etc. All selections are independent. We always choose the so called *cadlag* realizations, that is right continuous and with left hand limits.

We can define $(X(t))$ more formally as follows. Let (Y_n) be a Markov chain with state space $\mathbb{E}$ and transition probability matrix $\boldsymbol{P}^\circ$, $Y_0 = i_0$ and suppose that $(\eta_{ij})_{i,j}$ are independent random variables, also independent of the DTMC $\{Y_n\}$. We assume that $\eta_{ij} \sim \mathrm{Exp}(q_j)$. Let

$$\tau_n^{\mathrm{c}} = \eta_{0,Y_0} + \ldots + \eta_{n-1,Y_{n-1}}, \qquad n = 1, 2, \ldots,$$

$$N^{\mathrm{c}}(t) = \#\{n \geq 1 : \tau_n^{\mathrm{c}} \leq t\}$$

and

$$X(t) = Y_{N^{\mathrm{c}}(t)}, \qquad t \geq 0. \tag{1.3}$$

The *explosion time* is

$$\tau_\infty^{\mathrm{c}} = \lim_{n\to\infty} \tau_n^{\mathrm{c}} .$$

We assume that such the construction leads to a *regular* process that is $\{X(t)\}$ is well defined for all $t \geq 0$ (with $\mathbb{P}_{i_0}$-probability 1). [2] For processes defined by (1.3) this is equivalent that the process is *non-explosive* that is $\mathbb{P}_{i_0}(\tau_\infty^{\mathrm{c}} = \infty) = 1$. The chain (Y_n) is said to be an *embedded DTMC* . The process $X(t)$ defined as above is of course a CTMC. It is said a minimal CTMC defined by intensity matrix $\boldsymbol{Q}$. Since in these notes we do not other CTMC's with intensity matrix $\boldsymbol{Q}$ we omit giving details here.

If the process starts off an initial state i we say that the underlying probability measure is $\mathbb{P}_i$. If the distribution of $X(0)$ is $\boldsymbol{\mu}$, then we first choose i according to $\boldsymbol{\mu}$ and then we begin the construction as above. In this case we say that the minimal CTMC is defined by $(\boldsymbol{Q}, \boldsymbol{\mu})$. We use notations $\mathbb{P}_{\boldsymbol{\mu}}$, $\mathbb{E}_i$, $\mathbb{E}_{\boldsymbol{\mu}}$ in an obvious manner.

The transition probability function is $\boldsymbol{P}(t) = (p_{ij}(t))_{i,j\in\mathbf{E}}$, where

$$p_{ij}(t) = \mathbb{P}_i(X(t) = j).$$

For the regular processes , $\{p_{ij}(t),\ j = 0, 1, \ldots\}$ is a probability function for each $i = 0, 1, \ldots$ and $t \geq 0$. **In this case there is one to one correspondence between $\boldsymbol{P}(t)$ and $\boldsymbol{Q}$.**

Proposition 1.2 *For $j \neq i$*

$$\lim_{t\to 0+} \frac{p_{ij}(t)}{t} = q_{ij} .$$

[2] Formally a cadlag process is regular if the number of jumps is finite in finite intervals $[a, b] \subset \mathcal{T}$ a.s.

Proof We have

$$\begin{aligned} p_{ij}(t) &= \mathbb{P}_i(X(t)=j) \\ &= \sum_{n=0}^{\infty} \mathbb{P}_i(X(t)=j, N^c(t)=n) \\ &= \int_0^t q_{ij} e^{-q_i s} e^{-q_j(t-s)}\, ds + o(t) \ . \end{aligned}$$

□

Proposition 1.3 *The process $\{X(t), t \geq 0\}$ is a continuous time Markov chain, that is*

$$\mathbb{P}_{i_0}(X(t_1)=i_1, X(t_2)=i_2, \ldots, X(t_n)=i_n) \tag{1.4}$$

$$= p_{i_0 i_1}(t_1) p_{i_1 i_2}(t_2-t_1) \cdots p_{i_{n-1} i_n}(t_n - t_{n-1}) \tag{1.5}$$

The above proposition justifies the name of *continuous time Markov chains* (CTMC) .

Remark Markov property from Proposition 1.3 implies that the *Chapmann-Kolmogorov* (CK) equation holds, It is a classical problem in the theory of CTMCs to study relationships between intensity matrices $\boldsymbol{Q}$ and families of t.p.f.s $\{\boldsymbol{P}(t),\ t \geq 0\}$, Under our regularity (or non-explosion) assumption each intensity matrix defines uniquely $\{\boldsymbol{P}(t),\ t \geq 0\}$ and conversely. Removing the assumption of regularity makes the theory much more complicated, but we do not need this case in our study. In these notes we will study only regular CTMCs

Definition 1.4 We say that a process defined by $\boldsymbol{Q}$ is *irreducible* if $p_{ij}(t) > 0$ for all $t > 0$ and $i \neq j$.

When $\boldsymbol{P}(t)$ is uniquely determined by $\boldsymbol{Q}$ (and this is our case) it is said equivalently that $\boldsymbol{Q}$ is irreducible.

Proposition 1.5 *The following sentences are equivalent for CTMCs.*
(i) *CTMC defined by t.p.f.s $\boldsymbol{P}(t)$ is irreducible.*
(ii) *For some $t > 0$ we have $p_{ij}(t) > 0$, $i \neq j$.*
(iii) *$\boldsymbol{P}^\circ$ is irreducible in the sense considered for DTMCs.*

Proof Asmussen p. 50.

Theorem 1.6 *[Reuter's explosion criteria] A CTMC is regular if and only if the only non-negative bounded solution $\boldsymbol{x}$ of $\boldsymbol{Qx} = \boldsymbol{x}$ is $\boldsymbol{x} = \boldsymbol{0}$.*

Proof Asmussen p. 47.

Remark Typically Reuter's explosion criteria is given in the form: for all $\lambda > 0$ the only non-negative bounded solution $\boldsymbol{x}$ of $\boldsymbol{Qx} = \lambda\boldsymbol{x}$ is $\boldsymbol{x} = \boldsymbol{0}$. Note that they are equivalent. Suppose that Theorem 1.6 is true. Then we use the result of this theorem for $\boldsymbol{Q}/\lambda$. Note that if the evolution of a process defined by $(\boldsymbol{Q}, i)$ is given by (V.3.6), then the evolution of the process defined by $(\boldsymbol{Q}/\lambda, i)$ is given

$$\tau_n^{\mathrm{c}} = \lambda(\eta_{0,Y_0} + \ldots + \eta_{n-1,Y_{n-1}}), \qquad n = 1, 2, \ldots,$$
$$N^{\mathrm{c}}(t) = \#\{n \geq 1 : \tau_n^{\mathrm{c}} \leq t\}$$

and

$$X(t) = Y_{N^{\mathrm{c}}(t)}, \qquad t \geq 0.$$

Therefore the finiteness of explosion time is unchanged.

Problems

1.1 Show an example of an explosive CTMC.

1.2 For a CTMC $X(t)$ defined by $(\boldsymbol{Q}, \boldsymbol{\mu})$ let

$$Z = \sum_{n\geq 0} \frac{1}{q_{Y_n}}$$

Show that $Z = \infty$-$\mathbb{P}_\mu$ if and only if the chain is regular.

1.3 *Competing risks.* Let $\eta_{ij} \sim \mathrm{Exp}(q_{ij})$ for $j \neq i$ and

$$\begin{aligned} E_i &= \min_{j\neq i} \eta_{ij} \\ I_i &= \arg\min_{j\neq i} \eta_{ij} = \sum_{j\neq i} j1(\eta_{ij} = \eta_i) \end{aligned}$$

Show that

$$\mathbb{P}_i(\eta_i \in dt, I_i = j) = q_{ij} e^{-q_i t}\, dt, \qquad t \geq 0\ .$$

1.2 Recurrence, invariant distributions and GBE

For a CTMC $\{X(t)\}$ we define *escape time*

$$T_i^{\text{escape}} = \inf\{t \geq 0 : X(t) \neq i\},$$

and the *return time*

$$T_i = \inf\{t > T_i^{\text{escape}} : X(t) = i\}.$$

We say that the state i is

- *transient* if $\mathbb{P}_i(T_i < \infty) < 1$,
- *recurrent* if $\mathbb{P}_i(T_i < \infty) = 1$,
- *positive recurrent* if $\mathbb{E}_i[T_i] < 1$.

Theorem 1.7 *In an irreducible DTMC all states are either recurrent (positive recurrent) or transient.*

An irreducible and positive recurrent CTMC is called *ergodic*.

Proposition 1.8 *The following sentences are equivalent for a regular CTMC $X(t)$.*
(i) *State i CTMC $X(t)$ is recurrent (transient).*
(ii) *The set $\{t : X(t) = i\}$ is unbounded (unbounded).*
(iii) *The embedded DTMC Y_n is recurrent (transient).*

We will see that it is not true the equivalence between positive recurrence for $X(t)$ and positive recurrence for Y_n.

Computing stationary distributions for special models of CTMCs is one of important issues we study in these notes. Therefore we recall now results in this area. We start introducing three concepts, which in turn, will appear equivalent, under the irreducibility assumption.

Definition 1.9 A measure $\boldsymbol{\mu}$ on $\mathbb{E}$ (that is a nonnegative sequence of real numbers (μ_i)) is said to be an *invariant measure* if $\sum_i \mu_i p_{ij}(t) = \mu_j$, for all $j \in \mathbb{E}$ and $t \geq 0$. Furthermore it is said to be a *invariant distribution* . if $\sum_j \mu_j = 1$.

It is convenient to use the vector notations. We denote by $\boldsymbol{\nu}$ the row vector $(\nu_j)_{j\in\mathbf{E}}$ and $\boldsymbol{P}(t) = (p_{ij}(t))_{i,j\in\mathbf{E}}$. Then (ii) of the invariant distribution can be written as $\boldsymbol{\nu}\boldsymbol{P}(t) = \boldsymbol{\nu}$, for all $t \geq 0$.

Definition 1.10 We say that a probability function $\boldsymbol{\pi} = (\pi_j)_{\mathrm{j}\in\mathbf{E}}$ fulfils the *global balance equation* (GBE) if

$$\boldsymbol{\pi}\boldsymbol{Q} = \mathbf{0}.$$

Remark Rewrite the GBE in the following form:

$$\pi_i q_i = \sum_{j\in\mathbf{E}-i} \pi_j q_{ji}$$

which can be read the rate out state i ($= \pi_i q_i$) is equal the rate into state i ($=\sum_{j\neq i} \pi_j q_{ji}$).

In the following theorem we use the so called *regenerative property of CTMSs*, that is that, under $\mathbb{P}_i$, processes $(X(t))_{t\geq 0}$ and $(X(t))_{t\geq T_i}$ have the same distribution.

Theorem 1.11 *Let $(X(t))$ be an irreducible, non-explosive and recurrent CTMC defined by an intensity matrix $\boldsymbol{Q}$.*
(i) *Let i be an arbitrary state and define*

$$\mu_j = \mathbb{E}_i[\int_0^{T_i} 1(X(t) = j)\, dt] \ .$$

We have $0 < \mu_j < \infty$ for all $j \in \mathbb{E}$ and $\boldsymbol{\mu}$ is an invariant measure.
(ii) *The invariant measure $\boldsymbol{\mu}$ is unique up to a multiplicative factor.*
(ii) *Let i be an arbitrary state and define*

$$\mu_j = \mathbb{E}_i[\int_0^{T_i} 1(X(t) = j)\, dt] \ .$$

We have $0 < \mu_j < \infty$ for all $j \in \mathbb{E}$ and $\boldsymbol{\mu}$ is an invariant measure.
(iii) *If $\boldsymbol{\nu}$ is invariant for embedded chain $(Y_n)_{n\geq 0}$, then $\boldsymbol{\mu}$ defined by*

$$\mu_j = \frac{\nu_j}{q_j}$$

is an invariant measure for $X(t)$.
(iii) *The CTMC is positive recurrent if and only if*

$$\sum_j \mu_j < \infty \ .$$

Then $\boldsymbol{\pi}$ defined by $\pi_i = \mu_i / \sum_j \mu_j$ is the invariant distribution.

Proof Asmussen Th. II.4.2.

$$\begin{aligned}
\mu_j &= \mathbb{E}_i[\int_0^h + \int_h^{T_i} 1(X(t)=j)\,dt] \\
&= \mathbb{E}_i[\int_0^h 1(X(t)=j)\,dt + \mathbb{E}_i \int_h^{T_i} 1(X(t)=j)\,dt] \\
&= \mathbb{E}_i[\int_{T_i}^{T_i+h} 1(X(t)=j)\,dt + \mathbb{E}_i \int_h^{T_i} 1(X(t)=j)\,dt] \\
&= \mathbb{E}_i[\int_h^{T_i+h} 1(X(t)=j)\,dt] \\
&= \mathbb{E}_i[\int_0^{T_i} 1(X(t+h)=j)\,dt] \\
&= \mathbb{E}_i[\int_0^{\infty} 1(X(t+h)=j, T_i > t)\,dt] \ .
\end{aligned}$$

Now

$$\begin{aligned}
\mathbb{E}_i[\int_0^{\infty} 1(X(t+h)=j, T_i > t)\,dt &= \mathbb{E}_i \int_0^{\infty} p(h)_{X(t)j} 1(T_i > t)\,dt \\
&= \sum_{k\in\mathbf{E}} p_{kj}(h) \mathbb{E}_i \int_0^{\infty} 1(X(t)=k, T_i > t)\,dt \\
&= \sum_{k\in\mathbf{E}} \mu_k p_{kj}(h) \ .
\end{aligned}$$

□

Definition 1.12 An irreducible and positive recurrent CTMC is called *ergodic*.

The following theorem will give justification for consideration in next chapters.

Theorem 1.13 *An irreducible nonexplosive CTMC is ergodic if and only if one can find a probability function* $\boldsymbol{\mu}$*, which solves GBE* $\boldsymbol{\mu Q} = \mathbf{0}$*. In this case* $\boldsymbol{\pi}$ *is the stationary solution.*

Proof Asmussen p. 52. □

In comparison to Theorem 1.13 in the next proposition we do not require to known a priory about regularity.

Proposition 1.14 *A sufficient condition for ergodicity of an irreducible CTMC is the existence of a probability measure* $\boldsymbol{\pi}$ *that solves* $\boldsymbol{\pi Q} = \mathbf{0}$ *such that* $\sum_j \pi_j q_j < \infty$.

Theorem 1.15 (i) *If* $(X(t)$ *is ergodic and* $\boldsymbol{\pi}$ *the stationary distribution, then for all* i, j

$$\lim_{t\to\infty} p_{ij}(t) = \pi_j \ .$$

(ii) *If* $(X(t))$ *is recurrent but not ergodic, then for all* $i, j \in \mathbb{E}$

$$\lim_{t\to\infty} p_{ij}(t) = 0 \ .$$

Proof See Asmussen p. 54.

Recall that for a transient case we always have

$$\lim_{t\to\infty} p_{ij}(t) = 0 \ .$$

Notice that irreducibility and the existence of a probability solution of the GBE does not imply automatically that CTMC $X(t)$ is ergodic. For this we must have also regularity. The following example (see Asmussen p. 53) demonstrate this sentence. Thus take a transient transition probability matrix $\boldsymbol{P}^\circ$, which have an invariant measure $\boldsymbol{\nu}$. Then choose strictly positive $(q_i)_{i\in\mathbf{E}}$ such that numbers

$$\pi_j = \frac{\nu_j}{q_j}$$

sum up to 1. We leave to the reader to check that (π_j) fulfil the GBE but the transience of (Y_n) excludes the recurrence of $X(t)$. What went wrong it was the lack of regularity of $(X(t))$. The following theorem gives necessary sufficient condition for it.

It sometimes quite challenging to check the GBE and therefore, for special cases, we develop more friendly balance equations.

Problems

1.1 Show that the following all four cases are possible: $(X(t))$ is recurrent (positive recurrent) and (Y_n) is recurrent (positive recurrent).

1.3 Birth and death processes

By a birth and death process (B& D process) we mean a CTMC with $q_{ij} = 0$ except if $|i-j| \leq 1$. In these lecture notes we will meet B&D processes on $\mathbb{Z}$, $\mathbb{Z}_+$ or $0, \ldots, N$. We already know that not all B&D processes can be regular and this problem will be studied here for B&D processes on $\mathbb{Z}_+$. On $\mathbb{Z}_+$ the intensity matrix is

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdot \\ \mu_1 & -\lambda_1 - \mu_1 & \lambda_1 & 0 & \cdot \\ 0 & \mu_2 & -\lambda_2 - \mu_2 & \lambda_2 & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{1.6}$$

For irreducibility in the case of denumerable infinite state space $\mathbb{Z}_+ = \{0, 1, \ldots\}$ we have to assume $\lambda_0, \lambda_1, \ldots > 0, \mu_1, \mu_2, \ldots > 0$

For B&D processes we introduce

$$a_0 = 1 \qquad a_n = \frac{\lambda_0 \ldots \lambda_{n-1}}{\mu_1 \ldots \mu_n}, \qquad n \geq 1 \; . \tag{1.7}$$

Further on

$$D = \sum_{k=0}^{\infty} \frac{1}{\lambda_k a_k} \sum_{i=0}^{k} a_i$$

and

$$\sigma = \sum_{i=0}^{\infty} a_i \; . \tag{1.8}$$

Note that

$$D = \sum_{n=0}^{\infty} r_n,$$

where $r_0 = 1/\lambda_0$ and

$$r_n = \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n \lambda_{n-1}} + \cdots + \frac{\mu_n \cdots \mu_1}{\lambda_n \cdots \lambda_1 \lambda_0}$$

From Reuter's criterion (see Theorem 1.6) we obtain:

Theorem 1.16 *[Reuter's criterion for B&D processes] A necessary and sufficient condition of regularity is*

$$D = \sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n \lambda_{n-1}} + \cdots + \frac{\mu_n \cdots \mu_1}{\lambda_n \cdots \lambda_1 \lambda_0} \right] = \infty \ .$$

Proof We use Reuter's criterion. The consecutive lines of $\boldsymbol{Qx} = \boldsymbol{x}$ are

$$\begin{array}{rcl} x_0 & = & -\lambda_0 x_0 + \lambda_0 x_1 \\ \vdots & \vdots & \\ x_n & = & \mu_n x_{n-1} - (\lambda_n + \mu_n) x_n + \lambda_n x_{n+1} = x_n \\ \vdots & \vdots & \end{array}$$

Denote

$$r_n = \sum_{k=0}^{n} \frac{\mu_{k+1} \cdots \mu_n}{\lambda_k \cdots \lambda_n}$$

and note that

$$r_n = \sum_{k=0}^{n} f_k g_{k+1} \cdots q_n ,$$

where

$$f_n = \frac{1}{\lambda_n}, g_n = \frac{\mu_n}{\lambda - n} \ .$$

Letting $\Delta_n = x_n - x_{n-1}$ $(n = 1, 2, \ldots$ we get

$$\Delta_1 = f_0 x_0, \qquad \Delta_{n+1} = f_n x_n + g_n \Delta_n$$

and hence we obtain immediately, that if $x_0 = 0$, then $\boldsymbol{x} = \boldsymbol{0}$. Otherwise, if $x_0 > 0$ (let say $x_0 = 1$), then the solution $x_n > 0$ for all n. Now for $n = 0, 1, \ldots$

$$\Delta_{n+1} = \sum_{k=0}^{n} f_k g_{k+1} \cdots g_n x_k \left\{ \begin{array}{cc} \geq & r_n x_0 \\ \leq & r_n x_n \ . \end{array} \right.$$

Thus

$$\Delta_1 + \ldots \Delta_{n+1} \left\{ \begin{array}{cc} \geq & r_0 + \ldots + r_{n-1} x_0 \\ \leq & r_0 + \ldots + r_{n-1} x_{n-1} \ . \end{array} \right.$$

Since $\Delta_1 + \ldots + \Delta_n = x_n - x_0$, we have that $D = \sum_{j\geq 1} r_j = \infty$ yields $x_n \to \infty$ so the positive solution must be unbounded. On the other hand suppose that $D < \infty$. Since from the upper bound (V.3.6) yields

$$x_{n+1} \leq (1+r_n)x_n \leq \cdots \leq \prod_{k=0}^{n}(1+r_k)$$

and under $D < \infty$[3]

$$x_n \leq \prod_{k=0}^{\infty}(1+r_k) < \infty$$

we obtain that $\boldsymbol{x}$ is bounded. □

From now on we tacitly assume that considered B&D processes are regular.

Let $0 < \tau_1^{\mathrm{c}} < \tau_2^{\mathrm{c}} < \ldots$ are the consecutive jumps of the process. The embedded Markov chain $\{Y_n = X(\tau_n^{\mathrm{c}})\}$ is a state dependent Bernoulli random walk, that is a Markov chain with probability transition matrix

$$\boldsymbol{P}^{\circ} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdot \\ q_1 & 0 & p_1 & 0 & \cdot \\ 0 & q_2 & 0 & p_2 & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \tag{1.9}$$

where $p_n = 1 - q_n = \frac{\lambda_n}{\lambda_n + \mu_n}$. At the beginning we assume that $\lambda_i > 0, \mu_i > 0$. This yields that $0 < p_n < 1$ and that the chain Y_n is irreducible.

Proposition 1.17 *Transience of* $\{Y_n\}$ *is equivalent to*

$$\sum_{n\geq 0} \frac{1}{a_n \lambda_n} < \infty \tag{1.10}$$

Proof Asmussen [5] Prop. III.2.1. Note that

$$\begin{aligned} \sum_{n\geq 0} \frac{1}{a_n\lambda_n} &= \frac{1}{\lambda_0}\sum_{n=1}^{\infty}\frac{\mu_1\ldots\mu_n}{\lambda_1\ldots\lambda_n} \\ &= \frac{1}{\lambda_0}\sum_{n=1}^{\infty}\frac{q_1\ldots q_n}{p_1\ldots p_n}\,. \end{aligned}$$

[3] Use that $\log(1+r_n) \leq r_n$.

Corollary 1.18 *The CTMC $\{X(t)\}$ or equivalently the DTMC $\{Y_n\}$ is recurrent if and only if*

$$\sum_{n=1}^{\infty} \frac{\mu_1 \dots \mu_n}{\lambda_1 \dots \lambda_n} = \infty. \tag{1.11}$$

Proof Recurrence of $\{Y_n\}$ follows by Proposition 1.17 and (1.10). If the embedded chain $\{Y_n\}$ is recurrent then the process $\{X(t)\}$ is regular and also recurrent.

Lemma 1.19 *Irrespective of recurrence or transience there is the unique solution $\boldsymbol{\pi}$ up to proportionality of the GBE $\boldsymbol{\pi Q} = 0$:*

$$\pi_n = \pi_0 a_n, \qquad n \geq 0. \tag{1.12}$$

Proof We have to solve the following system of equations:

$$\begin{array}{ccl} 0 & = & -\lambda_0 \pi_0 + \mu_1 \pi_1 \\ 0 & = & \lambda_0 \pi_0 - (\lambda_1 + \mu_1)\pi_1 + \mu_2 \pi_2 \\ \vdots & \vdots & \vdots \\ 0 & = & \lambda_{i-1}\pi_{i-1} - (\lambda_i + \mu_i)\pi_i + \mu_{i+1}\pi_{i+1} \\ \vdots & \vdots & \vdots \end{array}$$

□

Recall σ define in (1.8).

Corollary 1.20 *The process $(X(t))$ is ergodic if and only if*

$$\sum_{n=1}^{\infty} \frac{\mu_1 \dots \mu_n}{\lambda_1 \dots \lambda_n} = \infty \quad \text{and} \quad \sigma < \infty \tag{1.13}$$

In this case the stationary distribution is

$$\pi_0 = \frac{1}{\sigma} \qquad \pi_n = \frac{1}{\sigma} a_n, \qquad n \in \mathbb{E}\,. \tag{1.14}$$

Proof Corollary 1.18 states that CTMC is recurrent if and only if

$$\sum_{n=1}^{\infty} \frac{\mu_1 \dots \mu_n}{\lambda_1 \dots \lambda_n} = \infty.$$

From Lemma 1.19 the invariant measure is finite if and only if $\sigma < \infty$. The existence of the stationary distribution is equivalent to ergodicity; see Theorem 1.11. □

Consider now the B&D process when the state space $\mathbb{E} = \{0, \dots, K\}$ is finite. Then we have to suppose that

$$\lambda_i > 0, \quad (i = 0, \dots, K-1) \qquad \lambda_K = 0, \qquad \mu_i > 0 \quad (i = 1, \dots, K).$$

Under these conditions the process $X(t)$ is irreducible and always ergodic with the stationary distribution as in (1.14), where now

$$\sigma = 1 + \sum_{n=1}^{K} \frac{\lambda_0 \dots \lambda_{n-1}}{\mu_1 \dots \mu_n}.$$

In the following subsections we survey the most typical examples.

Further on we need a special class of B&D processes.

Definition 1.21 A *queueing birth and death processes* , or queueing B&D process is any B&D process with $\lambda_n = \lambda$ for all n regardless the state space is finite or not.

Note that for each queueing B&D process $(X(t))$, if we define $N^{\mathrm{a}}(t)$ as the number of up-changes by time t, then $X(t) \le N^{\mathrm{a}}(t)$. We will show later that N^{a} is a Poisson process with intensity λ. Notice that in queueing context upward-changes are identified with job arrivals.

Remark Questions to be asked. For an irreducible CTMC: Relationship between

a. regular

b. recurrence

c. positive recurrence

d. stationary distribution

e. solution of the GBE

Begin with an example. Let birth intensities in B&D process are such that

$$\sum_{n\geq 0} \lambda_n^{-1} < \infty$$

Under $\mathbb{P}_{\boldsymbol{\mu}}$ the evolution of $X(t)$ is as follows. We start with an initial distribution $\boldsymbol{\mu}$. Then the process evolves as the pure birth process with birth intensities λ_k until the first explosion at T_0^0. It restarts from state 0 and proceeds to the next explosion T_0^1 and so on. This process is irreducible, positive recurrent with stationary distribution

$$\pi_n = \frac{\mathbb{E}_0[\int_0^{T_0^0} 1(X(t) = n)\, dt]}{\mathbb{E}_0[T_0]} .$$

Notice however that this process is not regular and furthermore the GBE has no probabilistic solution. Moreover it is not cadlag.

2 Reversibility.

Definition 2.1 Let $\mathcal{T} = \mathbb{R}, \mathbb{R}_+$ or $[0, A]$. The process $\{X(t),\ t \geq \mathcal{T}\}$ is *stationary (in narrow sense)* if for $t_1, \ldots, t_n$ and s such that $t_i + s \in \mathcal{T}$, $(i = 1, \ldots, n)$ $n = 1, 2, \ldots$

$$(X(t_1 + s), \ldots, X(t_n + s)) =_{\mathrm{d}} (X(t_1), \ldots, X(t_n)) .$$

In this section we tacitly assume that all processes X are stationary processes.

For a double ended stationary stochastic process $X(t)$ let

$$X_T^*(t) = X((T - t) - \circ)$$

Lemma 2.2 *If $(X(t))$ is stationary, then all processes $(X_T^*(t))_{t\in\mathbf{R}}$ have the same distribution.*

Proof As an exercise. □

Therefore from now on, without loss of generality we may take $T = 0$. Let $X^{\leftarrow}(t) = X(-t - \circ)$ ($-\circ$ denotes that we take a cadlag version of $\{X(-t)\}$). Process $X^{\leftarrow}$ is called a *reversed process* . First that taking a cadlag version and left continuous with right hand side limits version of a stationary process have the same distribution. Furthermore, it is convenient to study stationary processes as doubly-ended processes $(X(t),\ t \in \mathbb{R})$.

Lemma 2.3 *For each stationary process* $(X(t),\ t \in \mathbb{R}_+)$ *there exists one (in distributional sense) process* $(X_0(t),\ t \in \mathbb{R})$*, such that* $(X(t),\ t \in \mathbb{R}_+)$ *and* $(X_0(t),\ t \in \mathbb{R}_+)$ *have the same distribution.*

Exercise: Show it.

Definition 2.4 We say that $\{X(t),\ t \geq 0\}$ is *reversible* if the finite dimensional distributions of processes $(X(t))$ and $(X^{\leftarrow}(t))$ are the same, which means that processes X and $X^{\leftarrow}$ have the same distribution..

Every irreducible and positive recurrent CTMC can be made stationary assuming the initial distribution to be the stationary distribution. **Exercise: Show it.** We have to consider a CTMC $\{X(t),\ t \geq 0\}$ defined by $(\boldsymbol{\pi}, \boldsymbol{Q})$ where $\boldsymbol{\pi}$ is the stationary distribution.

In this section we will study a regular CTMC $(X(t))$ defined by its intensity matrix $\boldsymbol{Q}$. Since we want to study stationary processes we must assume that $\boldsymbol{Q}$ is ergodic. The consequence of this assumption (regularity) is that the transition matrix function $\boldsymbol{P}(t) = (p_{ij}(t))$ is uniquely determined by $\boldsymbol{Q}$. **We will tacitly assume in this section that considered processes are regular and ergodic.**

We are going to study the class of reversible processes CTMCs is a subclass of stationary CTMCs. For the intensity matrix and its stationary distribution $(\boldsymbol{\pi}, \boldsymbol{Q})$ define matrix $\tilde{\boldsymbol{Q}} = (\tilde{q}_{ij})_{i,j=0,1,\ldots}$ by

$$\tilde{q}_{ij} = \frac{\pi_j}{\pi_i} q_{ji}, \quad i, j = 0, 1, \ldots. \tag{2.1}$$

Intensity matrix defines transition probability matrix $\boldsymbol{P}(t)$. Let

$$\tilde{p}_{ij}(t) = \frac{\pi_j}{\pi_i} p_{ji}(t), \quad i, j = 0, 1, \ldots. \tag{2.2}$$

Lemma 2.5
(i) *Matrix* $\tilde{\boldsymbol{Q}}$ *is an intensity matrix. Furthermore it is regular and ergodic with the stationary distribution* $\boldsymbol{\pi}$.
(ii) $\tilde{\boldsymbol{P}}(t)$ *is a transition probability matrix defined by* $\tilde{\boldsymbol{Q}}$.
(iii) $\tilde{\boldsymbol{Q}}$ *is the intensity matrix of the reversed chain.*

Proof

□

For $\tilde{\boldsymbol{Q}}$ we define the embedded stochastic matrix $\tilde{\boldsymbol{P}}^{\circ}$ according to procedure given in (1.3).

We can proceed conversely.

Proposition 2.6
(i) $\tilde{\boldsymbol{Q}}$ *is the intensity matrix of the reversed process* $X^{\leftarrow}$.
(ii) *Suppose that for a given intensity matrix* $\boldsymbol{Q}$ *(regular and ergodic), there exists positive numbers* $(\pi_j)_{j\in\mathbf{E}}$ *summing up to 1 such that*

- $\tilde{\boldsymbol{Q}}$ *is an intensity matrix, where*

$$\tilde{q}_{ij} = \frac{\pi_j}{\pi_i} q_{ji}$$

Then $\boldsymbol{\pi}$ *is the stationary distribution and* $\tilde{\boldsymbol{Q}}$ *is the intensity matrix of the reversed CTMC.*

For the convenience we study the following processes as double-ended processes with time parameter $t \in \mathbb{R}$.

Recall that we say that a CTMC with intensity matrix $\boldsymbol{Q}$ is reversible if the processes $\{X(t),\ t \in \mathbb{R}\}$ and $\{X^{\leftarrow}(t),\ t \in \mathbb{R}\}$ are equal in distribution.

Proposition 2.7 (i) *Suppose that* $(X(t))_{t\geq 0}$ *is reversible with the stationary distribution* $\boldsymbol{\pi}$*. Then*

$$\pi_i p_{ij}(t) = \pi_j p_{ji}(t), \qquad i,j \in \mathbb{E}, \qquad t \geq 0 \tag{2.3}$$

from which

$$\text{(DBE)} \qquad \pi_i q_{ij} = \pi_j q_{ji} \ i,j \in \mathbb{E}.$$

(ii) *If there exist positive numbers* $\pi_i,\ i \in \mathbb{E}$ *summing up to* 1 *and such that*

$$\pi_i p_{ij}(t) = \pi_j p_{ji}(t), \qquad i,j \in \mathbb{E}, t \geq 0,$$

then the process is reversible with stationary distribution $\boldsymbol{\pi}$*.*
(ii) *If there exist positive numbers* $\pi_i,\ i \in \mathbb{E}$ *summing up to* 1 *and such that* $\pi_i q_{ij} = \pi_j q_{ji}$ *for all* $i,j \in \mathbb{E}$ *hold, then (2.3) is true, which means that the process is reversible with stationary distribution* $\boldsymbol{\pi}$*.*

Proof (i) Let $t > 0$. Since $(X(0), X(t)) =_{\mathrm{d}} (X(t), X(0))$ we have

$$\pi_i p_{ij}(t) = \Pr\{X(t) = j | X(0) = i\}\Pr\{X(0) = i\}$$

$$= \Pr(X(t) = i | X(0) = j)\Pr(X(0) = j) = \pi_j p_{ji}(t). \tag{2.4}$$

From (2.3) dividing by $t > 0$ and passing with $t \downarrow 0$ we obtain (DBE).
(ii) In matrix notation condition (2.3) is read:

$$\mathrm{diag}\{\pi_i,\ i \in \mathbb{E}\}\boldsymbol{P}(t) = \boldsymbol{P}^{\mathrm{T}}(t)\mathrm{diag}\{\pi_i,\ i \in \mathbb{E}\},\ t \geq 0 \tag{2.5}$$

and (DBE) is read

$$\mathrm{diag}\{\pi_i,\ i \in \mathbb{E}\}\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}}\mathrm{diag}\{\pi_i,\ i \in \mathbb{E}\}. \tag{2.6}$$

We have the following converse result to Proposition 2.7.

Proposition 2.8 *Suppose there exist sequence of positive numbers* $\pi_i,\ i \in \mathbb{E}$ *summing up to 1 and such that (DBE) holds. Then (2.3) is true, which means that the process is reversible with stationary distribution* $\boldsymbol{\pi}$*.*

Proof

From Proposition 2.8 it follows the following equivalent definition of a reversible CTMC. We say that a CTMC with intensity matrix $\boldsymbol{Q}$ is reversible, if there exits a sequence of strictly positive numbers $\pi_n,\ n = 0, 1, \ldots$, summing up to 1, such that

(DBE) $\pi_i q_{ij} = \pi_j q_{ji}$ for all $i \neq j$.

The system of equations (DBE) is called *detailed balance equation.*

We give now the so called *Kolmogorov Criterion.* By a *path* between $i, j \in \mathbb{E}$ we call $ii_1 \ldots i_n j$ if $q_{ii_1} q_{i_1 i_2} \ldots q_{i_n j} > 0$. The path is *closed* if $i = j$.

