TWO STARS

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Abstract. We investigate an operation * on the subsets of $\mathcal{P}(\mathbb{R})$. It is connected with the strong measure zero sets as well as the strongly meager sets. We give simple proofs of two theorems of Solecki. The first one says that the family of countable sets is a fixed-point of the operation ** (* applied twice). The second one says that there exists a translation invariant σ-ideal I on $\mathbb{R}$ such that I^* is the family of countable sets.

1. Introduction and notation

We will be investigating subsets of the real line $\mathbb{R}$ and families of such subsets. The family of all countable subsets of $\mathbb{R}$ will be of particular interest, so let us denote it by $\mathcal{C}$. Other important families are the family of sets of Lebesgue's measure zero, which we shall denote by $\mathcal{N}$, and the family of meager sets denoted by $\mathcal{M}$. We will focus on the algebraic structure of the real line. For $A, B \in \mathcal{P}(\mathbb{R})$ write $A + B = \{a + b \colon a \in A,\ b \in B\}$ and $-A = \{-a \colon a \in A\}$.

Definition 1. For $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$ let

$$\mathcal{A}^* = \{B \in \mathcal{P}(\mathbb{R}) \colon \forall A \in \mathcal{A}\ A + B \neq \mathbb{R}\}.$$

The operation * behaves nicely when applied several times. In particular, the following facts hold.

Fact 1. *For $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$ $\mathcal{A} \subseteq A^{**}$.* □

Fact 2. *For $\mathcal{A}, \mathcal{B} \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$ if $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{B}^* \subseteq \mathcal{A}^*$.* □

Fact 3. *For $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$ $\mathcal{A}^{***} = \mathcal{A}^*$.* □

One of the main motivations for investigating the above operation is connected with the family of strongly null sets and the result of Galvin, Mycielski and Solovay [2].

Definition 2. We say that $A \in \mathcal{P}(\mathbb{R})$ is *strongly null* if for every sequence $\langle \varepsilon_n \colon n \in \omega \rangle$ of positive real numbers there exists a sequence $\langle I_n \colon n \in \omega \rangle$ of open intervals such that $|I_n| \leq \varepsilon_n$ for each n and $A \subseteq \bigcup_{n<\omega} I_n$. The family of strongly null sets is denoted by $\mathcal{SN}$.

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Theorem 1 (Galvin, Mycielski, Solovay)**.** $\mathcal{M}^* = \mathcal{SN}$

Strongly null sets are related to the Borel conjecture (BC) which says that $\mathcal{SN} = \mathcal{C}$. By the result of Laver ([3]), BC is independent of ZFC. A measure counterpart of BC, the Dual Borel Conjecture (DBC), says that $\mathcal{N}^* = \mathcal{C}$. It is also independent (Carlson, [1]).

Note however, that if any of these conjectures holds, then the family of countable sets is closed under the operation **. In [4] Seredyński asked if this can be proved directly in ZFC.

Problem 1 (P 1368). Is it true that $\mathcal{C}^{**} = \mathcal{C}$?

The affirmative answer to his question was given by Solecki in [5]. However, his proof is quite sophisticated.

2. $\mathcal{C}$ is closed under **

Let us present another but astonishingly short solution to Problem 1.

Theorem 2. $\mathcal{C}^{**} = \mathcal{C}$.

Proof. By Fact 1, $\mathcal{C}^{**} \supseteq \mathcal{C}$. To show the reverse inclusion take any uncountable $U \subseteq \mathbb{R}$. Showing $U \notin \mathcal{C}^{**}$ amounts to finding $A \in \mathcal{C}^*$ such that

$$U + A = \mathbb{R}.$$

After shrinking U if neccessary, we may assume that $|U| = \omega_1$ and that there exists a Hamel basis $H \supseteq U$ such that $|H \setminus U| = 2^\omega$. In order to construct A enumerate all countable subsets of $\mathbb{R}$ in a sequence $(C_\alpha : \alpha < 2^\omega)$. Now, inductively construct $y_\alpha \in \mathbb{R}$ such that

$$(C_\alpha + y_\alpha - U) \cap (C_\beta + y_\beta - U) = \emptyset$$

for each $\alpha \neq \beta$. This can be done since $\bigcup_{\beta<\alpha} C_\beta + y_\alpha + C_\alpha$ is expressed by only $|\beta| \cdot \aleph_0$ elements of H, so at stage α we may pick y_α as the first unused element of $H \setminus U$. Let

$$A = \mathbb{R} \setminus \bigcup_{\alpha<2^\omega} (C_\alpha + y_\alpha)$$

Claim 1. $A \in \mathcal{C}^*$.

Proof. Any C_α can be translated via y_α outside of A. $\square$

Claim 2. $U + A = \mathbb{R}$.

Proof. Fix $t \in \mathbb{R}$. We need to show that $U + t$ meets A. Suppose, towards a contradiction, that $U + t \subseteq \bigcup_{\alpha<2^\omega} C_\alpha + y_\alpha$. As y_α were chosen so that $C_\alpha + y_\alpha - U$ are disjoint, there is at most one α such that $U + t$ meets $C_\alpha + y_\alpha$ and thus $U + t \subseteq C_\alpha + y_\alpha$ is countable. □

□

3. A translation invariant σ-ideal

In the paper [5] the previous result is established by showing that there exists a translation invariant σ-ideal I such that $\mathcal{C} = I^*$. Under either of BC or DBC this indeed is the case. The family $\mathcal{C}^*$, however, fails to be an ideal, namely it is not closed under finite unions because $\mathcal{C}^*$ contains the ideals of meager and of null sets, and it is well known that the real line is a union of a meager set and a null set. So the result doesn't trivially follow from Theorem 2. Let us present another construction of such an ideal.

