Queues and Communication Networks. Exercises

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0.1 Let $p^n_{ij} = \mathbb{P}_i(X_n = j)$ and $\boldsymbol{P}^{(n)} = (p^{(n)}_{ij})_{i,j=0,1,\ldots}$. Show that

$$\boldsymbol{P}^{(n+m)} = \boldsymbol{P}^n \boldsymbol{P}^m$$

and hence

$$\boldsymbol{P}^{(n)} = \boldsymbol{P}^n$$

for all $n = 1, 2, \ldots$.

0.2 Show that an irreducible DTMC with finite state space is positive recurrent.

0.3 Show that the entry g_{ij} in the potential matrix $\boldsymbol{G}$ is the expected number of visits to state j, given that the chain starts from state i.

0.4 Show that state i is transient if and only if

$$\sum_{n\geq 1} 1(X_n = i) < \infty, \qquad \mathbb{P}_i - \text{a.s.}$$

0.5 Show that, for 1-D random walk on $\mathbb{Z}$ with transition probability matrix

$$p_{i,i+1} = p, p_{i,i-1} = 1 - p$$

for all $i \in \mathbb{Z}$ is transient if $p \neq 1/2$, null recurrent for $p = 1/2$. Such the random walk is said sometimes a *Bernoulli random walk*.

0.6 Show that transition matrix in a Bernoulli radom walk is *double stochastic*, that is $\sum_i p_{ij} = \sum_j p_{ij} = 1$. Furthermore show that $\nu_i = 1$ and $\nu_i = p^n/(1-p)^n$ are invariant. (Asmussen p. 15). Notice that in the transient case an invariant measure are also possible, but they are not unique.

0.7 Show that random walk reflected at 0 with transition probability matrix

$$\begin{aligned} p_{i,i+1} &= p, \quad i \geq 0, \\ p_{i,i-1} &= 1-p, \quad i > 0, \\ p_{0,1} &= 1 \end{aligned}$$

is irreducible, positive recurrent if and only if $0 < p < 1/2$.

0.8 Consider a DTMC with transition probability matrix

$$\begin{aligned} p_{i,i+1} &= p_i, \quad i \geq 0, \\ p_{i,i-1} &= 1 - p_i, \quad i > 0, \\ p_{0,1} &= 1 \end{aligned}$$

Show that the chain is irreducible and positive recurrent if and only if $0 < p_i < 1$ and

$$\sum_{i \geq 1} \frac{p_0 \cdots p_{i-1}}{q_0 \cdots q_{i-1}}$$

where $q_i = 1 - p_i$.

0.9 Consider a transition probability matrix of form

$$\boldsymbol{P}^+ = \begin{pmatrix} 1 - b_0 & b_0 & 0 & 0 & \cdot \\ 1 - b_0 - b_1 & b_1 & b_0 & 0 & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 - \sum_{i=0}^{j} b_j & b_j & b_{j-1} & b_{j-2} & \cdot \\ \vdots & \vdots & \vdots & \vdots \& \vdots & \end{pmatrix}$$

where $b_i \geq 0$ and $\sum_{j=0}^{\infty} b_j = 1$. Show that, if $\lambda \sum_{j=1}^{\infty} j b_j < 1$, then the chain is positive ergodic and with the stationary distribution $\pi_n = (1-\delta)\delta^i$, $i = 01, 2, \ldots$ and δ is the positive solution $\hat{g}(x) = x$, where $\hat{g}(x) = \sum_{j=0}^{\infty} b_j x^j$ is the generating function of $\{b_j\}$.

0.10 Consider the random walk $(Y_n)_{n \in \mathbb{Z}_+}$ on $\mathbb{Z}^2$, where $Y_0 = (0,0)$, $Y_n = \sum_{j=1}^{n} \xi_j$ and $(\xi_j)_{j \in \mathbb{Z}_+}$ are i.i.d.