Definition 2.9 It is said that Kolmogorov criterion is fulfilled if for each closed path

$$q_{ii_1}q_{i_1i_2}\dots q_{i_ni} = q_{ii_n}q_{i_ni_{n-1}}\dots q_{i_1i}. \tag{2.7}$$

Theorem 2.10 *(i) The Kolmogorov criterion holds for reversible processes. (ii) Suppose that the Kolmogorov criterion holds. then there exists a sequence of positive numbers* $\{m_i,\ i \in \mathbb{E}\}$ *such that*

$$m_iq_{ij} = m_jq_{ji}, \qquad i,j \in \mathbb{E}. \tag{2.8}$$

If moreover $\sum_i m_i = 1$ *then* $\boldsymbol{\pi}$ *is the stationary distribution of* X*.*

Proof (i) Suppose that the process is reversible, that is (2.6) holds. Let $ii_1\dots i_ni$ be a closed path. Then

$$\begin{array}{rcl} \pi_iq_{ii_1} & = & \pi_{i_1}q_{i_1i} \\ \pi_{i_1}q_{i_1i_2} & = & \pi_{i_2}q_{i_2i_1} \\ \vdots & \vdots & \vdots \\ \pi_{i_n}q_{i_ni} & = & \pi_iq_{ii_n}. \end{array}$$

Multiplying both sides and dividing by $\pi_i\pi_{i_1}\dots\pi_{i_n}$ we get that (2.7) holds.
(ii) Fix a reference state $i \in \mathbb{E}$ and put $m_i = 1$. We now define m_j for $j \neq i$. Take a path $ji_r\dots i_1i$ from j to i. Such a path always exists in view of irreducibility. Put

$$m_j = \frac{q_{ii_1}q_{i_1i_2}\cdots q_{i_{r-1}i_r}q_{i_rj}}{q_{ji_r}q_{i_ri_{r-1}}\cdots q_{i_2i_1}q_{i_1i}}$$

We have to prove the definition is correct.
1^0. value of m_j does not depend on the chosen path. If $jj_s\dots j_1i$ is another path from j to i, then from (2.7)

$$q_{ii_1}q_{i_1i_2}\cdots q_{i_rj}\cdot q_{jj_s}q_{j_sj_{s-1}}\cdots q_{j_1i} = q_{ij_1}q_{j_1j_2}\cdot q_{j_sj}\cdot q_{ji_r}q_{i_ri_{r-1}}\cdots q_{i_1i}$$

and so

$$\frac{q_{ii_1}q_{i_1i_2}\cdots q_{i_rj}}{q_{ji_r}q_{i_ri_{r-1}}\cdots q_{i_1i}} = \frac{q_{ij_1}q_{j_1j_2}\cdot q_{j_sj}}{q_{jj_s}q_{j_sj_{s-1}}\cdots q_{j_1i}}.$$

2^0. We prove now that for (m_i) defined in part 1^0 we have (2.8), i.e. $m_i q_{ij} = m_j q_{ji},\ i,j \in \mathbb{E}$. Using Kolmogorov criterion we can check that

$$\frac{q_{jj_1}q_{j_1j_2}\cdots q_{j_{s-1}i_s}q_{j_si}}{q_{ij_s}q_{j_sj_{s-1}}\cdots q_{j_2j_1}q_{j_1j}}q_{ij} = \frac{q_{ii_1}q_{i_1i_2}\cdots q_{i_{r-1}i_r}q_{i_rj}}{q_{ji_r}q_{i_ri_{r-1}}\cdots q_{i_2i_1}q_{i_1i}}q_{ji}$$

3^0. $m_j > 0,\ j \in \mathbb{E}$. Let $A = \{k \in \mathbb{E}.\ m_k = 0\}$. Suppose that both A and $\mathbb{E} - A$ are nonempty. If $j \in \mathbb{E} - A$ and $k \in A$, then $m_j q_{jk} = m_k q_{kj} = 0$, so that $q_{jk} = 0$ whenever $j \in \mathbb{E} - A$ and $k \in A$. This means that no state in A can be reached from a state in $\mathbb{E} - A$, contradicting irreducibility. Now $\mathbb{E} - A$ is nonempty since $i \in \mathbb{E}$ (recall $m_1 = 1$). Hence A must be empty. □

Example 2.11 Each B&D process is reversible. We can prove it using the Kolmogorov criterion, or from formula (V.3.6)

$$\pi_{n+1} = \frac{1}{\sigma}\frac{\lambda_0\ldots\lambda_{n-1}\lambda_n}{\mu_1\ldots\mu_n\mu_{n+1}} = \pi_n\frac{\lambda_n}{\mu_{n+1}}$$

we obtain

$$\mu_{n+1}\pi_{n+1} = \pi_n\lambda_n\ .$$

Similarly we can consider the finite case from formula (V.3.6).

2.1 Quasi-reversibility

Through this section $\boldsymbol{Q}$ is an intensity matrix of an irreducible and nonexplosive CTMC. We look for the probabilistic solution of the

(GBE) $\boldsymbol{\pi}\boldsymbol{Q} = \boldsymbol{0}$.

Under some mild conditions the solution is the stationary distribution of the chain. In Proposition 2.8 we had that if a probability function $\boldsymbol{\pi}$ fulfils the *detailed balance equation* (DBE), that is

(DBE) $\pi_i q_{ij} = \pi_j q_{ji}$, for all $i \neq j$,

then $\boldsymbol{\pi}$ is the stationary distribution. However we know that this simple system of equation holds only for the reversibility case. Therefore we sometimes need to use something else.

The starting point is the guess of a summable to 1 sequence of strictly positive numbers $\boldsymbol{\pi}$ and for this sequence we define

$$\tilde{q}_{ij} = \frac{\pi_j}{\pi_i} q_{ji}, \quad i,j \in \mathbb{E}. \tag{2.9}$$

Notice that if $\boldsymbol{\pi}$ is indeed the stationary distribution, then the matrix $\tilde{\boldsymbol{Q}} = (\tilde{q}_{ij})_{i,j=0,1\ldots}$ is the intensity matrix of the reversed CTMC. However we do not know yet that this is the stationary distribution.

Definition 2.12 We say that a chain is *quasi-reversible* if for each $i \in \mathbb{E}$, there exists a partition $\{\mathcal{A}^i_j,\ j \in \mathcal{I}_i\}$ of set $\mathbb{E} - \{i\}$ such that $\sum_{k\in\mathcal{A}^i_j} \tilde{q}_{ik} = \sum_{k\in\mathcal{A}^i_j} q_{ik}$ for all $j \in \mathcal{I}_i$.

In this case we will say that $\boldsymbol{Q}$ is $(\mathcal{A}^i_j)_{j\in\mathcal{I}_i}-$ quasi-reversible.

Proposition 2.13 *Let $\boldsymbol{Q}$ be $(\mathcal{A}^i_j)_{j\in\mathcal{I}_i}-$ quasi-reversible and suppose there exists a strictly positive sequence $\boldsymbol{\pi} = (\pi_i)_{i\mathrm{i}\mathbf{E}}$ summable up to 1 such that*

$$\sum_{k\in\mathcal{A}^i_j} \tilde{q}_{ik} = \sum_{k\in\mathcal{A}^i_j} q_{ik}, \quad \text{for all } j \in \mathcal{I}_i, \tag{2.10}$$

where

$$\tilde{q}_{ji} = \frac{\pi_i}{\pi_j} q_{ij}, \qquad i,j \in \mathbb{E}\ .$$

Then the sequence $\boldsymbol{\pi}$ is stationary the stationary distribution and $\tilde{\boldsymbol{Q}}$ is the intensity matrix of the reversed chain.

Proof We show that $\boldsymbol{\pi}$ fulfils the GBE

$$\pi_i \sum_{j\in\mathbf{E}-\{i\}} q_{ij} = \sum_{j\in\mathbf{E}-\{i\}} \pi_j q_{ji}, \qquad i \in \mathbb{E}\ .$$

The LHS we rewrite as

$$\pi_i \sum_{j\in\mathbf{E}-\{i\}} q_{ij} = \pi_i \sum_{l\in I_i} \sum_{k\in\mathcal{A}^i_l} q_{ik}\ .$$

Now in view of (2.10)

$$\pi_i \sum_{l\in I} \sum_{k\in\mathcal{A}^i_l} q_{ik} = \pi_i \sum_{l\in I} \sum_{k\in\mathcal{A}^i_l} \tilde{q}_{ik} = \sum_{l\in I} \sum_{k\in\mathcal{A}^i_l} \pi_k q_{ki} = \sum_{j\in\mathbf{E}-\{i\}} \pi_j q_{ji}, \qquad i \in \mathbb{E}\ .$$

□

We can rewrite (2.10) in the form: for all $i \in \mathbb{E}$

$$\sum_{k\in\mathcal{A}_j^i} \pi_j q_{ji} = \pi_i \sum_{k\in\mathcal{A}_j^i} q_{ik}, \qquad j \in \mathcal{I}_i \ . \tag{2.11}$$

This explains why (2.10) or equivalently (2.11) is called *partial balance equation* (PBE or PBE($(\mathcal{A}_j^i)_{i,j}$) The message from Proposition 2.13 is that if PBE is true, then GBE holds, and thus we have ergodicity.

Example 2.14 We check PBE for the following families of partitions $(\mathcal{A}_j^i)$.
(i) We get DBE if $\mathcal{A}_j^i = \{j\}$, $j \in \mathcal{I}_i = \mathbb{E} - \{i\}$. Indeed PBE is then

$$\pi_i q_{ij} = \pi_j q_{ji}, \qquad i \neq j \ .$$

(ii) We get GBE if $\mathcal{A}_j^i = \mathbb{E} - \{i\}$. Indeed then

$$\pi_i \sum_{j\in\mathbf{E}-\{i\}} q_{ij} = \sum_{j\in\mathbf{E}-\{i\}} \pi_j q_{ji}, \qquad i \in \mathbb{E} \ .$$

We will end up with few small general remark on CTMCs, which might be useful later on. It is known that one of equivalent definitions of a Markov process is that *past* and *future* are conditionally independent on the present state. To make this statement more precise, for a CTMC $(X(s))_{s\in\mathbf{R}}$ and an instant $t \in \mathbb{R}$, denote $\mathcal{F}_{(-\infty,t)} = \sigma\{X(s), s \in (-\infty,t)\}$ and $\mathcal{F}_{(t,\infty)} = \sigma\{X(s), s \in (t,\infty)\}$. Then we have (see [4])

Lemma 2.15 *The following two sentences are equivalent.*
(i) *Process X is a CTMC.*
(ii) *For each t and $A_1 \in \mathcal{F}_{(-\infty,t)}$, $A_2 \in \mathcal{F}_{(t,\infty)}$*

$$\mathbb{P}(A_1 \cap A_2 | X(t)) = \mathbb{P}(A_1|X(t))\mathbb{P}(A_2|X(t)), \qquad \mathbb{P} - \text{a.s.} \ .$$

Lemma 2.16 *Suppose that for a CTMC X, $t \in \mathbb{R}$, $A_1 \in \mathcal{F}_{(-\infty,t)}$ and $A_2 \in \mathcal{F}_{(t,\infty)}$, we have $\mathbb{P}(A_i|X(t)) = \mathbb{P}(A_i)$, $\mathbb{P} - \text{a.s.}$. Then A_1, A_2 and $X(t)$ are independent.*

[4]Dac reference

Proof

$$\begin{aligned}
\mathbb{P}(A_1 \cap \{X(t) \in B\} \cap A_2) &= \int_{\{X(t)\in B\}} \mathbb{P}(A_1 \cap A_2 | X(t))\, d\mathbb{P} \\
&= \int_{\{X(t)\in B\}} \mathbb{P}(A_1 | X(t)) \mathbb{P}(A_2 | X(t))\, d\mathbb{P} \\
&= \int_{\{X(t)\in B\}} \mathbb{P}(A_1) \mathbb{P}(A_2)\, d\mathbb{P} \\
&= \mathbb{P}(X(t) \in B) \mathbb{P}(A_1) \mathbb{P}(A_2).
\end{aligned}$$

□

Comments. Anderson (1991) Asmussen (2003), Bremaud (1999) Billingsley (1985) Kelly (1979) Robert (2003)

Chao, Miyazawa and Pinedo, Serfozo

3 Exercises

3.1 Let $p^n_{ij} = \mathbb{P}_i(X_n = j)$ and $\boldsymbol{P}^{(n)} = (p^{(n)}_{ij})_{i,j=0,1,\ldots}$. Show that

$$\boldsymbol{P}^{(n+m)} = \boldsymbol{P}^n \boldsymbol{P}^m$$

and hence

$$\boldsymbol{P}^{(n)} = \boldsymbol{P}^n$$

for all $n = 1, 2, \ldots$.

3.2 Show that an irreducible DTMC with finite state space is positive recurrent.

3.3 Show that the entry g_{ij} in the potential matrix $\boldsymbol{G}$ is the expected number of visits to state j, given that the chain starts from state i.

3.4 Show that state i is transient if and only if

$$\sum_{n\geq 1} 1(X_n = i) < \infty, \qquad \mathbb{P}_i - \text{a.s.}$$

3.5 Show that, for 1-D random walk on $\mathbb{Z}$ with transition probability matrix

$$p_{i,i+1} = p, p_{i,i-1} = 1 - p$$

for all $i \in \mathbb{Z}$ is transient if $p \neq 1/2$, null recurrent for $p = 1/2$. Such the random walk is said sometimes a *Bernoulli random walk*.

3.6 Show that transition matrix in a Bernoulli random walk is *double stochastic*, that is $\sum_i p_{ij} = \sum_j p_{ij} = 1$. Furthermore show that $\nu_i = 1$ and $\nu_i = p^n/(1-p)^n$ are invariant. (Asmussen p. 15). Notice that in the transient case an invariant measure are also possible, but they are not unique.

3.7 Show that random walk reflected at 0 with transition probability matrix

$$\begin{aligned} p_{i,i+1} &= p, \quad i \geq 0, \\ p_{i,i-1} &= 1-p, \quad i > 0, \\ p_{0,1} &= 1 \end{aligned}$$

is irreducible, positive recurrent if and only if $0 < p < 1/2$.

3.8 Consider a DTMC with transition probability matrix

$$\begin{aligned} p_{i,i+1} &= p_i, \quad i \geq 0, \\ p_{i,i-1} &= 1 - p_i, \quad i > 0, \\ p_{0,1} &= 1 \end{aligned}$$

Show that the chain is irreducible and positive recurrent if and only if $0 < p_i < 1$ and

$$\sum_{i \geq 1} \frac{p_0 \cdots p_{i-1}}{q_0 \cdots q_{i-1}}$$

where $q_i = 1 - p_i$.

3.9 Consider a transition probability matrix of form

$$\boldsymbol{P}^+ = \begin{pmatrix} 1-b_0 & b_0 & 0 & 0 & \cdot \\ 1-b_0-b_1 & b_1 & b_0 & 0 & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-\sum_{i=0}^{j} b_j & b_j & b_{j-1} & b_{j-2} & \cdot \\ \vdots & \vdots & \vdots & \vdots \& \vdots & \end{pmatrix}$$

where $b_i \geq 0$ and $\sum_{j=0}^{\infty} b_j = 1$. Show that, if $\lambda \sum_{j=1}^{\infty} j b_j < 1$, then the chain is positive ergodic and with the stationary distribution $\pi_n = (1-\delta)\delta^i$, $i = 01, 2, \ldots$ and δ is the positive solution $\hat{g}(x) = x$, where $\hat{g}(x) = \sum_{j=0}^{\infty} b_j x^j$ is the generating function of $\{b_j\}$.

3.10 Consider the random walk $(Y_n)_{n \in \mathbb{Z}_+}$ on $\mathbb{Z}^2$, where $Y_0 = (0,0)$, $Y_n = \sum_{j=1}^{n} \xi_j$ and $(\xi_j)_{j \in \mathbb{Z}_+}$ are i.i.d.

3.11 Show an example of an explosive CTMC.

3.12 For a CTMC $X(t)$ defined by $(\boldsymbol{Q}, \mu)$ let

$$Z = \sum_{n \geq 0} \frac{1}{q_{Y_n}}$$

Show that $Z = \infty$-$\mathbb{P}_\mu$ if and only if the chain is regular.

3.13 *Competing risks.* Let $E_{ij} \sim \text{Exp}(q_{ij})$ for $j \neq i$ and

$$\begin{aligned} E_i &= \min_{j\neq i} E_{ij} \\ I_i &= \arg\min_{j\neq i} E_{ij} = \sum_{j\neq i} j1(E_{ij} = E_i) \end{aligned}$$

Show that

$$\mathbb{P}_i(E_i \in dt, I_i = j) = q_{ij}e^{-q_i t}\,dt, \qquad t \geq 0\,.$$

3.14 Show that the following all four cases are possible: $X(t)$ is recurrent (positive recurrent) and Y_n is recurrent (positive recurrent).

3.15 Show that a B&D process on $\boldsymbol{x}_+$ is irreducible iff $\lambda_0, \lambda_1, \ldots > 0$ and $\mu_1, \mu_2, \ldots > 0$.

3.16 (i) Consider a queueing B&D process with $\lambda_n = \lambda$ and $\mu_n = n\mu$. (ii) Show that the process is ergodic for any $\rho = \lambda/\mu > 0$ with the stationary distribution

$$\pi_n = \frac{\rho^n}{n!}\exp(-\rho).$$

(This is so called the $M/M/\infty$ service system).
Show that for the B&D process with

$$\lambda_{n+1} = \frac{\lambda}{n+1}, \quad \mu_n = \mu$$

the stationary distribution is

$$\pi_n = \frac{\eta^n}{n!}\exp(-\rho),$$

where $\rho = \lambda/\mu > 0$.

3.17 Consider a CTMC $\{X(t),\ 0 \leq t \leq T\}$ with transition probability function $(p_{ij}(t))$ and let $X^{\leftarrow} = X(T-t)$, $0 \leq t \leq T$. Show that $X^{\leftarrow}(t)_{0\leq t\leq T}$ is a nonhomogeneous CTMC with t.p.f.

$$\begin{aligned} p^{\leftarrow}_{ij}(s,t) &= \Pr(X^{\leftarrow}(t) = j | X^{\leftarrow}(s) = i) \\ &= \frac{\Pr(X(T-t) = j)}{\Pr(X(T-s) = i)} p_{ji}(t-s)\,. \end{aligned}$$

3.18 If $(X(t))$ is stationary, then all processes $(X^*_T(t))_{t\in\mathbf{R}}$ have the same distribution.

3.19 Show the procedure in the spirit of the definition of minimal CTMC how to generate a doubly ended stationary CTMC $(X(t))_{t\in\mathbf{R}}$.

3.20 Let $\boldsymbol{Q} = (q_{ij})_{i,j\in\mathbf{E}}$ be an intensity matrix of a reversible process and $\boldsymbol{\pi}$ its stationary distribution. Let $\mathbb{A} \subset \mathbb{E}$ and define a new intensity matrix $\tilde{\boldsymbol{Q}} = (\tilde{q}_{ij})_{i,j\in A}$ by $\tilde{q}_{ij} = q_{ij}$ for $i \neq j$. Show that if $\tilde{\boldsymbol{Q}}$ is irreducible, then it defines a reversible intensity matrix, which admits the stationary distribution $\tilde{\boldsymbol{\pi}}$

$$\tilde{\pi}_i = \frac{\pi_i}{\sum_{j\in A} \pi_j} .$$

3.21 Let $\alpha_i > 0$ $(i = 1, \ldots, m)$. Demonstrate that $\boldsymbol{Q}$ defined by

$$q_{ij} = \begin{cases} \alpha_j & , i \neq j \\ -\sum_{\nu\neq j} \alpha_\nu & , i = j \end{cases} .$$

is reversible and find the stationary distribution $\boldsymbol{\pi}$. Show an example that although the original intensity matrix $\boldsymbol{Q}$ is irreducible, the new $\tilde{\boldsymbol{Q}}$ is not.

3.22 Using Burke theorem argue that for m B&D queues in tandem $(\lambda, (\mu^k_n)_{n\geq 1}(k = 1, \ldots, m)$ the stationary distribution of $\boldsymbol{Q} = (Q_1, \ldots, Q_m)$ ($Q_i(t)$ is the number in the system at time t), has the product form solution $\pi_{\boldsymbol{n}} = \prod_{k=1}^m \pi^{(k)}_{n_k}$, where $\boldsymbol{\pi}^{(k)}$ is the stationary solution for singe B&D queue $(\lambda, (\mu_n)_{n\geq 1})$.

Bibliography

[1] Asmussen, S. (2003) *Applied Probability and Queues*. Springer, New York.

[2] Bremaud, P. (1999) *Markov Chains; Gibbs Fields, Monte Carlo Simulation, and Queues*. Springer, New York.

Chapter III

An outline of the theory of point processes

1 Poisson process

We begin with a basic notion in these notes.

Definition 1.1 A B&D process $(\Pi(t))_{t\geq 0}$ on $\mathbb{Z}_+$ is said to be *Poisson process* with intensity λ if $\lambda_n = \lambda$ and $\mu_n = 0$.

$$p_{ij}(t) = \begin{cases} \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t}, & j \geq i \\ 0, & \text{otherwise} \end{cases}$$

Unless it is said otherwise Poisson process Π is considered under probability $\mathbb{P}_0$.

Let $\eta_1, \ldots,$ be a sequence of i.i.d. random variables with common exponential distribution $\text{Exp}(\lambda)$. An equivalent definition of the Poisson process is given in the following theorem.

Theorem 1.2 *The following are equivalent sentences:*
(i) $\Pi(t)$ *is a Poisson process with intensity* λ*, starting at* $t = 0$ *from 0.*
(ii) $\Pi(t) = \#\{m > 0 : \eta_1 + \cdots + \eta_m \leq t\}$,
(ii) *Process* $\Pi(t)$ *has independent increments (that is increments over disjoint intervals are independent), starting from 0 and and*

$$\mathbb{P}_0(\Pi(t) - \Pi(s) = k) = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)} .$$

Consider a cadlag stochastic process $(N(t))_{t\geq 0}$ with values at $\mathbb{Z}$. With probability 1, jumps epochs define a denumerable subset of points, not having accumulation. It can be shown that for each Borel subset $B \subset \mathbb{R}_+$, if $N(B)$ is the number of points in B, then $N(B)_{B\in\mathcal{B}(\mathbf{R}_+)}$ is a stochastic process. Furthermore, with probability 1, realisations are locally finite point measures. Such the processes are called *point processes* (p.p.s). Note that instead of a p.p. with points in $\mathbb{R}_+$ we may define a p.p. process on other spaces like $\mathbb{R}$, $(-\infty, t)$, (t, ∞). Therefore it seems to be useful to have a general definition of a Poisson process on a space E. We assume that $E \in \mathcal{B}(\mathbb{R}^d)$.

Definition 1.3 It is said that a p.p. $(\Pi(B), B \in \mathcal{B}(E))$ is a Poisson process with intensity λ is

1. for disjoint sets $B_1, \ldots, B_n \in \mathcal{B}(E)$ random variables $\Pi(B_1), \ldots, \Pi_n(B_n)$ are independent,
2. $\Pi(B)$ is Poisson distributed with parameter $\lambda|B|$.

Suppose now that $\Pi(t)$ is a Poisson process with intensity λ and $(Z_n)_{n\geq}$ a random walk on $\mathbb{Z}$. The process

$$X(t) = \sum_{j=1}^{\Pi(t)} Z_n$$

is said to be a *compound Poisson process.* and denote it by CP(λ,$(p_n)_{n\in\mathbb{Z}}$), where $p_n = \mathbb{P}(Z_1 = n)$.

Let $\lambda_{\boldsymbol{i}\in\mathbb{Z}^d} \geq 0$ and

$$\lambda = \sum_{\boldsymbol{i}\in\mathbb{Z}^d} \lambda_{\boldsymbol{i}\in\mathbb{Z}^d} < \infty \ .$$

Furthermore let $(\Pi_{\lambda_{\boldsymbol{i}}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ be a family of independent Poisson processes with intensity $\lambda_{\boldsymbol{i}}$ respectively. We define a *marked Poisson process* $\Pi(\cdot \times \cdot)$ on $\mathbb{R}_+ \times \{1, 2, \ldots\}$ with intensity $\lambda \times \nu$, where $\nu = \sum_{\boldsymbol{i}} \lambda_{\boldsymbol{i}}/\lambda$ by

$$\Pi(A \times B) = \sum_{i\in B} \Pi_{\lambda_{\boldsymbol{i}}}(A) \ .$$

Definition 1.4 It is said that a CTMC X is a *continuous time Bernoulli random walk* or simply Bernoulli random walk if X is a B&D process on $\mathbb{Z}$ with $\lambda_n = \lambda$ and $\mu_n = \mu$.

We can prove the following important representation for Bernoulli random walks.

Proposition 1.5 *The following sentences are equivalent.*
(i) *X is a continuous time Bernoulli random walk.*
(ii) *X is a compound Poisson process $CP(\lambda+\mu, (p_1 = \frac{\lambda}{\lambda+\mu}, p_{-1} = \frac{\mu}{\lambda+\mu}))$.*
(iii) *$X = \Pi^\lambda - \Pi^m u$, where Π^λ and Π^μ are independent Poisson processes with intensities λ and μ respectively.*

The proof is left as an exercise.

Let $E_\lambda(dx) = \lambda e^{-\lambda x}\, dx$ be the exponential distribution with parameter λ. Let $\Omega = \mathbb{R}_+ \times \mathbb{R}_+ \times \cdots$, $\mathcal{F} = \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \cdots$, $\mathbb{P}^\lambda = E_\lambda \otimes E_\lambda \otimes \cdots$ define the basic probability space. Let for $(x_1, x_2, \ldots) \in \Omega$

$$\Pi(t) = \Pi(t;\omega) = \#\{m > 0 : x_1 + \cdots + x_m \le t\} .$$

Remark that $(\Pi(t))_{t\ge 0}$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P}^\lambda)$. Let $\mathcal{F}_t^\Pi$ be the σ-field generated by process Π up to time t and $\mathbb{P}^{\lambda|t}$ is the restriction of $\mathbb{P}^\lambda$ to $\mathcal{F}_t^\Pi$. Suppose now that we have two measures $\mathbb{P}^{\lambda_1}$ and $\mathbb{P}^{\lambda_2}$ on $(\Omega, \mathcal{F})$ and let $\mathbb{P}^{\lambda_1|t}$ and $\mathbb{P}^{\lambda_2|t}$ be their restrictions to $\mathcal{F}_t^\Pi$ respectively. The corresponding expectation operators are denoted by $\mathbb{E}^{\lambda_1|t}$ and $\mathbb{E}^{\lambda_2|t}$ respectively. In the following proposition we show the form of the Radon-Nikodym derivative or the *likelihood ratio process*

$$M(t) = M(t,\omega) = \frac{d\mathbb{P}^{\lambda_1|t}}{d\mathbb{P}^{\lambda_2|t}}(\omega) ,$$

which is a stochastic process adapted to $\mathcal{F}_t^\Pi$ such that for all $A \in \mathcal{F}_t^\Pi$,

$$\mathbb{P}^{\lambda_1|t}(A) = \mathbb{E}_{\lambda_2|t}[M(t); A] . \tag{1.1}$$

Since $A \in \mathcal{F}_t^\Pi$ we may write

$$\mathbb{P}^{\lambda_1}(A) = \mathbb{E}^{\lambda_2}[M(t); A] .$$

Since for $h \ge 0$ and $A \in \mathcal{F}_t^\Pi$

$$\mathbb{P}^{\lambda_1}(A) = \mathbb{E}_{\lambda_2}[M(t+h); A]$$

we have that $(M(t))_{t\ge 0}$ is a martingale.

Exer. Show it.

Our aim is to prove the following fact; see for example the monograph of Bremaud [2], page 165.

Proposition 1.6 *Two distributions* $\mathbb{P}_{\lambda_1|t}$ *and* $\mathbb{P}_{\lambda_2|t}$ *with intensities* λ_1 *and* λ_2 *respectively are absolute continuous and the likelihood ratio process*

$$M(t) = \frac{d\mathbb{P}^{\lambda_1,t}}{d\mathbb{P}^{\lambda_2,t}} = \left(\frac{\lambda_1}{\lambda_2}\right)^{\Pi(t)} \exp(\lambda_2 - \lambda_1)t. \tag{1.2}$$

Proof Clearly M is an $\mathcal{F}_t^\Pi$-adapted process. To demonstrate (1.1) define for $0 \le t_1 < \ldots < t_k = t$, $n_1 \le n_2 \le \ldots \le n_k$ and

$$A = \{\Pi(t_1) = n_1, \ldots, \Pi(t_k) = n_k\} \in \mathcal{F}_t^\Pi .$$

Notice that a family of such sets generates $\mathcal{F}_t^\Pi$. Then

$$\begin{aligned}
\mathbb{P}_{\lambda_1|t}(A) &= \frac{(\lambda_1 t_1)^{n_1}}{n_1!} e^{-\lambda_1 t_1} \ldots \frac{(\lambda_1(t_k - t_{k-1}))^{n_k - n_{k-1}}}{(n_k - n_{k-1})!} e^{-\lambda_1(t_k - t_{k-1})} \\
&= (\frac{\lambda_1}{\lambda_2})^{n_1} e^{(\lambda_2-\lambda_1)t_1} \frac{(\lambda_2 t_1)^{n_1}}{n_1!} e^{-\lambda_2 t_1} \ldots \\
&\qquad (\frac{\lambda_1}{\lambda_2})^{n_k - n_{k-1}} e^{(\lambda_2-\lambda_1)(t_k - t_{k-1})} \frac{(\lambda_2(t_k - t_{k-1}))^{n_1}}{(n_k - n_{k-1})!} e^{-\lambda_2(t_k - t_{k-1})} \\
&= \left(\frac{\lambda_1}{\lambda_2}\right)^{n_k} e^{(\lambda_2-\lambda_1)t} \mathbb{P}_{\lambda_2|t}(A)
\end{aligned}$$

A basic property of Poisson processes is that they have *independent and stationary increments*. These notions are defined first.

Definition 1.7 A real-valued stochastic process $(X(t))_{t\ge 0}$ is said to have (a) independent increments if for all $n = 1, 2, \ldots$ and $0 \le t_0 < t_1 < \ldots < t_n$, the random variables $X(t_0), X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent,
(b) stationary increments if for all $n = 1, 2, \ldots$, $0 \le t_0 < t_1 < \ldots < t_n$ and $h \ge 0$, the distribution of $(X(t_1+h) - X(t_0+h), \ldots, X(t_n+h) - X(t_{n-1}+h))$ does not depend on h.

An immediate consequence of the definition of a process X with stationary increments is that

$$\mathbb{E}\, X(t) = t \mathbb{E}\, X(1)$$

provided $\mathbb{E}\, |X(1)| < \infty$

Definition 1.8 For a p.p. N on $\mathbb{R}_+$ with stationary increments $\lambda = \mathbb{E}\, N(1)$ is called *intensity* of this p.p., provided that $\mathbb{E}\, N(1) < \infty$

Suppose Π is a Poisson process on $(0,\infty)$ with intensity λ with points at $0 \le \tau_1 < \tau_2 < \ldots$. Let $(Y_n)_{n\ge 1}$ be a sequence of i.i.d. $\mathbb{E}$-valued random variables with common probability function $(p_i)_{i\in\mathbf{E}}$. Define now a point process Ψ on $(0,\infty)\times\mathbb{E}$ by

$$\Psi(A\times B) = \sum_{n\ge 1} 1(\tau_n \in A, Y_n \in B)\ .$$

Process Ψ is called *marked Poisson process*$(\lambda,(p_i)_{i\in\mathbf{E}})$.

We have the following important result:

Theorem 1.9 (i) *Point processes* $\Psi(A\times\{j\})$ $(j\in\mathbb{E})$ *are independent.* (ii) *For each* $j\in\mathbb{E}$ *point process* $\Psi(A\times\{j\})$ *point process* $\Psi(A\times\{j\})$ *is Poisson with intensity* λp_j.

2 Basic notions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a basic probabilistic triple. In this section we deal only with the so called *simple point processes* (p.p.) on real line $\mathbb{R}$ or an interval $\mathcal{T} \subset \mathbb{R}$ (finite or not). It means that realizations are without multiple points. Thus a p.p. is a sequence of random variables $(T_n)_n$ such that

- $\tau_n \in \mathcal{T}$ for all n,
- $\tau_n < \tau_{n+1}$,
- sequence $(\tau_n)_n$ has no accumulation point.

Denote

$$N(B) = \sum_n 1(\tau_n \in B) .$$

In such the case we will say that a p.p. N is given or equivalently that $(\tau_n)_n$ is a p.p. N. Note that

$$N = \sum_n \delta_{\tau_n} .$$

Example 2.1 $0 < \tau_1 < \tau_2 < \ldots$ define a simple p.p. on $(0, \infty)$. In this case we write $N(t) = N((0,t]$. In view of our assumptions $(N(t))_{t\geq 0}$ is well defined for all $t \geq 0$, a nonnegative, cadlag and nondecreasing.

With a stochastic process $(X(t))_{t\geq 0}$, we relate a family of σ-fields $(\mathcal{F}_t)_{t\geq 0}$, called *filtration*; see Appendix. A stochastic process $(X(t))_{t\geq 0}$ is said to be $\mathcal{F}_t$*-adapted* if $X(t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.

Example 2.2 Let $\mathcal{F}_t^X = \sigma\{X(s), 0 \leq s \leq t\}$. Then $(\mathcal{F}_t^X)_{t\geq 0}$ is a filtration called *internal history* of X.

Lemma 2.3 *Suppose that Z is caglad and $\mathcal{F}_t$-adapted. Then $Z(t) = \lim_n Z^{(n)}(t)$, where*

$$Z^{(n)}(t) = \sum_{k\geq 0} Z(\frac{k}{2^n}) 1(k/2^n < t \leq (k+1)/2^n) \tag{2.3}$$

and $Z(\frac{k}{2^k})$ is $\mathcal{F}_{k/2^n}$–measurable.

Suppose now that N is:

P1 a p.p. on $(0,\infty)$,

P2 is adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$,

P3 $\mathbb{E}\, N(t) < \infty$ for all $t \geq 0$.

Further on we will tacitly assume that [P3] condition holds.

Definition 2.4 It is said that a p.p. N admits an $\mathcal{F}_t$-*stochastic intensity* $(\lambda(t))_{t\geq 0}$ if

C1 $(\lambda(t))_{t\geq 0}$ is nonnegative $\mathcal{F}_t$–adapted and caglad,

C2 locally integrable,[1] i.e. $\int_B \lambda(s)\, ds < \infty$ a.s. for all bounded Borel $B \subset (0,\infty)$,

fulfilling

$$\mathbb{E}\,[N(s,t]|\mathcal{F}_s] = \mathbb{E}\,[\int_s^t \lambda(v)\, dv|\mathcal{F}_s], \qquad 0 \leq s \leq t\ . \tag{2.4}$$

The process $\lambda(t)$ is called $\mathcal{F}_t$-*stochastic intensity.*

Proposition 2.5 *A p.p. N admits an $\mathcal{F}_t$-*stochastic intensity $(\lambda(t))_{t\geq 0}$ *fulfilling conditions C1-C2 if and only if*

$$M(t) = N(t) - \int_0^t \lambda(s)\, ds$$

is an $\mathcal{F}_t$-martingale.

Proof Suppose that N admits $\mathcal{F}_t$-*stochastic intensity* $(\lambda(t))_{t\geq 0}$. Clearly $M(t)$ is $\mathcal{F}_t$–adapted. Defining condition (2.4) means

$$\mathbb{E}\,[N(s,t] - \int_s^t \lambda(v)\, dv|\mathcal{F}_s] = 0, \qquad 0 \leq s \leq t\ .$$

Then for $s < t$

$$\begin{aligned}\mathbb{E}\,[M(t)|\mathcal{F}_s] &= \mathbb{E}\,[M(t) - M(s) + M(s)|\mathcal{F}_s] = M(s) + \mathbb{E}\,[M(t) - M(s)|\mathcal{F}_s] \\ &= M(s)\ .\end{aligned}$$

□

From definition 2.4 of $(\lambda(t))_{t\geq 0}$ we can easily conclude the following lemma.

[1] Do we need it?

