Inner ultrahomogeneous groups

(and definable ultrahomogeneity)

Tomasz Rzepecki

CAS Praha
Uniwersytet Wrocławski

Nesin Köyü,
May 20, 2023

Ultrahomogeneity

- Given a first order structure M, a *partial automorphism* is a partial function $M \to M$ which preserves all atomic formulas and their negations (essentially, preserves the truth value of relations and commutes with functions in the language).
- In particular for groups, a partial automorphism is just a function preserving all equations and their negations.
- We say that a first order structure is *ultrahomogeneous* if every *finite* partial automorphism of M can be extended to an automorphism of M.

Fraïssé classes

A class of finitely generated first order structures in a fixed language L, closed under isomorphism, is a *Fraïssé class* if it has the following properties:

- (essential) countability,
- hereditary property (HP),
- joint embedding property (JEP),
- amalgamation property (AP).

Fact

If M is countable ultrahomogeneous, then its age (class of isomorphism types of f.g. substructures) is a Fraïssé class. Conversely, a Fraïssé class has a well-defined, unique ultrahomogeneous "limit" whose age it is.

Inner ultrahomogeneity

From now on, we consider only (pure) groups.

Definition

We say that a group Γ is *inner ultrahomogeneous* if every finite partial automorphism of Γ extends to an *inner* automorphism of Γ.

Remark

Given ultrahomogeneous Γ, inner ultrahomogeneity is equivalent to a condition about $\mathtt{Age}(\Gamma)$ which we call *inner EPPA*.

Definition

We say that a class $\mathcal{K}$ of groups has *inner EPPA* if for every $K \in \mathcal{K}$ and every finite partial automorphism p of K, there is $L \geq K$ in $\mathcal{K}$ such that p is extended by conjugation by some $\alpha \in L$.

Finite and finitely presentable groups

- The class of finite groups is a Fraïssé class and it has inner EPPA.
- It follows that it has a limit (Hall's universal group) which is inner ultrahomogeneous (all this is essentially due to Phillip Hall).
- The class of finitely generated groups has HP, JEP and AP and inner EPPA, but it is not countable (there are continuum many 2-generated groups).
- The class of finitely presentable groups is countable, has JEP, AP and inner EPPA.
- However, it does not have the hereditary property.
- On the other hand, every f.g. subgroup of a f.p. group is recursively presentable (the converse is also true).
- The class of f.g. recursively presentable groups is Fraïssé with inner EPPA.

Theorem (Hall, Song, Siniora, Rz., many parts probably folklore...)

If Γ is the Hall's universal group or the limit of finitely presentable groups, then:

1. Γ *is simple and divisible,*
2. Γ *is not $\aleph_0$-categorical,*
3. *the formula $xy = yx$ has the independence property,*
4. *the commutator dimension is infinite (the formula $\forall z(xz = zx \to yz = zy)$ has the strict order property); in fact, $C(x) \subseteq C(y)$ if an only if $y \in \langle x \rangle$,*
5. $xy_1 = y_1x \wedge xy_2 \neq y_2x$ *has TP_2,*
6. Γ *has ample generic automorphisms.*

The goal is to understand how these properties can be derived from just inner ultrahomogeneity + some extra assumptions (some are needed: the trivial group and the group with two elements are inner ultrahomogeneous, and they fail all the conclusions).

Other examples

Some other examples of Fraïssé classes of groups with inner EPPA:

- given any countable transitive model $M \models \mathsf{ZFC}$, the class of finitely generated groups in M,
- any countable hereditary class of f.g. groups closed under amalgamated free products is a Fraïssé class, and if it is also closed under (finitary) HNN extensions, it has inner EPPA,
- in particular, we can start with any f.g. group (or countable class of groups) and close it under these operations,
- for example, if we start with a class of torsion-free groups, then as the limit, we obtain a torsion-free inner ultrahomogeneous group,
- it follows that every countable group is a subgroup of a countable inner ultrahomogeneous group.

Conjugacy and divisibility

Fact

If Γ is an inner ultrahomogeneous group, then conjugacy classes in Γ consist of all elements of given order.

Proof.

If $g_1, g_2 \in \Gamma$ have the same order, them $g_1 \mapsto g_2$ is a partial automorphism. □

Fact

If Γ is an ultrahomogeneous group, then an element $g \in \Gamma$ is n-divisible if and only if there is an element of order $n \cdot \mathrm{ord}(g)$.

Proof.

If h is of order $n \cdot \mathrm{ord}(g)$, then h^n is of order $\mathrm{ord}(g)$. Thus, we have $\sigma \in \mathrm{Aut}(\Gamma)$ with $\sigma(h^n) = g$. But then $\sigma(h)^n = g$. □

(Lack of) $\aleph_0$-categoricity

Proposition

If Γ is an infinite inner ultrahomogeneous group, then it is not $\aleph_0$-categorical.

Proof.

Suppose towards contradiction that Γ is $\aleph_0$-categorical. We will show that it has infinite exponent, contradicting categoricity.
Fix any N. By $\aleph_0$-categoricity and Ramsey, there is a non-constant indiscernible sequence $g_0, g_1, g_2, \ldots, g_N$ in Γ. By inner ultrahomogeneity, there is some $h \in \Gamma$ such that for $i < N$, $g_i^h = g_{i+1}$. It follows that the order of h is at least $N + 1$. □

Independence Property

Proposition

Suppose Γ is nontrivial, inner ultrahomogeneous and $\mathsf{Age}(\Gamma)$ is closed under $\times$.
Then the formula $xy = yx$ has the IP.