0.11 Show an example of an explosive CTMC.

0.12 For a CTMC $X(t)$ defined by $(\boldsymbol{Q}, \mu)$ let

$$Z = \sum_{n \geq 0} \frac{1}{q_{Y_n}}$$

Show that $Z = \infty$-$\mathbb{P}_\mu$ if and only if the chain is regular.

0.13 *Competing risks.* Let $E_{ij} \sim \text{Exp}(q_{ij})$ for $j \neq i$ and

$$\begin{aligned} E_i &= \min_{j \neq i} E_{ij} \\ I_i &= \arg\min_{j \neq i} E_{ij} = \sum_{j \neq i} j 1(E_{ij} = E_i) \end{aligned}$$

Show that

$$\mathbb{P}_i(E_i \in dt, I_i = j) = q_{ij} e^{-q_i t}\, dt, \qquad t \geq 0\,.$$

0.14 Show that the following all four cases are possible: $X(t)$ is recurrent (positive recurrent) and Y_n is recurrent (positive recurrent).

0.15 Show that a B&D process on $\boldsymbol{x}_+$ is irreducible iff $\lambda_0, \lambda_1, \ldots > 0$ and $\mu_1, \mu_2, \ldots > 0$.

0.16 (i) Consider a queueing B&D process with $\lambda_n = \lambda$ and $\mu_n = n\mu$. (ii) Show that the process is ergodic for any $\rho = \lambda/\mu > 0$ with the stationary distribution

$$\pi_n = \frac{\rho^n}{n!} \exp(-\rho).$$

(This is so called the $M/M/\infty$ service system).
Show that for the B&D process with

$$\lambda_{n+1} = \frac{\lambda}{n+1}, \quad \mu_n = \mu$$

the stationary distribution is

$$\pi_n = \frac{\eta^n}{n!} \exp(-\rho),$$

where $\rho = \lambda/\mu > 0$.

0.17 Consider a CTMC $\{X(t),\ 0 \leq t \leq T\}$ with transition probability function $(p_{ij}(t))$ and let $X^{\leftarrow} = X(T-t)$, $0 \leq t \leq T$. Show that $X^{\leftarrow}(t)_{0 \leq t \leq T}$ is a nonhomogeneous CTMC with t.p.f.

$$\begin{aligned} p^{\leftarrow}_{ij}(s,t) &= \Pr(X^{\leftarrow}(t) = j | X^{\leftarrow}(s) = i) \\ &= \frac{\Pr(X(T-t) = j)}{\Pr(X(T-s) = i)} p_{ji}(t-s)\,. \end{aligned}$$

0.18 If $(X(t))$ is stationary, then all processes $(X^*_T(t))_{t\in\mathbf{R}}$ have the same distribution.

0.19 Show the procedure in the spirit of the definition of minimal CTMC how to generete a doubly ended stationary CTMC $(X(t))_{t\in\mathbf{R}}$.

0.20 Let $\boldsymbol{Q} = (q_{ij})_{i,j\in\mathbf{E}}$ be an intensity matrix of a reversible process and $\boldsymbol{\pi}$ its stationary distribution. Let $\mathbb{A} \subset \mathbb{E}$ and define a new intensity matrix $\tilde{\boldsymbol{Q}} = (\tilde{q}_{ij})_{i,j\in A}$ by $\tilde{q}_{ij} = q_{ij}$ for $i \neq j$. Show that if $\tilde{\boldsymbol{Q}}$ is irreducible, then it defines a reversible intensity matrix, which admits the stationary distribution $\tilde{\boldsymbol{\pi}}$

$$\tilde{\pi}_i = \frac{\pi_i}{\sum_{j\in A}\pi_j} .$$

0.21 Let $\alpha_i > 0$ $(i = 1,\ldots,m)$. Demonstrate that $\boldsymbol{Q}$ defined by

$$q_{ij} \quad = \quad \begin{cases} \alpha_j & , i \neq j \\ -\sum_{\nu\neq j}\alpha_\nu & , i = j \end{cases} \quad .$$

is reversible and find the stationary distribution $\boldsymbol{\pi}$. Show an example that although the original intensity matrix $\boldsymbol{Q}$ is irreducible, the new $\tilde{\boldsymbol{Q}}$ is not.