Lemma 2.6 *If for $\mathcal{F}_s$–measurable random variable ζ, such that one of the conditions $\mathbb{E}\,[|\zeta|N(0,t]] < \infty$ or $\mathbb{E}\,[|\zeta|\int_0^t \lambda(v)\,dv] < \infty$ holds, then*

$$\mathbb{E}\,[\zeta N(s,t]] = \mathbb{E}\,[\zeta \int_s^t \lambda(v)\,dv], \qquad 0 \le s \le t\,. \tag{2.5}$$

Proof From definition 2.4 we see that(2.5) holds for $Z = 1(A)$, where $A \in \mathcal{F}_s$. Next for $Z \ge 0$ we define

$$\zeta^{(n)} = \sum_{i=0}^{\infty} \frac{i+1}{2^n} 1(\frac{i}{2^n} < \zeta \le \frac{i+1}{2^n}) + \sum_{i=-1}^{-\infty} \frac{i}{2^n} 1(\frac{i}{2^n} < \zeta \le \frac{i+1}{2^n}),$$

so $|\zeta^{(n)}| \le \zeta$ and $\lim_{n\to\infty} \zeta^{(n)} = \zeta$. Clearly we have

$$\mathbb{E}\,[\zeta^{(n)} N(s,t]] = \mathbb{E}\,[\zeta^{(n)} \int_s^t \lambda(v)\,dv], \qquad 0 \le s \le t\,.$$

The proof is completed by the use of the Lebesgue dominated-convergence theorem. □

Example 2.7 If $(\Pi(t))_{t\ge0}$ is a Poisson process with intensity l, then $\lambda(t) \equiv l$ is $\mathcal{F}_t^{\Pi}$-*stochastic intensity*. Namely for $s < t$

$$\begin{aligned} \mathbb{E}\,[\Pi(t)|\mathcal{F}_s^{\Pi}] &= \mathbb{E}\,[\Pi(t) - \Pi(s) + \Pi(s)|\mathcal{F}_s^{\Pi}] \\ &= \mathbb{E}\,[\Pi(t) - \Pi(s)|\mathcal{F}_s^{\Pi}] + \mathrm{E}[\Pi(s)|\mathcal{F}_s^{\Pi}] \\ &= \Pi(s) \end{aligned}$$

because $\Pi(t) - \Pi(s)$ is independent of $\mathcal{F}_s$ and $\Pi(s)$ is measurable with respect to $\mathcal{F}_s$.

In contrast to a general definition of stochastic integral the definition of the stochastic integral with respect to a simple p.p. is not complicated. Thus for a stochastic process X and a simple p.p. N we define *stochastic integral*

$$\int_0^t X(s)\,dN(ds) = \sum_{n\ge1} X(\tau_n)\;.$$

Remark that this definition is good only if N is a simple p.p..

The main result is the following proposition.

Proposition 2.8 *Suppose that p.p. N admits $\mathcal{F}_t^N$*-stochastic intensity $\lambda(t)$ *and $(X(t))_{t\geq 0}$ is $\mathcal{F}_t^N$–adapted and caglad. If*

$$\mathbb{E}\,[\int_0^t |X(s)|\lambda(s)\,ds] < \infty, \tag{2.6}$$

or

$$\mathbb{E}\,[\int_0^t |X(s)|\,dN(s)] < \infty, \tag{2.7}$$

then

$$\mathbb{E}\,[\int_0^t X(s)\,dN(s)] = \mathbb{E}\,[\int_0^t X(s)\lambda(s)\,ds]\;. \tag{2.8}$$

If moreover (2.6) holds for all $t \geq 0$, then

$$\int_0^t X(s)\,dN(s) - \int_0^t X(s)\lambda(s)\,ds \tag{2.9}$$

is an $\mathcal{F}_t^N$-martingale.

Proof 1°. Let $v+h \leq t$. Suppose that $X(v) = Z1(s < v \leq s+h)$, where Z is $\mathcal{F}_s$ measurable and

$$\mathbb{E}\,[|Z|\int_s^{s+h} \lambda(v)\,dv] < \infty\;.$$

Then

$$\begin{aligned}
\mathbb{E}\,[\int_0^t X(v)\,dN(v)] &= \mathbb{E}\,[ZN(v,v+h]] \\
&= \mathbb{E}\,[ZN(v,v+h]] = \mathbb{E}\,[Z\mathbb{E}\,[N((v,v+h])] \\
&= \mathbb{E}\,[Z\int_s^{s+h} \lambda(v)\,dv]\;.
\end{aligned}$$

2°. Assume that X is nonnegative and bounded, say by a constant K. By Lemma V.3.6

$$\mathbb{E}\,[\int_0^t X^{(n)}(s)\lambda(s)\,ds] = \sum_k \mathbb{E}\,[X(\frac{k}{2^k})\int_{k/2^n}^{(k+1)/2^n} \lambda(s)\,ds] < \infty$$

and so

$$\sum_k \mathbb{E}\,[X(\frac{k}{2^k})\int_{k/2^n}^{(k+1)/2^n} \lambda(s)\,ds] =$$

$$= \sum_k \mathbb{E}\,[X(\frac{k}{2^k})N(k/2^n,(k+1)/2^n]] = \mathbb{E}\,[\int_0^t X^{(n)}(s)\,dN(s)]\ .$$

Now by the Lebesgue dominated-convergence theorem the result follows.
3° Suppose that X is nonnegative fulfilling the conditions of the theorem. Define

$$X_K(t) = \min(X(t), K)\ .$$

Then $X_K(t)$ fulfills the conditions from part 2° and hence the (2.8) holds for such the function. The proof is completed by the use of the Lebesgue dominated-convergence theorem if we let $K \to \infty$.
4° Assume that X is a processes fulfilling the conditions of the theorem. Then define $X_+ = \max(X, 0)$ and $X_= - \min(X, 0)$ so $X = X_+ - X_-$. Processes X_+, X_- are nonnegative, caglad and $\mathcal{F}_t$-adapted, so we can use part 2°. □

2.1 Characterizations involving Poisson process

Consider first, for a motivation, the following characterization of CTMC X with state space $\mathbb{E}$ and intensity matrix $\boldsymbol{Q} = (q_{ij})_{i,j\in\mathbf{E}}$ as a solution of an SDE. Recall that $q_i = \sum_{j\in\mathbf{E}-\{i\}} q_{ij}$. As usual we suppose that X is cadlag and regular. We need the following ingredients for stating our representation:

- $(\Pi^i)_{i\in\mathbf{E}}$ – a family of independent Poisson processes; the i-Poisson process is with intenisty q_i,
- $(Y_n^i)_{n\geq 1, i\in\mathbf{E}}$ – a family of indepedent $\mathbb{E}$-valued random varrables; for each $i \in \mathbb{E}$ random variables $(Y_n^i)_{n\geq 1}$ are i.i.d. with common probability function $(p_{ij}^\circ)_{j\in\mathbf{E}}$,
- $X(0)$ – an $\mathbb{E}$-valued random variable,
- $X(0)$, $(\Pi^i)_{i\in\mathbf{E}}$, $(Y_n^i)_{n\geq 1}$ and $X(0)$ are independent.

Denote points of Π^i by $(\tau_n^i)_{n\geq 1}$ and define

$$\Psi^i(A\times B) = \sum_{n\geq 1} \delta_{(\tau_n^i, Y_n^i)}(A\times B)\ . \tag{2.10}$$

One can show that for each $j \in \mathbb{E}$, $\Psi^i((0,t] \times \{j\}$ is a Poisson process with intensity q_{ij}. Exer. Show it.

Proposition 2.9 *CTMC X is the cadlag solution of*

$$X(t) = X(0) + \int_0^t \sum_{i \in \mathbf{E}} \sum_{j \in \mathbf{E} - \{i\}} \mathit{1}(X(s - \circ) = i) j \, d\Psi^i(ds \times \{j\}) \; .$$

of Π.

Let

$$\mathcal{G}_t = \sigma\{\Psi^i((0,s] \times \{k\}), 0 \le s \le t, i, k \in \mathbb{E}\}$$

be another filtration. Note the following important fact:

- $(\Psi^i((s,t] \times B))_{t \ge s, B \subset \mathbf{E}}$ is independent of $\mathcal{G}_t$
- $\mathcal{F}_t^X \subset \mathcal{G}_t$ for all $t \ge 0$.

Suppose we have a left continuous and with right hand limits (caglad) process $(Z(t))_{t \ge 0}$ such that Z is $\mathcal{G}_t$-adapted.

The following result belongs to Bremaud. [2]

Theorem 2.10 *[Smoothing formula] If Z is caglad, $\mathcal{G}_t$–adapted process and $\mathbb{E} \int_0^\infty |Z(s)| \, d\Psi^i((0,s] \times \{j\}) < \infty$ or $\mathbb{E} \int_0^\infty |Z(s)| \, ds \infty$, then*

$$\mathbb{E} \int_0^\infty Z(s) \, d\Psi^i(s) = q_i p^\circ_{ij} \mathbb{E} \int_0^\infty Z(s) \, ds \; .$$

Proof 1° Assume first that Z process is nonnegative. Using $Z^{(n)}$ from Lemma 2.3 we write

$$\begin{aligned} \mathbb{E} \int_0^\infty Z^{(n)}(s) \, d\Pi(s) &= \sum_{k \ge 0} \mathbb{E} \, [\int_0^\infty Z(\frac{k}{2^n}) 1(k/2^n < s \le (k+1)/2^n) \, d\Pi(s) \\ &= \sum_{k \ge 0} \mathbb{E} \, [Z(\frac{k}{2^n}) \int_0^\infty 1(k/2^n < s \le (k+1)/2^n) \, d\Pi(s) \\ &= \sum_{k \ge 0} \mathbb{E} \, [Z(\frac{k}{2^n}) \Pi(k/2^n, (k+1)/2^n]] \, . \end{aligned}$$

[2] Dac odnosnik

Since $Z(\frac{k}{2^n}$ is independent of $\Pi(k/2^n,(k+1)/2^n]$ we have

$$\mathbb{E}\,[Z(\frac{k}{2^n})\Pi(k/2^n,(k+1)/2^n]] = q_i p^\circ_{ij}\frac{1}{2^n}\mathbb{E}\,Z(\frac{k}{2^n})\ .$$

Moreover

$$\lim_{n\to\infty}\sum_{k\geq 0}\frac{1}{2^n}Z(\frac{k}{2^n}) = \int_0^\infty Z(s)\,ds.$$

2° Suppose that Z fulfills the conditions of the theorem. The proof is completed by representing $Z = Z_+ - Z_-$. Clearly Z_+ and Z_- fulfill the condition of part 1°. □

For the next result recall that a Poisson process $(\Pi(t)_{t\geq 0}$ with intensity α is a process with independent, stationary increments and that increment $\Pi(s,t] = \Pi(t) - \Pi(s)$ has Poisson distribution with mean $\alpha(t-s)$.

Proposition 2.11 *A p.p.* Π *is a Poisson process with intensity* α *if and only if for each* $t,h\geq 0$ *and*

$$\mathbb{E}\,[e^{iuN(t,t+h)}|\mathcal{F}^\Pi_t] = \exp\{\alpha h(e^{iu}-1)\}\ . \tag{2.11}$$

or equivalently for each $t,h\geq 0$ *and* $A\in\mathcal{F}^\Pi_t$

$$\mathbb{E}\,[1(A)e^{iuN(t,t+h)}] = \mathbb{P}(A)\exp\{\alpha h(e^{iu}-1)\}\ .$$

Theorem 2.12 *[Watanabe theorem] Let* N *be a p.p. such that for some* $\alpha > 0$

$$M(t) = N(t) - \alpha t$$

is an $\mathcal{F}^N_t$*-martingale. Then* $N(t)$ *is a Poisson process with intensity* α*.*

Proof Proofs is divided into steps: Choose $t,h\geq 0$ and $A\in\mathcal{F}^\Pi_s$.
1°. Let (τ_j) are points of N. Write

$$\{\tau_j : a<\tau_j\leq b\} = \{\tau_1',\tau_2',\ldots,\tau_l\}$$

where

$$\tau_1' < \tau_2' < \ldots < \tau_l'\ .$$

Since

$$e^{iul} = 1 + \sum_{j=1}^{l}(e^{iu}-1)e^{iu(j-1)}$$

we have

$$\begin{aligned} e^{iuN(a,b]} &= 1 + \sum_{j:a<\tau_j\le b} (e^{iu}-1)e^{iuN(a,\tau_j)} \\ &= 1 + \int_{(a,b]} (e^{iu}-1)e^{iuN(a,s)}\, N(ds)\ . \end{aligned}$$

Hence for $A \in \mathcal{F}_s^N$ and applying the smoothing formula from Theorem 2.10 in the second equality we write

$$\begin{aligned} \mathbb{E}\,[1(A)e^{iuN(s,t]}] &= \mathbb{E}\,[1(A) + \int_{(s,t]} 1(A)(e^{iu}-1)e^{iuN(s,v)}\, N(dv)] \\ &= \mathbb{P}(A) + \alpha\mathbb{E}\,[\int_{(s,t]} (e^{iu}-1)e^{iuN(a,v)}\, dv] \\ &= \mathbb{P}(A) + \alpha\mathbb{E}\int_{(s,t]} 1(A)(e^{iu}-1)e^{iuN(s,v]}\, dv]\ . \quad (2.12) \end{aligned}$$

$$(2.13)$$

2°. Denote

$$x(t) = \mathbb{E}\,[1(A)e^{iuN(s,t]}], \qquad s \le t\ .$$

From (2.12) we have

$$x(t) = \mathbb{P}(A) + \alpha(e^{iu}-1)\int_{(s,t]} x(s)\, ds, \qquad s \le t\ .$$

The solution of this integral equation is $x(t) = \mathbb{P}(A)\exp\{\alpha(t-s)(e^{iu}-1)\}$, and the proof is completed. □

2.2 Streams induced by jumps of CTMC

Consider a regular, cadlag CTMC $(X(s))_{t\ge 0}$ with state space $\mathbb{E}$, intensity matrix $\boldsymbol{Q}$ and let $i \ne j$. By N^{ij} we denote a p.p. of jump instants of X from i to j. Formally it is

$$N^{ij}(t) = \sum_{0\le s\le t} 1(X(s-\circ) = i, X(s) = j)\ .$$

Let N be a p.p. generated by all jump instants of X, that is $N = \sum_{i\in\mathbf{E}}\sum_{i\in\mathbf{E}-\{j\}} N^{ij}$. We assume $\mathbb{E}\, N(t) < \infty$. Note that we choose filtration $\mathcal{F}_t^X$.

Remark Regularity is important because otherwise $N(t)$ is not defined for all $t \geq 0$. We also note that assumption $\mathbb{E}\, N(t) < \infty$ is not redundant; see ?????[3]

The following is a key lemma for proving the *Levy's formula* for CTMC.

Lemma 2.13 *For* $i \neq j$

$$N^{ij}(t) - \int_0^t 1(X(s) = i) q_{ij}\, ds$$

is an $\mathcal{F}_t^X$*–martingale.*

Proof We have to prove that

$$\mathbb{E}\,[N^{ij}(s,t]|\mathcal{F}_s^X] = \mathbb{E}\,[\int_s^t 1(X(v-\circ) = i) q_{ij}\, dv|\mathcal{F}_s^X]\ .$$

Recalling Proposition V.3.6 write

$$N^{ij}(s,t] = \int_s^t 1(X(v-\circ) = i)\Psi^i(dv \times \{j\})\ ,$$

where marked Poisson process Ψ^i was defined in (2.10). Since for fixed j, process $\Psi^i((0,t]\times\{j\})$ is Poisson with intensity q_{ij}, we have, by the *smoothing formula* (see Theorem 2.10)

$$\begin{aligned}
\mathbb{E}\,[N^{ij}(s,t]|\mathcal{F}_s^X] &= \mathbb{E}\,[\int_s^t 1(X(v-\circ) = i)\Psi(dv \times \{j\})|X(s)] \\
&= \mathbb{E}\,[\int_s^t 1(X(v-\circ) = i) q_{ij}|X(t)] \\
&= \mathbb{E}\,[\int_s^t 1(X(v-\circ) = i) q_{ij}\, dv|\mathcal{F}_t^X]\ .
\end{aligned}$$

□

The above result means that p.p. N^{ij} admits $\mathcal{F}_t^X$–stochastic intensity

$$\lambda(t) = 1(X(s-\circ) = i) q_{ij}\ .$$

The idea of the proof of the following celebrated formula of Levy stems from Pierre Bremaud books [2], page 5 and [], pa

[3] Do we know a counterexample?

Theorem 2.14 *[Levy's formula for CTMC] Let $f : \mathbb{E}\times\mathbb{E} \to \mathbb{R}$ be a function such that $f(i,i) = 0$ for $i \in \mathbb{E}$. If moreover for all $t \geq 0$*

$$\mathbb{E}\,[\int_0^t \sum_{j\in\mathbf{E}} |f(X(s),j)| q_{X(s),j}\, ds] < \infty, \tag{2.14}$$

then

$$\sum_{0<s\leq t} f(X(s-\circ), X(s)) - \int_0^t \sum_{j\in\mathbf{E}} f(X(s),j) q_{X(s),j}\, ds$$

is an $\mathcal{F}_t^X$–martingale.

Proof [Bremaud Elements p. 60, Bremaud, Point processes, p. 5.] Since X is a CTMC, it suffices to condition on $X(s)$ instead from $\mathcal{F}_s^X$. We start from an obvious identity

$$\begin{aligned} &\sum_{s<v\leq t} f(X(v-\circ), X(v)) \\ &= \sum_{j\in\mathbf{E}} \sum_{i\in\mathbf{E}-\{j\}} f(i,j)\, N^{ij}((s,t])\,. \end{aligned}$$

Now by Lemma 2.13

$$\mathbb{E}\,[f(i,j)\, N^{ij}((s,t])|X(s)] = \mathbb{E}\,[\int_{(s,t]} f(i,j) 1(X(v-\circ)=i) q_{ij}\, dv | X(s)]\,.$$

Hence

$$\begin{aligned} &\mathbb{E}\left[\sum_{s<v\leq t} f(X(s-\circ), X(s)) | X(s)\right] \\ &= \sum_{j\in\mathbf{E}} \sum_{i\in\mathbf{E}-\{j\}} \mathbb{E}\left[\int_{(s,t]} f(i,j) 1(X(v-\circ)=i) q_{ij}\, dv | X(s)\right] \\ &= \sum_{j\in\mathbf{E}} \sum_{i\in\mathbf{E}-\{j\}} \mathbb{E}\,[\int_{(s,t]} f(X(v-\circ,j) 1(X(v-\circ)=i) q_{X(v-\circ),j}\, dv | X(s)] \\ &= \sum_{j\in\mathbf{E}} \sum_{i\in\mathbf{E}-\{j\}} \mathbb{E}\left[\int_{(s,t]} f(X(v,j) 1(X(v)=i) q_{X(v),j}\, dv | X(s)\right]\,. \end{aligned}$$

□

Corollary 2.15 *Let for $j \in \mathbb{E}$, we denote by $N^j(t)$ the number of entrances of X into j during interval $(0,t]$. If*

$$\sum_{i \in \mathbf{E}-\{j\}} \int_0^t q_{ij} \mathbb{P}(X(s) = i)\, ds < \infty,$$

then

$$N^j(t) - \int_0^t q_{ij} 1(X(s) = i)\, ds$$

is an $\mathcal{F}_t^X$-martingale.

3 Exercises

3.1 Prove that a compound Poisson process is a regular CTMC and find its intensity matrix.

3.2 Suppose that an intensity matrix $\boldsymbol{Q}$ is given such that $\sup_i q_i < \infty$ Show that in this case CTMC $X(t)$ defined by $(\boldsymbol{Q}, i)$ can be represented as

$$X(t) = \sum_{n=0}^{\Pi(t)} Z_n$$

where $\Pi(t)$ is a Poisson process with intensity $\lambda \geq \sup_j q_j$, and Z_n is a DTMC. Find its transition probability matrix. (This procedure is known as *uniformization*)

3.3 Suppose that Π is a Poisson process on $\mathbb{R}$ with intensity λ and points $0 < \tau_1 < \tau_2 < \ldots$ and $(Z_n)_{n\geq 1}$ a sequence of 0-1 i.i.d. random variables with $\mathbb{P}(Z_1 = 1) = p$. Define now a new p.p. $N(t) = \sum_{n\geq 0} Z_n 1(\tau_n \leq t)$. Show that N is a Poisson process with intensity $p\lambda$. Conclude now part (ii) of Theorem 1.9.

3.4 Let $\alpha(t)$ be a locally integrable function. We say that Π is a Poisson process on $(0, \infty)$ if $\Pi(0, t]$ is a process with independent increments and

$$\mathbb{P}(\Pi(a, b] = k) = \frac{(\int_a^b \alpha(u)\, du)^k}{k!} e^{-\int_a^b \alpha(u)\, du}.$$

Show that $\alpha(t)$ is $\mathcal{F}^{\Pi}$-intensity function.

3.5 Let $X(t)$ be a B&D process (λ_n, μ_n) on $\mathbb{Z}_+$ such that $\mathbb{E}_{\boldsymbol{\nu}} X(0) < \infty$, where $\boldsymbol{\nu}$ is an initial distribution and define

$$A(t) = \sum_{0<s\leq t} 1(X(s) = X(s-\circ)+1), \qquad D(t) = \sum_{0<s\leq t} 1(X(s) = X(s-\circ)-1)$$

Consider now $M_1(t) = A(t) - \int_0^t \lambda_{X(s)}\, ds$ and $M_y 2(t) = D(t) - \int_0^t 1(X(s) > 0)\mu_{X(s)}\, ds$. Show that, if

$$\sum_{n=0}^{\infty} \lambda_n \int_0^t \mathbb{P}_{\boldsymbol{\nu}}(X(s) = n)\, ds < \infty$$

then M_1 and M_2 are $\mathcal{F}_t^X$–martingales. Show that the above condition is automatically fulfilled for B&D queues.

3.6 Let N be a p.p., which admits $\mathcal{F}^N$-stochastic intensity $\lambda(t)$. Show that if X is left continuous and with right hand limits, adapted $\mathcal{F}_t^N$, and

$$M(t) = N(t) - \int_0^t \lambda(s)\,ds,$$

then $\int_0^t X(s)\,dM(t)$ is a martingale. Show that the result does not hold for some cadlag process X.

3.7 Show that in Gordon-Newel networks, the intensity of the flow of jobs transferred from i to j is

$$\lambda(t) = 1(Q_i(t) = i)\mu_i p_{ij}\ .$$

Conclude that the throughput is

$$d_{ij} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \lambda(s)\,ds - \text{a.s.}\ .$$

3.8 Let $(\tau_i)_{i\geq 1}$ be a renewal (point) process on $(0,\infty)$, that is such that $(\tau_{i+1} - \tau_i)_{i\geq 0}$ is a sequence of i.i.d. random variables with a common distribution function $F(t)$; $\tau_0 = 0$. Suppose that F has the density function $f(t)$ and let $r(t) = f(t)/(1-F(t))$ be the corresponding hazard rate function. Let N be the corresponding point process. Show that the $\mathcal{F}_t^N$–stochastic intensity function is

$$\lambda(t) = \sum_{j\geq 0} r(t-\tau_j)1(\tau_j \leq t < \tau_{j+1})\ .$$

Bibliography

[1] Baccelli, F. and Bremaud, P. (1994) *Elements of Queueing Theory; Palm-Martingale Calculus and Stochastic Recurrences*. Springer, Berlin.

[2] Bremaud, P. (1981) *Point Processes and Queues*. Springer.

Chapter IV

Steady state analysis of Markovian queues

In this chapter we present Markovian queues. We consider the number of jobs process, which in this setting is a regular CTMC. In classical theory of queues this process was said to be *queue length process*. In each case the regularity of the studied process will be obvious and not be discussed. The point is that in the considered systems the number of jobs is less or equal than the number of arriving jobs, which in this and the next chapters is a Poisson process. To define a queueing system one has to specify the following elements:

- arrival process,
- service requirements,
- number of servers,
- queueing discipline,
- buffer size.

We will denote the arrival process by A, the service requirements by S_j, number of servers C, buffer size (together with serviced jobs) by N. Various queueing disciplines are studied. Among others

- *First Come First Served* (FCFS),
- *First Come Last Served* (FCLS),

- *Service in Random Order* (SIRO),
- *Processor Sharing* (PS).

All these disciplines belong to the class of *work conserving* disciplines. For example we show in details now the so called M/M/1/∞ (FCFS) queueing system. Arrivals are according to a Poisson process $\Pi(t)$ with rate λ, or in other words, inter-arrival times are i.i.d. with the common exponential distribution with parameter λ. Each job brings its service requirement S_i and we suppose that $\Pi(t)$ and (S_i) are independent and (S_i) are i.i.d. with the common exponential distribution function with parameter μ. Jobs enter the server in order of their arrivals and we assume that server cannot be idle if there is at least one job in the buffer. Jobs are served according to their arrivals. Since there is only one server, then the FCFS discipline is equivalent to *First In First Out* (FIFO) discipline.

We begin with the basic Markovian single server queue for which we study the number in the system process $Q(t)$. We consider only *work conserving* disciplines. We now list the basic notations:

- $A(s,t]$ the number of arrivals to the system in time interval $(s,t]$,
- $D(s,t]$ be the number of departures from the system in time interval $(s,t]$,
- $\mathcal{F}_t = \sigma\{A(s), (0 \le s \le t), (S_i)_{i=1}^{A(t)}\}$ be the history filtration up to time t. Notice that the knowledge of $\mathcal{F}_t$ yields the knowledge of the evolution of $Q(s)$ $(0 \le s \le t)$.

1 Queueing B&D processes

Consider now a queueing B&D X process with $\lambda_n = \lambda$ and μ_n. We suppose that all $\mu_n > 0$ and then the state space be $\mathbb{E} = \mathbb{Z}_+$. Let $(\Pi_m(t))_{m\ge 0}$ be a family independent Poisson processes with intensity μ_n, where $\mu_0 = \lambda$ independent of $Q(0)$. Let Q solves the following stochastic differential equation (SDE)

$$Q(t) = k + \Pi_0(t) - \int_0^t \sum_{n\ge 1} 1(Q(t-\circ) = n)\, \Pi_n(ds)\ . \tag{1.1}$$

One can show that $Q(t)$ is a unique solution such that $Q(0) = k$ and it is a representation of the considered queueing B&D process. From this representation it is also apparent that the process is regular because $Q(t) \leq Q(0) + \Pi_0(t)$. The concepts of stochastic integration with respect to a p.p. will be made precise in Appendix V.3.6.

In this chapter we will be interested in ergodic properties of the process X but also we will also study streams generated by this process. For the queueing B&D process we have two streams: arrival stream, that is upward-jump instants of the process X and departure stream, which is formed by downward-jumps of X. We will denote them by A and D respectively. Streams formed by all jumps of x are denoted by $N^{\mathrm{c}} = A \cup D$.

We are going now to state and prove the following important result for a queueing B&D processes. Let $(X(t))_{t\in\mathbf{R}}$ be a double-ended version of a queueing B&D processes. Let A (D) be the point process of (arrivals) departures. We denote by $N_{|I}$ the point process N restricted to set I.

Theorem 1.1 (i) *Departure instants form a Poisson p.p. D on $\mathbb{R}$ with rate λ.*
(ii) *For each t the p.p. of arrival instants $A_{|(t,\infty)}$ is independent of $X(t)$.*
rm (iii) For each t the p.p. of departure instants $D_{|(-\infty,t)}$ is independent of $X(t)$.

Using now Lemma II.2.16 we can state the following corollary.

Corollary 1.2 *X is reversible if and only if for each $t \in \mathbb{R}$ we have $D_{|(-\infty,t)}$,$X(t)$,$A_{|(t,\infty)}$ independent.*

Consider now a queueing B&D process with birth rates $\lambda_n = \lambda$ and death rates μ_n. We suppose that all $\mu_n > 0$ and then the state space be $\mathbb{E} = \mathbb{Z}_+$. Let $\Pi_\lambda, \Pi_{\mu_i}, i = 1, 2, \ldots$ be a family of independent Poisson processes with intensities $\lambda, \mu_1, \mu_2, \ldots$ respectively. Let $Q(t)$ be a cadlag process defined by

$$Q(t) = k + \Pi_\lambda(t) - \sum_{n\geq 1} \int_0^t 1(Q(s-\circ) = n)\, \Pi_{\mu_n}(ds) \tag{1.2}$$

We first show that there exists a unique cadlag jump process $X(t)$, which solves (1.2).

From this representation it is apparent that the process is regular ($X(t) \leq X(0) + \Pi_a(t)$). Exercise: Show it.

Proposition 1.3 *Upward-jump instants form a Poisson process with rate* λ.

We are going now to state and prove the following important result for a queueing birth and death processes. Upward-jumps are said to ba arrivals and downward jumps are said to be departure instants. Let $(X(t))_{t\in\mathbf{R}}$ be a double-ended version. Let A (D) be the point process of (arrivals) departures. We denote by $N_{|I}$ the point process N restricted to set t.

Theorem 1.4 [Burke theorem] (i)*Departure instants form a Poisson p.p.* D *with rate* λ.
(ii) *For each* t *the p.p. of arrival instants* $A_{|(t,\infty)}$ *is independent of* $X(t)$.
rm (iii) For each t *the p.p. of departure instants* $D_{|(-\infty,t)}$ *is independent of* $X(t)$.

Proof

1.1 M/M/1/N queue

We now define M/M/1/N system with finite buffer of size $N-1$. It is a single server queue with inter-arrival times T_i's exponentially distributed with parameter $\lambda > 0$ and exponentially distributed service requirement S_i with

$$B(x) = 1 - e^{-\mu x}, \qquad x \geq 0 \ .$$

There is one server and we assume that if the server is empty, and there is at least one job in the queue, then one of jobs present immediately enters for the service. From example without loss of generality we may assume FCFS discipline. Furthermore, if in the system there is N jobs (together with the one served), then incoming jobs are lost. Let $Q(t)$ denote the number of jobs at time t.

Theorem 1.5 *The process* $(Q(t)_{t\geq 0}$ *is a B & D process with* $\mathbb{E} = \{0, 1, \ldots, N\}$ *with* $N+1 \times N+1$ *intensity matrix*

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdot \\ \mu & -\lambda-\mu & \lambda & 0 & \cdot \\ 0 & \mu & -\lambda-\mu & \lambda & \cdot \\ 0 & 0 & \mu & -\lambda-\mu & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mu & -\mu \end{pmatrix} \tag{1.1}$$

Proof Suppose there is k jobs present. If $k = 0$, then the jump up, which is the nearest arrival happens after exponential time λ. Thus $q_{01} = \lambda$. If $0 < k < N$, then the jump up, which is the nearest arrival happens after exponential time with parameter λ. On the other hand the nearest jump down, which is the completion of the service is after exponential time with parameter μ. Thus $q_{k,k+1} = \lambda$ and $q_{k,k-1} = \mu$. Similarly we have to analyse case $k = N$. □

The intensity matrix $\boldsymbol{Q}$ is irreducible and of course ergodic (state space is finite). We denote $\rho = \lambda/\mu$.

Proposition 1.6 *The stationary distribution* $\boldsymbol{\pi}$ *is*

$$\pi_k = \frac{\rho^n}{K(N)} \tag{1.2}$$

where $\rho = \lambda/\mu$ *and*

$$K(N) = \frac{1 - \rho^{N+1}}{1 - \rho} \ .$$

Remark Suppose there is N jobs which move between two servers s_1 and s_2 in a loop as follows. Before each server there is (unlimited) buffer. Server at s_1 serves each job exponentially long with parameter μ and similarly at s_2 serves each job exponentially long with parameter λ. The number of jobs at server s_1 is a B&D process with intensity matrix (1.1).

$$\pi_N = \frac{\rho^N - \rho^{N+1}}{1 - \rho^{N+1}}$$

is the probability of the overflow.

Exercise. Find the stationary distribution for the embedded chain.

1.2 M/M/1 queue

We now define M/M/1 or sometimes denoted by M/M/1/∞ system with infinite buffer . As before it is a single server queue with inter-arrival times T_i's exponentially distributed with parameter $\lambda > 0$ and exponentially distributed service requirement S_i with

$$B(x) = 1 - e^{-\mu x}, \qquad x \geq 0$$

There is one server and we assume that if the server is empty, and there is at least one job in the queue, then one of jobs present immediately enters for the service. Such service disciplines are called *work conserving*

Our aim is to study the number of jobs in the system process $\{Q(t),\ t \geq 0\}$. Unless it is said otherwise all processes are cadlag.

$\Pi_\lambda(t)$ and $\Pi_\mu(t)$ are two independent Poisson processes with intensity λ and μ respectively.

It can be shown that process $Q(t)$ fulfils the following SDE:

$$dQ(t) = \Pi_\lambda(dt) - 1(Q(t-\circ) > 0)\Pi_\mu(dt)$$

with $Q(0) = x$ or equivalently

$$Q(t) = x + \Pi_\lambda(t) - \int_{(0,t]} 1(Q(t-\circ) > 0)\Pi_\mu(dt)$$

We can rewrite the above in the form

$$Q(t) = x + Z(t) + \int_0^t 1(Q(t-\circ) = 0)\Pi_\mu(dt), \tag{1.1}$$

where $Z(t) = \Pi_\lambda(t) - \Pi_\mu(t)$ is a CTBRW(λ, μ).

Exercise: Show that

$$M(t) = Q(t) - x - \lambda t + \mu \int_0^t 1(Q(s) > 0)ds$$

is a martingale.

The following point processes are of interest for studying of the process $Q(t)$:

A– the process defined by arrival epochs $(\tau^{\mathrm{a}})_{n\geq 1}$; in this case it is the stationary Poisson process with intensity λ,

We have jumps up of the process $Q(t)$ at arrival epochs $\{\tau^{\mathrm{a}}\}$. Jumps down $(\tau^{\mathrm{d}})_{n\geq 1}$ define departure epochs;

$$\tau^{\mathrm{d}}_0 = 0 \qquad \tau^{\mathrm{d}}_{n+1} = \inf\{t > \tau^{\mathrm{d}}_n : Q(t) < Q(t-)\}. \tag{1.2}$$

Theorem 1.7 *The process $(Q(t))_{t\geq 0}$ is a B & D process with intensity matrix*

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdot \\ \mu & -\lambda-\mu & \lambda & 0 & \cdot \\ 0 & \mu & -\lambda-\mu & \lambda & \cdot \\ 0 & 0 & \mu & -\lambda-\mu & \cdot \end{pmatrix} \tag{1.3}$$

Proof

As an immediate corollary we obtain the following form of the stationary distribution for the $M/M/1$ queue. We generalise this result later in different fashions.

Corollary 1.8 *The process* $\{Q(t)\}$ *is irreducible and regular. If*

$$\rho = \lambda/\mu > 1,$$

then the process is transient, otherwise it is recurrent. If moreover $\rho < 1$ *then the process is positive recurrent with the stationary distribution*

$$\pi_i = (1-\rho)\rho^i, \qquad i = 0, 1, \dots \tag{1.4}$$

CTMC $(Q(t))_{t\geq 0}$ with initial state i drawn with probability π_i is a stationary process. That is the distribution of the vector $Q(t_1+t), \dots, Q(t_k+t)$ does not depend on t. Note that

$$\mathbb{P}_\pi(Q=0) = 1-\rho \tag{1.5}$$

$$\mathbb{E}_\pi Q = \frac{\rho}{(1-\rho)} \tag{1.6}$$

$$\mathrm{Var}_\pi Q = \frac{\rho}{(1-\rho)^2}. \tag{1.7}$$

Remark Theorem 1.7 shows that we can describe the queueing process by its local characteristics. This is often important for working out computable solutions for system performance characteristics. In the future we frequently give a local description of processes of interest without justification. In most cases they are tedious and uninformative.

Remark Quantity ρ is sometimes called the *offered load*. In the case of M/M/1 queues ρ is also the *server utilisation*.

Consider now two independent Poisson processes $(\Pi_0(t))_{t\geq 0}$, and $(\Pi_1(t))_{t\geq 0}$ with intensities λ and μ respectively defined on a probability space $(\Omega, \mathcal{F}, \Pr)$. Qet $Z(t) = \Pi_0(t) - \Pi_1(t)$.