Proof.

Fix n. By hypothesis, we can find a cyclic group C such that $\langle g_1, \ldots, g_n \rangle = C \times C \times \cdots \times C \leq \Gamma$. Any permutation of the generators is a partial automorphism.
If σ is any such permutation and $h \in \Gamma$ is a corresponding witness to inner ultrahomogeneity, then h commutes with g_j if and only if $g_j \notin \mathsf{supp}(\sigma)$. □

Strict order property and TP_2

Lemma

Suppose Γ is inner ultrahomogeneous and $\mathtt{Age}(\Gamma)$ is closed under $\times$. Then given any $g \in \Gamma$ of order $n < \infty$, then for every $m|n$, $m > 1$, there is $k \in \Gamma$ commuting with g^m but not g.

Proof.

Since $\mathtt{Age}(\Gamma)$ is closed under $\times$, there is some $h \in \Gamma$ of order n, such that $\langle g, h\rangle = \langle g\rangle \times \langle h\rangle$. Let $g' = gh^{n/m}$. Then $(g')^m = g^m$, and by inner ultrahomogeneity, there is some k such that $g^k = g'$, and hence $(g^m)^k = (g')^m = g^m$. □

Lemma

Suppose Γ *is inner ultrahomogeneous and* $\mathtt{Age}(\Gamma)$ *is closed under* $\times$*. Then given any* $g \in \Gamma$ *of order* $n < \infty$*, then for every* $m|n$*,* $m > 1$*, there is* $k \in \Gamma$ *commuting with* g^m *but not* g*.*

Corollary

If, furthermore, Γ *has elements of arbitrarily large finite order, then* Γ *does not have finite commutator dimension (so we have the strict order property).*

Proof.

The hypothesis implies that for every n, there is a sequence $p_1, p_2, \ldots, p_n$ of primes and an element $g \in \Gamma$ of order $p_1 \cdots p_n$. Then commutators of $g, g^{p_1}, g^{p_1 p_2}, g^{p_1 p_2 p_3}, \ldots$ are progressively larger. □

Strict order property and TP_2

Lemma

Suppose Γ is inner ultrahomogeneous and $\mathtt{Age}(\Gamma)$ is closed under $\times$. Then given any $g \in \Gamma$ of order $n < \infty$, then for every $m|n$, $m > 1$, there is $k \in \Gamma$ commuting with g^m but not g.

Corollary

If, furthermore, Γ has elements of arbitrarily large finite order, then Γ does not have finite commutator dimension (so the formula $\forall z(xz = zx \to yz = zy)$ has SOP).

Corollary

If, furthermore, Γ has elements of arbitrary finite order, then the formula $xy_1 = y_1x \wedge xy_2 \neq y_2x$ has TP_2.

Ample generics

Proposition

Suppose Γ *is inner ultrahomogeneous and* $\mathrm{Age}(\Gamma)$ *is closed under* $\times$*. Then the class of (tuples of) partial automorphisms of* Γ*-structures has cofinal amalgamation.*
More precisely, given $G \leq G_1, G_2$ *in* $\mathrm{Age}(\Gamma)$ *and automorphisms* $\sigma, \sigma_1, \sigma_2$ *of the respective groups, with* $\sigma \subseteq \sigma_j$*, we can amalgamate* (G_1, σ_1) *and* (G_2, σ_2) *over* (G, σ)*.*
(Essentially, there is $K \geq G_1, G_2$ *in* $\mathrm{Age}(\Gamma)$ *and* $k \in K$ *such that* $g^k = \sigma_j(g)$ *for* $g \in G_j$*).*

Proof.

We may assume without loss of generality that σ_1, σ_2 are inner automorphisms given by g_1 and g_2.
If the orders of g_1 and g_2 are the same and $\langle g_j \rangle \cap G = \{e\}$, then $\langle G, g_1 \rangle \cong \langle G, g_2 \rangle$. Thus, we can amalgamate G_1 and G_2 over $\langle G, g_1 \rangle$ to K. k is the image of g_1 in the amalgam.
But using the hypothesis, we can force this to be true. □

Ample generics

Proposition

Suppose Γ is inner ultrahomogeneous and $\mathtt{Age}(\Gamma)$ is closed under $\times$. Then the class of (tuples of) partial automorphisms of Γ-structures has cofinal amalgamation.
More precisely, given $G \leq G_1, G_2$ in $\mathtt{Age}(\Gamma)$ and automorphisms $\sigma, \sigma_1, \sigma_2$ of the respective groups, with $\sigma \subseteq \sigma_j$, we can amalgamate (G_1, σ_1) and (G_2, σ_2) over (G, σ).
(Essentially, there is $K \geq G_1, G_2$ in $\mathtt{Age}(\Gamma)$ and $k \in K$ such that $g^k = \sigma_j(g)$ for $g \in G_j$).

Corollary

If Γ is inner ultrahomogeneous and $\mathtt{Age}(\Gamma)$ is closed under $\times$, then Γ has ample generic automorphisms (which are "locally inner").

Definable ultrahomogeneity

- The notion of inner ultrahomogeneity can be generalised.
- For example, we can assume that there is a (quantifier-free?) definable/interpretable group of automorphisms witnessing ultrahomogeneity.
- Or, we can ask that there be uniformly definable/interpretable family (possibly closed under composition) of automorphisms.
- Under suitable assumptions, we can, at least, use these notions to obtain similar non-tameness conclusions.