Theorem 3. *There exists a translation invariant σ-ideal I on $\mathbb{R}$ such that $I^* = \mathcal{C}$.*

Proof. The group $(\mathbb{R}, +)$ is clearly isomorphic to $\bigoplus_{i<2^\omega} \mathbb{Q}$ (via expressions in a Hamel basis). We will substitute $\mathbb{Z}_2$ for $\mathbb{Q}$ in order to simplify notation. In the original case the proof is analogous. After this substitution, the group is $([\kappa]^{<\omega}, \oplus)$, where $\kappa = 2^\omega$ and $\oplus$ is the symmetric difference. Since we may as well put $\kappa = 2^\omega + \omega_1$, let us carry out the entire construction in the group $([2^\omega + \omega_1]^{<\omega}, \oplus)$ (denoted by G).

Let Λ be the set of all injections of ω_1 into 2^ω. For $s \in G$ and $\lambda \in \Lambda$ define

$$\begin{aligned} ||s|| &= \min\{\alpha \in \omega_1 : (2^\omega + \alpha, 2^\omega + \omega_1) \cap s = \emptyset\} \\ ||s||_\lambda &= \min\{\alpha \in \omega_1 : \lambda[(\alpha, \omega_1)] \cap s = \emptyset\} \end{aligned}$$

For $Z \in \mathrm{NS}_{\omega_1}$ (nonstationary subsets of ω_1) let

$$A_Z = \{s \in G : ||s|| \in Z\}$$

and for $\lambda \in \Lambda$ let

$$B_\lambda = \{s \in G : ||s|| < ||s||_\lambda\}.$$

Now, define I to be the σ-ideal generated via translations and countable unions by the family of sets A_Z ($Z \in \mathrm{NS}_{\omega_1}$) and B_λ ($\lambda \in \Lambda$).

Lemma 1. *I is a proper σ-ideal.*

Proof. For $Z_i \in \mathrm{NS}_{\omega_1}$, $\lambda_i \in \Lambda$ and $s_i, t_i \in G$ $(i < \omega)$ find $\alpha \in \omega_1$ such that $\alpha \notin \bigcup_i Z_i$ and $\alpha > ||s_i||, ||t_i||, ||t_i||_{\lambda_i}$ (for all $i < \omega$). We claim that

$$\{2^\omega + \alpha\} \notin \bigcup_{i<\omega}(A_{Z_i} \oplus s_i) \cup \bigcup_{i<\omega}(B_{\lambda_i} \oplus t_i).$$

Indeed, if $\{2^\omega + \alpha\} \in A_{Z_i} \oplus s_i$, then $\{2^\omega + \alpha\} \oplus s_i \in A_{Z_i}$, contradicting $||\{2^\omega + \alpha\} \oplus s_i|| = \alpha \notin Z_i$. On the other hand, if $\{2^\omega + \alpha\} \in B_{\lambda_i} \oplus t_i$, then $\{2^\omega + \alpha\} \oplus t_i \in B_{\lambda_i}$, contradicting $||\{2^\omega + \alpha\} \oplus t_i|| = \alpha > ||t_i||_{\lambda_i} = ||\{2^\omega + \alpha\} \oplus t_i||_{\lambda_i}$ □

Lemma 2. $I^* = \mathcal{C}$

Proof. Let us take any $S \subseteq G$ of size ω_1 and show there is $C \in I$ such that $C \oplus S = G$. Without loss of generality S is a Δ-system with kernel $\emptyset$.

Case 1°. Uncountably many $s \in S$ meet $[2^\omega, 2^\omega + \omega_1)$.
Since any set of size ω_1 contains a nonstationary subset of size ω_1, we may find such $\mathrm{NS}_{\omega_1} \ni Z \subseteq \{||s|| : s \in S\}$. We claim that $A_Z \oplus S = G$. Indeed, for any $t \in G$ find $s \in S$ such that $||s|| \in Z$ and $||s|| > ||t||$. Then $||s \oplus t|| = ||s|| \in Z$, so $s \oplus t \in A_Z$ and $t \in A_Z \oplus s$.

Case 2°. Uncountably many $s \in S$ miss $[2^\omega, 2^\omega + \omega_1)$.
Find $\lambda \in \Lambda$ such that $\forall \alpha < \omega_1 \exists s \in S\ \lambda(\alpha) \in s$. Now, we claim that $B_\lambda \oplus S = G$. For any $t \in G$ let $s \in S$ be such that $||s||_\lambda > ||t||, ||t||_\lambda$. Then $||s \oplus t||_\lambda = ||s||_\lambda > ||t|| = ||s \oplus t||$ (the last equality holds because $s \cap [2^\omega, 2^\omega + \omega_1) = \emptyset$). In that case $s \oplus t \in B_\lambda$, so $t \in B_\lambda \oplus s$. □

□

References

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