Proposition 1.9 *The process*

$$Q(t) = \max(i + Z(t), \sup_{0\leq s\leq t}(Z(t) - Z(s)), \tag{1.8}$$

is a birth and death process with intensity matrix $\boldsymbol{Q}$ *given in (1.3) such that* $\Pr(Z(0) = i) = 1$.

Consider the number of jobs just before arrivals. That is we define a sequence of random variables $(Q_n^+)_{n\geq 0}$ by: $Q_0^+ = Q(0)$ and $Q_n^+ = Q(\tau^{\mathrm{a}}-)$, $n = 1, 2, \ldots$. It is said that $\{Q_n^+\}$ is an embedded queue length process.

Proposition 1.10 *$\{Q_n^+\}$ is a DTMC with transition matrix*

$$\boldsymbol{P}^+ = \begin{pmatrix} 1-b_0 & b_0 & 0 & 0 & \cdot \\ 1-b_0-b_1 & b_1 & b_0 & 0 & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-\sum_{i=0}^{j} b_j & b_j & b_{j-1} & b_{j-2} & \cdot \\ \vdots & \vdots & \vdots & \vdots\&\vdots & \end{pmatrix} \tag{1.9}$$

where $b_i = (1-\kappa)\kappa^i$ and $\kappa = 1/(1+\rho)$.

Proposition 1.11 *If $\rho < 1$ the chain is positive recurrent with the stationary distribution*

$$\pi_n^+ = (1-\rho)\rho^n, \qquad n = 0, 1, \ldots$$

Consider now a doubly-ended version of the stationary queue length process $(Q(t))_{t\in\mathbf{R}}$ defined by intensity matrix (1.3). For a given t and consider the p.p. A defined by jumps up (that is arrivals) and the p.p. D defined by jumps down (that is departures). Clearly A is a Poisson process. Clearly the process $Q(t)$ is reversible because it is a B&D process.

Theorem 1.12 *[Burke theorem]*
(i) *D is a Poisson process with rate λ.*
(ii) *For each t, the p.p. of departures on $(-\infty, t)$ is independent of $Q(t)$.*

Proof P.p. consisting of jumps down (that is departure epochs) of Q in $(-\infty, t)$ corresponds to jumps up (that is arrivals) of Q^* in (t, ∞). However $Q^*(t)$ is independent of arrivals after time t. In view of the reversibility departures prior t are independent of $Q(t)$. □

Before we state the next theorem on a formula for stochastic intensity if we change filtration into a smaller, one we need to define a notion which plays an important role in stochastic analysis. For a filtration $\mathcal{F}_t$ of subsets $(\Omega, \mathcal{F})$ we define a σ-field $\mathcal{P}(\mathcal{F}_t)$ over $(0, \infty) \times \Omega$, which is generated by rectangles of the form

$$(s, t] \times A, \qquad 0 \leq s \leq t, \qquad A \in \mathcal{F}_s\ .$$

$\mathcal{P}(\mathcal{F}_t)$ is called the $\mathcal{F}_t$–predictable σ-field over $(0,\infty)\times\Omega$. Consider now two measures on $((0,\infty)\times\Omega),\mathcal{P}(\mathcal{F}_t)$: $P_2(ds\times d\omega)$ and $P_1(ds\times d\omega)$, defined by

$$P_1((s,t]\times A) = \mathbb{E}\,[1(A)\int_s^t \alpha(v)\,dv], \qquad s<t, A\in\mathcal{G}_s,$$

$$P_2((s,t]\times A) = \mathbb{E}\,[1(A)\int_s^t 1\,dv], \qquad s<t, A\in\mathcal{G}_s\ .$$

The following theorem is from the book by Bremaud [2], page 32.

Theorem 1.13 *[Change of history for intensities] Let N admits $\mathcal{F}_t$-intensity $\alpha(t)$ and $(\mathcal{G}_t)_{t\ge 0}$ is another filtration such that*

$$\mathcal{F}_t^N \subset \mathcal{G}_t \subset \mathcal{F}_t\ , \qquad t\ge 0.$$

Then N admits a $\mathcal{G}_t$-intensity $\beta(t)$ which is restriction of $\frac{dP_1}{dP_2}(s,\omega)$ to $\mathcal{P}(\mathcal{G}_t)$.

We conclude looking at the Burke theorem from the point of view of martingale theory of p.p.s. Let $(Q(t))_{t\in\mathbf{R}}$ be a stationary number of jobs process in M/M/1 queue. and let $(D(t)_{t\ge 0}$ be the p.p.of departure epochs on $(0,\infty)$. Let $(\mathcal{F}_t^Q)$ be the internal history filtration of X. We have that
(i) process D is $\mathcal{F}_t^Q$–adapted and $\mathcal{F}_t^Q$–stochastic intensity is

$$\alpha(t) = 1(Q(t)>0)\mu, \qquad t\ge 0\ ,$$

(ii) process D admits constant $\mathcal{F}_t^D$–stochastic intensity $\beta(t)\equiv\lambda$.

Thus from Watanabe theorem D is a Poisson process with intensity λ.

Part (i) follows from Theorem III.2.14 if we take $f(i,j)=1(i=j+1)$. For part (ii) we have to use Theorem 1.13. Less formally we look for a $\mathcal{G}_t=\mathcal{F}_t^D$–adapted process $\beta(t)$ such that

$$\mathbb{E}\,1(A)\int_s^t \alpha(v)\,dv = \mathbb{E}\,1(A)\int_s^t \beta(v)\,dv, \qquad s<t, A\in\mathcal{F}_s^D\ .$$

From Burke theorem we see that for $A\in\mathcal{F}_s^D$

$$\mathbb{E}\,1(A)\int_s^t \mu 1(Q(v)>0)\,dv \;=\; \mathbb{E}\,1(A)\int_s^t \mathbb{E}\,[\mu 1(Q(v)>0)]$$

$$\mathbb{P}(A)\int_s^t \mu\rho = \mathbb{P}(A)\lambda(t-s),$$

because A and $Q(s)$ are independent.

Problems

1.1 Show that

$$Q(t) \;=_{\mathrm{d}}\; \max(i + Z(t), B_t),$$

where $Z(t) = \Pi_0(t) - \Pi_1(t)$ and $B(t) = \sup_{0\le s\le t} Z(s)$ (Example 7.4 in Asmussen (2003), p. 98).

1.2 Consider a transition probability matrix of form (1.9), where $b_i \ge 0$ and $\sum_{j=0}^{\infty} b_j = 1$. Show that, if $\lambda \sum_{j=1}^{\infty} j b_j < 1$, then the chain is positive ergodic and with the stationary distribution $\pi_n = (1-\delta)\delta^i$, $i = 01, 2, \ldots$ and δ is the positive solution $\hat{g}(x) = x$, where $\hat{g}(x) = \sum_{j=0}^{\infty} b_j x^j$ is the generating function of $\{b_j\}$.

1.3 $M/M/c$ queue

This is a queue with Poisson input and i.i.d. exponential service requirements. We suppose that $\lambda > 0$ is the arrival rate and μ is the service rate, that is μ^{-1} is the mean service requirement. In these queues ρ is the offer load and $a = \rho/c$ is the server utilisation. A queueing discipline is of work conserving type. We consider the number of jobs in the system process $Q(t)$. Similarly as in Section 1.2 on $M/M/1$ queues the process $Q(t)$ is a B&D process on $\mathbb{Z}_+$ with birth intensities $\lambda_n \equiv \lambda$ and the death intensities $\mu_n = \min(n, c)\mu$. It is quite clear that if $Q(t) = n$ (for $1 \le n \le c$) all present jobs are in the service, and due to supposed exponential service requirement, $n\mu\Delta + \mathrm{o}(\Delta)$ is the probability that in the time interval of length Δ the service of one job is completed. Similarly, for $k \ge c$ jobs present the corresponding probability is $c\mu\Delta + \mathrm{o}(\Delta)$, because there is only c servers available. [1] As usual ρ denotes the traffic intensity. Following Corollary II.1.20 we see that the process $Q(t)$ is ergodic if and only if $\rho < 1$. From Corollary II.1.20 we have the stationary distribution

$$\pi_n = \begin{cases} \pi_0 \frac{\rho^n}{n!} & \text{for } n \le c \\ \\ \pi_0 \frac{\rho^c}{c!} (\frac{\rho}{c})^{n-c} & \text{for } n > c \end{cases} \tag{1.10}$$

[1] Due to the lack of the memory property the nearest service is completed after time $Y = \min(\eta_1, \ldots, \eta_n)$, where $\eta_1, \ldots, \eta_n$ are independent and exponentially distributed with intensity μ. Thus $\mathbb{P}(Y \le t) = 1 - \exp(-n\mu t) = n\mu t + o(t)$.

where

$$\pi_0^{-1} = S \;=\; \sum_{n=0}^{c} \frac{\rho^n}{n!} + \frac{\rho^c}{c!}\frac{a}{1-a}$$

where $a = \rho/c$. We can easily prove (1.10) using the DBE

$$\min(n,c)\mu\pi_n = \lambda\pi_{n-1}, \qquad \pi_{-1} = 0\ ,$$

or

$$\min(n,c)\pi_n = \rho\pi_{n-1}, \qquad \pi_{-1} = 0\ .$$

We denote by Q the generic steady-state number of jobs. In the sequel we assume that the system is in the (time) stationary conditions. We list now few basic characteristics of M/M/c queues. For the stationary mean number of jobs in the system we have

$$\mathbb{E}_\pi Q = \pi_0(\sum_{n=1}^{c} \frac{\rho^n}{(n-1)!} + \frac{\rho^c}{c!}\left[\frac{ca}{1-a} + \frac{a}{(1-a)^2}\right]).$$

The steady-state number of jobs in the queue Q_q is defined by

$$Q_q = (Q-c)_+.$$

We have

$$\begin{aligned}\mathbb{E}_\pi Q_q &= \sum_{n=0}^{\infty}(n-c)_+\pi_n = \pi_0 \sum_{n=c+1}^{\infty}(n-c)\frac{\rho^c}{c!}(\frac{\rho}{c})^{n-c} \\ &= \frac{\pi_c a}{(1-a)^2}. \qquad (1.11)\end{aligned}$$

Similarly the generic steady-state number of busy servers is $Q_s = \min(c,Q)$, so

$$\begin{aligned}\mathbb{E}_\pi \min(c,Q) &= \sum_{j=0}^{c-1} j\pi_j + c\sum_{n=c}^{\infty}\pi_n \\ &= \pi_0\sum_{j=1}^{c-1}\frac{\rho^j}{(j-1)!} + c\sum_{n=c}^{\infty}\frac{\rho^c}{c!}a^{n-c}\pi_0 \\ &= \pi_0\sum_{j=1}^{c-1}\frac{\rho^j}{(j-1)!} + c\pi_0\frac{\rho^c}{c!}\frac{1}{1-a}\ .\end{aligned}$$

We can now pose a few interesting operational questions concerning of comparison between single server and many server queues. We have first to decide what are systems to be compared and comparison criteria. The simplest choice is the simple comparison for the same input data (that is in the systems the arrival intensity is λ and the mean service requirement is μ^{-1}). However more interesting seems to be comparison of systems with the same traffic intensity ρ/c, that is in the single server queue we take the service rate $c\mu$. For comparison criteria we usually consider some ordering between the number in system (in stationary conditions) $Q^{(c)}$ in the c-server queue and $Q^(1)$ in the single server queue. Another possibility is to consider the queue length $Q_q^{(c)}$ and $Q_q^{(1)}$ respectively.

Another performance characteristic is the *steady state delay probability*

$$C(c,\rho) = \mathbb{P}_\pi(Q \geq c) \; .$$

Function $C(c,\rho)$ is called *Erlang C formula.* We have

$$\begin{aligned} C(c,\rho) &= \frac{\frac{\rho^c}{c!}\frac{1}{1-a}}{\sum_{n=0}^{c}\frac{\rho^n}{n!} + \frac{\rho^c}{c!}\frac{a}{1-a}} \\ &= \frac{\frac{\rho^c}{c!}\frac{c}{c-\rho}}{\sum_{n=0}^{c}\frac{\rho^n}{n!} + \frac{\rho^c}{c!}\frac{\rho}{s-\rho}} \; . \end{aligned}$$

In the theory of queues there is also *Erlang B formula*

$$B(c,\rho) = \frac{\frac{\rho^c}{c!}}{\sum_{n=0}^{c}\frac{\rho^n}{n!}} \; .$$

We have the following relationship betweeen B and C:

$$C(c,\rho)^{-1} = a + (1-a)B(c,\rho)^{-1} \; .$$

The above can be easily seen. Thus

$$\begin{aligned} C(c,\rho)^{-1} &= \frac{\sum_{n=0}^{c}\frac{\rho^n}{n!} + \frac{\rho^c}{c!}\frac{a}{1-a}}{\frac{\rho^c}{c!}\frac{1}{1-a}} \\ &= (1-a)B(c,\rho)^{-1} + a \; . \end{aligned}$$

Throught out function C we can express some performance characteristics for M/M/c queues. For example we can ask for the point process of jobs which have to wait, that is jobs which upon their arrival find all servers busy. Thus let $N^{\mathrm{w}}(t)$ be the number of jobs which have to wait;

$$N^{\mathrm{w}}(t) = \int_0^t 1(Q(s-\circ) \geq c)\, d\Pi^{\lambda}(s)\ .$$

Suppose Q is stationary. Then $(N^{\mathrm{w}}(t))_{t\geq 0}$ is the process with stationary increments. Using xxxx we can compute the intensity of the point process which have to wait

$$\lambda^{\mathrm{w}} = \mathbb{E}_{\pi} N^{\mathrm{w}}(1) = E[\int_0^1 1(Q(s-\circ) \geq c)\, ds] = \lambda C(\rho, c)\ .$$

There are different asymptotic results concerning $C(c,\rho)$. One is when

$$c = \rho + \beta\sqrt{\rho},$$

for some β. Such the conditions are called Whitt-Halfin regime. In this case we have the following asymptotic.

Proposition 1.14 *Under the Halfin-Whitt regime*

$$B(c,\rho) \sim \frac{\phi(\beta)}{\Phi(\beta)\sqrt{\rho}}$$

and

$$C(c,\rho) \sim,$$

where $\Phi(x)$ and $\phi(x)$ are the distribution function and the density function of the standard normal distribution.

2 Systems with a finite number of jobs

In this section we consider two systems, which are modelled by a finite state space CTMC.

2.1 Engseth's Loss System

Suppose there are K terminals and N users where $K < N$. We assume that each user request a terminal with intensity λ and the time utilised by a user is exponentially distributed with parameter μ. Let $\rho = \lambda/\mu$.As usual we suppose that all variables are independent. The number of used terminals $(Q(t),\ t \geq 0)$ is a B&D queueing process with state space $\{0, \dots K\}$ and intensities

$$\lambda_n = (N-n)\lambda \qquad \mu_n = n\mu.$$

Hence

$$S = 1 + \binom{N}{1}\rho + \cdots + \binom{N}{K}\rho^K$$

and

$$\pi_n = \frac{\binom{N}{n}\rho^n}{1 + \binom{N}{1}\rho + \cdots + \binom{N}{K}\rho^K} \tag{2.12}$$

Let $\rho = p(1-p)^{-1}$. Clearly $0 < p < 1$. Then for $n \leq K$

$$\pi_n \quad = \quad \frac{\binom{N}{n}p^n(1-p)^{N-n}}{(1-p)^N + \binom{N}{1}p(1-p)^{N-1} + \cdots + \binom{N}{K}p^K(1-p)^{N-K}} \; .$$

It means that the distribution of the number of jobs can be represented as the conditional truncated Bernoulli distribution. Therefore, if now $Np \to \nu$, then

$$\pi_n \to \frac{\frac{\rho^n}{n!}}{1 + \rho + \cdots + \frac{\rho^K}{K!}} \; , \qquad n = 0, \dots, K \; .$$

2.2 Erlang Loss System

The former example can be formulated in terms of telephone exchange with N subscribers and K lines. In case the case of big N the stream of calls is approximated by a Poisson process. Suppose the arrival rate is λ. The holding times, as before, are i.i.d. exponentially distributed with parameter μ. Let $(Q(t))_{t\geq 0}$ be the number of busy lines process. Under our assumption it is again a B&D queueing process with $\lambda_n = \lambda$ and $\mu_n = n\mu$. The stationary distribution exists for any $\rho = \lambda/\mu > 0$ with

$$\pi_n = \frac{\frac{\rho^n}{n!}}{1 + \rho + \cdots + \frac{\rho^K}{K!}} \; , \qquad n = 0, \dots, K. \tag{2.13}$$

The $0-1$ process $1(Q(t) = K)$ is when all lines are busy. Consider the stream of accepted calls N^{w} and let

$$0 = \tau_0^{\mathrm{w}} < \tau_1^{\mathrm{w}} < \tau_2^{\mathrm{w}} \dots$$

are their arrival epochs. The first question is for the arrival rate of rejected calls $\overline{\lambda}^{\mathrm{w}}$. Suppose that Q is stationary. Similarly as in Section V.3.6 on many server queues we introduce the number $N^{\mathrm{loss}}(t)$ of lost calls

$$N^{\mathrm{loss}}(t) = \int_0^t 1(Q(s-\circ) = K)\, d\Pi^{\lambda}(s) \ .$$

Since $N^{\mathrm{w}}(0,t]$ is a process with stationary increments we can prove the following result.

Proposition 2.1

$$\overline{\lambda}^{\mathrm{w}} = \lambda B(K,\rho) \ , \tag{2.14}$$

Proof Use the fact that process

$$N^{\mathrm{w}}(t) = \int_0^t 1(Q(s-\circ) = K)\, d\Pi^{\lambda}(s) \ ,$$

is with stationary increments and next take the expectation

$$\begin{aligned} \mathbb{E}\, N^{\mathrm{w}}(t) &= \mathbb{E} \int_0^t 1(Q(s-\circ) = K)\, d\Pi^{\lambda}(s) \\ &= t\lambda \mathbb{P}_{\pi}(Q(s-\circ) = K) \ . \end{aligned}$$

For K server loss system with $\rho = \lambda/\mu$ we define

$$B(K,\rho) = \frac{\frac{\rho^K}{K!}}{1 + \rho + \cdots + \frac{\rho^K}{K!}}$$

Comments. Problem 2 is from [10]. Is this true that $B(K,\rho)$ is convex in K for $K \geq 1$? Problem is from [10].

3 Markovian network of single server queues

3.1 m-queues in series

The simplest network can be obtained arranging m queues in series. Without loss of generality we may suppose that $m = 2$. In this case the system has two *nodes*, each one is a single server queue with unlimited buffer and service requirements brought by a job are exponentially distributed with rate $\mu_1 > 0$, $\mu_2 > 0$. respectively. At the first, jobs arrive according to a Poisson process with rate $\lambda > 0$. A job leaving the first node (station) goes to node 2, from which it leaves the network. We suppose that at each node the service discipline is of work conserving type. Such a network is called *m-queues in series* or *tandem queue* . The arrivals and all service requirements are assumed to be independent. For such the network we study a two dimensional process $\boldsymbol{Q}(t) = (Q_1(t), Q_2(t))$, where $Q_i(t)$ is the number of jobs at the node i. From the just studied results on M/M/1 queues, it is quire apparent that each node can be considered as independent M/M/1 queueing systems. Recall that in the steady-state conditions departures from node 1 prior t, which constitute arrivals to node 2 are independent of $(Q_1(s), s \geq t)$ and that these arrivals to node 2 are according to a Poisson process with rate λ. Thus, in the steady–state conditions, these 2 nodes at each moment, they can be considered as independent M/M/1 queueing systems and so the joint steady state distribution of $\boldsymbol{Q}(t)$ is

$$\mathbb{P}_\pi(Q_1 = l_1, Q_2 = l_2) = (1-\rho_1)\rho_1^{l_1}(1-\rho_2)\rho_2^{l_2}$$

where $\rho_i = \lambda/\mu_i$. Remark however that it is not true that stochastic processes $(Q_1(t))$ and $(Q_2(t))$ are independent. Furthermore we can prove that for stationary $(\boldsymbol{Q}(t))_{t\geq 0}$, for each t departures by time $D_{|(-\infty,t)}, \boldsymbol{Q}(t), A_{|(t,\infty)}$ are independent. Formally we will study a CTMC with transition intensity matrix $\boldsymbol{Q} = (q_{\boldsymbol{i}\boldsymbol{j}})_{\boldsymbol{i},\boldsymbol{j}\in\mathbb{Z}^m}$ with

$$q_{\boldsymbol{i}\boldsymbol{j}} = \begin{cases} \lambda & \boldsymbol{i} \to \boldsymbol{j} = (i_1+1, i_2, \ldots, i_m)\ , \\ \mu_l & \boldsymbol{i} \to \boldsymbol{j} = (i_1, \ldots, i_l - 1, i_{l+1}+1, \ldots, i_m), \quad l < m\ , \\ \mu_m & \boldsymbol{i} \to \boldsymbol{j} = (i_1+1, i_2, \ldots, i_m - 1)\ . \end{cases}$$

Theorem 3.1 *$\boldsymbol{Q}(t)$ is a regular, positive recurrent CTMC if and only if $\rho_i < 1$ $(i = 1, \ldots, m)$. The stationary distribution is*

$$\pi_{\boldsymbol{i}} = \prod_{j=1}^{m}(1-\rho_j)\rho_j^{i_j}\ .$$

Proof Since $\Pi(t) \geq Q_1(t) + \ldots + Q_m(t)$, process $\boldsymbol{Q}(t)$ is obviously regular. The irreducibility is also obvious. By inspection we check that GBE holds. Therefore by Theorem V.3.6 the process is ergodic and $\boldsymbol{\pi}$ is its stationary distribution. □

3.2 Open Jackson network

We consider the following extension of the queue in tandem. There are

- m nodes (queueing stations),
- jobs arrive at node i according to a Poisson process Π_i with intensity λ_i,
- each node is of type $\cdot$/M/1 and jobs at the i-th node are served with with rate μ_i,
- after completion of the service in node i-th job goes to node j with probability p_{ij},
- arrivals, services and routes are independent

Let

$$\boldsymbol{P} = (p_{ij})_{i,j=1,\ldots,m} \ .$$

We must assume that $\boldsymbol{I} - \boldsymbol{P}$ is nonsingular and irreducible. Therefore there exists $(\boldsymbol{I} - \boldsymbol{P})^{-1} = \boldsymbol{I} + \boldsymbol{P} + \boldsymbol{P}^2 + \ldots$. This shows that $(\boldsymbol{I} - \boldsymbol{P})^{-1}$ have only positive entries. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ and $\bar{\boldsymbol{\lambda}} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)$. There exists a unique positive solution of the so called *traffic equation*

$$\bar{\boldsymbol{\lambda}} = \boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}}\boldsymbol{P}$$

of form

$$\bar{\boldsymbol{\lambda}} = \boldsymbol{\lambda}(\boldsymbol{I} - \boldsymbol{P})^{-1} \ .$$

Let $\bar{\rho}_i = \bar{\lambda}_i/\mu_i$. We need to define an $(m+1) \times (m+1)$ matrix $\boldsymbol{R} = (r_{ij})_{i,j=0,\ldots,m}$ by

$$r_{ij} = \begin{cases} 1 & i = j = 0 \\ 1 - p_{i1} - \ldots - p_{im} & i > 0, j = 0 \\ p_{ij} & i, j > 0 \end{cases}$$

The CTMC $\boldsymbol{Q}(t)$ associated with the considered Jackson network has the state space $\mathbb{E} = \mathbb{Z}_+^m$. Let $\boldsymbol{e}_i = (1(j = i))_{j=1,\ldots,m}$. The element of the state space is $(n_1, \ldots, n_m) \in \mathbb{Z}_+^m$, where n_i means the number of jobs at the node i.

We are now ready to write intensity matrix $\boldsymbol{Q}$. Thus outside off the diagonal of $\boldsymbol{Q}$ we have

$$
\begin{aligned}
q_{\boldsymbol{n},\boldsymbol{n}+\boldsymbol{e}_i-\boldsymbol{e}_j} &= \mu_j r_{ji}, \qquad \text{if } n_j > 0 && (3.15)\\
q_{\boldsymbol{n},\boldsymbol{n}-\boldsymbol{e}_j} &= \mu_j r_{j0}, \qquad \text{if } n_j > 0 && (3.16)\\
q_{\boldsymbol{n},\boldsymbol{n}+\boldsymbol{e}_i} &= \lambda_i && (3.17)\\
q_{\boldsymbol{n},\boldsymbol{n}'} &= \qquad \text{otherwise .} && (3.18)
\end{aligned}
$$

Definition 3.2 A CTMC with state space $\mathbb{E} = \mathbb{Z}_+^m$ and intensity matrix $\boldsymbol{Q}$ of form (3.15)–(3.18) is said to be a Jackson network JN$(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{P})$.

We now state the main theorem for Jackson networks. Since $\sum_{j=1}^m \Pi_j(t) \geq \sum_{j=1}^m Q_j(t)$, the process must be regular. It is also clear that the process is irreducible.

The stationary distribution appearing in the following theorem has the *product-form.*

Theorem 3.3 *If $\bar{\rho}_i < 1$ for all $i = 1, \ldots, m$, then the process $\boldsymbol{Q}(t)$ is ergodic and its stationary distribution is given by the* product formula

$$\pi_{\boldsymbol{n}} = \prod_{j=1}^m (1 - \bar{\rho}_j)\bar{\rho}_j^{n_j}, \qquad \boldsymbol{n} \in \mathbb{Z}_+^m$$

Proof According to prescription given in II.2.9 we now define new intensity matrix $\tilde{\boldsymbol{Q}}$ by:

$$
\begin{aligned}
\tilde{q}_{\boldsymbol{n},\boldsymbol{n}+\boldsymbol{e}_i-\boldsymbol{e}_j} &= \frac{\bar{\lambda}_i}{\bar{\rho}_j} r_{ij}, \qquad n_j > 0 && (3.19)\\
\tilde{q}_{\boldsymbol{n},\boldsymbol{n}-\boldsymbol{e}_j} &= \frac{\bar{\lambda}_j}{\bar{\rho}_j}, \qquad n_j > 0 && (3.20)\\
\tilde{q}_{\boldsymbol{n},\boldsymbol{n}+\boldsymbol{e}_i} &= \bar{\lambda}_i r_{i0} && (3.21)
\end{aligned}
$$

and otherwise we put $\tilde{q}_{\boldsymbol{n},\boldsymbol{n}'} = 0$ for $\boldsymbol{n} \neq \boldsymbol{n}'$. To demonstrate (3.19) we compute

$$\tilde{q}_{\boldsymbol{n},\boldsymbol{n}+\boldsymbol{e}_i-\boldsymbol{e}_j} = \frac{\pi_{\boldsymbol{n}+\boldsymbol{e}_i-\boldsymbol{e}_j}}{\pi_{\boldsymbol{n}}} q_{\boldsymbol{n}+\boldsymbol{e}_i-\boldsymbol{e}_j,\boldsymbol{n}} = \frac{\bar{\rho}_i}{\bar{\rho}_j}\mu_i r_{ij} = \frac{\bar{\lambda}_i}{\bar{\rho}_j} r_{ij}$$

To demonstrate (3.20) we write ...
To demonstrate (3.21) we compute

$$\tilde{q}_{\boldsymbol{n},\boldsymbol{n}+\boldsymbol{e}_i} = \frac{\pi_{\boldsymbol{n}+\boldsymbol{e}_i}}{\pi_{\boldsymbol{n}}} q_{\boldsymbol{n}+\boldsymbol{e}_i,\boldsymbol{n}} = \rho_i \mu_i r_{i0} = \bar{\lambda}_i r_{i0} \ .$$

We are now going to apply Proposition II.2.13. Thus for $\boldsymbol{n} \in \mathbb{E}$ the partition of $\mathbb{E} - \{\boldsymbol{n}\}$ is

$$\begin{aligned} \mathcal{A}_j^{\boldsymbol{n}} &= \{\boldsymbol{n}+\boldsymbol{e}_i-\boldsymbol{e}_j, i \neq j\} \cup \{\boldsymbol{n}-\boldsymbol{e}_j\}, \qquad j=1,\ldots,m \\ \mathcal{A}_0^{\boldsymbol{n}} &= \{\boldsymbol{n}+\boldsymbol{e}_i, i=1,\ldots,m\}, \\ \mathcal{A}_{m+1}^{\boldsymbol{n}} &= \mathbb{E} - (\bigcup_{j=0}^{m} \mathcal{A}_j^{\boldsymbol{n}} \cup \{\boldsymbol{n}\}) \ . \end{aligned}$$

Notice that $\mathcal{A}_j^{\boldsymbol{n}}$ is responsible for all departures from node j if the system is in state $\boldsymbol{n}$, while $\mathcal{A}_0^{\boldsymbol{n}}$ is responsible for all arrivals to node j if the system is in state $\boldsymbol{n}$. Now

$$\sum_{\boldsymbol{n}' \in \mathcal{A}_j^{\boldsymbol{n}}} q_{\boldsymbol{n},\boldsymbol{n}'} = \sum_{i \in \{1,\ldots,m\}-\{j\}} \mu_j r_{ji} + \mu_j r_{j0} = \mu_j \tag{3.22}$$

$$\sum_{\boldsymbol{n}' \in \mathcal{A}_j^{\boldsymbol{n}}} \tilde{q}_{\boldsymbol{n},\boldsymbol{n}'} = \sum_{\boldsymbol{n}' \in \mathcal{A}_j^{\boldsymbol{n}}} \frac{\bar{\lambda}_i}{\bar{\rho}_j} p_{ij} + \frac{\lambda_j}{\bar{\rho}_j} = \mu_j \tag{3.23}$$

Similarly

$$\sum_{\boldsymbol{n}' \in \mathcal{A}_0^{\boldsymbol{n}}} q_{\boldsymbol{n},\boldsymbol{n}'} = \sum_{i=1}^{m} \lambda_i$$

and

$$\begin{aligned}
\sum_{\boldsymbol{n}' \in \mathcal{A}_0^{\boldsymbol{n}}} \tilde{q}_{\boldsymbol{n},\boldsymbol{n}'} &= \sum_{i=1}^{m} \bar{\lambda}_i r_{i0} \\
&= \sum_{i=1}^{m} \bar{\lambda}_i (1 - \sum_{j=1}^{m} p_{ij}) \\
&= \sum_{j=1}^{m} \underbrace{(\bar{\lambda}_j - \sum_{i=1}^{m} \bar{\lambda}_i p_{ij})}_{=\lambda_j} \\
&= \sum_{i=1}^{m} \lambda_i \, .
\end{aligned}$$

Theorem 3.4 (i) *The departure process $\boldsymbol{D}$ out of the network from the nodes are independent Poisson processes with intensities $\bar{\lambda}_i r_{i0}$.*
(ii) *In the steady-state conditions, the state $\boldsymbol{Q}(t)$ at time t is independent from departures $\boldsymbol{D}_{|(-\infty)}$ before t and arrivals $\boldsymbol{A}_{|(t,\infty)}$ after time t.*

Proof $\tilde{\boldsymbol{Q}}(s) = \boldsymbol{Q}(-s - \circ)$ is a Jackson network NJ$(\tilde{\boldsymbol{\lambda}}, \boldsymbol{\mu}, \tilde{\boldsymbol{P}})$, called here J̃N, where

$$\begin{aligned}
\tilde{\boldsymbol{\lambda}} &= (\bar{\lambda} r_{10}, \ldots, \bar{\lambda} r_{m0}) \\
\tilde{\boldsymbol{P}} &= \frac{\bar{\lambda}_i}{\bar{\lambda}_j} p_{ji}
\end{aligned}$$

Thus arrivals at the NJ$(\tilde{\boldsymbol{\lambda}}, \boldsymbol{\mu}, \tilde{\boldsymbol{P}})$ are independent Poisson processes with intensities $\bar{\lambda}_i r_{i0}$ $(i = 1, \ldots, m)$ respectively. This proves part (i). Now arrivals at the ÑJ after time t are independent of $X(t)$ but they are departures for the original JN. Clearly arrivals after t at the original network are independent Poisson processes independent also of $X(t)$. The proof is completed if we apply Lemma [**??A1A2X??**]. □

3.3 Gordon-Newel network

We consider the following closed network in which N jobs are travelling among nodes and no job arrive/departure from/to outside. There are

- m nodes (queueing stations),
- each node is of type $\cdot$/M/1 and jobs at the i-th node are served with with rate $\mu_i > 0$,
- after completion of the service in node i-th job goes to node j with probability p_{ij},
- services and routes are independent.

Let

$$\boldsymbol{P} = (p_{ij})_{i,j=1,\ldots,m} .$$

We must assume that $\boldsymbol{P}$ is irreducible stochastic matrix and hence, because of finite dimension, there exists the unique stationary distribution $\bar{\boldsymbol{\lambda}}$, that is fulfilling $\bar{\boldsymbol{\lambda}} = \bar{\boldsymbol{\lambda}}\boldsymbol{P}$. Therefore there exists Let $\bar{\rho}_i = \bar{\lambda}_i/\mu_i$. As before we consider a CTMC $\boldsymbol{Q}(t) = (Q_1(t), \ldots, Q_m(t))$ of the number of jobs at each node process. In view of our assumptions the chain is irreducible and since the state space $\boldsymbol{E} = \{\boldsymbol{n} \in \mathbb{Z}_+^m : n_1 + \ldots + n_m\}$ is finite, it is also ergodic. Similarly as Theorem V.3.6 we can prove the following product-form of the stationary distribution.

Theorem 3.5 *Process $\boldsymbol{Q}(t)$ is ergodic and its stationary distribution is given by the* product formula

$$\pi_{\boldsymbol{n}} = \pi_{\boldsymbol{0}} \prod_{j=1}^{m} (1 - \bar{\rho}_j)\bar{\rho}_j^{n_j}, \qquad \boldsymbol{n} \in \mathbb{E} .$$

Remark Suppose that $(\boldsymbol{Q}(s))_{s\in\mathbf{R}}$. Notice that, in contrast with Jackson networks, we do not have independence of $Q_i(t)$ $(i = 1, \ldots, m)$, because $Q_1(t) + \ldots + Q_m(t) = N$. In contrast with open networks, computing $\pi_{\boldsymbol{0}}$ is not immediate. Function

$$G(N, m) = (\pi_{\boldsymbol{0}})^{-1} = \sum_{\boldsymbol{n}\in\mathbf{E}} \prod_{j=1}^{m} \rho_j^{n_j}$$

is called a *partition function.* It can be computed recursively

$$G(j, l) = G(j, l-1) + \rho_l G(j-1, l) , \tag{3.24}$$

with initial conditions $G(j, 1) = \rho_1^j$ $j \geq 0$ and $G(0, l) = 1$ $(l \geq 1)$.

Important performance characteristics can be computed from the product form formula. Let

$$U_i(N,m) = \mathbb{P}_{\boldsymbol{\pi}}(Q_i > 0)$$

The average number of jobs transferred from node i to node j in one unit time is called *throughput* $d_{ij}(N,m)$. One can demonstrate that

$$d_{ij}(N,m) = U_i(N,m)\mu_i p_{ij} \ .$$

4 Multi-class Queue

Consider the following generalisation of the classical $M/M/1$ queue allowing jobs being from different classes. Thus let $\mathbb{J} = \{1,\dots,J\}$ be a family of job types and jobs of each class arrive at the system according to a Poisson process with intensity $\lambda_i > 0$ ($i \in \mathbb{J}$). Jobs of the i-th type are served independently from other jobs with the duration of the service exponentially distributed with parameter $\mu_i > 0$. Streams of different types of jobs and service requirements are independent. The total arrival intensity is

$$\lambda = \sum_{i\in\mathbb{J}} \lambda_i \tag{4.1}$$

Let the elementary element of the state space is $(k_1 \dots k_n)$, where $k_i \in \mathbb{J}$. The string $(k_1 \dots k_n)$ describes all n jobs being present in the system. We suppose that positions of jobs in the system are distinguishable. Each time we have to define, where a coming job can join the queue and which customers are served. The empty system is denoted by 0 and it is also identified with the string $(k_1 \dots k_n)$, wherein $n = 0$. The state space of the state process $\{\boldsymbol{K}(t),\ t \ge 0\}$ is

$$\mathbb{K}^* = \bigcup_{n=0}^{\infty} \{(k_1 \dots k_n), k_i \in \mathbb{J}\} \tag{4.2}$$

Let $\rho_k = \lambda_k/\mu_k$ and $\rho = \sum_k \rho_k$.