0.22 Using Burke theorem argue that for m B&D queues in tandem $(\lambda, (\mu^k_n)_{n\geq 1}(k = 1,\ldots,m)$ the stationary distribution of $\boldsymbol{Q} = (Q_1,\ldots,Q_m)$ $(Q_i(t)$ is the number in the system at time $t)$, has the product form solution $\pi_{\boldsymbol{n}} = \prod_{k=1}^m \pi^{(k)}_{n_k}$, where $\boldsymbol{\pi}^{(k)}$ is the stationary solution for singe B&D queue $(\lambda, (\mu_n)_{n\geq 1})$.

0.1 Let $\alpha(t)$ be a locally integrable function. We say that Π is a Poisson process on $(0,\infty)$ if $\Pi(0,t]$ is a process with independent increments and

$$\mathbb{P}(\Pi(a,b]=k)=\frac{(\int_a^b \alpha(u)\,du)^k}{k!}e^{-\int_a^b \alpha(u)\,du}.$$

Show that $\alpha(t)$ is $\mathcal{F}^\Pi$-intensity function.

0.2 Let $X(t)$ be a B&D process (λ_n,μ_n) such that $\mathbb{E}_{\boldsymbol{\nu}}X(0)<\infty$, where $\boldsymbol{\nu}$ is an initial distribution and define $M_1(t)=A(t)-\int_0^t \lambda_{X(s)}\,ds$ and $M_2(t)=D(t)-\int_0^t 1(X(s)>0)\mu_{X(s)}\,ds$. Show that, if

$$\sum_{n=0}^\infty \lambda_n\int_0^t \mathbb{P}_{\boldsymbol{\nu}}(X(s)=n)\,ds<\infty$$

then M_1 and M_2 are $\mathcal{F}_t^X$–martingales. Show that the above condition is automatically fulflled for B&D queues.

0.3 Let N be a p.p., which admits $\mathcal{F}^N$-stochastic intensity $\lambda(t)$. Show that if X is left continuous and with right hand limits, adapted $\mathcal{F}_t^N$, and

$$M(t)=N(t)-\int_0^t \lambda(s)\,ds,$$

then $\int_0^t X(s)\,dM(t)$ is a martingale. Show that the result does not hold for some cadlag process X.

0.4 Show that in Gordon-Newel networks, the intensity of the flow of jobs transferred from i to j is

$$\lambda(t)=1(Q_i(t)=i)\mu_i p_{ij}\ .$$

Conclude that the throughput is

$$d_{ij}=\lim_{t\to\infty}\frac{1}{t}\int_0^t \lambda(s)\,ds-\text{a.s.}\ .$$

0.5 Let $(\tau_i)_{i\geq 1}$ be a renewal (point) process on $(0,\infty)$, that is such that $(\tau_{i+1}-\tau_i)_{i\geq 0}$ is a sequence of i.i.d. random variables with a common distribution function $F(t)$; $\tau_0=0$. Suppose that F has the density function $f(t)$ and let $r(t) = f(t)/(1-F(t))$ be the corresponding hazard rate function. Let N be the corresponding point process. Show that the $\mathcal{F}_t^N$–stochastic intensity function is

$$\lambda(t) = \sum_{j\geq 0} r(t-\tau_j)1(\tau_j \leq t < \tau_{j+1}) \; .$$

0.1 Let $(\Pi_m(t))_{m\geq 0}$ be a family independent Poisson processes with intensity μ_n, where $\mu_0 = \lambda$ independent of $X(0)$.
(i) Show that

$$X(t) = X(0) + \Pi_0(\lambda t) - \int_0^t \sum_{n\geq 1} 1(X(t-\circ) = n)\, \Pi_n(dt)$$

is a regular queueing B&D process. Find its intensity matrix.
(ii) Show that there is a unique solution of the SDE above.

