O-minimal structures

In the early eighties L. van den Dries noticed that many properties of semialgebraic sets and maps could be derived from a few simple axioms. This observation led to the definition of o-minimality. An infinite first order structure \( \mathcal{M} = (M, \leq, \ldots) \) expanding a linear ordering \((M, \leq)\) is said to be o-minimal iff every subset of \(M\) definable in \(\mathcal{M}\) is a finite union of intervals. Surprisingly, this simple definition turned out to have numerous deep consequences of topological, algebraic and structural character.

The course is intended to serve as an introduction to the subject of o-minimality. The first part of the course will be mainly devoted to discussion of topological properties of sets and functions definable in o-minimal structures and will be mostly based on L. van den Dries’ book *Tame topology and o-minimal structures*. The second part of the course will have more model-theoretic character and will cover several fundamental results published in various papers.

I am going to assume that the students attending the lectures will be familiar with basic notions of model theory. The approximate program of the course is outlined below.

1. Ordered fields, real closed fields and semialgebraic sets and maps.
2. O-minimality – basic notions, facts and examples.
3. Definable unary functions.
5. Dimension and Euler characteristics for definable sets.
6. Point-set topology in o-minimal structures.
7. Differentiation in o-minimal expansions of real closed fields, \(C^n\)-cell decomposition and its consequences.
8. Miller’s dichotomy theorem.
10. Small o-minimal theories and Vaught’s conjecture.
11. Groups and fields definable in o-minimal structures.
12. Definability of types in o-minimal theories.

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