In this section we derive the stationary distributions of $K(t)$ under different queueing disciplines. Actually two types of stationary distributions will appear. In a few cases we obtain the stationary distribution of the state process

$$\pi(\boldsymbol{k}) = \pi(k_1 \dots k_n) = (1-\rho)\rho_{k_1}\cdots\rho_{k_n}.$$

However the characteristic we are frequently looking for is the number $Q_k(t)$ at time t of jobs process of class k. Suppose that $\mathbb{J} = \{1, \ldots, J\}$. Let for $\boldsymbol{k} \in \mathbb{K}^*$ and $i \in \mathbb{J}$

$$l_i(\boldsymbol{k}) = \#\{j : \; k_j = i\}, \qquad l(\boldsymbol{k}) = \sum_i l_i(\boldsymbol{k}). \tag{4.3}$$

Under the stationary distribution as above we have

$$\sum_{l_1(\boldsymbol{k})=m_1,\ldots,l_J(\boldsymbol{k})=m_J} \pi(\boldsymbol{k}) = (1-\rho)\frac{(m_1 + \ldots m_J)!}{m_1! \ldots m_J!}\rho^{m_1} \ldots \rho_J^{m_J} \tag{4.4}$$

and

$$\pi_n = \sum_{l(\boldsymbol{k})=n} \pi_{\boldsymbol{k}} = (1-\rho)\rho^n. \tag{4.5}$$

Consider now the case when

$$\pi(k_1 \ldots k_n) = \exp(-\rho)\frac{\rho_{k_1} \cdots \rho_{k_n}}{n!} \; . \tag{4.6}$$

Then

$$\begin{aligned} \mathbb{P}(Q_1 = n_1, \ldots, Q_K = n_k) &= \sum_{l_1(\boldsymbol{k})=n_1,\ldots,l_K(\boldsymbol{k})=n_K} \pi(\boldsymbol{k}) \\ &= \prod_{j=1}^{K} e^{-\rho_j}\frac{\rho^{n_j}}{n_j!} \end{aligned}$$

and

$$\pi_n = \sum_{l(\boldsymbol{k})=n} \pi_{\boldsymbol{k}} = e^{-\rho}\frac{\rho^n}{n!}. \tag{4.7}$$

Problems

4.1 Prove formulas (4.7) and (4.5).

4.1 Job flows in networks

4.2 $\sum_k M_k/M/1$-FCFS Queue

Suppose now that jobs are served in order of their arrivals that is the service discipline is FCFS. In this case it is essential for the presented results to assume $\mu_n \equiv \mu$. We suppose that an arriving job join jobs present in the system from the right and that the job from the left is serviced. Thus after joining the system in a state $(k_1 \dots k_n)$ by a job of class k a the new state is $(k_1 \dots k_n k)$. Similarly the job from the left end of $(k_1 \dots k_n)$ is being serviced and after the service completion the new state is $(k_2 \dots k_n)$.

Under this assumption the process $(\boldsymbol{K}(t))_{t\geq 0}$ of the state at time t defined by the evolution of the $\sum_i M_i/M/1$ queue is a regular jump Markov process. We now study possible changes of the process. Thus from $(k_1 \dots k_n)$ we have up changes to $(k_1 \dots k_n k)$ upon an arrival of a new job at the system, and if this job is of type k, this happens with intensity λ_k $(k \in \mathbb{J})$. On the other side we have a down change from $(k_1 \dots k_n)$ to $(k_2 \dots k_n)$ upon a completion of the service of job of type k_1 and this happens with intensity μ_{k_1}. All other changes have intensity zero, because they require at least two jumps of the process. Let $\boldsymbol{Q}$ denote the intensity matrix of the process.

Question: Find conditions for the recurrence of the process.

Let $\rho_i = \frac{\lambda_i}{\mu}$ and $\rho = \sum_i \rho_i$.

Proposition 4.1 *If $\rho < 1$, then the process $K(t)$ is ergodic with stationary distribution*

$$\pi(k_1 \dots k_n) = (1-\rho)\rho_{k_1} \cdots \rho_{k_n} \tag{4.8}$$

Proof We have to find the stationary distribution $\boldsymbol{\pi}$, which is the unique solution of $\boldsymbol{\pi Q} = \boldsymbol{0}$. Writing this in the form of the full balance equation we have

$$\pi(k_1 \dots k_n)(\sum_k \lambda_k + \mu) = \sum_k \pi(k k_1 \dots k_n)\mu + \pi(k_1 \dots k_{n-1})\lambda_{k_n} \tag{4.9}$$

By inspection we can verify that $\pi(k_1 \dots k_n) = \pi(0)\rho_{k_1} \dots \rho_{k_n}$. The LHS of (4.9) is now

$$\begin{aligned} &\lambda\pi(0)\rho_{k_1} \dots \rho_{k_n} + \mu\pi(0)\rho_{k_1} \dots \rho_{k_n} \\ = &\sum_k \mu_k \pi(0)\rho_k \rho_{k_1} \dots \rho_{k_n} + \lambda_{k_n}\pi(0)\rho_{k_1} \dots \rho_{k_{n-1}} \\ = &\sum_k \lambda_k \pi(0)\rho_{k_1} \dots \rho_{k_n} + \mu\pi(0)\rho_{k_1} \dots \rho_{k_n} \end{aligned}$$

We now have to compute $\pi(0)$. Thus

$$S = \sum_{n=0}^{\infty} \sum_{k_i \in \mathbf{K}^*} \pi(0)\rho_{k_1} \dots \rho_{k_n} = \pi(0)(1 + \sum_{n=1}^{\infty} (\sum_k \rho_k)^n) = \frac{\pi(0)}{(1-\rho)}.$$

The above is finite if $\rho < 1$ and sums up to 1 if $\pi(0) = (1 - \rho)$.

4.3 $\sum_k M_k / \sum_k M_k /1$-LCFS Queue.

Consider now the following service discipline. A new job is being served immediately upon its arrival and the job served before stays the next to the line. We suppose that an arriving job joins jobs present in the system from the left end and that the job from the left end is serviced.

Thus we have the following changes. From $(k_1 \dots k_n)$ we have up changes upon an arrival of a new job at the system to state $kk_1 \dots k_n$, and if this job is of type k, this happens with intensity λ_k $(k \in \mathbb{J})$. On the other side we have a down change from $(k_1 \dots k_n)$ to $(k_2 \dots k_n)$ upon a completion of the service of job of type k_1 and this happens with intensity μ_{k_1}. All other changes have intensity zero, because they require at least two jumps of the process. Let $\boldsymbol{Q}$ denote the intensity matrix of the process.

Question: Find conditions for the recurrence of the process.

Proposition 4.2 *If $\rho < 1$ then the process $K(t)$ is ergodic with stationary distribution*

$$\pi(k_1 \dots k_n) = (1-\rho)\rho_{k_1} \cdots \rho_{k_n} \tag{4.10}$$

Proof . We have to find the stationary distribution $\boldsymbol{\pi}$, which is the unique solution of $\boldsymbol{\pi Q} = \mathbf{0}$. Writing this in the form of the full balance equation we have

$$\pi(k_1 \dots k_n)(\sum_k \lambda_k + \mu_{k_1}) = \sum_k \pi(kk_1 \dots k_n)\mu_k + \pi(k_2 \dots k_n)\lambda_{k_1} \tag{4.11}$$

By inspection we can verify that $\pi(k_1 \dots k_n) = \pi(0)\rho_{k_1} \dots \rho_{k_n}$. The rest is as in the last subsection. The LHS of (4.9) is now

$$\begin{aligned} &\lambda\pi(0)\rho_{k_1} \dots \rho_{k_n} + \mu_{k_1}\pi(0)\rho_{k_1} \dots \rho_{k_n} \\ &= \sum_k \mu_k \pi(0)\rho_k\rho_{k_1} \dots \rho_{k_n} + \lambda_{k_1}\pi(0)\rho_{k_2} \dots \rho_{k_n} \\ &= \sum_k \pi(kk_1 \dots k_n)\mu_k + \pi(k_2 \dots k_n)\lambda_{k_1} \end{aligned}$$

The rest is the same as in Proposition 4.1.

4.4 $\sum_k M_k/\sum_k M_k/1$-PS Queue.

Consider now the system working under processor sharing (PS) discipline. This is when the server shares the whole its potential equally to all jobs present in the system. More formally, if in the time interval $[t_0, t_1]$ there are n jobs in the system, then each of them will receive $n^{-1}(t_1 - t_0)$ units of service in that time interval. We also add a technical assumption that an arriving job is equally likely to join the queue in any position.

It is helpful to introduce two operators $T_i^- : \mathbb{K}^* \to \mathbb{K}^*$ and $T_{ik}^+ : \mathbb{K}^* \to \mathbb{K}^*$. The first operator T_i^- $(i = 1, \ldots n)$, acting on $(k_1 \ldots k_n)$, it deletes the i-th in order element. We count position i $(i = 1, \ldots n+1)$ of $(k_1 \ldots k_n)$ as follows: position 1 is the left of k_1, position 2 between k_1 and k_2 and so on. Now we define operator T_{ik}^+, which acting on $(k_1 \ldots k_n)$ add $k \in \mathbb{J}$ at position i.

We have the following changes with positive intensity. From $(k_1 \ldots k_n)$ we have up changes upon an arrival of a new job at the system to state $T_{ik}^+(k_1 \ldots k_n)$ $(i = 1, \ldots, n+1, k \in \mathbb{J})$ with intensity $\lambda_k/(n+1)$. On the other side we have a down change from $(k_1 \ldots k_n)$ to $T_i(k_1 \ldots k_n)$ upon a completion of the service of job of type k_i of the i-element and this happens with intensity μ_{k_i}/n. All other changes have intensity zero, because it requires at least two jumps of the process. The full balance equation is

$$\pi(k_1 \ldots k_n)(\sum_k \lambda_k + \frac{1}{n}\sum_{i=1}^{n} \mu_{k_i})$$

$$= \frac{1}{n+1}\sum_{k \in \mathbb{J}} \pi(T_{ik}^+(k_1 \ldots k_n))\mu_k + \frac{1}{n}\sum_{i=1}^{n} \lambda_{k_i}\pi(T_i^-(k_1 \ldots k_n).$$

Proposition 4.3 *If $\rho < 1$ then the process $K(t)$ is ergodic with stationary distribution*

$$\pi(k_1 \ldots k_n) = (1-\rho)\rho_{k_1} \cdots \rho_{k_n} \tag{4.12}$$

4.5 $\sum_k M_k/\sum_k M_k/\infty$ Queue

In this system jobs from the class $k \in \mathbb{J}$ obtain immediate service, which length is exponentially distributed with intensity μ_k. From $(k_1 \ldots k_n)$ we have upward-changes upon an arrival of a new job at the system to state $T_{ik}^+(k_1 \ldots k_n)$ $(i = 1, \ldots, n+1, k \in \mathbb{J})$ with intensity $\lambda/(n+1)$. On the other side we have a down-change from $(k_1 \dots k_n)$ to $T_i^-(k_1 \dots k_n)$ upon a completion of the service of job of type k_i of the i-element and this happens with intensity μ_{k_i}. All other changes have intensity zero, because it requires at least two jumps of the process. The full balance equation is

$$\pi(k_1 \dots k_n)(\sum_k \lambda_k + \sum_{i=1}^n \mu_{k_i})$$

$$= \sum_{k \in \mathbb{J}} \pi(T_{ik}^+(k_1 \dots k_n))\mu_{k_i} + \frac{1}{n}\sum_{i=1}^n \lambda_{k_i}\pi(T^-(k_1 \dots k_n).$$

Proposition 4.4 *For each $\rho > 0$*

$$\pi(k_1 \dots k_n) = \exp(-\rho)\frac{\rho_{k_1} \cdots \rho_{k_n}}{n!} \tag{4.13}$$

Denote by $L_k(t) = \#\{i : K_i(t) = k\}$ the number of jobs of class $k \in \mathbb{J}$ at time t.

Corollary 4.5 *In stationary conditions $L_1, L_2, \dots$ are independent random variables and L_k has the Poisson distribution with mean ρ_k.*

4.6 $\sum_k M_k / \sum_k M_k / K$-Loss System

Consider now the Erlang loss system with jobs of different classes. The system is blocked if there is K jobs. The state space is now

$$\mathbb{K}^* = \bigcup_{n=0}^{K} \{(k_1 \dots k_n),\ k_i \in \mathbb{J}\}.$$

The evolution of the system is similar to the considered before infinite-server system, with the change that there is no upward-changes to states with $K+1$ jobs. From $(k_1 \dots k_n)$, where $n < K$, we have upward-changes upon an arrival of a new job at the system to state $T_{ik}^+(k_1 \dots k_n)$ $(i = 1, \dots, n+1,\ k \in \mathbb{J})$ with intensity $\lambda_k/(n+1)$. On the other side we have a down-change from $(k_1 \dots k_n)$ to $T_i^-(k_1 \dots k_n)$ upon a completion of the service of job of type k_i of the i-element and this happens with intensity μ_{k_i}. All other changes have intensity zero, because it requires at least two jumps of the process. The full balance equation is

$$\pi(k_1 \dots k_n)(\sum_k \lambda_k I(n < K) + \sum_{i=1}^{n} \mu_{k_i})$$

$$= \sum_{k \in \mathbb{J}} \pi(T_{ik}^{+}(k_1 \dots k_n))\mu_{k_i} I(n < K) + \frac{1}{n}\sum_{i=1}^{n} \lambda_{k_i} \pi(T^{-}(k_1 \dots k_n)).$$

Proposition 4.6 *For each* $\rho > 0$

$$\pi(k_1 \dots k_n) = \frac{\frac{\rho_{k_1} \cdots \rho_{k_n}}{n!}}{1 + \rho + \dots + \frac{\rho^K}{K!}}, \qquad (k = 0, \dots, K) \tag{4.14}$$

Suppose there are J classes of jobs. Let $L_k(t) = \#\{i : K_i(t) = k\}$. For the process $\{(L_1(t), \dots, L_J(t))\}$ the state space is

$$\mathcal{L} = \{(n_1, \dots, n_J) \in \mathbb{Z}_+ : \ n_1 + \cdots + n_J \le K\}.$$

Corollary 4.7 *In stationary conditions* $L_1, \dots, L_J$ *are independent random variables and for* $(n_1, \dots, n_J) \in \mathcal{L}$

$$\Pr\{(L_1, \dots, L_J) = (n_1, \dots, n_J)\} = S^{-1} \prod_{j=1}^{J} \frac{\rho_j^{n_j}}{n_j!},$$

where

$$S = \sum_{\boldsymbol{n}: \sum_{j=1}^{J} n_j \le K} \prod_{j=1}^{J} \frac{\rho_j^{n_j}}{n_j!}.$$

We can generalise the system as follows. As before there are K servers in the system and no waiting room. There are J classes of jobs (thus $\mathbb{J} = \{1, \dots, J\}$), and job of class j require M_j servers (simultaneously), which they hold by exponentially distributed time with parameter μ_j. Any job who, upon arrival, does not find enough idle servers is lost (blocked).

Corollary 4.8 *In stationary conditions* $L_1, \dots, L_J$ *are independent random variables and*

$$\Pr\{(L_1, \dots, L_J) = (n_1, \dots, n_J)\} = S^{-1} \prod_{j=1}^{J} \frac{\rho_j^{n_j}}{n_j!}, \qquad \boldsymbol{n} \in \mathcal{S},$$

where

$$S = \sum_{\boldsymbol{n}\in\mathcal{S}} \prod_{j=1}^{J} \frac{\rho_j^{n_j}}{n_j!},$$

where

$$\mathcal{L} = \left\{(n_1,\ldots,n_J):\ \sum_{j=1}^{J} n_j M_j \le K\right\}.$$

Comments. For generalised Erlang loss system see [7].

4.7 Symmetric Queue.

Consider the queue with infinite state space $\mathbb{K}^*$ as in (4.2). and N is the maximal number of jobs in the system. Desciption of the queue is as follows:

- jobs of type k arrive according to a Poisson stream with rate λ_k,
- service requirements of class k job is exponentially distributed with intensity μ_k;
- $\psi(n)$ is the total service effort, if there is n jobs in the system,
- the proportion of service that is directed at the job in position i is $(\gamma(n,l),\ l=1,\ldots,n)$, where $\sum_{l=1}^{n}\gamma(n,l)=1$,
- if there is n jobs in the system, then the new arriving job is moved to position i $(i=1,\ldots,n)$ with probability $(\beta(n,l),\ l=1,\ldots,n)$,
- arrival steams and service requirements are independent.

Definition 4.9 It is said the the system is symmetric if $\delta(n,l)=\beta(n,l)$.

For the symmetric queue we have the following intensity matrix:

- from $(k_1\ldots k_n)$ to $T^+_{ik}(k_1\ldots k_n)$ $(i=1,\ldots,n+1, k\in\mathbb{J})$ with intensity $\gamma(n+1,i)\lambda_k I(n<K)$,
- and from $(k_1\ldots k_n)$ to $T^-_i(k_1\ldots k_n)$ with intensity $\psi(n)\gamma(n,i)\mu_{k_i}$.

All other changes have intensity zero, because they require at least two jumps of the process.

We now show systems, which are symmetric.

LIFO $\beta(i,n) = 1(i = n)$ and $\phi(n) = 1$,

PS $\beta(i,n) = 1/n$ and $\phi(n) = 1$,

IS $\beta(i,n) = 1/n$ and $\phi(n) = n$.

We now show the stationary distribution for $\boldsymbol{K}(t)$,

Proposition 4.10 *Let*

$$\sigma = 1 + \sum_{n=1}^{K} \sum_{k_1 \ldots k_n} \frac{\rho_{k_1} \ldots \rho_{k_n}}{\psi(1) \ldots \psi(n)} < \infty. \tag{4.15}$$

Then the process $\boldsymbol{K}(t)$ is ergodic with the stationary distribution

$$\pi_{(k_1 \ldots k_n)} = \sigma^{-1} \frac{\rho_{k_1} \ldots \rho_{k_n}}{\psi(1) \ldots \psi(n)}. \tag{4.16}$$

4.8 $M/M/1$ Queue with Feedback

Jobs arrive at the system according to a Poisson process with intensity λ. After having received a service, a job may either leave the system or be fed back. When a job has completed his i-th service, he departs from the system with probability $1 - p(i)$ and is fed back with probability $p(i)$. Fed back jobs return instantaneously, joining the end of the queue. A job, who is visiting the queue for the i-th time will be called a *type-i job*. The service discipline is FCFS. It is assumed that the successive service requirements of a job are independent, negative exponentially distributed, random variables with parameter μ. Let

$$q(0) = 1, \qquad q(i) = \prod_{j=0}^{i-1} p(j), \qquad (i = 1, 2, \ldots), \tag{4.17}$$

with $p(0) = 1$. Let $\rho_k = \lambda q(k)/\mu$. The traffic intensity is

$$\rho = \frac{\lambda}{\mu} \sum_{i=1}^{\infty} q(i) = \sum_{i=1}^{\infty} \rho_i. \tag{4.18}$$

Let L_i be the number of i-type jobs and $L = \sum_i L_i$.

Proposition 4.11

$$\Pr\{L_1 = n_1, L_2 = n_2, \ldots\} = (1-\rho)\left(\sum_{i=1}^{\infty} n_i\right)! \prod_{i=1}^{\infty} \frac{\rho^{n_i}}{n_i!} \tag{4.19}$$

and

$$\Pr\{L = j\} = (1-\rho)\rho^j, \qquad (j = 0, 1, \ldots). \tag{4.20}$$

Problems

4.1 Show that in the $M/M/1$ with feedback

$$\mathbf{E}\left\{\prod_{i=1}^{\infty} z_i^{L_i}\right\} = \frac{1-\rho}{1 - \sum_{i=1}^{\infty} \rho_i z_i}.$$

Comments. Feedback queues were used to study PS queues in [2].

5 Reversibility and quasi-reversibility for multi-class queues.

We already know that ergodic B&D queues are reversible.

Proposition 5.1 *The state process $(\boldsymbol{K}(t))$ of a stationary $\sum_k M_k/M/1$ or symmetric queue with exponentially distributed service requirements is reversible.*

Consider the number of jobs $Q_k(t)$ at time t of class k $(1 \le k \le K)$. Note that the process $\{(Q_k(t),\ k = 1, \ldots, K\}$ need not be Markov.(Exer. Argue!) The arrival process N_k^{a} consists of upward-changes of the process $\{Q_k(t),\ t \ge 0\}$ and the departure process N_k^{d} of jobs of class k consists of downward-changes of the process $\{Q_k(t),\ t \ge 0\}$. By $N_{|I}$ we denote the restriction of the point process N to interval I. Let $N^{\mathrm{a}} = (N_k^{\mathrm{a}},\ k \in \mathbb{I})$ and $N^{\mathrm{d}} = (N_k^{\mathrm{d}},\ k \in \mathbb{I})$

Corollary 5.2 *For each $t \ge 0$, $(Q_k(t),\ k \in \mathbb{I})$ is independent of $N^{\mathrm{a}}_{|(0,t]}$. Similarly $(Q_k(t),\ k \in \mathbb{I})$ is independent of $N^{\mathrm{d}}_{|[t,\infty)}$.*

It can be proved using stronger results that $N^{\mathrm{a}}_{|(0,t]}$, $N^{\mathrm{d}}_{|[t,\infty)}$ and $(Q_k(t),\ k = 1,\dots,K)$ are mutually independent. A queue with such a property is said to be *quasi-reversible.*

Comments. [4], [9]

6 M/M/k; shortest queue

7 Queues with vacations

Comments. Doshi, B.T. [3] Kramer, M. [5]

8 Exercises

8.1 Let $(\Pi_m(t))_{m\geq 0}$ be a family independent Poisson processes with intensity μ_n, where $\mu_0 = \lambda$ independent of $X(0)$.
(i) Show that

$$X(t) = X(0) + \Pi_0(\lambda t) - \int_0^t \sum_{n\geq 1} 1(X(t-\circ) = n)\, \Pi_n(dt)$$

is a regular queueing B&D process. Find its intensity matrix.
(ii) Show that there is a unique solution of the SDE above.

8.2 For the M/M/1/N find the stationary distribution for the embedded DTMC $(Y_n)_n$.

8.3 Let Q be the number of jobs process in M/M/1 system.

$$M(t) = Q(t) - x - \lambda t + \mu \int_0^t 1(Q(s) > 0) ds$$

is a martingale.

8.4 Suppose that $(\Pi_0(t))_{t\geq 0}$ and $(\Pi_1(t))_{t\geq 0}$ are independent Poisson process with intensities λ and μ respectively. Show that if Q is the number of jobs in the system process in the M/M/1 queue with $Q(0) = i$, then

$$Q(t) \;=_{\mathrm{d}}\; \max(i + Z(t), L(t)),$$

where $Z(t) = \Pi_0(t) - \Pi_1(t)$ and $L(t) = \sup_{0\leq s\leq t} Z(s)$ (Example 7.4 in Asmussen (2003), p. 98).

8.5 Prove that the number of jobs in the system $Q(t)$ process in $M/M/c$ queue is positive recurrent if and only if $\rho < 1$.

8.6 Show that for the $M/M/c$ queue the stationary mean number of jobs in the system

$$\mathbb{E}_\pi Q = \frac{1}{\sigma}\left\{\sum_{n=1}^{c-1} \frac{\eta^n}{(n-1)!} + \frac{\eta^c}{c!}\left[\frac{\rho}{(1-\rho)^2} + \frac{m}{1-\rho}\right]\right\},$$

the stationary probability that the system is empty

$$\pi_0 = \frac{1}{\sigma},$$

and the stationary probability that all servers are busy

$$\pi_c + \pi_{c+1} + \cdots = \frac{1}{\sigma}\frac{\rho^c}{c!}\frac{c}{c-\rho}.$$

Show that the queue size process in the $M/M/c$ queue is $Q_q(t) = (Q(t)-c)_+$ and find its stationary distribution (that is $\pi^q_n = \lim_{t\to\infty} \Pr(Q_q(t) = n)$).

8.7 In M/M/K/K loss system show that the stationary overflow rate $\beta(K, \lambda, \mu) = \lambda - \lambda^{\mathrm{a}}$, where λ^{a} is the steady-state rate of accepted calls fulfills

$$\beta(K, \lambda, \mu) = \lambda B(K, \rho).$$

8.8 Let Q_1, Q_2 are independent random variables representing the number of customers in the loss system with parameters K_1, K_2, λ_1, λ_2 and μ_1, μ_2 and Q with parameters $K_1 + K_2$, $\lambda_1 + \lambda_2$ and $\mu_1 + \mu_2$ respectively. Show that

$$Q_1 + Q_2 <_r Q,$$

where $<_r$ denotes the monotone likelihood-ratio ordering. Conclude that

$$\beta(K_1 + K_2, \lambda_1 + \lambda_2, \mu_1 + \mu_2) \le \beta(K_1, \lambda_1, \mu_1) + \beta(K_2, \lambda_2, \mu_2).$$

8.9 Suppose there are K terminals and N users where $K < N$. and assume that each user requests a terminal with intensity λ and the time utilized by a user is exponentially distributed with parameter μ. Let $\rho = \lambda/\mu$. As usual we suppose that all variables are independent. Show that under a work conserving dysciple, the number of used terminals $(Q(t),\ t \ge 0)$ is a B&D queueing process with state space $\{0, \dots K\}$ and intensities

$$\lambda_n = (N-n)\lambda \qquad \mu_n = n\mu.$$

Furthermore show that the stationary distribution

$$\pi_n = \frac{\binom{N}{n}\rho^n}{1 + \binom{N}{1}\rho + \cdots + \binom{N}{K}\rho^K}, \tag{8.1}$$

where $\rho = \lambda/\mu$.

8.10 Show that a vector process $\boldsymbol{X}(t) = (X_1(t), \ldots, X_m(t))$ of independent reversible processes $(X_j(t))$ is reversible.

8.11 [Bramson] Two single server queueing systems are supposed to share a common buffer of size $N-2$. Jobs for the $i = 1,2$ system are arriving at the server according to a Poisson processes Π_i with intensity λ_i and their service requirements $(S_n^{(i)})_n$ are i.i.d. exponentially distributed $\mathrm{Exp}(\mu_i)$. We assume that Π_i, $(S_n^{(i)})_n$ $(i = 1,2)$ are independent. If the buffer is full the arriving jobs are lost. Using the result of Exercise II.3.21 and 8.10 show that the stationary distribution of $\boldsymbol{Q} = (Q_1, Q_2)$; Q_i is the number of jobs

$$\pi_{\boldsymbol{n}} = \pi_{00}\rho_1^{n_1}\rho_2^{n_2}, \qquad n_1 + n_2 \le N \ .$$

where $\rho_i = \lambda_i/\mu_i$ and

$$\pi_{0,0} = (\sum_{0\le n_1+n_2\le N} \rho_1^{n_1}\rho_2^{n_2})^{-1} \ .$$

8.12 [Robert, p. 88] Consider the following loss system. The network has 3 nodes, which are vertices of graph $\{(1,2),(1,3)\}$. Jobs of type i arrives according to a Poisson process with rate λ_i and bring their service requirements $S_n^{(i)}$ $(i = 1,2)$. Arrivals and service requirements are independent. Jobs of type 1 arrive at node 1 and occupe a link along route $(1,2)$. Similarly jobs of type 2 arrive at node 3 and occupe a link along route $(3,1),(1,2)$. At route $(1,2)$ there are N_1 links available and at route $(3,1)$ there are N_2 links available. Let $\mathbb{E} = \{\boldsymbol{n} \in \mathbb{Z}_+^2 : n_1 + n_2 \le N_1, n_2 \le N_2\}$. Show that

$$\pi_{\boldsymbol{n}} = \pi_{00}\frac{\rho_1^{n_1}}{n_1!}\frac{\rho_2^{n_2}}{n_2}, \qquad \boldsymbol{n} \in \mathbb{E},$$

where

$$\pi_{00} = (\sum_{\boldsymbol{n}\in\mathbf{E}} \frac{\rho_1^{n_1}}{n_1!}\frac{\rho_2^{n_2}}{n_2!})^{-1} \ .$$

Notice that

$$\pi_{\boldsymbol{n}} = \mathbb{P}(\Pi^{\rho_1} = n_1, \Pi^{\rho_2} = n_2 | \Pi^{\rho_1} + \Pi^{\rho_2} \le N_1, \Pi^{\rho_2} \le N_2) \ .$$

8.13 [Kelly, p. 29] Suppose that the stream of jobs arriving at a two-server queue is Poisson with rate λ and each job brings its service requirement (S_n), where (S_n) are i.i.d. exponentially distributed with paramenter 1. Server 1 works with rate μ_1 and server 2 with rate μ_2. Arrivals and service requirements are independent. If a job arrives to find both the servers free it is allocated to the server who has benn free for the longest time. Show that the queue is not reversible. Find the stationary distribution.

8.14 Show that the number of jobs at nodes process $\boldsymbol{Q}(t)$ in a system of m queues in tandem is not reversible.

8.15 Let $\Pi_0(t), \ldots, \Pi_m(t)$ are independent Poisson processes with intensities $\lambda, \mu_1, \ldots, \mu_m$ respectively. Show that the number of jobs process $\boldsymbol{Q}$ in m queues in tandem is the solution of SDE

$$\boldsymbol{Q}(t) = \boldsymbol{k} + \boldsymbol{Z}(t) + \boldsymbol{L}(t)\,,$$

where $\boldsymbol{Z}(t) = (\Pi_0(t) - \Pi_1(t), \Pi_1(t) - \Pi_2(t), \ldots, \Pi_{m-1}(t) - \Pi_m(t))$ and

$$\boldsymbol{L}(t) = (\int_0^t 1(Q_i(s-) = 0)\, d\Pi_i(s))_{i=1,\ldots,m}\ .$$

The process $\boldsymbol{L}(t) = \boldsymbol{0}$ until time τ, where $\tau = \inf\{t : \min_{i=1,\ldots,m} k_i + Z_i(t) = 0$. Note that τ is the (first) *collision time* for the process $\boldsymbol{\xi}(t) = (k_1 + \ldots + k_m + \Pi_0(t), k_2 + \ldots + k_m + \Pi_1(t), \ldots, \Pi_m(t))$.

8.16 Show that the stationary process $\boldsymbol{Q}(t)$ in queues in tandem is not reversible.

8.17 [Asmussen] Show that the time-reversed process $\tilde{\boldsymbol{Q}}(t)$ of $\boldsymbol{Q}(t)$ in Gordon-Newel network is again a Gordon-Newel network with routing matrix $\tilde{\boldsymbol{P}} = (\tilde{p}_{ij})$, where $\bar{\lambda}_i \tilde{p}_{ij} = \bar{\lambda}_j p_{ji}$.

8.18 [Robert] Consider the Gordon-Newel network with a pure ring structure, i.e. $p_{i,i+1} = 1$ for $i = 1, \ldots, m-1$ and $p_{m1} = 1$. Compute the stationary distribution.

8.19 [Robert] Consider the following modification of the Jackson network, wherein server at node i has a speed $\phi_i(n_i)$ when there are n_i customers present in node i. We suppose that $\phi_i(k) > 0$ for all $k \geq 1$ and

$\phi_i(0) = 0$. The new intensity matrix is obtained from the standard one by replacing μ_i by $\mu_i\phi_i(n_i)$. Show that if for all $i = 1, \ldots, m$

$$A_i = 1 + \sum_{n_i=1}^{\infty} \left(\frac{\rho_i^{n_i}}{\prod_{k=1}^{n_i} \phi_i(k)} \right) < \infty$$

where $\rho_i = \bar{\lambda}_i/\mu_i$, then the network is ergodic with stationary distribution

$$\pi\boldsymbol{n} = \prod_{i=1}^{m} \pi_i(n_i),$$

where

$$\pi_i(n_i) = \frac{1}{A_i} \frac{\rho_i^{n_i}}{\prod_{k=1}^{n_i} \phi_i(k)} \ .$$

8.20 Show that in Gordon-Newel networks, the intensity of the flow of jobs transferred from i to j is

$$\lambda(t) = 1(Q_i(t) = i)\mu_i p_{ij} \ .$$

Conclude that the throughput is

$$d_{ij} = \lim_{t\to\infty} \frac{1}{t}\lambda(s)\, ds - \text{a.s.} \ .$$

8.21 Let $(Q(t))_{t\geq 0}$ be the stationary number of jobs in M/M/c. and let $N^{\mathrm{w}}(t)$ be the number of jobs arriving in $(0, t)$ which have to wait;

$$N^{\mathrm{w}}(t) = \int_0^t 1(Q(s - \circ) \geq c)\, d\Pi^{\lambda}(s) \ .$$

Then $(N^{\mathrm{w}}(t))_{t\geq 0}$ is the process with stationary increments. Show that N^{w} is a process with stationary increments.

8.22 Prove that under the Halfin-Whitt regime

$$B(c, \rho) \sim \frac{\phi(\beta)}{\Phi(\beta)\sqrt{\rho}}$$

and find the corresponding asymptotic for $C(c, \rho)$. Hint. Let Π^ρ be distributed as Poisson with mean ρ. For the asymptotics of $\mathbb{P}(\Pi^\rho = c) = \mathbb{P}(\Pi^\rho = \rho + \beta\sqrt{\rho})$ use Stirling formula and for $\Pi^\rho \leq c) = \mathbb{P}(\Pi^\rho \leq \rho + \beta\sqrt{\rho})$ use the central limit theorem. In particular you must show that

...

Bibliography

[1] Asmussen, S. (2003) *Applied Probability and Queues*. Springer, New York.

[2] Van der Berg, J. L. & Boxma, O. J. (1991) The $M/G/1$ queue with processor sharing and its relation to feedback queue. *Queueing Systems* **9**, 365–402.

[3] Doshi, B.T. (1986) Queueing systems with vacations – a survay. *QUESTA*, **1**, 29–66.

[4] Kelly, F.P. (1979) *Reversibility and Stochastic Networks*. Wiley, Chichester.

[5] Kramer, M. (1989) Stationary distributions in a queueing system with vacation times and limited service. *QUESTA*, **4**, 57–68.

[6] Chao, X., Miyzazawa, M. and Pinedo, M. (1999) *Queueing Networks; Customers, Signals and Product Form Solutions*. Wiley, New York.

[7] Reiman, M. I. (1991) A critically loaded multiclass Erlang loss system. *Queueing systems*, **9**, 65–82.

[8] Robert, Ph. (2003) *Stochastic Networks and Queues*.Springer, Berlin.

[9] Walrand, J. (1988) *An Introduction to Queueing Networks*. Prentice Hall, Englewood Cliffs.

[10] Smith, D. R. & Whitt, W. (1981) Resource sharing for system efficiency in traffic systems. *Bell System Technical Journal*, **60**, 39–55.