0.2 For the M/M/1/N find the stationary distribution for the embedded DTMC $(Y_n)_n$.

0.3 Let Q be the number of jobs process in M/M/1 system.

$$M(t) = Q(t) - x - \lambda t + \mu \int_0^t 1(Q(s) > 0) ds$$

is a martingale.

0.4 Suppose that $(\Pi_0(t))_{t\geq 0}$ and $(\Pi_1(t))_{t\geq 0}$ are independent Poisson process with intensities λ and μ respectively. Show that if Q is the number of jobs in the system process in the M/M/1 queue with $Q(0) = i$, then

$$Q(t) \;=_{\mathrm{d}}\; \max(i + Z(t), L(t)),$$

where $Z(t) = \Pi_0(t) - \Pi_1(t)$ and $L(t) = \sup_{0\leq s\leq t} Z(s)$ (Example 7.4 in Asmussen (2003), p. 98).

0.5 Prove that the number of jobs in the system $Q(t)$ process in $M/M/c$ queue is positive recurrent if and only if $\rho < 1$.

0.6 Show that for the $M/M/c$ queue the stationary mean number of jobs in the system

$$\mathbb{E}_\pi Q = \frac{1}{\sigma} \left\{ \sum_{n=1}^{c-1} \frac{\eta^n}{(n-1)!} + \frac{\eta^c}{c!} \left[\frac{\rho}{(1-\rho)^2} + \frac{m}{1-\rho} \right] \right\},$$

the stationary probability that the system is empty

$$\pi_0 = \frac{1}{\sigma},$$

and the stationary probability that all servers are busy

$$\pi_c + \pi_{c+1} + \cdots = \frac{1}{\sigma}\frac{\rho^c}{c!}\frac{c}{c-\rho}.$$

Show that the queue size process in the $M/M/c$ queue is $Q_q(t) = (Q(t)-c)_+$ and find its stationary distribution (that is $\pi^q_n = \lim_{t\to\infty} \Pr(Q_q(t) = n)$).

0.7 In M/M/K/K loss system show that the stationary overflow rate $\beta(K, \lambda, \mu) = \lambda - \lambda^{\mathrm{a}}$, where λ^{a} is the steady-state rate of accepted calls fulfills

$$\beta(K, \lambda, \mu) = \lambda B(K, \rho).$$

0.8 Let Q_1, Q_2 are independent random variables representing the number of customers in the loss system with parameters K_1, K_2, λ_1, λ_2 and μ_1, μ_2 and Q with parameters $K_1 + K_2$, $\lambda_1 + \lambda_2$ and $\mu_1 + \mu_2$ respectively. Show that

$$Q_1 + Q_2 <_r Q,$$

where $<_r$ denotes the monotone likelihood-ratio ordering. Conclude that

$$\beta(K_1 + K_2, \lambda_1 + \lambda_2, \mu_1 + \mu_2) \le \beta(K_1, \lambda_1, \mu_1) + \beta(K_2, \lambda_2, \mu_2).$$

0.9 Suppose there are K terminals and N users where $K < N$. We assume that each user request a terminal with intensity λ and the time utilized by a user is exponentially distributed with parameter μ. Let $\rho = \lambda/\mu$. As usual we suppose that all variables are independent. Show that under a work conserving dysciple, the number of used terminals $(Q(t),\ t \ge 0)$ is a B&D queueing process with state space $\{0, \ldots K\}$ and intensities

$$\lambda_n = (N-n)\lambda \qquad \mu_n = n\mu.$$

Hence

$$\sigma = 1 + \binom{N}{1}\eta + \cdots + \binom{N}{K}\eta^K$$

and

$$\pi_n = \sigma^{-1}\frac{\binom{N}{n}\eta^n}{1 + \binom{N}{1}\eta + \cdots + \binom{N}{K}\eta^K}, \tag{0.1}$$

where $\rho_i = \lambda_i/\mu_i$.