Chapter V

Transient analysis of Markovian queues

1 Transient analysis of finite CTMC's

We consider an irreducible CTMC $(X(t))_{t\geq 0}$ with state space $0,\ldots,m$ and $\boldsymbol{Q}$ is an intensity matrix. From Appendix .4 we know that

$$\boldsymbol{P}(t) = e^{t\boldsymbol{Q}}\ .$$

Then $P^\circ = \frac{1}{q}\boldsymbol{Q} + \boldsymbol{I}$, where $q \geq \max q_i$, is a primitive stochastic matrix. We now use the following fact. We have θ is an eigenvalue of $\boldsymbol{A}$, if and only if $c\theta + 1$ is an eigenvalue of $c\boldsymbol{A} + \boldsymbol{I}$. Therefore eigenvalue 1 of $\boldsymbol{P}^\circ$ corresponds to 0 of $\boldsymbol{Q}$ and since other of $\boldsymbol{P}^\circ$ are locate inside the circle of modulus 1, then for all other then 0 eigenvalues of $\boldsymbol{Q}$ the real part is strictly negative. Thus let $0 = \theta_0, \theta_1, \ldots, \theta_m$ where $0 = \theta_0 > \Re\theta_1 \geq \ldots \geq \Re\theta_m$ be the set of eigenvalues of $\boldsymbol{Q}$.

Let $\boldsymbol{x}_1$ be the left eigenvector of $\boldsymbol{Q}$ corresponding to $\theta_1 = 0$. Notice that because the chain is irreducible (and so ergodic) $\boldsymbol{x}_1 = \boldsymbol{\pi}$ is the stationary distribution (without loss of generality we may assume that $\boldsymbol{\pi}$ is normalized that sum of components is 1). Furthermore by inspection we have that $\boldsymbol{e}$ is the right eigenvector. Since $\boldsymbol{x}_1\boldsymbol{\xi}_1 = \boldsymbol{e}\boldsymbol{\pi} = (\pi_j)_{i,j=1}^m$, using spectral representation from Appendix (.7) we see that

$$e^{t\boldsymbol{Q}} - (\pi_j)_{i,j=1}^m = \sum_{j=2}^m e^{t\theta_j}\boldsymbol{x}_j\boldsymbol{\xi}_j\ ,$$

where $\boldsymbol{x}_j$ and $\boldsymbol{\xi}_j$ are left and right eigenvector corresponding to θ_j, provided eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are independent.

We will show that in some interesting for us cases we are able to give explicit solution. For this we must find the spectrum, that is we have to determine all the eigenvalues.

1.1 M/M/1/N queue

In this section we consider the number of jobs process $Q(t)$ in M/M/1/N system with finite buffer of size $N-1$, with arrival intenisty λ and service intensity μ and FCFS queueing dyscipline. In this case $\boldsymbol{Q}$ as it was shown in Section IV.1.1 is

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdot \\ \mu & -\lambda-\mu & \lambda & 0 & \cdot \\ 0 & \mu & -\lambda-\mu & \lambda & \cdot \\ 0 & 0 & \mu & -\lambda-\mu & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mu & -\mu \end{pmatrix}$$

In the next propisition we compute eigenvalues. In the proof we follow Takacs [18], Chapter 1.1.

Proposition 1.1 *Eigenvalues θ_j $(j = 0, \ldots, N$ are*

$$\theta_j = 2\sqrt{\lambda\mu}\cos\left(\frac{j\pi}{N+1}\right) - (\lambda+\mu)$$

Let $\boldsymbol{x}$ be an eigenvector corresponding to θ, that is $(\boldsymbol{Q}+\theta\boldsymbol{I})\boldsymbol{x} = \mathbf{0}$. That is the following system of linear equations hold

$$-(\lambda+\theta)x_0 + \lambda x_1 = 0 \qquad (1.1)$$
$$\mu_{i-1}x_{j-1} - (\lambda+\mu+\theta)x_j + \lambda x_{j+1} = 0 \qquad i = 1, \ldots, N-1$$
$$\mu x_{N-1} - (\mu+\theta)x_N = 0 \qquad (1.2)$$

Equations

$$\mu_{i-1}x_{j-1} - (\lambda+\mu+\theta)x_j + \lambda x_{j+1} = 0 \qquad j = 1, \ldots, N-1$$

form a recurrence relation of the second order and the general solution is of form $C_1 a^j + C_2 a^j$, where a fulfills the characteristic equation

$$\lambda a^2 - (\theta + \lambda = \mu)a + \mu = 0 \ ,$$

which have the roots:

$$a = \frac{\theta + \lambda + \mu \pm \sqrt{(\theta + \lambda + \mu)^2 - 4\lambda\mu}}{2\lambda} \ .$$

Substituting

$$\cos y = \frac{\lambda + \mu + \theta}{2\sqrt{\lambda\mu}},$$

we can rewrite the above in the form

$$\begin{aligned} a &= \frac{\theta + \lambda + \mu \pm \sqrt{(\theta + \lambda + \mu)^2 - 4\lambda\mu}}{2\lambda} \\ &= \frac{\frac{\theta+\lambda+\mu}{2\sqrt{\lambda\mu}2\sqrt{\lambda\mu}\pm 2\sqrt{\lambda\mu}\sqrt{\left(\frac{\theta+\lambda+\mu}{2\sqrt{\lambda\mu}}\right)^2 - 1}}}{2\lambda} \\ &= (\frac{\mu}{\lambda})^{1/2} e^{\pm \mathrm{i}\theta} \ . \end{aligned}$$

Hence

$$\begin{aligned} x_j &= C_1 a_1^j + C_2 a_2^j = (\frac{\mu}{\lambda})^{j/2}(C_1 e^{\mathrm{i}jy} + C_2 e^{-\mathrm{i}jy}) \\ &= (\frac{\mu}{\lambda})^{j/2}((C_1 + C_2)\cos(jy) + \mathrm{i}(C_1 - C_2)\sin(jy)) \end{aligned}$$

Substituting further $B = C_1 + C_2$ and $A = \mathrm{i}(C_1 - C_2)$ we obtain for $j = 0, 1, \ldots, N$

$$x_j = \left(\frac{\mu}{\lambda}\right)^{j/2} (A \sin jy + B \cos jy), \qquad (j = 0, 1, \ldots, N) \ .$$

Next (1.1) is fulfilled if

$$A = 1 - \left(\frac{\lambda}{\mu}\right)^{1/2}, \qquad B = -\left(\frac{\lambda}{\mu}\right)^{1/2} \ .$$

Now substitute to (1.2) TO BE CONTINUED.

1.2 Relaxation time for finite state CTMCs

Consider first a finite state space DTMC, defined by a *stochastic matrix* $\boldsymbol{P} = (p_{ij})_{i,j=1,\ldots,l}$. Recall that then all the entries $p_{ij} \geq 0$ and that $\sum_{j=1}^{l} p_{ij} = 1$ for all i. We say that $\boldsymbol{P}$ is *substochastic*, if for at least one row, say i_0, $\sum_{j=1}^{l} p_{i_0 j} < 1$. Recall that a probability vector $\boldsymbol{\pi}$, that is with nonnegative entries, summing up to 1, is a stationary distribution, if it solves $\boldsymbol{\pi}\boldsymbol{P} = \boldsymbol{\pi}$.

Corollary 1.2 *If $\boldsymbol{P}$ is a primitive stochastic matrix, then*
(a) $\theta_{\mathrm{PF}} = 1, \boldsymbol{x}_1 = \boldsymbol{e}$ *and* $\boldsymbol{\xi}_1 = \boldsymbol{\pi}$*;*
(b) $|\theta_i| < 1$ *for* $i = 2, \ldots, l$*.*

Proof We first remark that if θ is an eigenvalue, then $|\theta| \leq 1$, because from equation $\boldsymbol{P}\boldsymbol{x} = \theta\boldsymbol{x}$ we obtain

$$\max_j |x_j| \geq \sum_j p_{ij}|x_j| \geq |\theta||x_i|$$

for all $i = 1, \ldots, l$. Now choose $i = i_0$ such that $x_i = \max_j |x_j|$. Then $|\theta| \leq 1$. Next by inspection we get that $\boldsymbol{P}\boldsymbol{e} = \boldsymbol{e}$. Hence 1 is an eigenvalue of $\boldsymbol{P}$ and $\boldsymbol{e}, \boldsymbol{\pi}$ are right and left eigenvectors for this eigenvalue, respectively. Also, $\theta_1 = 1$, i.e. 1 is the eigenvalue with the largest modulus. □

We will reserve the letter $\boldsymbol{\pi}$ for the Perron-Frobenius left eigenvector, which is a probability function, for the case of primitive stochastic or substochastic matrices. Similarly we reserve $\boldsymbol{h}$ for the Perron-Frobenius right eigenvector.

We denote entries of $\boldsymbol{P}^n = (p_{ij}^{(n)})_{i,j=1,\ldots,l}$. and conclude with the following:

Corollary 1.3 *If $\boldsymbol{P}$ is a stochastic primitive matrix, then for some $k \in \mathbb{Z}_+$*

$$\begin{aligned} |p_{ij}^{(n)} - \pi_j| &= O(n^k|\theta_2|^n) \\ &= o(c^n) \end{aligned} \tag{1.3}$$

as $n \to \infty$, where $c > |\theta_2|$.

Suppose now that $\boldsymbol{Q}$ is an intensity matrix of an irreducible CTMC. Then $P^\circ = \frac{1}{q}\boldsymbol{Q} + \boldsymbol{I}$, wjere $q \geq \max q_i$ is a primitive stochastic matrix. We now use the following fact. We have θ is an eigenvalue of $\boldsymbol{A}$, if and only if $c\theta + 1$ is an eigenvalue of $c\boldsymbol{A} + \boldsymbol{I}$. Therefore eignevalue 1 of $\boldsymbol{P}^\circ$ corresponds to 0 of

$\boldsymbol{Q}$ and since others of $\boldsymbol{P}^\circ$ are locate inside the circle of modulus 1, then for all other then 1 eigenvalues of $\boldsymbol{Q}$ the real part is strictly negative. Thus let $0 = \theta_1, \theta_2, \ldots, \theta_n$ where $0 = \theta_1 > \Re\theta_2 \geq \ldots \geq \Re\theta_n$ be the set of eigenvalues of $\boldsymbol{Q}$. Let $\boldsymbol{\pi}$ be the left eigenvector of $\boldsymbol{Q}$ corresponfing to $\theta_1 = 0$, that is the stationary distribution (without loss of generality we may assume that $\boldsymbol{\pi}$ is normalized that sum of components is 1).

Corollary 1.4 *If $\boldsymbol{Q}$ is an irreducible intensity matrix, then for some $k \in \mathbb{Z}_+$*

$$\begin{aligned} |p_{ij}(t) - \pi_j| &= O(n^k e^{t\Re\theta_2}) \\ &= o(e^{-ct}) \end{aligned} \tag{1.4}$$

as $n \to \infty$, where $c > -\Re\theta_2$.

Definition 1.5 Quantity $1/(-\Re\theta_2)$ is called *relaxation time*

2 Continuous time Bernoulli random walk

The following theory is more advanced and will be used to study transient properties of M/M/1 systems in Chapter 3.6. We study now the B&D process, which under $\mathbb{P}_i$ evolves as $Z(t) = i + \Pi^\lambda(t) - \Pi^\mu(t)$,where $\Pi^\lambda(t)$ and $\Pi^\mu(t)$ are independent Poisson process with intensitites $\lambda > 0$ and $\mu > 0$ respectively. We have the following story about the process. As sometimes this process is called a taxistand process We have the following story about the process. Passengers and taxis arrive independent at a taxistand; each of streams is according to a Poisson process $(\Pi^\lambda(t))$ and $(\Pi^\mu(t))$. The state of the system is the number of passengers is the queue at the taxistand; if it is positive this is a queue of passengers, otherwise of taxis. Assuming that at time $t = 0$ the state is $k \in \mathbb{Z}$, our process is described by $Z(t) = k + \Pi^\lambda(t) - \Pi^\mu(t)$. It is not difficult to prove that $(Z(t))_{t\geq 0}$ is a CTMC with intensity matrix $\boldsymbol{Q} = (q_{ij})$ given by

$$q_{ij} = \begin{cases} \lambda & \text{for } j = i+1 \\ -\lambda - \mu & \text{for } j = i \\ \mu & \text{for } j = i-1 \end{cases} \tag{2.5}$$

Recall from Section 3.6 that such the process is said to be a continuous time Bernoulli random walk (CTBRW) Consecutive jumps define the point process $(\tau_n^{\mathrm{c}})_{n\geq 1}$ with

$$\tau_{n+1}^{\mathrm{c}} = \inf\{t > \tau_n^{\mathrm{c}} : Z(t) \neq Z(t-)\},$$

where $\tau_0^c = 0$. The embedded Markov chain $(Y_n = Z(\tau_n^c)\}$ has a very simple structure. It is a random walk with transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdots & 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \cdot & \cdots \\ \cdots & 0 & 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \cdots \\ \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} \tag{2.6}$$

and the initial state k. Since $\lambda > 0$ and $\mu > 0$, the process is irreducible. Denote $\rho = \lambda/\mu$.

Using known facts from the random walk theory we can prove that with $\mathbb{P}_k$-probability 1

$$\lim_{t\to\infty} Z(t) = \begin{cases} -\infty, & \text{for } \rho < 1 \\ \infty, & \text{for } \rho > 1 \end{cases} \tag{2.7}$$

$$\limsup_{t\to\infty} Z(t) = \infty, \qquad \text{for } \rho = 1. \tag{2.8}$$

By $\mathbb{P}_k^{\lambda,\mu}$ we denote the distribution of a jump Markov process $(Z(t))_{t\geq 0}$ governed by the intensity matrix $\boldsymbol{Q}$ and starting at $t = 0$ from k. Formally we can define a supporting probability space as follows: The basic probability space is $\Omega = \mathbb{Z}_+ \times \Omega_1 \times \Omega_2$, where $\Omega_i = \mathbb{R}_+$, $\mathcal{F} = \mathcal{B}(\Omega)$ is the σ-field generated by cylindrical sets. An element from Ω is $\omega = (z, \boldsymbol{x}, \boldsymbol{y})$. The probability measure $\mathbb{P}_\nu^{\lambda,\mu} = \nu \times \mathbb{P}^\lambda \times \mathbb{P}^\mu$, where ν is a distribution on $\mathbb{Z}$ and $\mathbb{P}^\lambda = \mathbb{E}\, E^\lambda \times \mathbb{E}^{\,\lambda} \times \ldots$ and similarly $\mathbb{P}^\mu = \mathbb{E}\, E^\mu \times \mathbb{E}^{\,\mu} \times \ldots$, where $E_a(A) = \int_A a \exp(-ax)\, dx$. Define now processes

$$\Pi_1(t) = \Pi_1(\omega, t) = \#\{m > 0 : x_1 + \cdots + x_m \leq t) \tag{2.9}$$

$$\Pi_2(t) = \Pi_2(\omega, t) = \#\{m > 0 : y_1 + \cdots + y_m \leq t) \tag{2.10}$$

$$Z(t) = Z(t, \omega) = k + \Pi_1(t, \omega) - \Pi_2(t, \omega), \tag{2.11}$$

where $\omega = (k, \boldsymbol{x}, \boldsymbol{y})$. Note that under $\mathbb{P}_n^{\lambda,\mu}$, processes $\Pi_1(t)$ and $\Pi_2(t)$ are independent Poisson processes with intensities λ and μ respectively and $Z(0) = k$.

For studying the taxistand problem we need a standardized Bessel's functions:

$$i_n(t) = \exp\left(-(\lambda + \mu)t\right) \rho^{n/2} I_n(2\sqrt{\lambda\mu}t), \tag{2.12}$$

where $I_n(x)$ is the n-th modified Bessesl function; see Appendix .3 The role of functions $i_n(t)$ explain the following proposition, wherein we show a formaula for transition probability function $p_{ij}^Z(t)$.

Proposition 2.1 *For $n, m \in \mathbb{Z}$*

$$\mathbb{P}_m(Z(t) = n) = i_{n-m}(t). \tag{2.13}$$

Proof Clearly $\mathbb{P}_m(Z(t) = n) = \mathbb{P}_0(Z(t) = n - m)$. Now

$$\begin{aligned}\mathbb{P}_0(Z(t) = n) &= \mathbb{P}_0(\Pi_1(t) - \Pi_2(t) = n) \\ &= \sum_{k=0}^{\infty} \frac{(\lambda t)^{k+n}}{(k+n)!} \exp(-\lambda t) \frac{(\mu t)^k}{k!} \exp(-\mu t) \\ &= \exp\left(-(\lambda + \mu)t\right) \rho^{n/2} \sum_{k=0}^{\infty} \frac{\left(\frac{2t\sqrt{\lambda\mu}}{2}\right)^{n+2k}}{(k+n)!k!} = i_n(t)\end{aligned}$$

□

In the sequel we denote

$$\kappa = \sqrt{\lambda\mu}. \tag{2.14}$$

and consider now on $(\Omega, \mathcal{F})$ a new probability measure $\mathbb{P}_0^{\kappa,\kappa}$. Note that under $\mathbb{P}_0^{\kappa,\kappa}$, processes Π_1 and Π_2 are independent Poisson process with the common intensity κ and $(Z(t))$ is a symmetric Bernoulli random walk, starting from 0. Recall that two distributions of Poisson point processes on $[0, t]$ with intensities λ_1 and λ_2 respectively are absolute continuous and denoting by $\mathbb{P}^{\lambda_i|t}$ the distribution of the Poisson process on $[0, t]$ with intensity λ_i we have

$$\frac{d\mathbb{P}^{\lambda_1|t}}{d\mathbb{P}^{\lambda_2|t}}(\omega) = \left(\frac{\lambda_1}{\lambda_2}\right)^{\Pi(t)} e^{(\lambda_2-\lambda_1)t}.$$

where $\Pi(t)$ denotes the number of point of Π in $[0, t]$; see Section III.1 Putting $\lambda_1 = \lambda$ and $\lambda_2 = \kappa$ we have the right hand side in (2)

$$\rho^{\frac{\Pi(t)}{2}} \exp\left((\kappa - \lambda)t\right). \tag{2.15}$$

Thus

$$\frac{d\mathbb{P}^{\lambda|t} \times \mathbb{P}^{\mu|t}}{d\mathbb{P}^{\kappa|t} \times \mathbb{P}^{\kappa|t}}(\omega) = \rho^{\frac{\Pi_1(t)-\Pi_2(t)}{2}} \exp\left((2\kappa - \lambda - \mu)t\right). \tag{2.16}$$

From Proposition 2.1 we obtain immediately the following corollary.

Corollary 2.2

$$\tilde{\mathbb{P}}_0(Z(t) = n) = \exp(-2\kappa t) I_n(2\kappa t). \tag{2.17}$$

Lemma 2.3 *For $k \le l$ and $l \ge 0$*

$$\tilde{\mathbb{P}}_0(\sup_{0\le s\le t} Z(s) \ge l, Z(t) = k) = \tilde{\mathbb{P}}_0(Z(t) = 2l - k) = \exp(-2\kappa t) I_{2l-k}(2\kappa t).$$

Proof Use reflection principle and Corollary 2.2. □

Remark In the remaining cases that is $k > l$ or $l < 0$ we have easily

$$\tilde{\mathbb{P}}_0(\sup_{0\le s\le t} Z(s) \ge l, Z(t) = k) = \tilde{\mathbb{P}}_0(Z(t) = k).$$

Lemma 2.4 *For $k \le l$ and $l \ge 0$*

$$\mathbb{P}_0(\sup_{0\le s\le t} Z(s) \ge l, Z(t) = k) = \rho^{k-l} i_{2l-k}(t). \tag{2.18}$$

and hence for $k \le 0$

$$\mathbb{P}_0(\sup_{0\le s\le t} Z(s) = 0, Z(t) = k) = \rho^k (i_{-k}(t) - \rho^{-1} i_{-k+2}(t)). \tag{2.19}$$

Proof We have from (2.16) that

$$\begin{aligned} &\mathbb{P}_0(\sup_{0\le s\le t} Z(s) \ge l, Z(t) = k) \\ &= \rho^{\frac{k}{2}} \exp\left((2\kappa - \lambda - \mu)t\right) \tilde{\mathbb{P}}_0(\sup_{0\le s\le t} Z(s) \ge l, Z(t) = k). \end{aligned} \tag{2.20}$$

By Lemma 2.3 we have that the last term equals to

$$\rho^{\frac{k}{2}} \exp\left((2\kappa - \lambda - \mu)t\right) \exp(-2\kappa t) I_{2l-k}(2\kappa t)$$

which yields (2.18). To prove (2.19) note

$$\{\sup_{0\le s\le t} Z(s) = 0\} = \{\sup_{0\le s\le t} Z(s) \ge 0\} - \{\sup_{0\le s\le t} Z(s) \ge 1\}$$

and hence by (2.18)

$$\mathbb{P}_0(\sup_{0\le s\le t} Z(s) \ge 0, Z(t) = k) - \mathbb{P}_0(\sup_{0\le s\le t} Z(s) \ge 1, Z(t) = k)$$

$$\exp\left(-(\lambda + \mu)t\right) \rho^{\frac{k}{2}} I_{-k}(2\kappa t) - \exp\left(-(\lambda + \mu)t\right) \rho^{\frac{k}{2}} I_{-k+2}(2\kappa t)$$

$$= \rho^k \left(\exp\left(\lambda+\mu\right)t\right)\rho^{-\frac{k}{2}} I_{-k}(2\kappa t) - \exp\left(\lambda+\mu\right)t\right)\rho^{-1}\rho^{\frac{-k+2}{2}} I_{-k+2}(2\kappa t)\right).$$

We use the result of Lemma 2.4 to show the density of the passage time from 0 to 1. Formally we define

$$T_j = \inf\{t > 0 : Z(t) = j\} \tag{2.21}$$

and our aim is to find $F_{0j}(t) = \mathbb{P}_0(T_j \le t)$. Notice that by the strong Markov property we have

$$F_{0n} = F_{01} * \cdots * F_{01}.$$

Proposition 2.5 *The first passage time from* 0 *to* 1 *has density*

$$\begin{aligned} f_{01}(t) &= \lambda \exp\left(-(\lambda+\mu)t\right)\left[I_0(2\kappa t) - I_2(2\kappa t)\right] \\ &= \frac{\rho^{1/2}}{t}\exp\left(-(\lambda+\mu)t\right) I_1(2\kappa t). \end{aligned} \tag{2.22}$$

Proof We have

$$\begin{aligned} &\mathbb{P}_0(T_1 \ge t) = \\ &= \mathbb{P}_0(\sup_{0\le s\le t} Z(s) = 0) \\ &= \sum_{k\le 0} \mathbb{P}_0(\sup_{0\le s\le t} Z(s) = 0, Z(t) = k) = \sum_{k\le 0} \rho^k (i_{-k}(t) - \rho^{-1} i_{-k+2}(t)) \\ &= \sum_{k=0}^{\infty} \rho^{-k} i_k(t) - \rho^{-1}\sum_{k=0}^{\infty} \rho^{-k} i_{k+2}(t) = \sum_{k=0}^{\infty} \rho^{-k} i_k(t) - \rho \sum_{k=2}^{\infty} \rho^{-k} i_k(t). \end{aligned}$$

Let

$$C_N(t) = \sum_{k=N}^{\infty} \rho^{-k} i_k(t) = \exp(-(\lambda+\mu)t)\sum_{k=N}^{\infty} \rho^{-\frac{k}{2}} I_k(2\kappa t).$$

Hence

$$f_{01}(t) = -\frac{d}{dt}\mathbb{P}_0(T_1 > t) = \rho C_2'(t) - C_0'(t).$$

Using that

$$I_k'(t) = \frac{1}{2}(I_{k-1} + I_{k+1})$$

(see app.specialf) we have

$$C_N'(t) = -(\lambda+\mu)C_N(t) + 2\kappa \exp(-(\lambda+\mu)t)\sum_{k=N}^{\infty} \rho^{-\frac{k}{2}} I_k'(2\kappa t)$$

$$= -(\lambda+\mu)C_N(t) + \kappa\exp(-(\lambda+\mu)t)\sum_{k=N}^{\infty}\rho^{-\frac{k}{2}}I_{k-1}(2\kappa t)$$

$$+\kappa\exp(-(\lambda+\mu)t)\sum_{k=N}^{\infty}\rho^{-\frac{k}{2}}I_{k+1}(2\kappa t)$$

$$= -(\lambda+\mu)C_N + \mu C_{N-1} + \lambda C_{N+1} = -\lambda\rho^{-N}i_N(t) + \mu\rho^{-(N-1)}i_{N-1}(t).$$

Hence

$$f_{01}(t) = \rho(\mu\rho^{-1}i_1(t) - \lambda\rho^{-2}i_2(t)) - \mu\rho i_{-1}(t) + \lambda i_0(t) = -\lambda\rho^{-1}i_2(t) + \lambda i_0(t)$$

$$= \lambda(i_0(t) - \rho^{-1}i_2(t)) = \exp(-(\lambda+\mu)t)\lambda(I_0(2\kappa t) - I_2(2\kappa t)).$$

Using the identity (see Appendix .3)

$$I_{n-1}(t) - I_{n+1}(t) = \frac{2n}{t}I_n(t)$$

we have that

$$f_{01}(t) = \lambda\exp(-\lambda+\mu)t)\frac{1}{\kappa t}I_1(2\kappa t) = \frac{\rho^{\frac{1}{2}}}{t}\exp(-(\lambda+\mu)t)I_1(2\kappa t).$$

□

Remark If $\rho < 1$ then the process Z_t drifts to $-\infty$ and it is not clear at the first sight whether f_{01} is a proper density. Indeed one can show that in such the case $\int f(t)\,dt = \rho$.

Problems

2.1 Prove that $Z(t)$ is a regular CTMC, with state space $\mathbb{E} = \{\ldots,-1,0,1,\ldots\}$ and intensity matrix $\boldsymbol{Q}'$ given in (2.5).

2.2 Prove that the process $(Z(t))$ is strong Markov.

2.3 Show that $\mathbb{P}_0(T_1 < \infty) = \rho \wedge 1$.

Comments. Asmussen (2003),

3 Transient behavior of $M/M/1$ queue

3.1 Busy Period

The busy period is the time interval from the epoch of transition from 0 to 1 to the nearest to the right epoch of coming back to zero. Thus its distribution equals to the distribution of $\mathbb{P}_1(T_0 \le x)$. Note that this is a proper distribution provided $\rho \le 1$.

Proposition 3.1 *The busy period distribution has density*

$$f_{10}(t) = \frac{\rho^{-1/2}}{t} \exp\left(-(\lambda+\mu)\right) I_1(2\kappa t).$$

Its defective with defect equal to ρ^{-1} for $\rho > 1$.

Proof This is as in (2.22) with λ changed to μ and μ changed to λ. □

The above proposition is useless for getting moments of the busy period. We now show a very simple method allowing us to compute the Laplace transform and hence moments of the busy period. We denote the generic busy period by G. Let

$$\hat{f}_{10}(\alpha) = \mathbf{E}\exp(-\alpha G) = \int_0^\infty \exp(-\alpha t) f_{10}(t)\, dt \tag{3.1}$$

Proposition 3.2 *The Laplace transform of the busy period is*

$$\hat{f}_{10}(\alpha) = \frac{1}{2\lambda}\left(\lambda+\mu+\alpha - \sqrt{(\lambda+\mu+\alpha)^2 - 4\lambda\mu}\right). \tag{3.2}$$

In particular if $\rho < 1$ then

$$\mathbf{E}[G] = \frac{1}{\mu(1-\rho)} \qquad \mathrm{Var}\,[G] = \frac{1+\rho}{\mu^2(1-\rho)^3}. \tag{3.3}$$

Proof Using the strong Markov property we immediately obtain that the Laplace transform of the passage time from 2 to 0 is $\hat{f}_{20}(\alpha) = (\hat{f}_{10}(\alpha))^2$. Now let $\tau = \min\{t : \quad L(t) \neq 1\}$ be the epoch of the first jump of $L(t)$ to the right of zero. We have by Exercise II.3.13 on competing risks

$$\mathbb{P}_1(Q(\tau) = 2, \tau \in B) = \lambda \int_B \exp\left(-(\lambda+\mu)t\right)\, dt \tag{3.4}$$

and

$$\mathbb{P}_1(Q(\tau)=0, \tau \in B) = \mu \int_B \exp\left(-(\lambda+\mu)t\right)\, dt \; .$$

Now conditioning

$$\begin{aligned}
\hat{f}_{10}(\alpha) &= \mathbb{E}_1 \exp\left(-\alpha T_0\right) = \mathbb{E}_1 \left[\mathbb{E}_1 \left[\exp\left(-\alpha T_0\right) | Q(\tau), \tau\right]\right] \\
&= \int_0^\infty e^{-\alpha x} \mu e^{-(\lambda+\mu)x}\, dx + \int_0^\infty e^{-\alpha(x+T_2)} \lambda e^{-(\lambda+\mu)x}\, dx \\
&= \frac{\mu}{\alpha+\lambda+\mu} + \frac{\lambda}{\alpha+\lambda+\mu} \hat{f}_{10}^2(\alpha) \; .
\end{aligned}$$

Solving equation we obtain (3.2). □

Corollary 3.3 *For $\rho > 0$*

$$\frac{\mathbb{E}_0 \exp\left(-\alpha T_n\right)}{\mathbb{E}_0 \exp\left(-\alpha T_{-n}\right)} = \rho^n. \tag{3.5}$$

Proof

$$\frac{\mathbb{E}_0 \exp\left(-\alpha T_1\right)}{\mathbb{E}_0 \exp\left(-\alpha T_{-1}\right)} = \rho \tag{3.6}$$

and the principle of mathematical induction. □

Recalling the result of Exercise (2.3) we can write (3.6) in the form

$$\begin{aligned}
\mathbb{E}_0[e^{-\alpha T_1} | T_1 < \infty] &= \mathbb{E}_0 e^{-\alpha T_{-1}} \\
&= \tilde{\mathbb{E}}\, e^{-\alpha T_1} \; ,
\end{aligned} \tag{3.7}$$

where $\tilde{\mathbb{E}}$ is the expectation operator for the BRW(μ, λ).

3.2 Transition Functions

Theorem 3.4 *For $\rho > 0$*

$$\mathbb{P}_n(Q(t) < m) = \mathbb{P}_n(Z(t) < m) - \rho^m \mathbb{P}_n(Z(t) < -m) \tag{3.8}$$

Proof Write

$$\begin{aligned}
&\mathbb{P}_n(Z(t) < m) \\
&\quad = \mathbb{P}_n(Q(t) < m, Z(t) < m) + \mathbb{P}_n(Q(t) \geq m, Z(t) < m) \\
&\quad = \mathbb{P}_n(Q(t) < m) + \mathbb{P}_n(Q(t) \geq m, Z(t) < m)
\end{aligned}$$

By (IV.1.8)

$$\begin{aligned}
&\mathbb{P}_n(Q(t) \geq m, Z(t) < m) \\
&\quad = \mathbb{P}(Z(t) - \inf_{0\leq s\leq t} 0 \wedge Z(s) \geq m, Z(t) < m) \\
&\quad = \sum_{k<m} \mathbb{P}_n(k - m \geq \inf_{0\leq s\leq t} 0 \wedge Z(s), Z(t) = k). \qquad (3.9)
\end{aligned}$$

To compute

$$\mathbb{P}_n(k - m \geq \inf_{0\leq s\leq t} 0 \wedge Z(s), Z(t) = k)$$

note that the process Z in time interval $[0, t]$ starts at zero from n, hits lavel $k - m$, then hits lavel k $(k - m < k)$ and is at t in state k. Using the strong Markov property

$$\mathbb{P}_n(k-m \geq \inf_{0\leq s\leq t} 0\wedge Z(s), Z(t) = k) = \int_0^t F_{n\,k-m} * F_{k-m\,k}(ds)\mathbb{P}_k(Z(t-s) = k).$$

By Corollary 3.3 $F_{k-m\,k} = \rho^m F_{k-m\,k-2m}$ and from the space homogeneity of Z we have $\mathbb{P}_k(Z(t - s) = k) = \mathbb{P}_{k-2m}(Z(t - s) = k - 2m)$

$$\begin{aligned}
&\int_0^t F_{n\,k-m} * F_{k-m\,k}(ds)\mathbb{P}_k(Z(t - s) = k) \\
&\quad = \int_0^t F_{n\,k-m} * F_{k-m\,k-2m}(ds)\mathbb{P}_{k-2m}(Z(t - s) = k - 2m) \\
&\mathbb{P}_n(Z(t) = k - 2m). \qquad (3.10)
\end{aligned}$$

The proof is completed after substituting to (3.9) and summing up. □

Corollary 3.5 *If $\rho < 1$ then for $m = 0, 1, \ldots$*

$$\lim_{t\to\infty} \mathbb{P}_n(Q(t) < m) = 1 - \rho^m \qquad (3.11)$$

If $\rho \geq 1$ then the limit is 0.

In the folowing corollary we show transition probability function of CTMC Q.

Corollary 3.6

$$
\begin{aligned}
p_{ij}^{Q}(t)\, \mathbb{P}_m(Q(t)=n) \\
&= \exp\left(-(\lambda+\mu)t\right)\left[\rho^{\frac{n-m}{2}} I_{n-m}(2t\kappa) + \right. \\
&\qquad \left. +\rho^{\frac{n-m-1}{2}} I_{n+m+1}(2t\kappa) + (1-\rho)\,\rho^{n} \sum_{k=-\infty}^{-(n-m+2)} \rho^{\frac{k}{2}} I_k(2t\kappa)\right].
\end{aligned}
$$

4 Collision times for Poison processes and queues

4.1 Karlin-McGregor theorem

Let $\boldsymbol{X}_t = (X_1(t), X_2(t))$ be a vector of two independent Poisson process with intensities λ_1, λ_2 starting at $t = 0$ from $\boldsymbol{x} = (x_1, x_2)$. Then $X_t^2 - X_t^2$ is one dimensional CTRW starting at $t = 0$ from $x = x_2 - x_1 > 0$ with birth and death intensities λ_1, λ_2. In this case the collision time τ is just the time of hitting zero line (or the first passage time to 0). For such the case we can write the Laplace transform for the first hitting time (see Feller XIV.6)

$$M(s,x) = \mathbb{E}_{\boldsymbol{x}} e^{-s\tau} = \left(\frac{\lambda_1 + \lambda_2 + s - \sqrt{(\lambda_1 + \lambda_2 + s)^2 - 4\lambda_1\lambda_2}}{2\lambda_2} \right)^x . \tag{4.12}$$

Thus the density of the first hitting time T_0 of BRW(λ, μ) is

$$f_x(t) = \left(\frac{\lambda_1}{\lambda_2} \right)^{\frac{x}{2}} \frac{x}{t} I_x(2\sqrt{\lambda_1\lambda_2} t) e^{-(\lambda_1+\lambda_2)t} , \tag{4.13}$$

where $I_x(t)$ is the modified Bessel function.

Proposition 4.1 (a) *If $\lambda_1 > \lambda_2$, then*

$$\begin{aligned} &\mathbb{P}_{\boldsymbol{x}}(\tau > t) = \\ &= x \left(\frac{\lambda_1}{\lambda_2} \right)^{\frac{x}{2}} \frac{1}{2\sqrt{\pi}(\lambda_1\lambda_2)^{1/4}(\sqrt{\lambda_1} - \sqrt{\lambda_2})^2} t^{-3/2} e^{-t(\sqrt{\lambda_1}-\sqrt{\lambda_2})^2} (1 + o(1)) \end{aligned} \tag{4.14}$$

(b) *If $\lambda_1 = \lambda_2$, then*

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = \frac{x}{\sqrt{\pi\lambda}} t^{-\frac{1}{2}} (1 + o(1)) . \tag{4.15}$$

(c) *If $\lambda_1 < \lambda_2$, then*

$$\lim_{t\to\infty} \mathbb{P}_{\boldsymbol{x}}(\tau > t) = 1 - \left(\frac{\lambda_1}{\lambda_2} \right)^x . \tag{4.16}$$

Proof For the proof of (a) we use the asymptotic expansion of Bessel function (.14) in (4.13). Notice that if $g(t) \to 1$, then $\int_t^\infty f(s)g(s)\,ds \sim \int_t^\infty f(s)\,ds$. Hence

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{x}}(\tau > t) &= x\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{x}{2}} \int_t^\infty s^{-1} I_x(2\sqrt{\lambda_1\lambda_2}s)e^{-(\lambda_1+\lambda_2)s}\,ds \\
&= x\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{x}{2}} \frac{1}{2\sqrt{\pi}(\lambda_1\lambda_2)^{1/4}} \int_t^\infty s^{-\frac{3}{2}} e^{-(\sqrt{\lambda_1}-\sqrt{\lambda_2})^2 s}\,ds(1+o(1))\ .
\end{aligned}$$

Now we substitute new variable $u = (\sqrt{\lambda_1} - \sqrt{\lambda_2})^2 s$, which yields

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{x}}(\tau > t) &= x\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{x}{2}} \frac{\sqrt{\lambda_1}-\sqrt{\lambda_2}}{2\sqrt{\pi}(\lambda_1\lambda_2)^{1/4}} \\
&\quad \times \int_{t(\sqrt{\lambda_1}-\sqrt{\lambda_2})^2}^\infty u^{-\frac{3}{2}} e^{-u}\,ds(1+o(1))\ .
\end{aligned}$$

Using the asymptotic expansion (.15) of $\Gamma(x,t)$ at $t = \infty$ we derive

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{x}}(\tau > t) &= x\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{x}{2}} \frac{1}{2\sqrt{\pi}(\lambda_1\lambda_2)^{1/4}(\sqrt{\lambda_1}-\sqrt{\lambda_2})^2} \\
&\quad \times t^{-3/2} e^{-t(\sqrt{\lambda_1}-\sqrt{\lambda_2})^2}(1+o(1))\ .
\end{aligned}$$

For the case (b) we use asymptotic expansion (.14) of Bessel function in (4.13). Thus we can write (here we put $\lambda = \lambda_1 = \lambda_2$)

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{x}}(\tau > t) &= \int_t^\infty \left(\frac{\lambda}{\lambda}\right)^{\frac{x}{2}} \frac{x}{s} I_x(2\sqrt{\lambda\lambda}s)e^{-(\lambda+\lambda)s}\,ds \\
&= \int_t^\infty \frac{x}{s} I_x(2\lambda s)e^{-2\lambda s}\,ds \\
&= \int_t^\infty \frac{x}{s}\frac{1}{\sqrt{2\pi 2\lambda s}}(1+o(1))\,ds = \frac{x}{\sqrt{\pi\lambda}} t^{-\frac{1}{2}}(1+o(1))\ .
\end{aligned}$$

For the case (c) $\lambda_1 < \lambda_2$, by examination of (4.12) we see that in this case the hitting time variable has an atom at infinity. Thus

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{P}_{\boldsymbol{x}}(\tau > t) &= 1 - M(0,x) = 1 - \left(\frac{\lambda_1+\lambda_2-\sqrt{(\lambda_1+\lambda_2)^2-4\lambda_1\lambda_2}}{2\lambda_2}\right)^x \\
&= 1 - \left(\frac{\lambda_1}{\lambda_2}\right)^x\ .
\end{aligned}$$

We start first with a classical ballot problem. Suppose that n votes are casts in sequel for two candidates 1 and 2. Suppose that initially there are a^1 votes for candidate 1 and a^2 for the second. Let $\boldsymbol{\xi}_i$ equal to (1,0) if the i-th vote is for candidate 1 and (0,1) otherwise. By an n-path from $\boldsymbol{a}$ to $\boldsymbol{b}$ we mean a sequence $(\boldsymbol{a}, \boldsymbol{a}+\boldsymbol{\xi}_1+\cdots+\boldsymbol{\xi}_n = \boldsymbol{b})$. Clearly $n = b^2-a^2+b^1-a^1$, there are b^1-a^1 of outcomes (1,0) and b^2-a^2 of outcomes (0,1). and everything has sense provided $n>0$. There are $\binom{n}{b^1-a^1}$ such the paths.

Suppose now $a^1<a^2$ and $b^1<b^2$ We want count the number of n-paths from $\boldsymbol{a}$ to $\boldsymbol{b}$ such that at least once the candidates drew. It means that a path has to touch the diagonal $\mathcal{D} = ((i,i))_{i\in\mathbb{Z}_+}$. We use the so called *reflection principle*. For this we introduce a *reflected n-path* $(\boldsymbol{a}^*, \boldsymbol{a}^*+\boldsymbol{\xi}_1^*+\boldsymbol{\xi}_\iota^*+\boldsymbol{\xi}_{\iota+1}\cdots+\boldsymbol{\xi}_n = \boldsymbol{b})$, where $\boldsymbol{\xi}_j^* = (\xi_j^2, \xi_j^1)$ and $\iota = \min\{j : \boldsymbol{a}^*+\boldsymbol{\xi}_1^*+\ldots+\boldsymbol{\xi}_\iota^* \in \mathcal{D}\}$ is the first time the coordinates equalise; see Fig. . Note the one to one correspondence between $(\boldsymbol{a}, \boldsymbol{a}+\boldsymbol{\xi}_1+\cdots+\boldsymbol{\xi}_n = \boldsymbol{b})$ and $(\boldsymbol{a}^*, \boldsymbol{a}^*+\boldsymbol{\xi}_1^*+\boldsymbol{\xi}_\iota^*+\boldsymbol{\xi}\iota+1\cdots+\boldsymbol{\xi}_n = \boldsymbol{b})$. Thus the number of n-paths touching or crossing the diagonal is $\binom{n}{b^1-a^2}$.

Suppose now we ask for the number of n-paths from $\boldsymbol{a}$ to $\boldsymbol{b}$ not touching the diagonal ($a^1<a^2$ and $b^1<b^2$). Then $\binom{n}{b^1-a^1}-\binom{n}{b^1-a^2}$ is the sought for number. We now rewrite this in the form

$$\begin{aligned}\binom{n}{b^1-a^1} - \binom{n}{b^1-a^2} &= \frac{n!}{(b^1-a^1)!(b^2-a^2)!} - \frac{n!}{(b^2-a^1)!(b^1-a^2)!}\\ &= n!\det\left[(\frac{1}{(b^i-a^j)!})_{i,j=1,2}\right].\end{aligned}$$

Consider now a continuous time version of the ballot problem, wherein votes are cast according to two independent Poisson processes; for candidate 1 with intensity μ and 2 with intensity λ. As before at the entry $s=0$ there are already a^1 and a^2 votes for candidates 1 and 2 respectively and we aim at the expiration time $s=t$ to have b^1 and b^2 votes for candidates 1 and 2 respectively. Suppose that $a^1<a^2$ and we ask for the probability that the second leads before the first all the time.

Consider first a random n-path from $\boldsymbol{a}$ to $\boldsymbol{b}$; $(\boldsymbol{\xi}_j)_{j=1,\ldots,n}$ are i.i.d. random variables with distribution

$$\mathbb{P}(\boldsymbol{\xi}=(1,0)) = \frac{\mu}{\lambda+\mu} \qquad \mathbb{P}(\boldsymbol{\xi}=(0,1)) = \frac{\lambda}{\lambda+\mu}.$$

The probability that the second path all the time is strictly over the first one is

$$n!\det\left[(\frac{1}{(b^i-a^j)!})_{i,j=1,2}\right](\frac{\lambda}{\lambda+\mu})^{b^2-a^2}(\frac{\mu}{\lambda+\mu})^{b^1-a^1}.$$

Suppose now that $X^1(t) = a^1 + \Pi^{\mu}(t)$ and $X^2(t) = a^2 + \Pi^{\lambda}(t)$ and $a_1 < a_2$, where Π^{λ} and Π^{μ} are independent.. We ask for the probability that X^1 and X^2 do not collide by time t. This probability is clearly

$$e^{-(\lambda+\mu)t}\frac{((\lambda+\mu)t)^n}{n!} \times n! \det\left[\left(\frac{1}{(b^i-a^j)!}\right)_{i,j=1,2}\right]\left(\frac{\lambda}{\lambda+\mu}\right)^{b^2-a^2}\left(\frac{\mu}{\lambda+\mu}\right)^{b^1-a^1} =$$

$$= e^{-(\lambda+\mu)t}\det\left[\left(\frac{1}{(b^i-a^j)!}\right)_{i,j=1,2}\right](\lambda t)^{b^2-a^2}(\mu t)^{b^1-a^1} .$$

We state Karlin McGregor theorem for the case of B&D processes. Thus consider n independent regular B&D processes $(X_i(t))_{t\geq 0}$ on $\mathbb{Z}$ with the same transition probability function $p_{ij}(t)$. We suppose that $X_i(0) = x_i$, where $\boldsymbol{x} \in \mathcal{W}$. Such the processes have a strong Markov property.[1]. Two processes X_t^i and $X_j(t)$ collide at $\tau_{ij} = \min\{t > 0 : \ X_i(t) = X_j(t)\}$. The time of the collision is $\tau = \min_{1\leq i<j\leq m}\tau_{ij}$. For $i \neq j$ two processes X_t^i and $X_j(t)$ collide at $\tau_{ij} = \min\{t > 0 : \ X_i(t) = X_j(t)\}$. The time of the collision is $\tau = \min_{1\leq i<j\leq m}\tau_{ij}$.

Theorem 4.2 *[Karlin–McGregor] For $\boldsymbol{i}, \boldsymbol{j} \in \mathcal{W}$*

$$\begin{aligned}&\mathbb{P}(\boldsymbol{X}(t) = \boldsymbol{j}, \tau > t | \boldsymbol{X}(0) = \boldsymbol{i}) = \\ &= \begin{cases} \det\left[(p_{i_k j_l}(t))_{k,l=1,\ldots,m}\right], & t \geq 0, \boldsymbol{i} \leq \boldsymbol{j} \\ 0 & \text{otherwise}\end{cases} .\end{aligned}$$

Consider now n independent Poisson processes $(\Pi_i(t))_{t\geq 0}$ with intensities λ_i respectively and let $X_i(t) = x_i + \Pi_i(t)$, where $i_1 < i_2 < \ldots < i_m$ are integers.

In the next theorem we assume that $\lambda_i \equiv \lambda$, that is $(\Pi_i(t))_{t\geq 0}$ $(i = 1, \ldots, m)$ are independent Poisson processes with the same transition function. Let

$$\mathcal{W} = \{\boldsymbol{i} \in \mathbb{Z}^n : i_1 < \ldots < i_m\} .$$

Clearly for $0 \leq t < \tau$ process $\boldsymbol{X}(t) \in \mathcal{W}$, we use here in further on the vector notation $\boldsymbol{X}(t) = (X_1(t), \ldots, X_m(t))$.

[1] strong Markov, referencje

Corollary 4.3 *For* $\boldsymbol{i}, \boldsymbol{j} \in \mathcal{W}$

$$\begin{aligned}
&\mathbb{P}(\boldsymbol{X}(t) = \boldsymbol{j}, \tau > t | \boldsymbol{X}(0) = \boldsymbol{i}) = \\
&= \begin{cases} \det\left[(\frac{(\lambda t)^{j_l - i_k}}{(j_l - i_k)!} e^{-\lambda t})_{k,l=1,\ldots,m} \right], & t \geq 0, \boldsymbol{i} \leq \boldsymbol{j} \\ 0 & \text{otherwise} \end{cases} .
\end{aligned}$$

From Corollary 4.3 we can draw an interesting result on a classical problem from combinatorics on m dimensional ballot problem. Suppose $\boldsymbol{i} \leq \boldsymbol{j} \in \mathbb{Z}^d$. By an n-path on $\mathbb{Z}^d$ from $\boldsymbol{i}$ to $\boldsymbol{j}$ we mean a sequence

$$(\boldsymbol{i}, \boldsymbol{i} + \boldsymbol{a}_1, \boldsymbol{i} + \boldsymbol{a}_1 + \boldsymbol{a}_2, \ldots, \boldsymbol{i} + \boldsymbol{a}_1 + \cdots + \boldsymbol{a}_n = \boldsymbol{j}) \, ,$$

where $\boldsymbol{a}_i \in \mathbb{Z}^d$ is a vector one entry equal to 1 and other 0. Clearly the number of such the paths is

$$\frac{n!}{k_1! \ldots k_m!},$$

where $\boldsymbol{k} = \boldsymbol{j} - \boldsymbol{i}$ and $k_1 + \cdots + k_m = n$. We now ask for paths such that

$$\boldsymbol{i} + \boldsymbol{a}_1 + \cdots + \boldsymbol{a}_l \in \mathcal{W}, \qquad 1 \leq l \leq n \, ,$$

that is paths without collision or noncolliding paths.

Corollary 4.4 *The number of noncolliding n-paths from $\boldsymbol{i}$ to $\boldsymbol{j}$ is*

$$n! \det[\left(\frac{1}{(j_l - i_k)!} \right)_{l,k=1,\ldots,m}]$$

Proof Suppose that $\boldsymbol{i} \leq \boldsymbol{j}$. For Poisson processes of Corollary 4.3, recalling that $\sum_{j=1}^m (j_l - i_l) = n$, we have

$$\begin{aligned}
&\mathbb{P}(\boldsymbol{X}(t) = \boldsymbol{j}, \tau > t | \boldsymbol{X}(0) = \boldsymbol{i}) = \\
&= \det\left[(\frac{(-\lambda t)^{j_l - i_k}}{(j_l - i_k)!} e^{-\lambda t})_{k,l=1,\ldots,m} \right] \\
&= e^{-m\lambda t} \frac{(m\lambda t)^n}{n!} (\frac{1}{m})^n n! \det\left[(\frac{1}{(j_l - i_k)!})_{k,l=1,\ldots,m} \right] .
\end{aligned}$$

4.2 Dieker-Warren theorem

For a while we will restrict attention to transition function $p_{00}(t) = \mathbb{P}_0(Q(t) = 0)$ in M/M/1. Let $(\Pi^\lambda(t))_{t\geq 0}$ and $(\Pi^\mu(t))_{t\geq 0}$ are independent Poisson process with intensities $\lambda > 0$ and $\mu > 0$ respectively. Process $(Z(t))_{t\geq 0}$, where $Z(t) = \Pi^\lambda(t) - \Pi^\mu(t)$ is a Bernoulli random walk BRW(λ, μ) starting from 0 at $t = 0$.

Lemma 4.5 *If $Q(t) = 0$, then*

$$\begin{aligned} Q(t) &= \sup_{0\leq s\leq t}(Z(t) - Z(s)) && (4.17) \\ &=_{\mathrm{d}} \sup_{0\leq s\leq t} Z(s) && (4.18) \end{aligned}$$

Proof Clearly

$$\begin{aligned} \max_{0\leq s\leq t}(Z(t) - Z(s)) &= Z(t) - \min_{0\leq s\leq t} Z(s) \\ &= Z(t) - \max_{0\leq s\leq t}(-Z(s)) \; . \end{aligned}$$

We know that Q fulfils (3.6). Thus

$$\begin{aligned} & Z(t) + \int_0^t 1(\max_{0\leq v\leq s-\circ}(Z(s) - Z(v)) = 0)\, d\Pi^\mu(s) \\ = \; & Z(t) + \int_0^t 1(Z(s-\circ) = \min_{0\leq v\leq s-\circ})\, d\Pi^\mu(s) \\ = \; & Z(t) + \max_{0\leq s\leq t}(-Z(s)) \end{aligned}$$

□

Thus

$$\begin{aligned} p_{00}(t) &= \mathbb{P}(\sup_{0\leq s\leq t} Z(s) = 0) && (4.19) \\ &= \mathbb{P}(T_1 > t) \; . && (4.20) \end{aligned}$$

Assume in the sequel $\rho < 1$. Then $Z(t) \to -\infty$ a.s. and in this case the distribution of T_1 is defective and

$$\mathbb{P}(T_1 = \infty) = 1 - \rho \; .$$

Hence

$$p_{00}(t) \quad = \quad 1-\rho+\rho\mathbb{P}(T_1>t|T_1<\infty) \tag{4.21}$$

Since from (3.6) we can conclude

$$\mathbb{P}^{\lambda,\mu}(T_1>t|T_1<\infty)=\mathbb{P}^{\mu,\lambda}(T_1>t)\ .$$

Furthermore

$$\mathbb{P}^{\mu,\lambda}(T_1>t)=\mathbb{P}_{0,1}(\tau>t)$$

where $\mathbb{P}_{0,1}(\tau>t)$ is the probability of collision of two independent Poisson processes Π^{μ} and Π^{λ}. Using (4.1) part (a) we have the following result.

Proposition 4.6 *In M/M/1 queue, if $\rho<1$, then from (4.14)*

$$p_{00}(t)-\pi_0=\rho\frac{\lambda}{2\sqrt{\pi}(\lambda\mu)^{3/4}(\sqrt{\mu}-\sqrt{\lambda})}t^{-3/2}e^{-t(\sqrt{\mu}-\sqrt{\lambda})^2}(1+o(1))\ .$$

Comments. It is a particular case of (4.34) from Cohen [8] for $i=j=0$. Similar formula was derived by Asmussen [5] in Theorem 8.12, however it is diffuclt to compare constants. In Robert [14] it was shown (see Proposition 5.8) that

$$d(t)=\frac{1}{2}\sum_{n=0}^{\infty}|\mathbb{P}_x(Q(t)=n)-(1-\rho)\rho^n|\le\left(\sqrt{\left(\frac{\mu}{\lambda}\right)^x}+1\right)e^{-(\sqrt{\lambda}-\sqrt{\mu})^2}\ .$$

and a weaker asymptotic bounds

$$\lim_{t\to\infty}\frac{1}{t}\log d(t)=-(\sqrt{\lambda}-\sqrt{\mu})^2\ .$$

Problems

4.1 Prove that

$$\mathbb{P}_m\{Q(t)=n\}$$

$$= (1-\rho)\rho^n - \exp\left(-(\lambda+\mu)t\right)(1-\rho)\rho^n \sum_{k=-(n+1)}^{\infty} (\rho^{\frac{k-m}{2}} I_{k-m}(2t\kappa)$$

$$+\exp\left(-(\lambda+\mu)t\right)\left[\rho^{\frac{n-m}{2}} I_{n-m}(2t\kappa\ ) + \rho^{\frac{n-m-1}{2}} I_{n+m+}(2t\kappa)\right].$$

4.2 Prove that for collision time of two Poisson processes

$$\mathbb{P}_{01}(\tau > t)$$

$(\lambda=\lambda_1=\lambda_2)$ prove we can write

$$\begin{aligned}
\mathbb{P}(\tau > t) &= \int_t^\infty \left(\frac{\lambda}{\lambda}\right)^{\frac{1}{2}} \frac{1}{s} I_1(2\sqrt{\lambda\lambda}s)e^{-(\lambda+\lambda)s}\,ds \\
&= \int_t^\infty \frac{1}{s} I_x(2\lambda s)e^{-2\lambda s}\,ds \\
&= \int_t^\infty \frac{1}{s}\frac{1}{\sqrt{2\pi 2\lambda s}}(1+o(1))\,ds = \frac{2x}{\sqrt{2\pi 2\lambda}} t^{-\frac{1}{2}}(1+o(1))\ .
\end{aligned}$$

Suppose $\lambda<\mu$. Prove that

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{P}_{0,1}(\tau > t) &= = 1-\left(\frac{\lambda+\mu-\sqrt{(\lambda+\mu)^2-4\lambda\mu}}{2\mu}\right) \\
&= 1-\left(\frac{\lambda}{\mu}\right)\ .
\end{aligned}$$

4.3 Let

$$A_n(t) = I_{n+m}(\kappa t) - 2\rho^{\frac{1}{2}} I_{m+n+1}(\kappa t) + \rho I_{n+m+2}(\kappa t).$$

Show that

$$\mathbb{P}_m\{Q(t)=n\}$$

$$= \rho^{\frac{n-m}{2}} \exp(-(\lambda+\mu)t) I_{n-m}(\kappa t) + \mu\rho^{\frac{n-m}{2}} \int_0^t \exp(-(\lambda+\mu)s) A_n(s)\,ds.$$

4.4 Prove that for $\rho\le 1$ the distribution $\mathbb{P}_1\{T_0\le x\}$ is proper.

Comments. The idea of the proof of Theorem 3.4 comes from [**?**]; see also [**?**]. A systematic study of the transient behaviour of $M/M/1$ queues was done by Abate and Whitt in [1], [2], [3], [4].

A proof of the result in Problem 2 was given by [**?**] and [**?**].

Bibliography

[1] Abate, J. & Whitt, W. (1987) Transient behavior of the M/M/1 queue: starting at the origin. *Queueing systems: Theory and applications* **2**, 41-65.

[2] Abate, J. & Whitt, W. (1988) Approximations for the $M/M/1$ busy-period distribution. In *Queueing Theory and Its Applications* Eds. O.J. Boxma & R. Syski, North-Holland, 149–191.

[3] Abate, J. & Whitt, W. (1989) Calcultating time-dependent performance measures for the $M/M/1$ queues. *IEEE Transactions on Communications* **37**,1102–1104.

[4] Abate, J., Kijima, M. & Whitt, W. (1991) Decomposition of the $M/M/1$ transition function. *Queueing Systems* **9**, 323-336.

[5] Asmussen, S. *Applied Probability and Queues*

[6] Böhm, W., Jain, J.L. and Mohanty, S.G. (1993) On zero-avoidning transition probabilities of an r-node tandem queuue: a combinatorial approach. *J. Appl.Probab.* **30**, 737–730.

[7] Böhm, W. and Mohanty, S.G. (1997) On the Karlin-McGregor theorem and applications. *The Annals of Applied Probability* **7**, 314–325.

[8] Cohen, J.W. (1969) *The Single Server Queue* North Holland, Amsterdam.

[9] Feller, W. 1971) An Introduction to Probability Theory and its Applications, vol. II. 2nd ed. wiley, New York.

[10] Karlin, S. and McGegor, J.G. (1957) Coincedence probabilities. *Pacific J. Math.* **9**, 1141–1164.

[11] Massey, W.A. A transient analysis of the M/M/1 queueu without transforms.

[12] Massey, W.A. (1987) Calculating exit times for series Jackson networks. *J. Appl. Probab.* **27**, 226–234.

[13] Puchała, Z. The asymptotics of the collision time tail distribution for Brownian motion, Poisson process and continuous time random walk. PhD thesis, Wroclaw, June 2007.

[14] Robert, Ph. (2003) *Stochastic Networks and Queues.*Springer, Berlin.

[15] Seneta, E. (1981) *Non-negative Matrices and Markov Chains.* Sec. Ed. Springer, Berlin.

[16] Stanley, R.P. (1986) *Enumerative Combinatorics* **1**. Wadsworth & Brooks, Belmont CA.

[17] Sumita, U. (1984) On conditional passage time structure of birth-death processes. *J. Appl. Probab.* **21**, 10–21.

[18] Takacs, L. (1962) *Introduction to the Theory of Queues* Oxford University Press, New York.

Appendix

1 Discrete time Markov chains

We consider a sequence of random variables, say $Y_0, Y_1, \dots$ assuming values in a denumerable state space $\mathbb{E}$. In these notes we encounter typically cases: $\mathbb{Z}$, $\mathbb{Z}_+$, $\{0, \dots, M\}$, $\mathbb{Z}_+^d$.

.1 Transition probability matrix

We call a matrix $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbf{E}}$ a *stochastic matrix* if

- $p_{ij} \geq 0$, $i, j \in \mathbb{E}$,
- $\sum_{j \in \mathbf{E}} p_{ij} = 1$, for all i.

Let $\boldsymbol{\nu} = (\nu_j)_{j \in \mathbf{E}}$ be a probability function.

Definition .1 A sequence of random variables $Y_0, Y_1, \dots$ assuming values at $\mathbb{E}$, is said to be a discrete time Markov chain, with state space $\mathbb{E}$, initial distribution $\boldsymbol{\nu}$ and transition probability matrix $\boldsymbol{P} = (p_{ij})_{i,j=0,1\dots}$ if

$$\begin{aligned} \mathbb{P}(Y_0 = i_0) &= \nu_{i_0}, & (.1)\\ \mathbb{P}(Y_0 = i_0, Y_1 = i_1) &= \nu_{i_0} p_{i_0 i_1}, & (.2)\\ \vdots &= \vdots & (.3)\\ \mathbb{P}(Y_0 = i_0, \dots, Y_n = i_n) &= \nu_{i_0} p_{i_0 i_1} \dots p_{i_{n-1} i_n} & (.4) \end{aligned}$$

for all $i_0, \dots, i_n$.

Transition matrix $\boldsymbol{P}$ is *irreducible* if for all $i \neq j$ there exists $n \geq 1$ such hat $p_{ij}^{(n)} > 0$. We say that a DTMC (Y_n) with probability transition matrix $\boldsymbol{P}$ is irreducible, if $\boldsymbol{P}$ is irreducible.

In these notes we will use the notation $\mathbb{P}_{\boldsymbol{\nu}}$ to indicate the initial condition $\boldsymbol{\nu}$ of the chain. In particular we use $\mathbb{P}_i$ if $\boldsymbol{\nu}$ is concentrated at i. Notice that $\mathbb{P}_{\boldsymbol{\nu}}(X_{n+1} = j | X_n = i) = \mathbb{P}_i(X_1 = j) = p_{ij}$, provided $\mathbb{P}_{\boldsymbol{\nu}}(X_n = i) > 0$.

.2 Recurrence and transience criteria

For a DTMC $(X_n)_{n \in \mathbf{E}}$ we define the *return time* to state i

$$R_i = \inf\{n \geq 1 : Y_n = i\}.$$

We say that the state i is

- *transient* if $\mathbb{P}_i(R_i < \infty) < 1$,
- *recurrent* if $\mathbb{P}_i(R_i < \infty) = 1$,
- *null recurrent* if $\mathbb{P}_i(R_i < \infty) = 1$ but $\mathbb{E}_i[R_i] = \infty$,
- *positive recurrent* if $\mathbb{E}_i[R_i] < \infty$.

The *potential matrix* $\boldsymbol{G}$ associate with transition matrix $\boldsymbol{P}$ is

$$\boldsymbol{G} = \sum_{n \geq 0} \boldsymbol{P}^n \ .$$

Theorem .2 *In an irreducible DTMC all states are either recurrent (positive recurrent) or transient.*

Proof Bremaud, p. 100.

An irreducible and positive recurrent DTMC is called *ergodic*.

Theorem .3 (i) *State i is recurrent if and only if*

$$\sum_{n \geq 0} p_{ii}^{(n)} = \infty \ .$$

In the next definition we deal with measures on $\mathbb{E}$, which are in this case sequences of nonnegative numbers. Thus a measure $\boldsymbol{\mu} = (x_j)_{j \in E}$ is *invariant* if

$$\mu_i = \sum_{j \geq 0} \mu_j p_{ji}, i \in \mathbb{E} \ . \qquad (.5)$$

Furthermore, if $\sum_j \mu_j = 1$, then we say that $\boldsymbol{\mu}$ is an invariant distribution. Clearly if $\sum_j \mu_j < \infty$, then $\nu_i = \mu_i / \sum_{\mu_j}$ is an invariant distribution.

Theorem .4 *An irreducible homogeneous Markov chain is positive recurrent if and only if there exists an invariant distribution. Moreover, the stationary distribution is, when it exists, unique and* $\boldsymbol{\pi} > 0$.

Proof Bremaud p. 104.

Theorem .5 *If* ν *is an invariant distribution of an irreducible positive recurrent chain, then*

$$\nu_i = \frac{1}{\mathbb{E}_i[R_i]} \ .$$

Theorem .6 *Let* $(Y_j)_{j\in\mathbb{Z}_+}$ *be an irreducible and recurrent DTMC with transition matrix* $\boldsymbol{P}$.
(i) *Let* i *be an arbitrary reference state and define*

$$\mu_j = \mathbb{E}_i[\sum_{j=1}^{R_i} 1(Y_j = i)] \ .$$

We have $0 < \mu_j < \infty$ *for all* $j \in \mathbb{E}$ *and* $\boldsymbol{\mu}$ *is an essentially invariant measure (that is unique up to a scale factor).*
(ii) *The chain is positive recurrent if and only if*

$$\sum_j \mu_j < \infty \ .$$

Then $\boldsymbol{\nu}$ *defined by* $\nu_i = \nu_i / \sum_j \mu_j$ *is the invariant distribution.*

Remark Part (i) is called a *regenerative form of invariant measure.* Notice that the chain $(Y_j)_{j\in\mathbb{Z}_+}$ starts off i, then returns to i constitute a renewal process.

We now give some classic criteria for positive recurrence.

Theorem .7 *[Foster's criterium] Consider an irreducible DTMC with transition probability matrix* $\boldsymbol{P}$. *If there exists a function* $f : \mathbb{E} \to \mathbb{R}$ *such that* $\inf_{i\in\mathbf{E}} f_i > -\infty$, *a finite subset* F *and* $\epsilon > 0$ *such that*

(FC i) $\sum_{k\in\mathbf{E}} p_{ik} f_k < \infty$, *for all* $i \in F$,

(FC ii) $\sum_{k\in\mathbf{E}} p_{ik} f_k \le f_i - \epsilon$, *for all* $i \notin F$,

then the chain is positive recurrent.

As a corollary we obtain the following useful result, called Pakes's lemma.

Lemma .8 *[Pakes] Consider an irreducible DTMC with state space* $\mathbb{E} = \{0, 1, \ldots$ *and transition probability matrix* $\boldsymbol{P}$*. If for all* $i \in \mathbb{E}$

(PL i) $\mathbb{E}_i X_1 < \infty$,

(PL ii) $\limsup_{i\uparrow\infty} \mathbb{E}_i[X_1 - i] < 0$,

then the chain is positive recurrent.

The proof of the following criteria can be found in Robert (2003), page 2.18.

Theorem .9 *If there exist a function* $f : \mathbb{E} \to \mathbb{R}_+$ *and constants* K*,* $\gamma > 0$ *such that* $\sup\{f(x) : \; x \in \mathbb{E}\} \geq K$ *and*
(a) $\mathbb{E}_x(f(X_1) - f(x)) \geq \gamma$*, when* $f(x) \geq K$*,*
(b) $\sup_{x\in\mathbf{E}} \mathbb{E}_x(|f(X_1) - f(x)|^2) < \infty$*,*
then the Markov chain $\{X_n\}$ *is transient.*

.3 Theory of random walk on $\mathbb{Z}^d$

We say that $(Y_n)_{n\in\mathbb{Z}_+}$ is a *random walk* on $\mathbb{Z}^d$ if

1. $Y_0 = (0, 0)$, $Y_n = \sum_{j=1}^n \xi_j$,
2. $\xi_j \in \mathbb{Z}^d$ and $(\xi_j)_{j\geq 1}$ are i.i.d.

We say that a random walk is *simple* if $\mathbb{P}(\xi_j = \boldsymbol{e}_i) = \mathbb{P}(\xi_j = -\boldsymbol{e}_i) = 1/(2d)$, for each unit vector $\boldsymbol{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$.

Problems

1.1 Let $p^n_{ij} = \mathbb{P}_i(X_n = j)$ and $\boldsymbol{P}^{(n)} = (p^{(n)}_{ij})_{i,j=0,1,\ldots}$. Show that

$$\boldsymbol{P}^{(n+m)} = \boldsymbol{P}^n \boldsymbol{P}^m$$

and hence

$$\boldsymbol{P}^{(n)} = \boldsymbol{P}^n$$

for all $n = 1, 2, \ldots$.

1.2 Show that an irreducible DTMC with finite state space is positive recurrent.

1.3 Show that the entry g_{ij} in the potential matrix $\boldsymbol{G}$ is the expected number of visits to state j, given that the chain starts from state i.

1.4 Show that state i is transient if and only if

$$\sum_{n\geq 1} 1(X_n = i) < \infty, \qquad \mathbb{P}_i - \text{a.s.}$$

1.5 Show that, for 1-D random walk on $\mathbb{Z}$ with transition probability matrix

$$p_{i,i+1} = p, p_{i,i-1} = 1 - p$$

for all $i \in \mathbb{Z}$ is transient if $p \neq 1/2$, null recurrent for $p = 1/2$. Such the random walk is said sometimes a *Bernoulli random walk*.

1.6 Show that transition matrix in a Bernoulli random walk is *double stochastic*, that is $\sum_i p_{ij} = \sum_j p_{ij} = 1$. Furthermore show that $\nu_i = 1$ and $\nu_i = p^n/(1-p)^n$ are invariant. (Asmussen p. 15). Notice that in the transient case an invariant measure are also possible, but they are not unique.

1.7 Show that random walk reflected at 0 with transition probability matrix

$$\begin{aligned} p_{i,i+1} &= p, \quad i \geq 0, \\ p_{i,i-1} &= 1-p, \quad i > 0, \\ p_{0,1} &= 1 \end{aligned}$$

is irreducible, positive recurrent if and only if $0 < p < 1/2$.

1.8 Consider a DTMC with transition probability matrix

$$\begin{aligned} p_{i,i+1} &= p_i, \quad i \geq 0, \\ p_{i,i-1} &= 1-p_i, \quad i > 0, \\ p_{0,1} &= 1 \end{aligned}$$

Show that the chain is irreducible and positive recurrent if and only if $0 < p_i < 1$ and

$$\sum_{i\geq 1} \frac{p_0 \cdots p_{i-1}}{q_0 \cdots q_{i-1}}$$

where $q_i = 1 - p_i$.

1.9 Consider the random walk $(Y_n)_{n\in\mathbb{Z}_+}$ on $\mathbb{Z}^2$, where $Y_0 = (0,0)$, $Y_n = \sum_{j=1}^n \xi_j$ and $(\xi_j)_{j\in\mathbb{Z}_+}$ are i.i.d.

2 Spectral theory of Matrices

Sequences $(x_1,\ldots,x_l)$ are denoted by $(x_j)_{j=1,\ldots,l}$ or simply (x_j). We use the following matrix and vector notations. Matrices, actually we deal with square matrices only, we denote by bold face capital letters, e.g. $\boldsymbol{A} = (a_{ij})_{i,j=1,\ldots,l}$ is an $l \times l$ matrix with entries a_{ij}. The *transposition* of any matrix $\boldsymbol{A} = (a_{ij})$ is denoted by $\boldsymbol{A}^\top$, i.e. $\boldsymbol{A}^\top = (a_{ji})$, but we rarely use it. For this we denote row vectors by Greek small case bold face letters, e.g. $\boldsymbol{\xi} = (\xi_j)_{j=1,\ldots,l}$, and column vectors by small case, bold face Roman letters, like $\boldsymbol{x}^\top = (x_j)_{j=1,\ldots,l}$, that is

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}.$$

The row vector of ones is $\boldsymbol{\varepsilon} = (1,\ldots,1)$, by $\boldsymbol{\varepsilon}_i$ we denote the (row) vector having zeros at all components with the exception of the i-th component, which is equal to 1, i.e.

$$\boldsymbol{\varepsilon}_i = (\underbrace{0,\ldots,0}_{i-1},1,0,\ldots)\,.$$

Similarly the column vector of ones is $\boldsymbol{e}$, and $\boldsymbol{e}_i = \boldsymbol{\varepsilon}^\top$. We always denote by $\mathbf{0}$ a vector or matrix having all entries equal to zero. We will use the following relations. We write $\boldsymbol{A} \geq \mathbf{0}$ if for all $a_{ij} \geq 0$, $\boldsymbol{A} > \mathbf{0}$ if for all $a_{ij} > 0$ and $\boldsymbol{A} \neq \mathbf{0}$ if there exists a nonzero entry $a_{ij} \neq 0$. Matrix $\boldsymbol{A}$ is said to be *nonnegative* if $\boldsymbol{A} \geq \mathbf{0}$ We say that $\boldsymbol{\rho} = (\rho_j)$ is a *probability vector* if $\rho_i \geq 0$ for all i and $\sum_i \rho_i = 1$.