0.10 Show that a vector process $\boldsymbol{X}(t) = (X_1(t), \ldots, X_m(t))$ of independent reversible processes $(X_j(t))$ is reversible.

0.11 [Bramson] Two single server queueing systems are supposed to share a common buffer of size $N-2$. Jobs for the $i = 1,2$ system are arriving at the server according to a Poisson processes Π_i with intensity λ_i and their service requirements $(S_n^{(i)})_n$ are i.i.d. exponentially distributed $\mathrm{Exp}(\mu_i)$. We assume that Π_i, $(S_n^{(i)})_n$ $(i = 1,2)$ are independent. If the buffer is full the arriving jobs are lost. Using the result of Exercise 0.21 and 0.10 show that the stationary distribution of $\boldsymbol{Q} = (Q_1, Q_2)$; Q_i is the number of jobs

$$\pi_{\boldsymbol{n}} = \pi_{00}\rho_1^{n_1}\rho_2^{n_2}, \qquad n_1 + n_2 \leq N\ .$$

where $\rho_i = \lambda_i/\mu_i$ and

$$\pi_{0,0} = (\sum_{0 \leq n_1 + n_2 \leq N} \rho_1^{n_1}\rho_2^{n_2})^{-1}\ .$$

0.12 [Robert, p. 88] Consider the following loss system. The network has 3 nodes, which are vertices of graph $\{(1,2),(1,3)\}$. Jobs of type i arrives according to a Poisson process with rate λ_i and bring their service requirements $S_n^{(i)}$ $(i = 1,2)$. Arrivals and service requirements are independent. Jobs of type 1 arrive at node 1 and occupe a link along route $(1,2)$. Similarly jobs of type 2 arrive at node 3 and occupe a link along route $(3,1),(1,2)$. At route $(1,2)$ there are N_1 links available and at route $(3,1)$ there are N_2 links available. Let $\mathbb{E} = \{\boldsymbol{n} \in \mathbb{Z}_+^2 : n_1 + n_2 \leq N_1, n_2 \leq N_2\}$. Show that

$$\pi_{\boldsymbol{n}} = \pi_{00}\frac{\rho_1^{n_1}}{n_1!}\frac{\rho_2^{n_2}}{n_2}, \boldsymbol{n} \in \mathbb{E},$$

where

$$\pi_{00} = (\sum_{\boldsymbol{n} \in \mathbf{E}} \frac{\rho_1^{n_1}}{n_1!}\frac{\rho_2^{n_2}}{n_2!})^{-1}\ .$$

Notice that

$$\pi_{\boldsymbol{n}} = \mathbb{P}(\Pi^{\rho_1} = n_1, \Pi^{\rho_2} = n_2 | \Pi^{\rho_1} + \Pi^{\rho_2} \leq N_1, \Pi^{\rho_2} \leq N_2)\ .$$

0.13 [Kelly, p. 29] Suppose that the stream of jobs arriving at a two-server queue is Poisson with rate λ and each job brings its service requirement (S_n), where (S_n) are i.i.d. exponentially distributed with paramenter 1. Server 1 works with rate μ_1 and server 2 with rate μ_2. Arrivals and service requirements are independent. If a job arrives to find both the servers free it is allocated to the server who has benn free for the longest time. Show that the queue is not reversible. Find the stationary distribution.

0.14 Show that the number of jobs at nodes process $\boldsymbol{Q}(t)$ in a system of m queues in tandem is not reversible.

0.15 Show that the number of jobs process $\boldsymbol{Q}$ in m queues in tandem is the solution of SDE

$$\boldsymbol{Q}(t) = \boldsymbol{k} + \boldsymbol{Z}(t) + \boldsymbol{L}(t)\,,$$

where $\boldsymbol{Z}(t) = (\pi_0(t) - \pi_1(t), \pi_1(t) - \pi_2(t), \ldots, \pi_{m-1}(t) - \pi_m(t))$ and

$$\boldsymbol{L}(t) = (\int_0^t 1(Q_i(s-) = 0)\, d\pi_i(s))_{i=1,\ldots,m}\,.$$

The process $\boldsymbol{L}(t) = \boldsymbol{0}$ until time τ, where $\tau = \inf\{t : \min_{i=1,\ldots,m} k_i + Z_i(t) = 0$. Note that τ is the (first) *collision time* for the process $\boldsymbol{\xi}(t) = (k_1 + \ldots + k_m + \pi_0(t), k_2 + \ldots + k_m + \pi_1(t), \ldots, \pi_m(t))$.

0.16 Show that the stationary process $\boldsymbol{Q}(t)$ in queues in tandem is not reversible.

0.17 [Asmussen] Show that the time-reversed process $\tilde{\boldsymbol{Q}}(t)$ of $\boldsymbol{Q}(t)$ in Gordon-Newel network is again a Gordon-Newel network with routing matrix $\tilde{\boldsymbol{P}} = (\tilde{p}_{ij})$, where $\bar{\lambda}_i \tilde{p}_{ij} = \bar{\lambda}_j p_{ji}$.

0.18 [Robert] Consider the Gordon-Newel network with a pure ring structure, i.e. $p_{i,i+1} = 1$ for $i = 1, \ldots, m-1$ and $p_{m1} = 1$. Compute the stationary distribution.

0.19 [Robert] Consider the following modification of the Jackson network, wherein server at node i has a speed $\phi_i(n_i)$ when there are n_i customers present in node i. We suppose that $\phi_i(k) > 0$ for all $k \geq 1$ and

$\phi_i(0) = 0$. The new intensity matrix is obtained from the standard one by replacing μ_i by $\mu_i\phi_i(n_i)$. Show that if for all $i = 1, \ldots, m$

$$A_i = 1 + \sum_{n_i=1}^{\infty} \left(\frac{\rho_i^{n_i}}{\prod_{k=1}^{n_i} \phi_i(k)} \right) < \infty$$

where $\rho_i = \bar{\lambda}_i/\mu_i$, then the netowrk is ergodic with stationary distribution

$$\pi_{\boldsymbol{n}} = \prod_{i=1}^{m} \pi_i(n_i),$$

where

$$\pi_i(n_i) = \frac{1}{A_i} \frac{\rho_i^{n_i}}{\prod_{k=1}^{n_i} \phi_i(k)} \ .$$

0.20 Show that in Gordon-Newel networks, the intensity of the flow of jobs transferred from i to j is

$$\lambda(t) = 1(Q_i(t) = i)\mu_i p_{ij} \ .$$

Conclude that the throughput is

$$d_{ij} = \lim_{t\to\infty} \frac{1}{t} \lambda(s)\, ds - \text{a.s.} \ .$$

0.21 Let $(Q(t))_{t\geq 0}$ be the stationary number of jobs in the system. and let $N^{\mathrm{w}}(t)$ be the number of jobs arriving in $(0, t)$ which have to wait;

$$N^{\mathrm{w}}(t) = \int_0^t 1(Q(s - \circ) \geq c)\, d\Pi^{\lambda}(s) \ .$$

Then $(N^{\mathrm{w}}(t))_{t\geq 0}$ is the process with stationary increments. Show that N^{w} is a process with stationary increments.

0.22 Prove that under the Halfin-Whitt regime

$$B(c, \rho) \sim \frac{\phi(\beta)}{\Phi(\beta)\sqrt{\rho}}$$

and find the corresponding asymptotic for $C(c, \rho)$. Hint. Let Π^{ρ} be distributed as Poisson with mean ρ. For the asymptotics of $\mathbb{P}(\Pi^{\rho} = c) = \mathbb{P}(\Pi^{\rho} = \rho + \beta\sqrt{\rho})$ use Stirling formula and for $\Pi^{\rho} \leq c) = \mathbb{P}(\Pi^{\rho} \leq \rho + \beta\sqrt{\rho})$ use the central limit theorem.