.1 Spectral theory of nonnegative matrices

Assume that $\boldsymbol{A}$ is an $(l \times l)$ matrix, that $\boldsymbol{x}$ is an l-dimensional vector with at least one component different from zero, and that θ is a real or complex number. A matrix of any dimension all of whose entries are 0 is denoted by $\mathbf{0}$. If

$$\boldsymbol{A}\boldsymbol{x} = \theta\boldsymbol{x}, \tag{.1}$$

then θ is said to be an *eigenvalue* of $\boldsymbol{A}$ and $\boldsymbol{x}$ is said to be a *right eigenvector* corresponding to θ. Writing (.1) as $(\boldsymbol{A}-\theta\boldsymbol{I})\boldsymbol{x}=\boldsymbol{0}$, from the theory of linear algebraic equations we get that the eigenvalues are exactly the solutions to the *characteristic equation*

$$\det(\boldsymbol{A}-\theta\boldsymbol{I}) = 0\,. \tag{.2}$$

A nonzero vector $\boldsymbol{\xi}$ which is a solution to

$$\boldsymbol{\xi}\boldsymbol{A} = \theta\boldsymbol{\xi} \tag{.3}$$

is called a *left eigenvector* corresponding to θ. It is easy to see that for each eigenvalue θ, a solution $\boldsymbol{\xi}$ to (.3) always exists because (.2) implies that $\det((\boldsymbol{A}-\theta\boldsymbol{I})^\top)=0$, i.e. there exists a nonzero (column) vector $\boldsymbol{\xi}^\top$ such that $(\boldsymbol{A}-\theta\boldsymbol{I})^\top\boldsymbol{\xi}^\top=\boldsymbol{0}$, which is equivalent to (.3).

Note that (.2) is an algebraic equation of order l, i.e. there are l eigenvalues $\theta_1,\ldots,\theta_l$, which can be complex and some of them can coincide (multiple eigenvalues). We always assume that the eigenvalues $\theta_1,\ldots,\theta_l$ are numbered such that

$$|\theta_1| \ge |\theta_2| \ge \ldots \ge |\theta_l|\,.$$

Let $\boldsymbol{X}=(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_l)$ be an $l\times l$ matrix consisting of right (column) eigenvectors,

$$\boldsymbol{\Xi} = \begin{pmatrix} \boldsymbol{\xi}_1 \\ \vdots \\ \boldsymbol{\xi}_l \end{pmatrix}$$

an $l\times l$ matrix consisting of left eigenvectors $\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_l$, and $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_l)$ the vector of eigenvalues. There results the equation

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\,\mathrm{diag}(\boldsymbol{\theta})\,, \tag{.4}$$

where $\mathrm{diag}(\boldsymbol{\theta})$ denotes the diagonal matrix with diagonal elements $\theta_1,\ldots,\theta_l$ and all other elements equal to zero. A direct consequence of (.4) is

$$\boldsymbol{A}^n\boldsymbol{X} = \boldsymbol{X}\,\mathrm{diag}(\theta_1^n,\ldots,\theta_l^n)\,. \tag{.5}$$

We make a number of observations.

- If all eigenvectors $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_l$ are linearly independent, then $\boldsymbol{X}^{-1}$ exists. In this case, we can put $\boldsymbol{\Xi}=\boldsymbol{X}^{-1}$.

- Under the linear independence assumption $\boldsymbol{A} = \boldsymbol{X}\,\mathrm{diag}(\boldsymbol{\theta})\boldsymbol{X}^{-1} = \boldsymbol{X}\,\mathrm{diag}(\boldsymbol{\theta})\Xi$ and, consequently,

$$\boldsymbol{A}^n = \boldsymbol{X}(\mathrm{diag}(\boldsymbol{\theta}))^n\boldsymbol{X}^{-1} = \boldsymbol{X}(\mathrm{diag}(\boldsymbol{\theta}))^n\Xi. \tag{.6}$$

- From (.6) we get

$$\begin{aligned}\boldsymbol{A}^n &= (\boldsymbol{x}_1,\ldots,\boldsymbol{x}_l)(\mathrm{diag}(\boldsymbol{\theta}))^n\begin{pmatrix}\boldsymbol{\xi}_1\\ \vdots\\ \boldsymbol{\xi}_l\end{pmatrix}\\ &= (\theta_1^n\boldsymbol{x}_1,\ldots,\theta_l^n\boldsymbol{x}_l)\begin{pmatrix}\boldsymbol{\xi}_1\\ \vdots\\ \boldsymbol{\xi}_l\end{pmatrix},\end{aligned}$$

which yields the *spectral representation* of $\boldsymbol{A}^n$, i.e.

$$\boldsymbol{A}^n = \sum_{i=1}^{l}\theta_i^n\boldsymbol{x}_i\boldsymbol{\xi}_i\,. \tag{.7}$$

We make two remarks. The first is that since $\Xi = \boldsymbol{X}^{-1}$ we have in (.7) that $\boldsymbol{\xi}_i\boldsymbol{x}_i = 1$. Secondly we stress that the crucial assumption for the validity of (.7) is that the eigenvectors $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_l$ are linearly independent. The following lemma gives a simple sufficient condition.

Lemma .1 *If the eigenvalues $\theta_1,\ldots,\theta_l$ are distinct, then $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_l$ are linearly independent. Moreover, if the left eigenvectors $\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_l$ are defined via $\Xi = \boldsymbol{X}^{-1}$, then*

$$\boldsymbol{\xi}_i\boldsymbol{x}_j = \begin{cases}1 & \text{if } i = j,\\ 0 & \text{if } i \neq j.\end{cases} \tag{.8}$$

We will always assume that if $\boldsymbol{x}_1$ and $\boldsymbol{\xi}_1$ are right and left eigenvectors corresponding to θ_{PF}, then $\boldsymbol{\xi}_1\boldsymbol{x} = 1$.

.2 Perron–Frobenius Theorem

Let $\boldsymbol{A}$ be a nonnegative $l \times l$ matrix.

Definition .2 We say that a matrix $\boldsymbol{A}$ is *primitive* if for some k, $\boldsymbol{A}^k > \boldsymbol{0}$.

Recall that $\boldsymbol{A}$ is *irreducible* if $\sum_n \boldsymbol{A}^n > \mathbf{0}$. Let $\boldsymbol{A} = (a_{ij}^{(n)})$. Let d_i be the greatest common divisor of $\{n : a_{ii}^{(n)} > 0\}$. If $d = d_i$ is the common period for all $i = 1, \ldots, l$, then d is said to be the period of $\boldsymbol{A}$. If $d = 1$, then we say $\boldsymbol{A}$ is *aperiodic*. It is known that irreducible $\boldsymbol{A}$ has a common period.

Problems

2.1 Show that $\boldsymbol{A}$ is primitive if and only if it is irreducible and aperiodic.

In many application it is important to know the position of the dominant eigenvalue, and features of the corresponding right and left eigenvectors. These properties are listed in the following important result, called the *Perron–Frobenius theorem.*

Theorem .3 *If $\boldsymbol{A}$ is a nonnegative primitive, then*
(a) *θ_1 is strictly positive and of multiplicity 1 and moreover $|\theta_1| > |\theta_i|$ for $i = 2, \ldots, l$;*
(b) *the right and left eigenvectors $\boldsymbol{x}_1, \boldsymbol{\xi}_1$ have all components strictly positive and are unique up to constant multiples;*
(c) *if $\mathbf{0} \le \boldsymbol{B} \le \boldsymbol{A}$ is another nonnegative matrix with an eigenvalue θ', then $|\theta'| \le \theta_{\mathrm{PF}}$.*

The *proof* of Theorem .3 can be found, for example, in Chapter 1 of Seneta (1981). The eigenvalue θ_1 of a regular matrix $\boldsymbol{A}$ is called the *Perron–Frobenius eigenvalue*, and therefore we denote it sometimes as θ_{PF}. Another name is *spectral radius.*

From Perron-Frobenius theorem we can learn about asymptotic behaviour of powers $\boldsymbol{A}^n$, for $n \to \infty$. Let $(g(n))_n$ be a sequence. We denote by O(g(n)) a sequence such that for some numbers $0 < \alpha < \beta < \infty$: $\alpha g(n) \le O(g(n)) \le \beta g(n)$, for all n.

Theorem .4 *Assume that a nonnegative $\boldsymbol{A}$ is primitive. Then for some $k \in \mathbb{Z}_+$ (which can be detected)*
(a) *for $\theta_2 \ne 0$ and $n \to \infty$,*

$$\boldsymbol{A}^n = \theta_{\mathrm{PF}}^n \boldsymbol{x}_1 \boldsymbol{\xi}_1 + O(n^k |\theta_2|^n)\,; \tag{.9}$$

(b) *for $\theta_2 = 0$ and $n \ge l - 1$,*

$$\boldsymbol{A}^n = \theta_{\mathrm{PF}}^n \boldsymbol{x}_1 \boldsymbol{\xi}_1\,. \tag{.10}$$

The *proof* of this theorem can also be found in Chapter 1 of Seneta (1981).

We now consider matrices which are not aperiodic but still the assumption of irreducibility holds. In this case Theorem .3 holds with the following modification in point (a): θ_1 is strictly positive and of multiplicity 1 and moreover $|\theta_1| \geq |\theta_i|$ for $i = 2, \dots, l$.[2]

More specific result we can obtain for $\boldsymbol{A}$ being a cyclic matrix with period d, that is the matrix which after a renumeration of row and columns respectively has the form

$$\begin{pmatrix} \mathbf{0} & \boldsymbol{A}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{A}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{A}_d & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Theorem .5 *For a cyclic matrix $\boldsymbol{A}$ with period $d > 1$, there are exactly d eigenvalues θ with $|\theta| = \theta_{\mathrm{PF}}$. These values are roots of the equation $\theta^d - (\theta_{\mathrm{PF}})^d = 0$.*

3 Special functions

Definition .1 We say that a function has the asymptotic expansion

$$f(t) = \sum_{k=0}^{\infty} c_k t^{-k}$$

at ∞ if

$$f(t) - \sum_{k=0}^{n} c_k t^{-k} = o(t^{-n})$$

for all $n = 0, 1, \dots$.[Wong Section 1.3, page 4.

.1 Modified Bessel function

The n-th modified Bessel function is defined as follows.

$$I_n(x) = \begin{cases} \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n}}{k!(k+n)!} & \text{for } n \in \mathbb{Z}_+ \\ I_{-n}(x) & \text{for } -n \in \mathbb{Z}_+ \end{cases} \qquad (.11)$$

[2] Seneta, p.22

We have

$$\exp\left(\frac{y}{2}\left(x+\frac{1}{x}\right)\right)=\sum_{n=-\infty}^{\infty}x^nI_n(y). \tag{.12}$$

Exer. Hint. Use the probabilistic method. Notice that by (3.6) $1=\sum_{n=-\infty}^{\infty}i_n(t)$. One can prove that

$$\lim_{t\to\infty}t^{\frac{1}{2}}\exp(-t)I_n(t)=\frac{1}{\sqrt{2\pi}}. \tag{.13}$$

It is true see e.g. Watson [**?**], page 203, Abramowitz and Stegun [1] ???

$$I_n(y)=\frac{\exp(y)}{\sqrt{2\pi y}}\left[1+\left(\frac{1}{8}-\frac{n^2}{2}\right)\frac{1}{y}+O\left(\frac{1}{y^2}\right)\right]. \tag{.14}$$

The following identities can be found in Abramowitz and Stegun [1], Section 9.6.

$$I_{n-1}(t)-I_{n+1}(t)=\frac{2n}{t}I_n(t)$$
$$I_k'(t)=\frac{1}{2}(I_{k-1}(t)+I_{k+1}(t))$$

.2 Asymptotic expansion of incomplete Gamma function

We define incomplete Gamma function by

$$\Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}\,dt.$$

Following Abramowiz and Stegun [1], formula 6.5.32 we have the following asymptotic expansion:

$$\Gamma(a,t)=t^{a-1}e^{-t}(1+(a-1)t^{-1}+O(\frac{1}{t^{-2}}) \tag{.15}$$

4 Transition Probability Function

.1 Transition semi-groups

The main object to study is a semi-group of $N\times N$ matrices $(\boldsymbol{P}(t))_{t\geq 0}$, that is fulfilling $\boldsymbol{P}(t+s)=\boldsymbol{P}(t)\boldsymbol{P}(s)$. It is said that semi-group is continuous if

$\boldsymbol{P}(0)=\boldsymbol{I}$ and $\lim_{h\downarrow 0}\boldsymbol{P}(h)=\boldsymbol{I}$. Similalry as in the one dimensional case the functional exquation $f(t+s)=f(t)f(s)$ with some regularity assumptions has the only solution of form e^{at}. If $\boldsymbol{A}$ is an $N\times N$ matrix, then

$$e^{\boldsymbol{A}}=\sum_{n\geq 0}\frac{\boldsymbol{A}^n}{n!}$$

is said to be a *matrix exponential* of $\boldsymbol{A}$. The defining sequence is always convergent.

Lemma .1 *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $N\times N$ commuting matrices, then*

$$e^{\boldsymbol{A}+\boldsymbol{B}}=e^{\boldsymbol{A}}e^{\boldsymbol{B}},$$

and

$$\frac{\mathrm{d}}{\mathrm{dt}}e^{t\boldsymbol{A}}=\boldsymbol{A}e^{t\boldsymbol{A}}=e^{t\boldsymbol{A}}\boldsymbol{A}\ .$$

We have the following result for the matrix case.

Proposition .2 *If the semi-group of $N\times N$ matrices $(\boldsymbol{P}(t))_{t\geq 0}$ is continuous, then there exists a matrix $\boldsymbol{A}$ such that*

$$\boldsymbol{P}(t)=e^{\boldsymbol{A}t},\qquad t\geq 0\ .$$

Proof The idea is to show that the derivative $\boldsymbol{P}^{'}(0+)$ exists and denote it by $\boldsymbol{A}$. In the next we show that

$$\boldsymbol{P}^{'}(t)=\boldsymbol{A}\boldsymbol{P}(t)=\boldsymbol{P}(t)\boldsymbol{A}$$

which has the solution $\exp(t\boldsymbol{A})$. □

The main result is stated for a continuous *transition semi-group* of finite $N\times N$ matrices $(\boldsymbol{P}(t))_{t\geq 0}$, that is fulfilling

- $\boldsymbol{P}(t)$ is a stochastic matrix, i.e. $p_{ij}(t)\geq 0$, $\sum_{1\leq j\leq N}p_{ij}(t)=1$ for all $1\leq i\leq N$,
- $\boldsymbol{P}(t+s)=\boldsymbol{P}(t)\boldsymbol{P}(s)$, for all $s,t\geq 0$,
- $\boldsymbol{P}(0)=\boldsymbol{I}$,

- $\lim_{h\downarrow 0} \boldsymbol{P}(h) = \boldsymbol{I}$.

Recall that $\boldsymbol{Q}$ is an intensity matrix if $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j\neq i} q_{ij} = 0$.

Theorem .3 *If $(\boldsymbol{P}(t))_{t\geq 0}$ is a transition semi-group of finite $N \times N$ matrices, then*

$$\boldsymbol{P}(t) = \exp(\boldsymbol{Q}t), \qquad t \geq 0,$$

where $\boldsymbol{Q}$ is an intensity matrix. The correspondence between transition semi-groups $\boldsymbol{P}(t)$ and intensity matrix $\boldsymbol{Q}$ is one to one.

Proof of Theorem .3. Suppose that $\exp(t\boldsymbol{Q})$ is the solution the continuous semi-group $(\boldsymbol{P}(t))_{t\geq 0}$. Since it is a transition semi-group, then

$$\boldsymbol{P}(t)\boldsymbol{e} = \boldsymbol{e}$$

, where $\boldsymbol{e} = (1, \ldots, 1)^T$. Clearly

$$\boldsymbol{P}(h) - \boldsymbol{I} = \boldsymbol{Q}h + \sum_{n\geq 2} \frac{\boldsymbol{Q}^n h^n}{n!} = \boldsymbol{Q}h + o(h) \ .$$

Moreover

$$0 = (\boldsymbol{P}(h) - \boldsymbol{I})\boldsymbol{e} = \boldsymbol{Q}\boldsymbol{e}h + \sum_{n\geq 2} \frac{\boldsymbol{Q}^n \boldsymbol{e} h^n}{n!} = \boldsymbol{Q}\boldsymbol{e}h + o(h) \ .$$

Hence $\boldsymbol{Q}\boldsymbol{e} = \boldsymbol{0}$ and

$$0 \leq \lim_{h\to 0} \frac{p_{ij}(h)}{h} = q_{ij} \ .$$

Thus $\boldsymbol{Q}$ is an intensity matrix.

Lamperti Stochastic Processes Roger-Williams v.1, p. 228.

5 Continuous-Time Martingales

The aim of this section is to recall some selected aspects of continuous-time martingales.

.1 Stochastic Processes and Filtrations

Under the notion of a *stochastic process* we understand a collection of random variables $(X(t))_{t\in\mathcal{T}}$ on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here $\mathcal{T}$ is an ordered space of parameters. Typically in these notes $\mathcal{T} \subset \mathbb{R}$ and in particular $\mathcal{T} = \mathbb{N}, \mathbb{Z}, \mathbb{R}_+$ or $\mathcal{T} = \mathbb{R}$.

Formally, a stochastic process is a mapping $X : \mathcal{T} \times \Omega \to \mathbb{R}$, but in general we do not require the measurability of this mapping. If $\mathcal{T}$ is a subset of $\mathbb{R}$ and X is measurable with respect to the product-σ-algebra $\mathcal{B}(\mathcal{T}) \otimes \mathcal{F}$, then we say the stochastic process $\{X(t), t \in \mathcal{T}\}$ is *measurable*.

In this section we always assume that $\mathcal{T} = \mathbb{R}_+$. Then the set $\mathcal{T}$ of parameters plays the role of time and so we speak about *continuous-time stochastic processes*. For each fixed $\omega \in \Omega$, the function $t \mapsto X(t, \omega)$ is called a *sample path* or *trajectory*; however, we usually drop the dependence on $\omega \in \Omega$. We will mostly deal with processes

- *càdlàg* that is with right-continuous realizations $g : \mathbb{R}_+ \to \mathbb{R}$ with left-hand limits

- càglàd that is with left-continuous realizations $g : \mathbb{R}_+ \to \mathbb{R}$ with right-hand limits.

Note that the process $(X(t))_{t\geq 0}$ is measurable if it is cadlag; see, for example, Lemma 2.1.1 in Last and Brandt (1995).

Let $t \geq 0$. By the *internal history of* $\{X(t)\}$ *up to time* t we mean the smallest σ-algebra $\mathcal{F}_t^X$ containing the events $\{\omega : (X(t_1, \omega), \ldots, X(t_n, \omega)) \in B\}$ for all Borel sets $B \in \mathcal{B}(\mathbb{R}^n)$, for all $n = 1, 2, \ldots$ and arbitrary sequences $t_1, t_2, \ldots, t_n$ with $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq t$. Note that for $0 \leq t \leq t'$

- $\mathcal{F}_t^X \subset \mathcal{F}$,

- $\mathcal{F}_t^X \subset \mathcal{F}_{t'}^X$,

- $X(t)$ is measurable with respect to $\mathcal{F}_t^X$.

The family of σ-fileds $(\mathcal{F}_t^X)$ is called the *history* of the process $\{X(t)\}$. An arbitrary family $\{\mathcal{F}_t, t \geq 0\}$ of σ-fields such that $\mathcal{F}_t \subset \mathcal{F}$, $\mathcal{F}_t \subset \mathcal{F}_{t'}$ for all $t, t' \in \mathcal{T}$ with $t \leq t'$ is called a *filtration*. We say that the process $(X(t))_{t\geq 0}$ is *adapted* to the filtration $\{\mathcal{F}_t, t \in \mathcal{T}\}$ if $X(t)$ is measurable with respect to $\mathcal{F}_t$, for all $t \in \mathcal{T}$.

.2 Stopping Times

A random variable τ taking values in $\mathbb{R}_+ \cup \{\infty\}$ is said to be a *stopping time* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ (or equivalently an $\{\mathcal{F}_t\}$-stopping time) if the event $\{\tau \leq t\}$ belongs to $\mathcal{F}_t$, for all $t \geq 0$. We define $\mathcal{F}_{t+} = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$. Note that $\mathcal{F}_{t+}$ is a σ-algebra because the intersection of any family of σ-algebras is a σ-algebra. If $\mathcal{F}_{t+} = \mathcal{F}_t$ for all $t \in \mathbb{R}_+$, we say that the filtration $\{\mathcal{F}_t,\ t \geq 0\}$ is *right-continuous.* In this case we have the following equivalent definition of a stopping time.

Lemma .1 *The random variable τ is an $\{\mathcal{F}_{t+}\}$-stopping time if and only if $\{\tau < t\} \in \mathcal{F}_t$ for all $t \geq 0$. In particular, if $\{\mathcal{F}_t\}$ is a right-continuous filtration, then τ is an $\{\mathcal{F}_t\}$-stopping time if and only if $\{\tau < t\} \in \mathcal{F}_t$ for all $t \geq 0$.*

Proof If τ is an $\{\mathcal{F}_{t+}\}$-stopping time, then $\{\tau < t\} \in \mathcal{F}_t$ since $\{\tau < t\} = \bigcup_{n=1}^{\infty}\{\tau \leq t - n^{-1}\} \in \mathcal{F}_t$. Conversely suppose that the random variable τ has the property that $\{\tau < t\} \in \mathcal{F}_t$ for all $t \geq 0$. Then $\{\tau \leq t\} = \bigcap_{n=1}^{\infty}\{\tau < t + n^{-1}\} \in \mathcal{F}_{t+}$. The second part of the statement is now obvious. □

Throughout the present section we assume that the stochastic process $\{X(t), t \geq 0\}$ is càdlàg. Let $B \in \mathcal{B}(\mathbb{R})$ and define the *first entrance time* τ^B of $\{X(t)\}$ to the set B by

$$\tau^B = \begin{cases} \inf\{t : X(t) \in B\} & \text{if } X(t) \in B \text{ for some } t \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

The question whether τ^B is a stopping time is not obvious. A positive answer can only be given under additional assumptions, for example on B or on the filtration $\{\mathcal{F}_t\}$. We now discuss this problem in more detail for sets of the form $B = (u, \infty)$ and $B = [u, \infty)$, where $u \in \mathbb{R}$. Let $\tau(u) = \inf\{t \geq 0 : X(t) > u\}$ denote the first entrance time of $\{X(t)\}$ to the open interval (u, ∞), where we put $\inf \emptyset = \infty$ as usual. For the interval $[u, \infty)$ it is more convenient to consider the *modified first entrance time*

$$\tau^*(u) = \inf\{t \geq 0 : X(t-0) \geq u \text{ or } X(t) \geq u\}\ . \qquad (.1)$$

Theorem .2 *Let $u \in \mathbb{R}$. If the process $\{X(t)\}$ is adapted to a filtration $\{\mathcal{F}_t\}$, then $\tau(u)$ is an $\{\mathcal{F}_{t+}\}$-stopping time and $\tau^*(u)$ is an $\{\mathcal{F}_t\}$-stopping time. In particular, if $\{\mathcal{F}_t\}$ is right-continuous then $\tau(u)$ is an $\{\mathcal{F}_t\}$-stopping time too.*

Proof Since the trajectories of $(X(t))$ are cadlag, we have

$$\{\tau(u) < t\} = \bigcup_{q \in \mathbb{Q}_t} \{X(q) > u\} \in \mathcal{F}_t \qquad (.2)$$

for each $t \geq 0$, where $\mathbb{Q}_t$ is the set of all rational numbers in $[0, t)$. Hence $\tau(u)$ is an $\{\mathcal{F}_{t+}\}$-stopping time by the result of Lemma .1. Furthermore, $\{\tau^*(u) \leq t\} = \bigcap_{n \in \mathbb{N}} \{\tau(u - n^{-1}) < t\} \cup \{X(t) \geq u\}$. Thus, (.2) implies that $\{\tau^*(u) \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$. □

Remarks 1. The proof of Theorem .2 can easily be extended in order to show that the first entrance time τ^B to an arbitrary open set B is an $\{\mathcal{F}_{t+}\}$-stopping time. Moreover, it turns out that τ^B is a stopping time for each Borel set $B \in \mathcal{B}(\mathbb{R}_+)$ provided that some additional conditions are fulfilled. In connection with this we need the following concept. We say that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *complete* if for each subset $A \subset \Omega$ for which an event $A' \in \mathcal{F}$ exists with $A \subset A'$ and $\mathbb{P}(A') = 0$, we have $A \in \mathcal{F}$. We now say that the filtration $\{\mathcal{F}_t, t \geq 0\}$ is *complete* if the probability space is complete and $\{A \in \mathcal{F} : \mathbb{P}(A) = 0\} \subset \mathcal{F}_0$. If the filtration $\{\mathcal{F}_t\}$ is right-continuous and complete, $\{\mathcal{F}_t\}$ is said to fulfil the *usual conditions*. Furthermore, if $\{\mathcal{F}_t\}$ fulfils the usual conditions and if $\{X(t)\}$ is adapted to $\{\mathcal{F}_t\}$, then τ^B is an $\{\mathcal{F}_t\}$-stopping time for each $B \in \mathcal{B}(\mathbb{R}_+)$. A proof of this statement can be found, for example, in Dellacherie (1972), p. 51. We mention, however, that in some cases it can be difficult to show that a given filtration is right-continuous.

2. Theorem .2 indicates that the first entrance time $\tau(u)$ is not always a stopping time, unless the considered filtration is right-continuous. An example where this problem appears can easily be found if the underlying probability space is large enough. Consider the process $\{X(t)\}$ on the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = D(\mathbb{R}_+)$ and $\mathcal{F} = \mathcal{B}(D(\mathbb{R}_+))$. Then, $\{\tau(u) \leq t\} \notin \mathcal{F}_t^X$ for each $t > 0$, i.e. $\tau(u)$ is not a stopping time with respect to the history $\{\mathcal{F}_t^X\}$ of $\{X(t)\}$. Indeed, the two sample paths given in Figure .1 show that from the knowledge of the process $\{X(t)\}$ up to time t it is not possible to recognize whether $\tau(u) \leq t$ or $\tau(u) > t$.

.3 Martingales, Sub- and Supermartingales

Suppose that the stochastic process $(X(t))_{t \geq 0}$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$ and that $\mathbb{E}\,|X(t)| < \infty$ for all $t \geq 0$. We say that $(X(t))$ is an $(\mathcal{F}_t)$-*martingale* if with

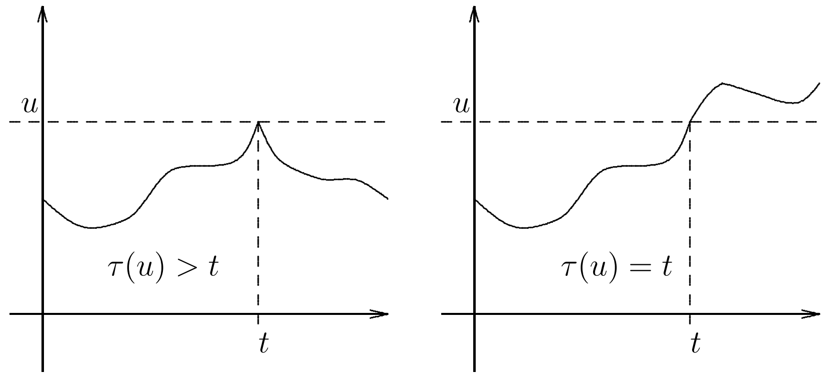

Figure .1: Two paths coinciding till time t

probability 1

$$\mathbb{E}\left(X(t+h) \mid \mathcal{F}_t\right) = X(t)\,, \tag{.3}$$

for all $t, t+h \in \mathcal{T}$ with $h \geq 0$. Similarly, $(X(t))$ is called a *submartingale* if

$$\mathbb{E}\left[X(t+h) \mid \mathcal{F}_t\right] \geq X(t)\,, \tag{.4}$$

and a *supermartingale* if

$$\mathbb{E}\left[X(t+h) \mid \mathcal{F}_t\right] \leq X(t)\,, \tag{.5}$$

for all $t, t+h \in \mathcal{T}$ with $h \geq 0$.

Examples

1. Let $(X(t))_{t\geq 0}$ be a process with stationary and independent increments.[3] If $\mathbb{E}\,|X(1)| < \infty$, then the process $M(t) = X(t) - t\mathbb{E}\,X(1)$ is a martingale with respect to the filtration $(\mathcal{F}_t^X)_{t\geq 0}$. We leave the proof of this fact to the reader. We only remark that it suffices to show that $\mathbb{E}\left[M(t+h) \mid M(t_1), \ldots, M(t_n), M(t)\right] = M(t)$ whenever $0 \leq t_1 < t_2 < \ldots < t_n < t < t+h$.

2. Let $(X(t))_{t\geq 0}$ be a process with stationary and independent increments with *Lévy exponent* $\kappa(s)$. If $\kappa(s)$ is finite, then process

$$M(t) = \exp(sX(t) - t\kappa(s)), \qquad t \geq 0$$

is an $\mathcal{F}_t^X$-martingale. This martingale is said to be *Wald martingale.* furthermore $\mathbb{E}\,M(0) = 1$.

[3]Gdzies dac definicje process with stationary and independent increments oraz ...

3. The current example indicates the close relationship between martingales and the concept of the infinitesimal generator in the theory of Markov processes. Consider a CTMC $(Z(t))_{t\geq 0}$ with finite state space $\mathbb{E} = \{1, 2, \ldots, \ell\}$ and intensity matrix $\boldsymbol{Q}$. Then, for each vector $\boldsymbol{b} = (b_1, \ldots, b_\ell) \in \mathbb{R}^\ell$, the process $\{X(t), t \geq 0\}$ with

$$X(t) = b_{Z(t)} - b_{Z(0)} - \int_0^t (\boldsymbol{Q}\boldsymbol{b}^\top)_{Z(v)} \, \mathrm{d}v \,, \qquad t \geq 0 \tag{.6}$$

is an $\{\mathcal{F}_t^Z\}$-martingale, where the integral in (.6) is defined pathwise. In order to demonstrate this fact, write for $t, h \geq 0$

$$\begin{aligned}
&\mathbb{E}\left(X(t+h) \mid \mathcal{F}_t^Z\right) = \mathbb{E}\left(X(t+h) \mid Z(t)\right) \\
&= \;\; X(t) + \mathbb{E}\left(b_{Z(t+h)} - b_{Z(t)} - \int_t^{t+h} (\boldsymbol{Q}\boldsymbol{b}^\top)_{Z(v)} \, \mathrm{d}v \;\middle|\; Z(t)\right).
\end{aligned}$$

Because $\{Z(t)\}$ is homogeneous we have

$$\begin{aligned}
&\mathbb{E}\left(b_{Z(t+h)} - b_{Z(t)} - \int_t^{t+h} (\boldsymbol{Q}\boldsymbol{b}^\top)_{Z(v)} \, \mathrm{d}v \;\middle|\; Z(t) = i\right) \\
&= \;\; \mathbb{E}\left(b_{Z(h)} - b_{Z(0)} - \int_0^{h} (\boldsymbol{Q}\boldsymbol{b}^\top)_{Z(v)} \, \mathrm{d}v \;\middle|\; Z(0) = i\right).
\end{aligned}$$

Thus it suffices to show that, for all $i \in E$,

$$\mathbb{E}\left(b_{Z(h)} \mid Z(0) = i\right) - b_i = \int_0^h \mathbb{E}\left((\boldsymbol{Q}\boldsymbol{b}^\top)_{Z(v)} \mid Z(0) = i\right) \mathrm{d}v \tag{.7}$$

since $\mathbb{E}\left(\int_0^h (\boldsymbol{Q}\boldsymbol{b}^\top)_{Z(v)} \, \mathrm{d}v \mid Z(0) = i\right) = \int_0^h \mathbb{E}\left((\boldsymbol{Q}\boldsymbol{b}^\top)_{Z(v)} \mid Z(0) = i\right) \mathrm{d}v$. However, recalling from Theorem [**??sol.kol.the??**] that the matrix transition function $\{\boldsymbol{P}(v), v \geq 0\}$ of $(Z(t))_{t\geq 0}$ is given by $\boldsymbol{P}(v) = \exp(\boldsymbol{Q}v)$, we have

$$\mathbb{E}\,\left(b_{Z(h)} \mid Z(0) = i\right) = \boldsymbol{e}_i \exp(\boldsymbol{Q}h)\boldsymbol{b}^\top \tag{.8}$$

and

$$\mathbb{E}\left((\boldsymbol{Q}\boldsymbol{b}^\top)_{Z(v)} \;\middle|\; Z(0) = i\right) = \boldsymbol{e}_i \exp(\boldsymbol{Q}v)\boldsymbol{Q}\boldsymbol{b}^\top, \tag{.9}$$

where $\boldsymbol{e}_i$ is the ℓ-dimensional (row) vector with all components equal to 0 but the i-th equal to 1. Using (.8) and (.9) we see that (.7) is equivalent to

$$\boldsymbol{e}_i \exp(\boldsymbol{Q}h)\boldsymbol{b}^\top - b_i = \int_0^h \boldsymbol{e}_i \exp(\boldsymbol{Q}v)\boldsymbol{Q}\boldsymbol{b}^\top \, \mathrm{d}v \,. \tag{.10}$$

The latter can be verified by differentiation and by using Lemma [**??mat.exp.diff??**]. So far, we have shown that the process $(X(t))_{t\geq 0}$ given by (.6) is an $\mathcal{F}_t^Z$-martingale.

The following converse statement is true. Suppose for the moment that $\boldsymbol{Q}'$ is an arbitrary $\ell \times \ell$ matrix, i.e. not necessarily the intensity matrix of the Markov process $\{Z(t)\}$. Moreover, assume that the process $(X'(t))$ with

$$X'(t) = b_{Z(t)} - b_i - \int_0^t (\boldsymbol{Q}'\boldsymbol{b}^\top)_{Z(v)}\,\mathrm{d}v\,, \qquad t \geq 0 \tag{.11}$$

is an $\mathcal{F}_t^Z$-martingale for each vector $\boldsymbol{b} \in \mathbb{R}^\ell$ and for each initial state $Z(0) = i$ of $\{Z(t)\}$, Then, analogously to (.10), we have

$$\boldsymbol{e}_i\boldsymbol{P}(h)\boldsymbol{b}^\top - b_i = \int_0^h \boldsymbol{e}_i \exp(\boldsymbol{Q}v)\boldsymbol{Q}'\boldsymbol{b}^\top\,\mathrm{d}v\,. \tag{.12}$$

On the other hand, using Theorem [**??exi.tra.int??**] we see that

$$\lim_{h\to 0} \frac{\boldsymbol{e}_i\boldsymbol{P}(h)\boldsymbol{b}^\top - \boldsymbol{e}_i\boldsymbol{b}^\top}{h} = \boldsymbol{e}_i\boldsymbol{Q}\boldsymbol{b}^\top$$

for all $i = 1, \ldots, \ell$ and $\boldsymbol{b} \in \mathbb{R}^\ell$. This means that $\boldsymbol{Q}'$ must be equal to the intensity matrix $\boldsymbol{Q}$ of Z.

4. For two $\mathcal{F}_t$-martingales X and Y, the process $X + Y$ is also an $\mathcal{F}_t$-martingale. The proof of this fact is left to the reader.

5. If the random variable Z is measurable with respect to $\mathcal{F}_0$ for some filtration $\mathcal{F}_t$, and if $\mathbb{E}\,|Z| < \infty$, then the process Y defined by $Y(t) \equiv Z$ is an $\mathcal{F}_t$-martingale. Moreover if X is another $\mathcal{F}_t$-martingale, then the process $(ZX(t))_{t\geq 0}$ is an $\mathcal{F}_t$-martingale, provided $\mathbb{E}\,[ZX(0)] = 0$. We leave the proofs of these simple properties to the reader.

Bibliography

[1] Abramowitz, M. and Stegun, I.A. (1964) *Handbook of Mathematical Functions with Formulas, Graps, and Mathematical Tables* Dover Publications, New York

Index